The pure descent statistic on permutations

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Abstract

We introduce a new statistic based on permutation descents which has a distribution given by the Stirling numbers of the first kind, *i.e.*, with the same distribution as for the number of cycles in permutations. We study this statistic on the sets of permutations avoiding one pattern of length three by giving bivariate generating functions. As a consequence, new classes of permutations enumerated by the Motzkin numbers are obtained. Finally, we deduce results about the popularity of the pure descents in all these restricted sets.

Keywords: Stirling number, permutation, descent, Dyck path, popularity.

1 Introduction and notations

Let S_n be the set of permutations of length n, *i.e.*, all one-to-one correspondences from $[n] = \{1, 2, \dots, n\}$ into itself. We represent a permutation $\pi \in S_n$ in one-line notation, $\pi = \pi_1\pi_2 \dots \pi_n$ where $\pi_i = \pi(i)$, $1 \le i \le n$. Moreover, if $\sigma = \sigma(1)\sigma(2)\dots\sigma(n)$ is a length n permutation then the *product* $\sigma \cdot \pi$ is the permutation $\sigma(\pi_1)\sigma(\pi_2)\dots\sigma(\pi_n)$. In S_n, a *k-cycle* $\pi = \langle i_1, i_2, \dots, i_k\rangle$ is a length n permutation verifying $\pi(i_1) = i_2, \pi(i_2) = i_3, \dots, \pi(i_{k-1}) = i_k, \pi(i_k) = i_1$ and $\pi(j) = j$ for $j \in [n]\backslash\{i_1, \dots, i_k\}$. For $1 \le k \le n$, we denote by $C_{n,k}$ the set of all n-length permutations which admit a decomposition in a product of k disjoint cycles. The cardinality of $C_{n,k}$ is given by the signless Stirling numbers of the first kind $c(n, k)$ satisfying the relation:

$$c(n, k) = (n-1)c(n-1, k) + c(n-1, k-1) \tag{1}$$

with the initial conditions $c(n, k) = 0$ if $n \le 0$ or $k \le 0$, except $c(0, 0) = 1$. See for instance [16, 18] and the sequence A132393 in the Sloane's on-line encyclopedia of integer sequences [15]. These numbers are also usually defined by means of the following combinatorial identity (see for instance [8, 9, 16])

$$\sum_{k=1}^{n} c(n,k)x^k = x(x+1)(x+2)\cdots(x+n-1). \tag{2}$$

Also, they enumerate n-length permutations π having k left-to-right maxima, *i.e.*, values $i \in [n]$ such that $\pi_j < \pi_i$ for every $j < i$. Indeed, erasing the parentheses from the *standard representation* of a permutation π in a product of disjoint cycles (each cycle is written with its largest element first and the cycles are in increasing order of their largest element), one constructs a bijection on S_n (see [16]) that transports cycles into left-to-right maxima. See [1, 2] for efficient generating algorithms of the sets of permutations with a given number of cycles or left-to-right maxima.

Surprisingly and to our knowledge, these two linked statistics on permutations are the only examples studied in the literature that interpolate the Stirling numbers of the first kind. In this paper, we introduce a new statistic on permutations (called *pure descent statistic*) which has the same distribution as the number of cycles. Its definition is based on the well known Eulerian statistic of descents in permutations (a descent in π is a value $i \in [n-1]$ such that $\pi_i > \pi_{i+1}$). In Section 2, we define the notion of pure descent and we show how their distribution on permutations realizes an interpolation of the Stirling numbers of the first kind. In Section 3, we focus our study on the sets of permutations avoiding one pattern of length three by giving bivariate generating functions with respect to the length of permutation and the number of pure descents. Finally, we deduce the total number of pure descents in all permutations of these restricted sets, which is usually called the *popularity* [5, 13].

2 Pure descent statistics

Let π be a permutation in S_n. Let us recall that a *descent* in π is a value $i \in [n-1]$ such that $\pi_i > \pi_{i+1}$. In the case where there is no $j < i$ such that $\pi_j \in [\pi_{i+1}, \pi_i]$, we call it a *pure descent*. We denote by $D_{n,k}$ the set of permutations of length n with k pure descents. For instance, in $\pi = 231645$, the descents are 2 and 4, and 4 is the only one pure descent (see Figure 1).

Theorem 1. For $0 \le k \le n-1$, the number of n-length permutations with k pure descents is given by the signless Stirling number of the first kind $c(n, k+1)$.

Proof. Let us define the insertion operator ψ from $S_{n-1} \times [n]$ to S_n by $\psi(\pi, j) = \pi'$ where

$$\pi'_i = \begin{cases} \pi_i & \text{if } \pi_i < j, \\ \pi_i + 1 & \text{if } \pi_i \ge j, \\ j & \text{if } i = n. \end{cases}$$

For instance, if $\pi = 2413$ then we have $\psi(\pi, 3) = 25143$ and $\psi(\pi, 2) = 35142$.

Let $\pi \in D_{n-1,k}$ be a permutation of length $n-1$ with k pure descents. If $j \ne \pi_{n-1}$ then the insertion operator provides a permutation $\pi' = \psi(\pi, j)$ that lies into $D_{n,k}$. Indeed, the

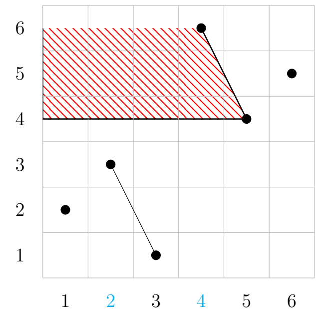

Figure 1: The value 4 is a pure descent in $\pi = 231645$, while 2 is a descent but not a pure descent.

pure descents of π are preserved in π' and no pure descents are created. Setting $d_{n,k} = |D_{n,k}|$, there are $(n-1)d_{n-1,k}$ permutations $\pi' \in S_n$ with k pure descents satisfying $\pi'_n \neq \pi'_{n-1} - 1$.

Now, let $\pi \in D_{n-1,k-1}$ be a permutation of length $n-1$ with $k-1$ pure descents. If $j = \pi_{n-1}$ then the insertion provides $\pi' = \psi(\pi, j)$ that lies into $D_{n,k}$. Indeed, the pure descents of π are preserved in π', and a new pure descent is created between π'_{n-1} and $\pi'_n = \pi'_{n-1} - 1$. Hence, there are $d_{n-1,k-1}$ permutations $\pi' \in S_n$ with k pure descents for which $\pi'_n = \pi'_{n-1} - 1$.

This induces the recurrence relation $d_{n,k} = (n-1)d_{n-1,k} + d_{n-1,k-1}$ for $n, k \geq 0$ with the initial conditions $d_{n,k} = 0$ if $n \leq 0$ or $k < 0$, except $d_{0,-1} = 1$. Using relation (1), we obtain $d_{n,k} = c(n, k+1)$. $\square$

Now, we define recursively a bijection ϕ on S_n that transports the number of pure descents into the number of cycles. Let π be a permutation in S_n and σ the permutation in S_{n-1} such that $\psi(\sigma, \pi_n) = \pi$, *i.e.*, σ is obtained from π by deleting the element π_n on the right, and after a normalization in S_{n-1}. Then we set $\phi(1) = 1$ and

$$\phi(\pi) = \begin{cases} \phi(\sigma) \cdot \langle n, n \rangle & \text{if } \pi_n = \pi_{n-1} - 1, \\ \phi(\sigma) \cdot \langle \pi_n, n \rangle & \text{if } \pi_n < \pi_{n-1} - 1, \\ \phi(\sigma) \cdot \langle \pi_n - 1, n \rangle & \text{if } \pi_n > \pi_{n-1}. \end{cases}$$

For instance, if $\pi = 2731645 \in D_{7,2}$ then we have $\sigma = 263154$, $\pi_7 = 5$ and $\phi(\pi) = \langle 1, 2 \rangle \cdot \langle 3, 3 \rangle \cdot \langle 1, 4 \rangle \cdot \langle 3, 5 \rangle \cdot \langle 6, 6 \rangle \cdot \langle 4, 7 \rangle = \langle 1, 4, 7, 2 \rangle \cdot \langle 3, 5 \rangle \cdot \langle 6, 6 \rangle = 4157362$ has a decomposition in three disjoint cycles.

Theorem 2. The map ϕ defined above is a bijection on S_n. Moreover, for $k \geq 0$, $\phi(\pi)$ has $k+1$ cycles if and only if π has k pure descents.

Proof. Due to the recursive definition of ϕ, it is straightforward to see that ϕ is injective and surjective. Furthermore, $\phi(\pi)$ consists of a product of 2-cycles $\Pi_{i=1}^{n} \langle p_i, i \rangle$ where $p_1 p_2 \ldots p_n$ appears in the literature as the transposition array of $\phi(\pi)$ (see for instance [3]). Lemma 1 in [3] proves that the number of cycles of $\phi(\pi)$ is the number of indices i, $1 \leq i \leq n$, such that $p_i = i$. According to the recursive definition of $\phi(\pi)$, the number of these indices minus one (we do not consider $i = 1$) is exactly the number of pure descents in π; indeed, if $i \neq 1$ then $p_i = i$ means that after the ith step of the induction, the current permutation $\sigma = \sigma_1\sigma_2 \ldots \sigma_{n-1}\sigma_n$ satisfies $\phi(\sigma) = \langle p_1, 1\rangle \cdot \langle p_2, 2\rangle \cdots \langle p_i, i\rangle$ with $p_i = i$. Using the definition of ϕ, σ is such that $\sigma_i = \sigma_{i-1} - 1$, which implies that $i - 1$ is a pure descent in π since π has no value π_j, $j < i$, such that $\pi_i < \pi_j < \pi_{i-1}$. All these arguments are reversible, so the converse is also available. $\square$

An immediate consequence of these last two results is the following corollary about the *popularity* [5] of the pure descents among S_n, *i.e.*, the total number of pure descents in all permutations of length n.

Corollary 1. The popularity of the pure descents among the set S_n is given by the generalized Stirling number (see A001705 in [15])

$$n! \cdot (H_n - 1)$$

where $H_n = \sum_{k=1}^{n} \frac{1}{k}$ is the n-th harmonic number.

Proof. Using Theorem 1, the total number of pure descents in all permutations of length n is given by $a(n) = \sum_{k=0}^{n-1} k \cdot d(n, k)$. Since $d(n, k) = c(n, k+1)$, we deduce $a(n) = \sum_{k=1}^{n} (k-1) \cdot c(n, k)$, and $a(n) = \sum_{k=1}^{n} k \cdot c(n, k) - \sum_{k=1}^{n} c(n, k)$. The first sum equals the total number of cycles in all permutations of length n, it is enumerated by $n!H_n$ (see A000254 in [15]); the second sum is $n!$. We obtain the result. $\square$

Notice that the total number of pure descents in all permutations A001705 equals the total number of cycles of length at least two in all permutations, while the distributions of these two statistics are different.

3 Pure descents in permutations avoiding one pattern of length three

A permutation π of length k is a *pattern* of a permutation $\sigma \in S_n$ if there is a subsequence of σ which is order-isomorphic to π, *i.e.*, a subsequence $\sigma_{i_1} \ldots \sigma_{i_k}$ of σ with $1 \leq i_1 < \cdots < i_k \leq n$ and such that $\sigma_{i_\ell} < \sigma_{i_m}$ whenever $\pi_\ell < \pi_m$. A permutation σ that does not contain π as a pattern is said to *avoid* π. For example, $\sigma = 2413$ avoids the pattern 123. We denote by $S_n(\pi)$ the set of permutations of length n avoiding the pattern π. See for instance [6, 11, 14, 17].

In this section, we provide bivariate generating functions for the distributions of the pure descents on the sets of permutations avoiding one pattern of length three. Table 1 focuses on the cases of permutations that do not contain any pure descent. Notice that Proposition 2.1 in [10], which enumerates permutations avoiding the pattern 213 and the generalized pattern $\bar{2}{-}31$, is a particular case of our result for permutations having no pure descents and avoiding the pattern 213.

Pattern	Sequence	Sloane	$a_n, 1 \leq n \leq 11$
$\{\}$	$(n-1)!$	A000142	$1, 1, 2, 6, 24, 120, 720, \ldots$
$\{231\}$	1	A000124	$1, 1, 1, \ldots$
$\{123\}$		A007477	$1, 1, 1, 2, 3, 6, 11, 22, 44, 90, 187$
$\{132\}, \{213\}, \{312\}$	Motzkin	A001006	$1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188$
$\{321\}$		A035929	$1, 1, 2, 6, 19, 61, 200, 670, 2286, 7918, 27770$

Table 1: Number of permutations with no pure descents and avoiding one pattern of length three.

3.1 The pattern $\alpha \in \{132, 213, 312\}$

The following theorem proves that the pure descent statistic has the same distribution on the sets $S_n(\alpha)$ of permutations avoiding a pattern $\alpha \in \{132, 213, 312\}$.

Theorem 3. Let $A(x, y) = \sum_{n \geq 0, k \geq 0} a_{n,k} x^n y^k$ be the bivariate generating function where the coefficient of $x^n y^k$ is the number $a_{n,k}$ of permutations of length n with k pure descents and avoiding the pattern α, $\alpha \in \{132, 213, 312\}$. Then, we have

$$A(x, y) = \frac{1 + x - xy - \sqrt{x^2y^2 + 2\,x^2y - 3\,x^2 - 2\,xy - 2\,x + 1}}{2x}.$$

For instance, the first terms of $A(x, y)$ are $1 + x + x^2 + 2\,x^3 + 4\,x^4 + 9\,x^5 + 21\,x^6 + 51\,x^7 + x^2y + 2\,x^3y + 6\,x^4y + 16\,x^5y + 45\,x^6y + x^3y^2 + 3\,x^4y^2 + 12\,x^5y^2 + x^4y^3$ (see Table 2).

Proof. Case $\alpha = 132$: let π be a permutation in $S_n(132)$. If π is not empty, then it can be written $\pi = \sigma n \gamma$ where $\gamma \in S_k(132)$ for some k, $0 \leq k \leq n-1$, and σ is obtained from a permutation in $S_{n-k-1}(132)$ by adding k on all these entries. We distinguish two cases: (i) σ is empty, and (ii) σ is not empty.

For the case (i), the permutation π is of the form $\pi = n\gamma$ where $\gamma \in S_{n-1}(132)$; if γ is not empty then γ has one pure descent less than π (π contains the pure descents of γ and the pure descent created by n at the first position). Hence, the generating function for these permutations is given by $x + xy(A(x, y) - 1)$.

For the case (ii), the entry n does not create any pure descent in π. Thus, the generating function for these permutations is given by $x(A(x, y) - 1)A(x, y)$.

Combining the two cases, the following functional equation provides the result

$$A(x, y) = 1 + x + xy(A(x, y) - 1) + x(A(x, y) - 1)A(x, y).$$

Case $\alpha = 312$: let π be a permutation in $S_n(312)$. If π is not empty, then it can be written $\pi = \sigma 1 \gamma$ where $\sigma 1 \in S_k(312)$ for some k, $1 \leq k \leq n$, and γ is obtained from a permutation in $S_{n-k}(312)$ by adding k on all these entries. We distinguish two cases: (i) γ is empty, and (ii) γ is not empty. Let $A_1(x, y)$ (resp. $A_2(x, y)$) be the generating function for the set of permutations π satisfying (i) (resp. (ii)). Obviously, we have $A(x, y) = 1 + A_1(x, y) + A_2(x, y)$.

For the case (i), the permutation π is of the form $\pi = \sigma 1$, where σ is obtained from a permutation in $S_{n-1}(312)$ by adding 1 to all these entries. If σ belongs to the case (i), then σ has one pure descent less than π (π contains the pure descents of σ and the pure descent created by the last value of σ and 1); otherwise, σ and π have the same number of pure descents. Hence, the generating function for these permutations satisfies the functional equation $A_1(x,y) = xyA_1(x,y) + x(A_2(x,y)+1)$.

For the case (ii), $\sigma 1$ satisfies the case (i), and γ can satisfy the two cases. Thus, the generating function for these permutations satisfies the functional equation $A_2(x,y) = A_1(x,y)(A(x,y)-1)$.

Combining the two cases, we obtain the following system of functional equations:

$$\begin{cases} A_1(x,y) &= xyA_1(x,y) + x(A_2(x,y)+1) \\ A_2(x,y) &= A_1(x,y)(A_1(x,y)+A_2(x,y)). \end{cases}$$

Thus, we obtain $A_1 = \frac{1+x-xy-\sqrt{x^2y^2+2\,x^2y-3\,x^2-2\,xy-2\,x+1}}{2(1+x-xy)}$, and $A_2 = \frac{1-x-2xy-2x^2+x^2y+x^2y^2+(xy-1)\sqrt{x^2y^2+2\,x^2y-3\,x^2-2\,xy-2\,x+1}}{2x(1+x-xy)}$ which completes the proof.

Case $\alpha = 213$: let π be a permutation in $S_n(213)$. If π is not empty, it can be written $\pi = \sigma 1 \gamma$ where $1\gamma \in S_k(213)$ for some k, $1 \leq k \leq n$, and σ is obtained from a permutation in $S_{n-k}(213)$ by adding k on all these entries. We distinguish two cases: (i) γ is empty and (ii) otherwise. Let $A_1(x,y)$ (resp. $A_2(x,y)$) be the generating function for the set of permutations π satisfying (i) (resp. (ii)). Obviously, we have $A(x,y) = 1+A_1(x,y)+A_2(x,y)$.

For the case (i), the permutation π is of the form $\pi = \sigma 1$, where σ is obtained from a permutation in $S_{n-1}(213)$ by adding 1 to all these entries. If σ belongs to the case (i), then σ has one pure descent less than π (π contains the pure descents of σ and the pure descent created by the last value of σ and 1); otherwise, σ and π have the same number of pure descents. Hence, the generating function for these permutations satisfies the functional equation $A_1(x,y) = xyA_1(x,y) + x(A_2(x,y)+1)$.

For the case (ii), $\sigma 1$ satisfies the case (i), and γ can satisfy the two cases. Thus, the generating function for these permutations satisfies the functional equation $A_2(x,y) = A_1(x,y)(A(x,y)-1)$. We conclude *mutatis mutandis* as for the pattern 312. $\square$

$k \backslash n$	1	2	3	4	5	6	7	8
0	1	1	2	4	9	21	51	127
1		1	2	6	16	45	126	357
2			1	3	12	40	135	441
3				1	4	20	80	315
4					1	5	30	140
5						1	6	42
6							1	7

Table 2: Coefficients $a_{n,k}$ for $1 \leq n \leq 8$ and $0 \leq k \leq 6$.

Corollary 2. For $\alpha \in \{132, 213, 312\}$ and $n \geq 1$, the permutations in $S_n(\alpha)$ with no pure descent are enumerated by the Motzkin numbers (see A001006 in [15]).

Proof. We obtain the result by calculating $A(x,0) = \frac{1+x-\sqrt{-3\,x^2-2\,x+1}}{2x}$, which is the generating function for the Motzkin numbers. $\square$

Corollary 3. For $\alpha \in \{132, 213, 312\}$ and $n \geq 1$, the popularity of the pure descents among the set $S_n(\alpha)$ is given by the binomial coefficient $\binom{2n-2}{n-2}$ (see A001791 in [15]).

Proof. We need to calculate the numbers $\sum\limits_{k=0}^{n-1} k.a_{n,k}$ for $n \geq 0$, which are the coefficients of x^n in $\frac{\partial A(x,y)}{\partial y}\rfloor_{y=1}$. A simple calculation provides $\frac{\partial A(x,y)}{\partial y}\rfloor_{y=1} = \frac{1-2x-\sqrt{1-4x}}{2\sqrt{1-4x}}$ which induces the result. $\square$

3.2 The pattern 231

Since any descent in a permutation avoiding 231 is also a pure descent, the set of permutations of length n with no pure descents and avoiding the pattern 231 is clearly reduced to the identity permutation $12\ldots n$.

Theorem 4. Let $B(x,y) = \sum_{n\geq 0, k\geq 0} b_{n,k} x^n y^k$ be the bivariate generating function where the coefficient of $x^n y^k$ is the number $b_{n,k}$ of permutations of length n with k pure descents and avoiding the pattern 231. Then, we have

$$B(x,y) = \frac{1 - x + xy - \sqrt{1 - 2x - 2xy + x^2 - 2x^2y + x^2y^2}}{2xy}.$$

For instance, the first terms of $B(x,y)$ are $1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^2y + 3\,x^3y + 6\,x^4y + 10\,x^5y + 15\,x^6y + x^3y^2 + 6\,x^4y^2 + 20\,x^5y^2 + x^4y^3$ (see the Narayana numbers A001263 in [15]).

Proof. Let π be a permutation in $S_n(231)$. If π is not empty, then it can be written $\pi = \sigma n \gamma$ where $\sigma \in S_k(231)$ for some k, $0 \leq k \leq n-1$, and γ is obtained from a permutation in $S_{n-k-1}(231)$ by adding k on all these entries.

In the case where γ is empty, the generating function for these permutations is given by $xB(x,y)$ (π and σ have the same number of pure descents); otherwise, the generating function is given by $xyB(x,y)(B(x,y)-1)$ (π contains the pure descents of σ and the pure descent created by n and the first value of γ).

Hence we deduce the functional equation:

$$B(x,y) = 1 + xB(x,y) + xyB(x,y)(B(x,y)-1),$$

and a simple calculation gives the result. $\square$

Corollary 4. The popularity of the pure descents among the set $S_n(231)$, $n \geq 1$, is given by the binomial coefficient $\binom{2n-1}{n-2}$ (see A002054 in [15]).

Proof. Using Theorem 4, the result is given by the coefficient of x^n in $\frac{\partial B(x,y)}{\partial y}\rfloor_{y=1} = \frac{1-3x-(1-x)\sqrt{1-4x}}{2x\sqrt{1-4x}}$. $\square$

3.3 The pattern $\alpha \in \{321, 123\}$

In this section, we investigate the distribution of pure descents on the sets $S_n(\alpha)$ for $\alpha \in \{321, 123\}$, but we proceed differently than we did in the subsections 3.1 and 3.2. Using classical bijections between $S_n(123)$ (or $S_n(321)$) and Dyck paths, the method consists in translating the pure descents on permutations into some specifical patterns on Dyck paths on which we study the distributions.

A *Dyck path* of semilength n, $n \geq 0$, is a lattice path starting at $(0,0)$, ending at $(2n,0)$, and never going below the x-axis, consisting of up steps $U = (1,1)$ and down steps $D = (1,-1)$. Let $\mathcal{P}_n$ be the set of all Dyck paths of semilength n. The *first return decomposition* of a Dyck path P is the unique decomposition $P = UQDR$ where Q and R are two (possibly empty) Dyck paths.

Given a permutation $\pi \in S_n(321)$, we consider the path on the graphical representation of π with up and right steps along the edges of the squares that goes from lower-left corner to the upper-right corner and leaving all the points (i, π_i), $i \in [n]$, to the right and remaining always as close to the diagonal $y = x$ as possible (the path can possibly reach the diagonal but never crosses it). See Figure 2 for an example of this construction. Let us define the Dyck path of length $2n$ (called Dyck path associated with π) obtained from this lattice path by reading an up-step U every time the path moves up, and a down-step D every time the path moves to the right. This already known construction (see for instance [12]) induces a bijection χ from $S_n(321)$ to $\mathcal{P}_n$. The following lemma shows how this bijection translates pure descents of permutations into some particular configurations of Dyck paths.

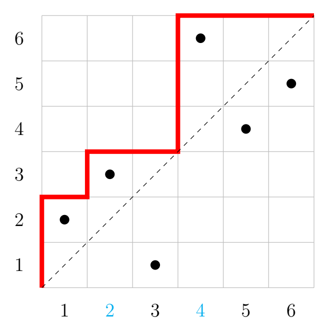

Figure 2: The Dyck path $\chi(\pi) = UUDUDDUUUDDD$ associated to the permutation $\pi = 231645 \in S_6(321)$.

Lemma 1. Let π be a permutation in $S_n(321)$. Then, π has a pure descent in position d, $d \geq 1$, if and only if its associated Dyck path $\chi(\pi)$ is of the form $\gamma U^k DD\delta$, $k \geq 2$, where γ is a Dyck path of semilength $d-1 \geq 0$ and δ is a word in $\{U, D\}^*$.

Proof. Let π be a permutation in $S_n(321)$ and let d be a pure descent in π. Let $\lambda_1 < \lambda_2 < \cdots < \lambda_\ell$ be the left-to-right maxima of π, *i.e.*, the values π_i such that $\pi_j < \pi_i$ for $j < i$. So, we can write $\pi = \lambda_1 w_1 \ldots \lambda_\ell w_\ell$ where w_j are some words possibly empty so that the word $w_1 \ldots w_\ell$ is increasing.

So, the descents in π occur necessarily between λ_j and the first letter of w_j whenever w_j is not empty. Thus, the descents in π are in one to one correspondence with the occurrences of UDD in $\chi(\pi)$. Since π avoids 321, all integers j such that $j < \pi_{d+1}$ must be placed before π_d, and every other integer must stay on the right of π_{d+1}, since d is a pure descent. This implies that $\gamma = \chi(\pi_1 \dots \pi_{d-1})$ is a Dyck path and $\chi(\pi)$ is of the form $\gamma U^k DD\delta$ for some $k \geq 2$ and δ is a word in $\{U, D\}^*$. See Figure 3 for an illustration of this proof. ☐

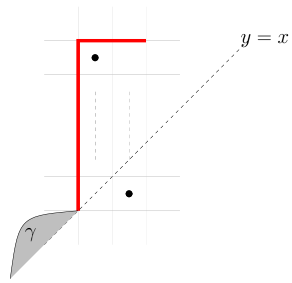

Figure 3: The configuration $\gamma U^k DD$, $k \geq 2$, for a pure descent in $S_n(321)$.

Theorem 5. Let $F(x, y) = \sum_{n\geq 0, k\geq 0} f_{n,k} x^n y^k$ be the bivariate generating function where the coefficient of $x^n y^k$ is the number $f_{n,k}$ of permutations of length n with k pure descents and avoiding the pattern 321. Then, we have

$$F(x, y) = \frac{1 - \sqrt{1 - 4x}}{3x - 2x^2 + 2x^2 y - xy + x(y - 1)\sqrt{1 - 4x}}.$$

For instance, the first terms of $F(x, y)$ are $1 + x + x^2 + 2\,x^3 + 6\,x^4 + 19\,x^5 + 61\,x^6 + 200\,x^7 + x^2 y + 3\,x^3 y + 7\,x^4 y + 18\,x^5 y + 53\,x^6 y + x^4 y^2 + 5\,x^5 y^2$.

Proof. Using Lemma 1, we need to provide the bivariate generating function for Dyck paths of semilength n with respect to the number of different decompositions of the form $\gamma U^k DD\delta$, $k \geq 2$, where γ is a Dyck path and δ is a word in $\{U, D\}^*$. For short, the occurrence of $U^k DD$ in this decomposition will be called a *crochet*.

Now, we consider the first return decomposition of a Dyck path $P = UQDR$ where Q and R are two Dyck paths.

We distinguish two cases: (i) P starts with an occurrence of $U^k DU$ for some $k \geq 1$, or $P = UD$; and (ii) otherwise. Let $F_1(x, y)$ (resp. $F_2(x, y)$) be the bivariate generating function for the set of permutations π satisfying (i) (resp. (ii)). Obviously, we have $F(x, y) = F_1(x, y) + F_2(x, y)$.

In the first case, the Dyck paths $P = UQDR$ and R have the same number of crochets (the crochets of Q do not appear in P). Hence, the generating function $F_1(x, y)$ for these paths satisfies the functional equation $F_1(x, y) = x(F_1(x, 1) - x + 1)F(x, y)$.

In the second case, P is either empty, or P is a Dyck path starting with a crochet $U^k DD$ for some $k \geq 2$. Then, for the non trivial case, $P = UQDR$ has one pure descent more than

R (a crochet is created at the beginning of the path). Hence, the generating function $F_2(x, y)$ for these paths satisfies the functional equation $F_2(x, y) = 1 + xy(F_2(x, 1) - 1 + x)F(x, y)$.

So, we obtain the functional equation:

$$F(x, y) = 1 + x(F_1(x, 1) - x + 1)F(x, y) + xy(F_2(x, 1) - 1 + x)F(x, y), \tag{3}$$

where $F_1(x, 1) + F_2(x, 1) = C(x) = \frac{1-\sqrt{1-4x}}{2x}$ is the Catalan generating function for the sets $S_n(321)$, $n \geq 0$.

By fixing $y = 1$ in (3), we obtain $F_1(x, 1) = \frac{(1-x)(1-2x-\sqrt{1-4x})}{2x}$ and $F_2(x, 1) = \frac{2-3x+(x-2)\sqrt{1-4x}}{1-\sqrt{1-4x}}$ which allows to entirely determine the generating function $F(x, y)$. $\square$

Corollary 5. The popularity of the pure descents among the set $S_n(321)$, $n \geq 1$, is given by the generating function

$$\frac{1 - 3x + (x - 1)\sqrt{1 - 4x}}{2x}$$

(see A000245 in [15]).

Proof. Using Theorem 5, the result is given by $\frac{\partial F(x,y)}{\partial y}\rfloor_{y=1} = \frac{1-3x+(x-1)\sqrt{1-4x}}{2x}$. $\square$

Finally, we study the distribution of the pure descents on the sets $S_n(123)$. From a permutation $\pi \in S_n(123)$, we consider the path on the graphical representation of π with right and down steps along the edges of the squares that goes from upper-left corner to the lower-right corner and leaving all the points (i, π_i), $i \in [n]$, to the left and remaining always as close to the diagonal $y = n - x + 1$ as possible (the path can possibly reach the diagonal but never crosses it). See Figure 4 for an example of this construction. Let us define the Dyck path of length $2n$ (called Dyck path associated with π) obtained from this lattice path by reading an up-step U every time the path moves right, and a down-step D every time the path moves down. This construction induces a bijection Ω from $S_n(123)$ to $\mathcal{P}_n$. The following lemma shows how this bijection translates pure descents of permutations into some particular configurations of Dyck paths. Notice that Lemma 2 is a refinement for pure descents of the Proposition 1 in [4].

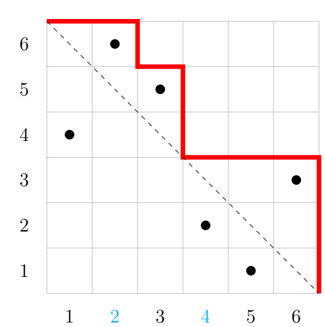

Figure 4: The Dyck path $\Omega(\pi) = UUDUDDUUUDDD$ associated to the permutation $\pi = 465213 \in S_6(123)$.

Lemma 2. Let π be a permutation in $S_n(123)$. Then, π has a pure descent in position d, $d \geq 1$, if and only if its associated Dyck path $\Omega(\pi)$ has one of the three following forms:

(a) $\gamma UDUD\delta$, where γ and δ are two words in $\{U,D\}^*$ and γ contains $d-1$ steps U,

(b) $\gamma UUU\delta$, where γ and δ are two words in $\{U,D\}^*$ and γ contains $d-1$ steps U,

(c) $\gamma UDU\delta$, where γ is a Dyck path of semilength $d-1 \geq 0$, and $U\delta$ is a Dyck path of semilength at least two.

Proof. Using Proposition 1 in [4], Barnabei et al. prove that the descents of a permutation $\pi \in S_n(123)$ are translated by Ω, modulo a symmetry, into the occurrences of UUU and DU in a Dyck path and vice versa.

Now let us study what happens whenever the descent is pure. Let $\lambda_1 > \lambda_2 > \cdots > \lambda_\ell$ be the right-to-left maxima of π, *i.e.*, the values π_i such that $\pi_i > \pi_j$ for $i < j$. So, we can write $\pi = w_1\lambda_1 \ldots w_\ell\lambda_\ell$ where w_j are some words possibly empty so that the word $w_1 \ldots w_\ell$ is decreasing.

The pure descents appear in the three following configurations (see Figure 5):

(C1) between two consecutive right-to-left maxima with consecutive values,

(C2) between two consecutive letters of a word w_i,

(C3) between the right-to-left maximum λ_i and the first letter x of w_{i+1} (whenever w_{i+1} is non-empty), such that there does not exist any value π_j lying in $[x, \lambda_i]$ on the left of λ_i.

Translating configurations (C1) and (C2) in terms of Dyck paths using the map Ω, we obtain easily the following correspondences:

- a pure descent satisfying (C1) is mapped into an occurrence of $UDUD$,
- a pure descent satisfying (C2) is mapped into an occurrence of UUU.

For the case (C3), since a pure descent d is obviously a descent, it is mapped with an occurrence of $UDU\delta$ where $U\delta$ is a Dyck path.

Moreover, if d is a pure descent, then there is no $j < d$ such that $\pi_{d+1} < \pi_j < \pi_d$. As π_d is a right-to-left maximum, we have $\pi_d \geq d$; since π_{d+1} is not right-to-left maximum there is $\pi_k > \pi_{d+1}$ for $k > d+1$, so the avoidance of 123 implies that all values on the left of π_{d+1} are greater than π_{d+1}. Finally, for all $j < d$, we have $\pi_j > \pi_d \geq d$, which implies that $\pi_d = d$. Hence, $\pi_1 \ldots \pi_{d-1}$ is obtained from a permutation of $S_{d-1}(123)$ by adding d to all its values. This implies that $\Omega(\pi_1 \ldots \pi_{d-1})$ is a Dyck path and $\Omega(\pi)$ is of the form $\gamma UDU\delta$ where γ is a Dyck path of semilength $d-1$ and $U\delta$ is also a Dyck path. Finally, the fact that π_{d+1} is not a right-to-left maximum ensures that this case is not the same as (C1) and thus, we necessarily have $U\delta$ is of semilength at least two. $\square$

Theorem 6. Let $G(x,y) = \sum_{n\geq 0, k\geq 0} g_{n,k}x^n y^k$ be the bivariate generating function where the coefficient of $x^n y^k$ is the number $g_{n,k}$ of permutations of length n with k pure descents and avoiding the pattern 123. Then, we have

$$G(x,y) = \frac{x^3y^3 - 2x^3y^2 + x^3y - 3x^2y^2 + 3x^2y + 3xy - x - 1 + (xy - x - 1)R(x,y)}{x^3y^3 - x^3y^2 - 3x^2y^2 + x^2y + 2x^2 + 3xy - 1 + (xy-1)R(x,y)}$$

where

$$R(x,y) = \sqrt{x^4y^4 - 2x^4y^3 + x^4y^2 - 4x^3y^3 + 4x^3y^2 + 4x^3y + 6x^2y^2 - 4x^3 - 2x^2y - 4x^2 - 4xy + 1}.$$

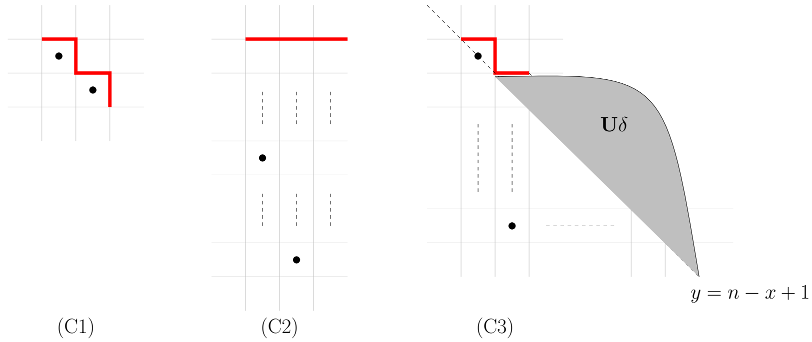

Figure 5: The three configurations (C1), (C2) and (C3) for a pure descent in $S_n(123)$.

For instance, the first terms of $G(x,y)$ are $1 + x + x^2 + x^3 + 2\,x^4 + 3\,x^5 + 6\,x^6 + 11\,x^7 + x^2y + 3\,x^3y + 5\,x^4y + 13\,x^5y + 26\,x^6y + x^3y^2 + 6\,x^4y^2 + 15\,x^5y^2 + x^4y^3$.

Proof. Using Lemma 2, we need to provide the bivariate generating function for the number of Dyck paths of semilength n with respect to the number of configurations (C1), (C2) and (C3).

We distinguish two cases: (i) the Dyck path starts with UD, and (ii) otherwise.

Let $H(x,y,z)$ (resp. $H^{UD}(x,y,z)$, $H^{UU}(x,y,z)$) be the trivariate generating function where the coefficient of $x^ny^kz^\ell$ is the number of Dyck paths of semilength n (resp. starting with UD, resp. starting with UU), having k configurations (C1) or (C2) and ℓ configurations (C3). Obviously, we have $H(x,y,z) = 1 + H^{UD}(x,y,z) + H^{UU}(x,y,z)$.

Using the first return decomposition, we obtain the system of functional equations (4):

$$\begin{cases} H(x,y,z) & = 1 + H^{UD}(x,y,z) + H^{UU}(x,y,z) \\ H^{UD}(x,y,z) & = x + xzH^{UU}(x,y,z) + xyH^{UD}(x,y,z) \\ H^{UU}(x,y,z) & = xyH^{UU}(x,y,1)H(x,y,z) + xH^{UD}(x,y,1)H(x,y,z). \end{cases} \tag{4}$$

Indeed, let P be a non empty Dyck path having its first return decomposition $P = UQDR$ where Q and R are two Dyck paths. When Q is empty (P starts with UD): if R starts with UD, then P and R have the same number of configurations (C2) and (C3), and P has one configuration (C1) more than R. Hence the generating function for this subcase is $xyH^{UD}(x,y,z)$; if R starts with UU, then P and R have the same number of configurations (C1) and (C2), and P has one configuration (C3) more than R. Hence the generating function for this subcase is $xzH^{UU}(x,y,z)$.

When Q is not empty, the juxtaposition with R does not create any new configuration (C1), (C2) or (C3): if Q starts with UU, then P has one configuration (C2) more than the total number of configurations (C2) in Q and R; the configurations (C3) of Q do not appear in P. Hence the generating function for this subcase is $xyH^{UU}(x,y,1)H(x,y,z)$. In the same way, we obtain the generating function $xH^{UD}(x,y,1)H(x,y,z)$ whenever Q starts with UD.

Resolving this system of equations (4), after fixing $z = y$, we obtain the expected bivariate generating function $G(x,y) = H(x,y,y)$. $\square$

Corollary 6. The popularity of the pure descents among the set $S_n(123)$, $n \geq 1$, is given by the generating function

$$\frac{4x^2 - 9x + 2 + (5x - 2)\sqrt{1 - 4x}}{8x - 2}$$

(see A129869 in [15]).

Proof. Using Theorem 6, the result is given by $\frac{\partial G(x,y)}{\partial y}\rfloor_{y=1} = \frac{4x^2-9x+2+(5x-2)\sqrt{1-4x}}{8x-2}$. ☐

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