APPROXIMATE INFERENCE
FOR
DETERMINANTAL POINT PROCESSES

Jennifer Gillenwater
Advised by Ben Taskar and Emily Fox
Joint work with Alex Kulesza

OUTLINE

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Motivation & Background

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1. **Dimensionality Reduction**

GILLENWATER, KULESZA, AND TASKAR (EMNLP 2012)

2. **MAP Estimation**

GILLENWATER, KULESZA, AND TASKAR (NIPS 2012)

3. **Likelihood Maximization**

GILLENWATER, KULESZA, FOX, AND TASKAR (NIPS 2014)

MOTIVATION & BACKGROUND

SOCIAL NETWORK MARKETING

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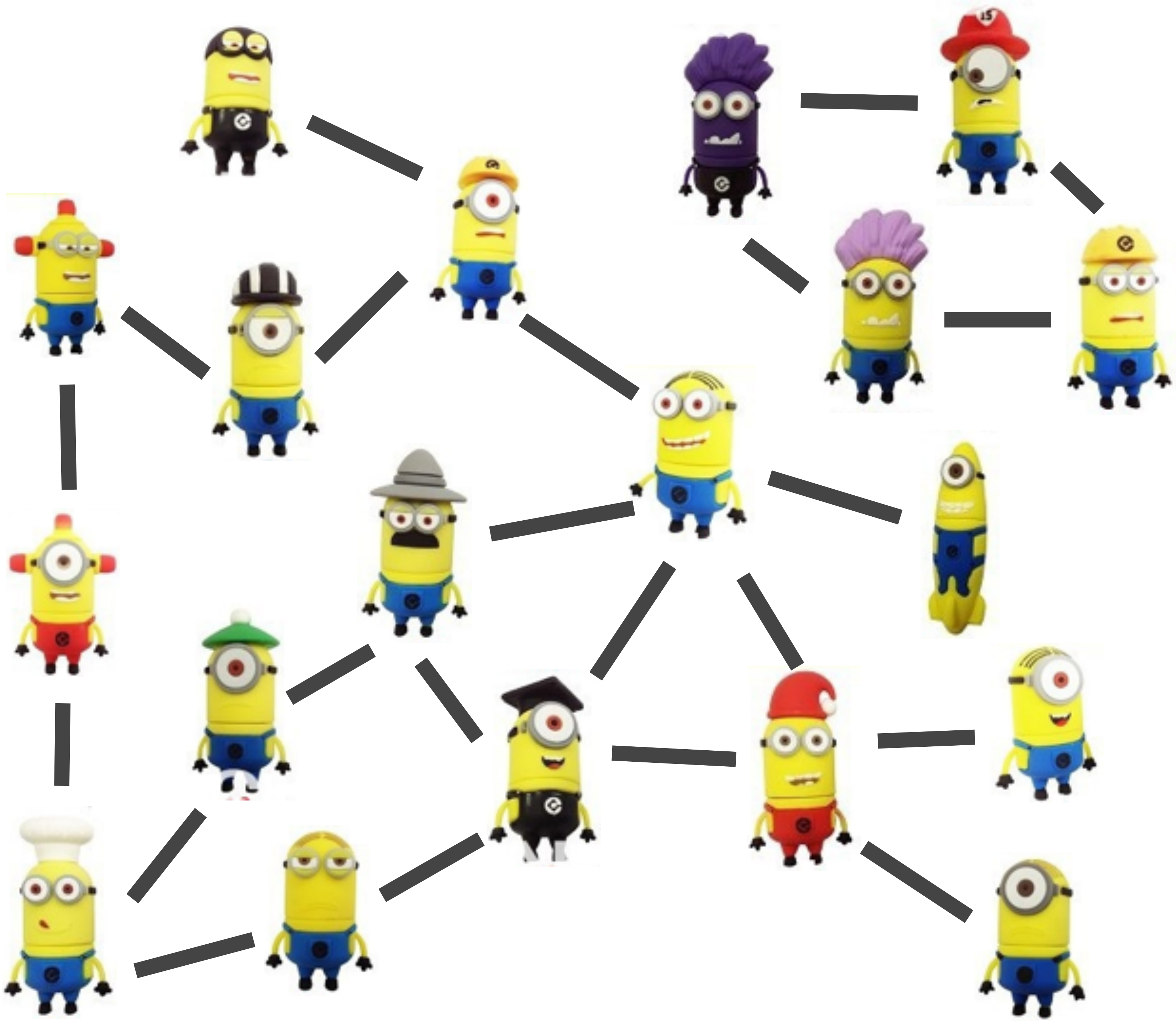
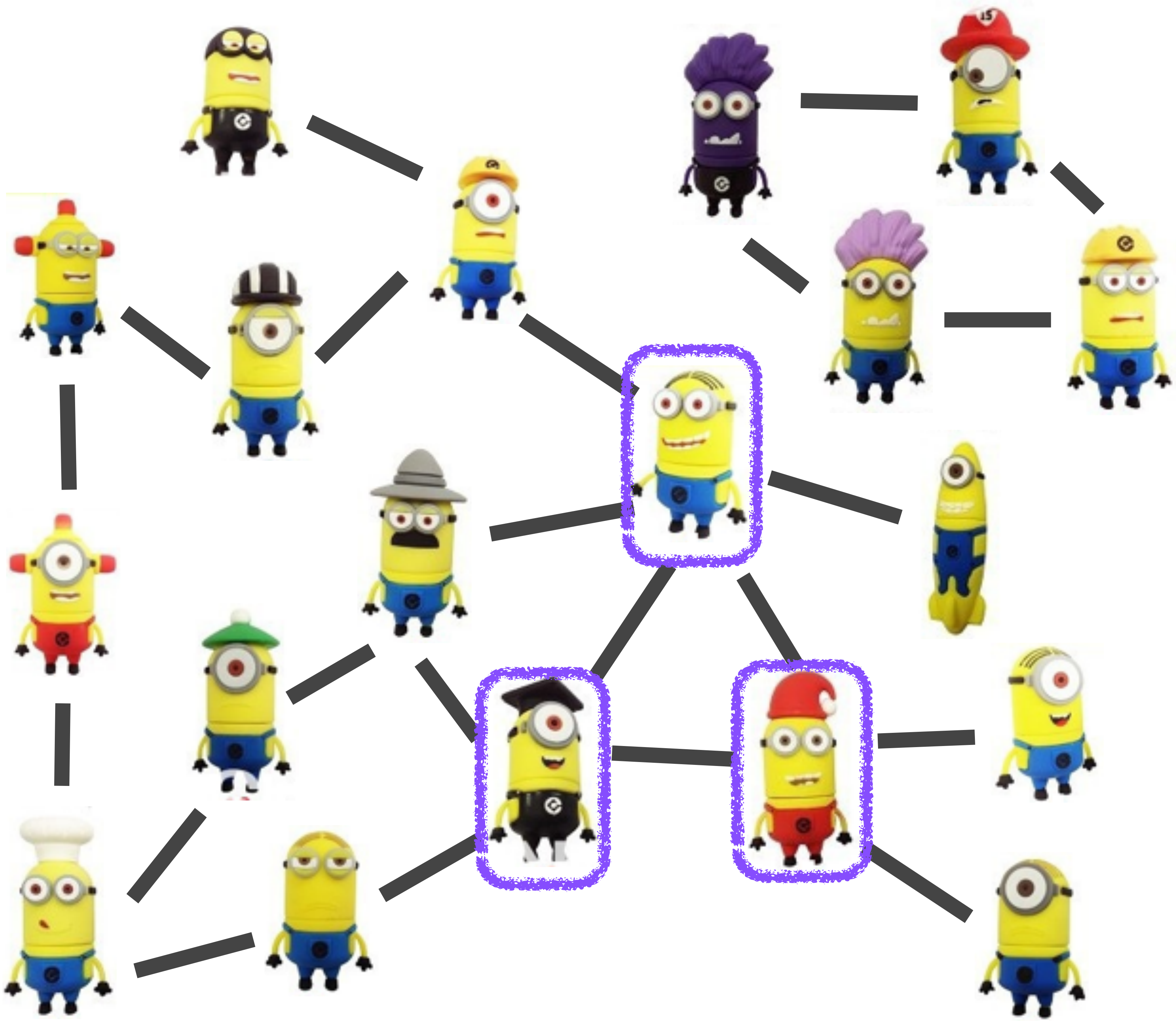


SOCIAL NETWORK MARKETING



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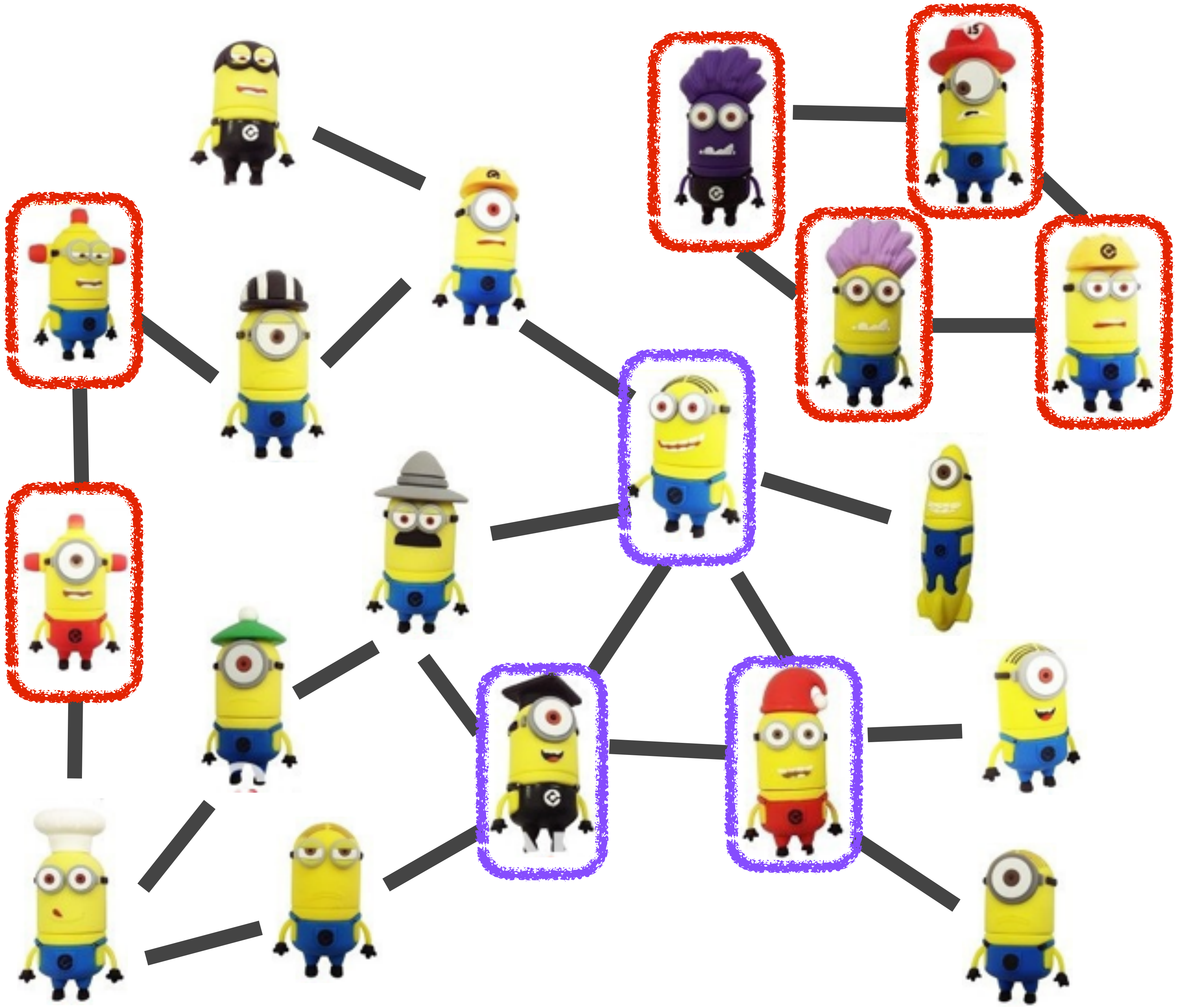

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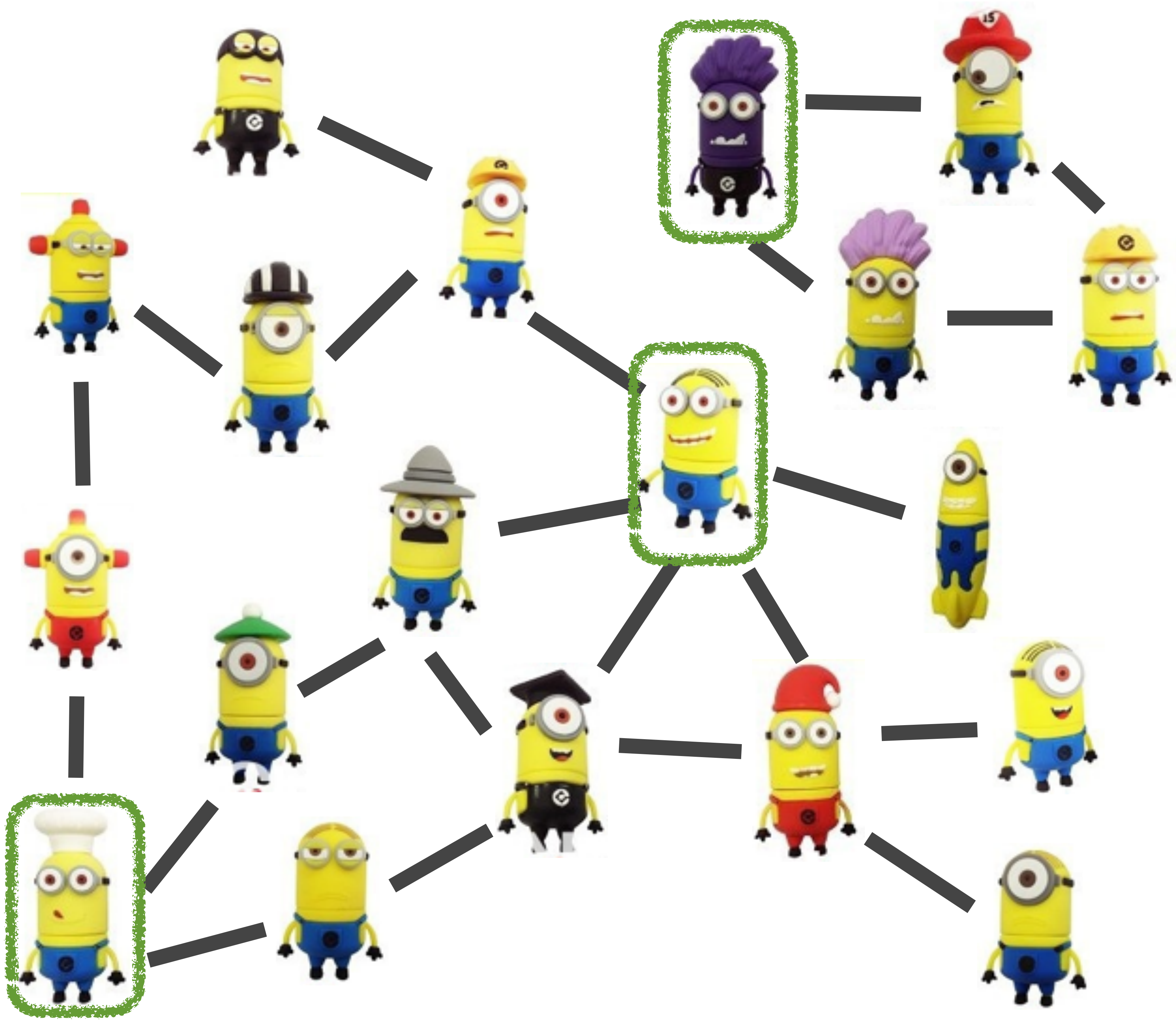

VIDEO TRACKING

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1

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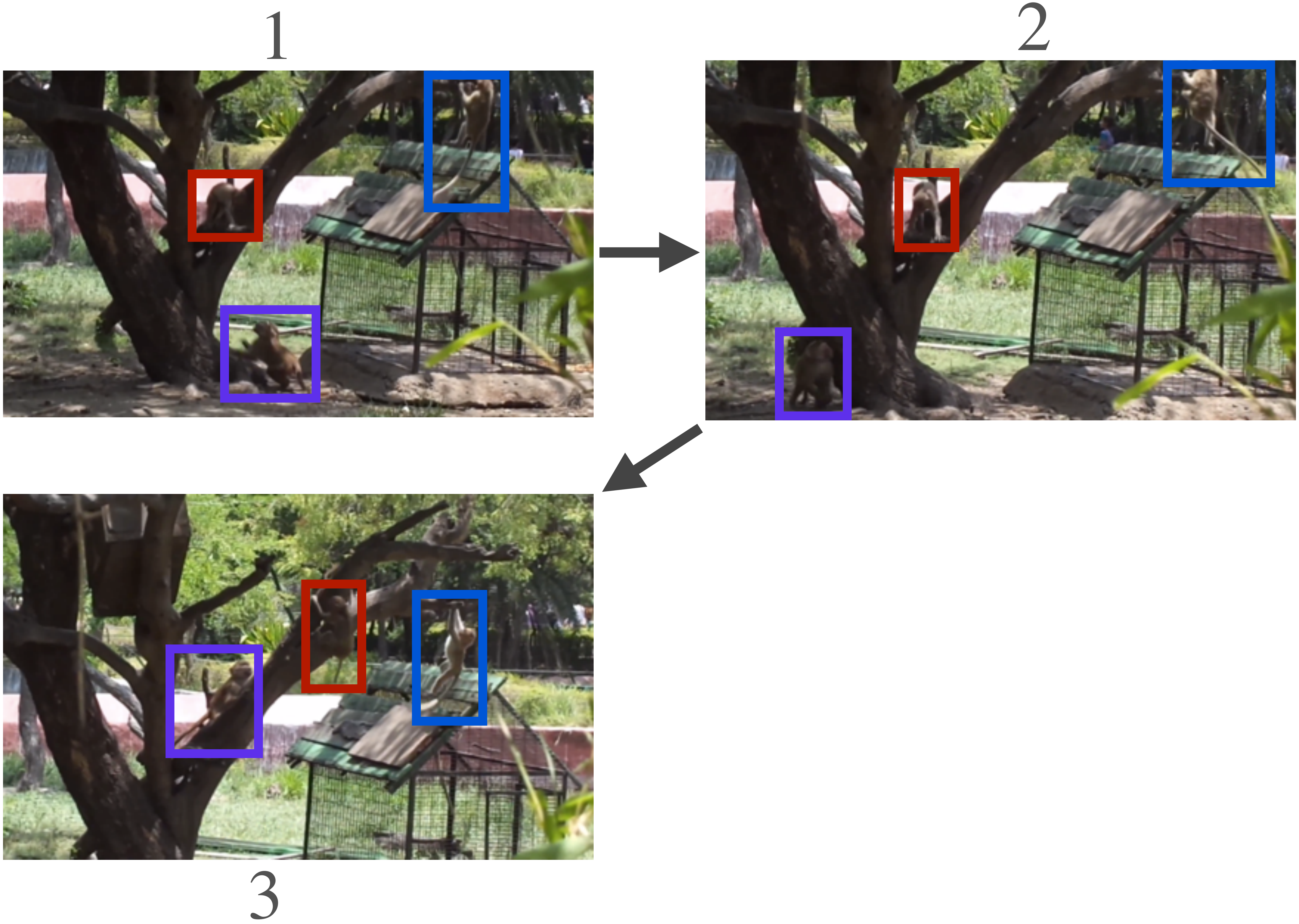
1

2

VIDEO TRACKING



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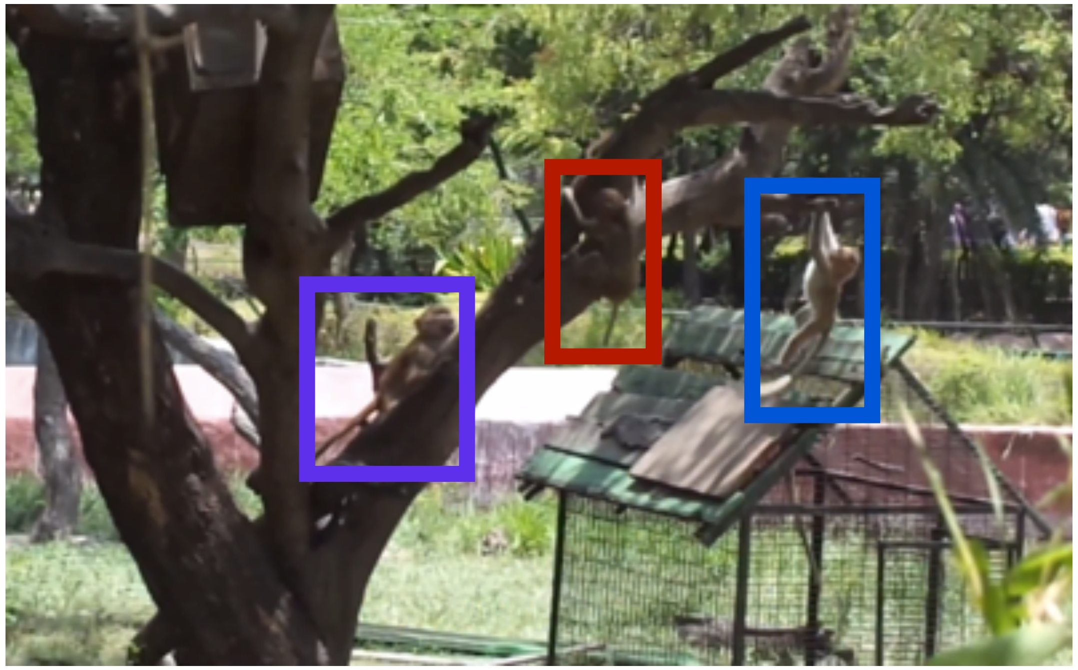

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SUBSET SELECTION

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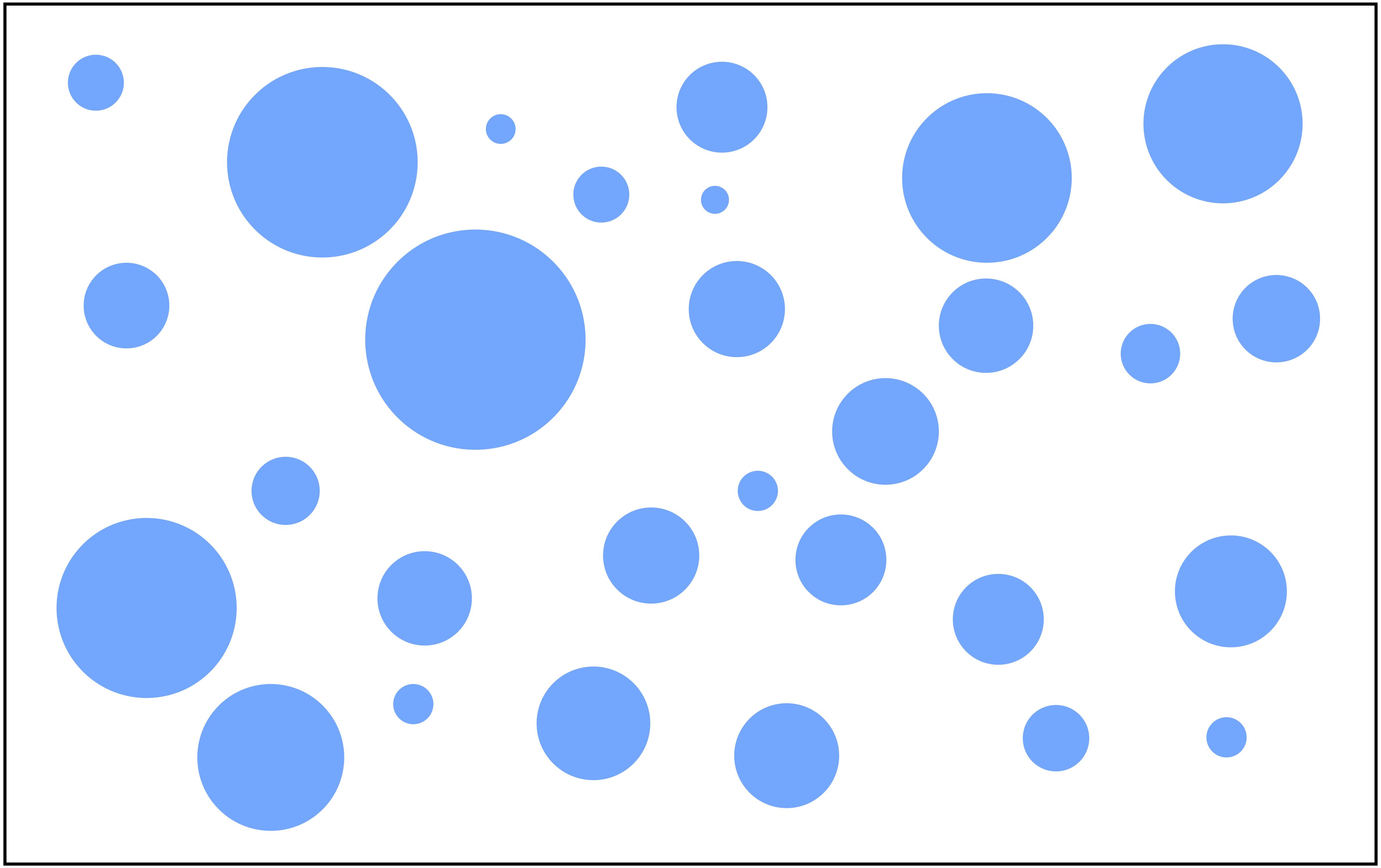

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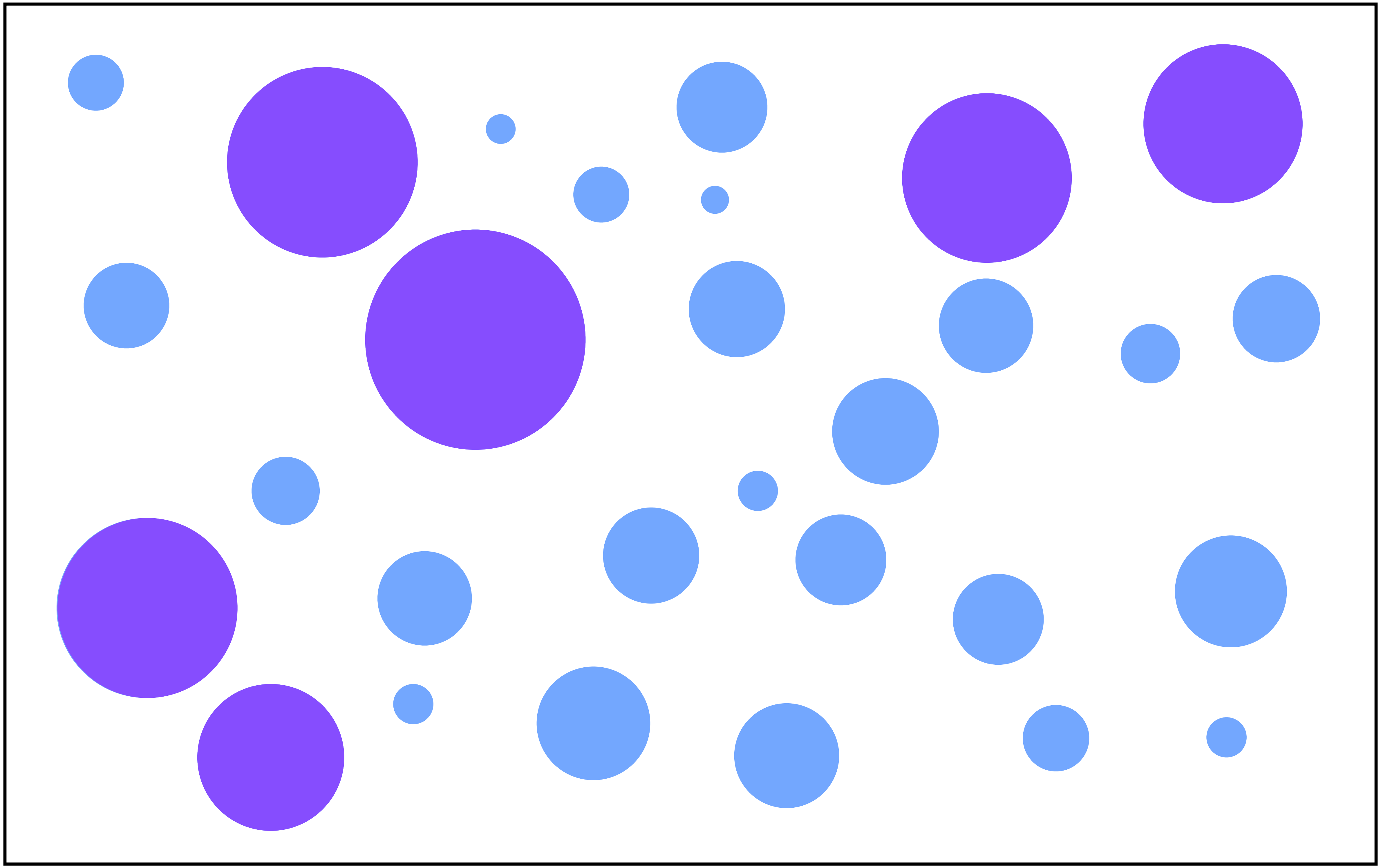

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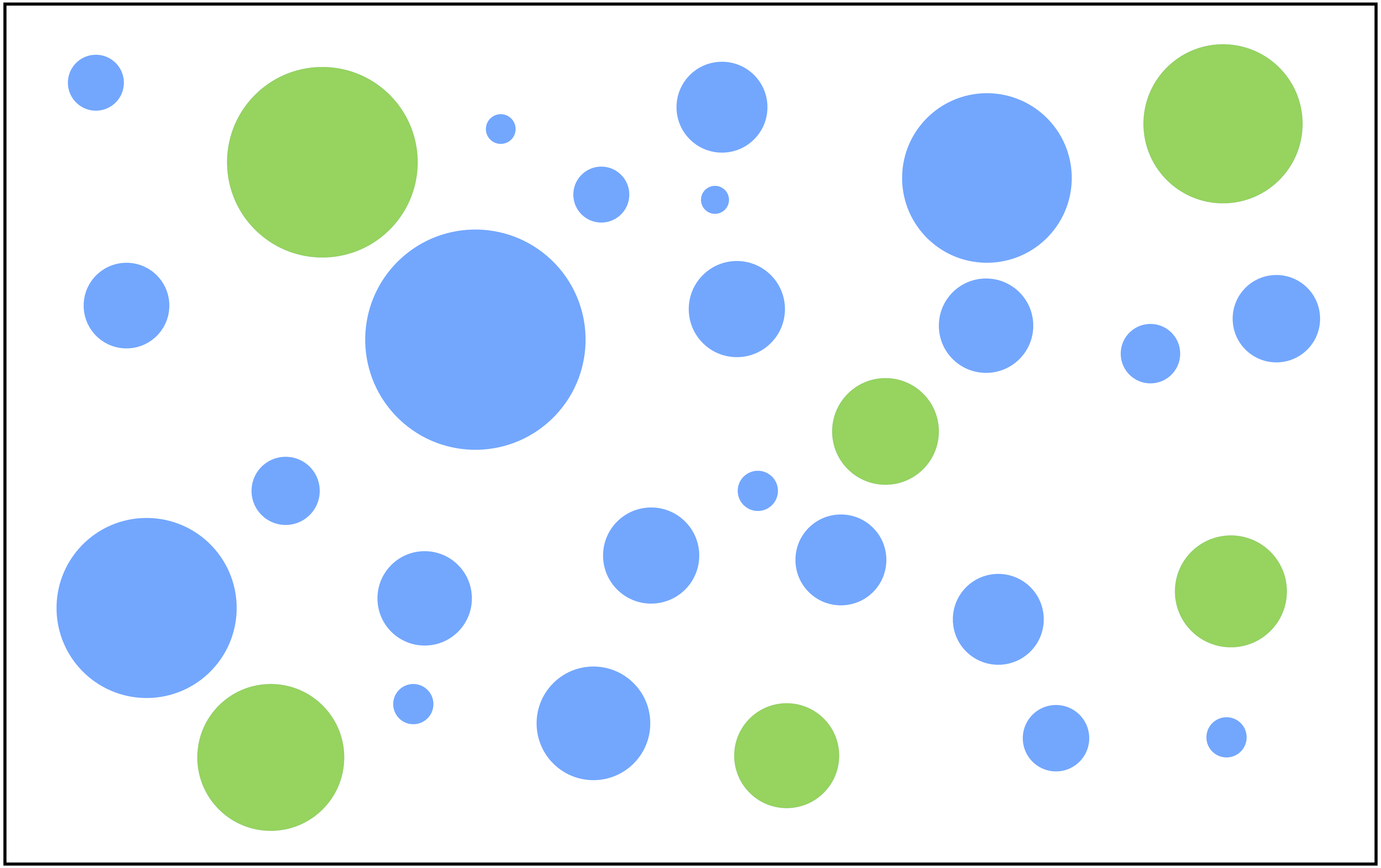

AREA AS SET-GOODNESS

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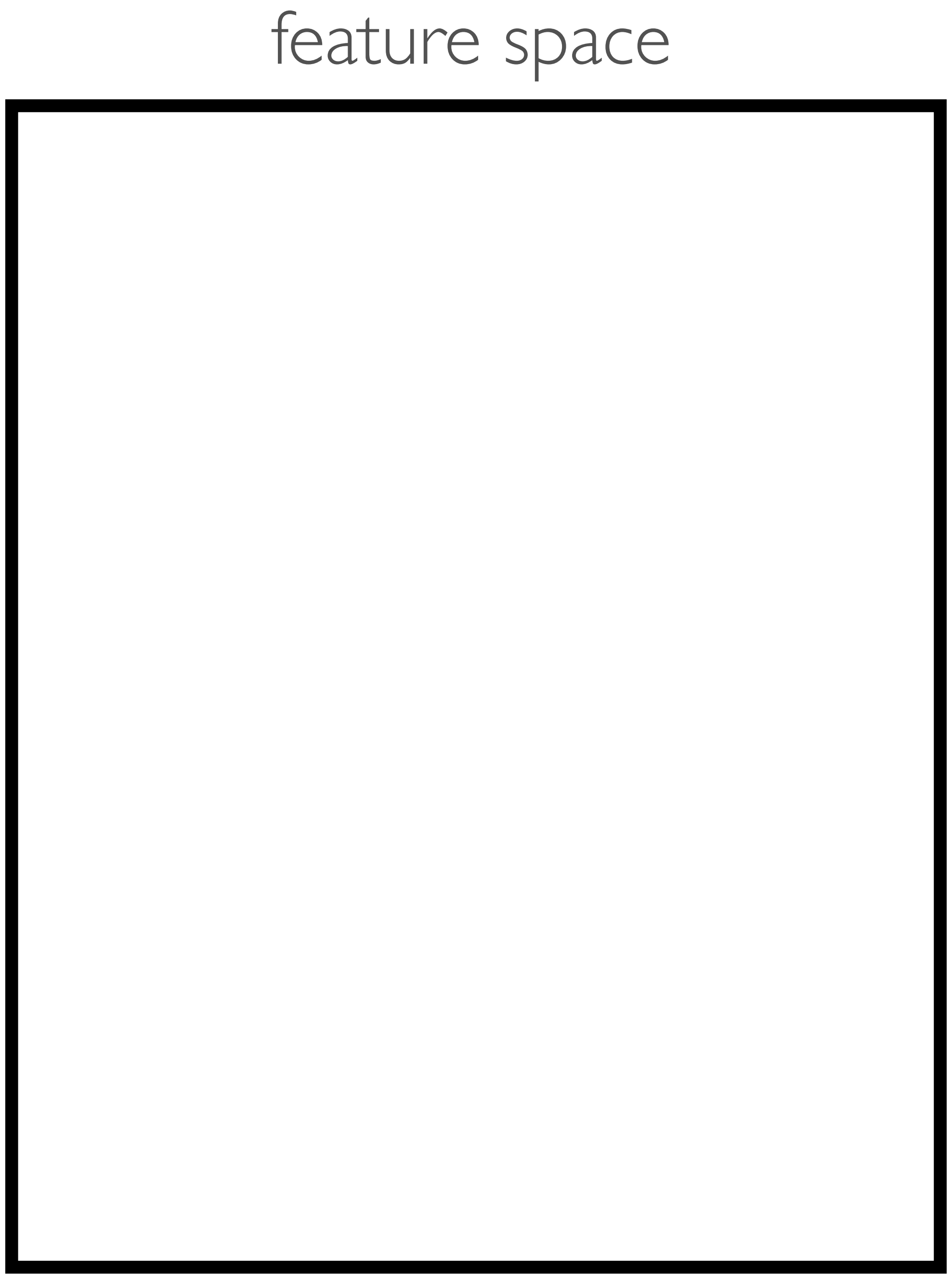

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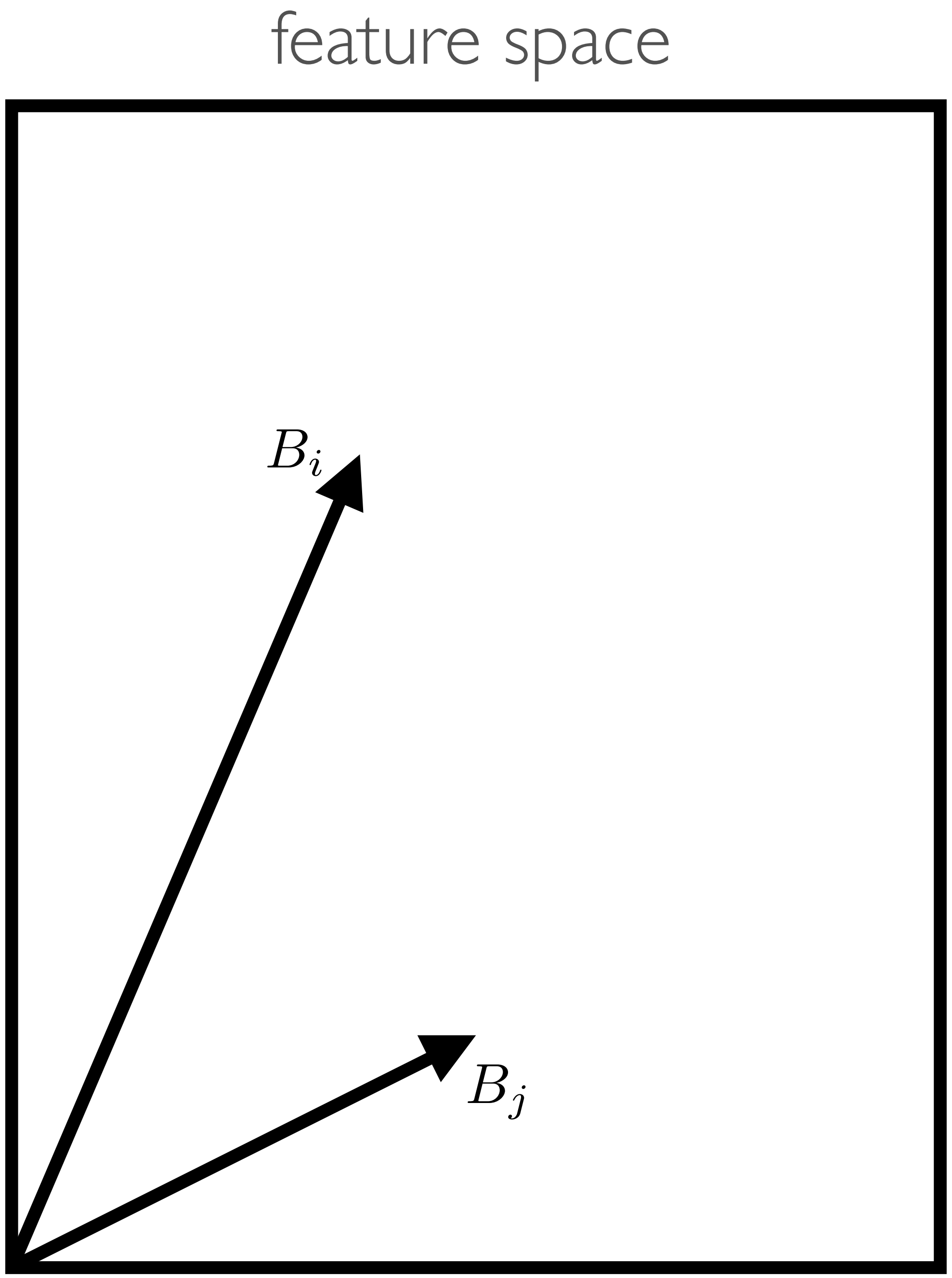
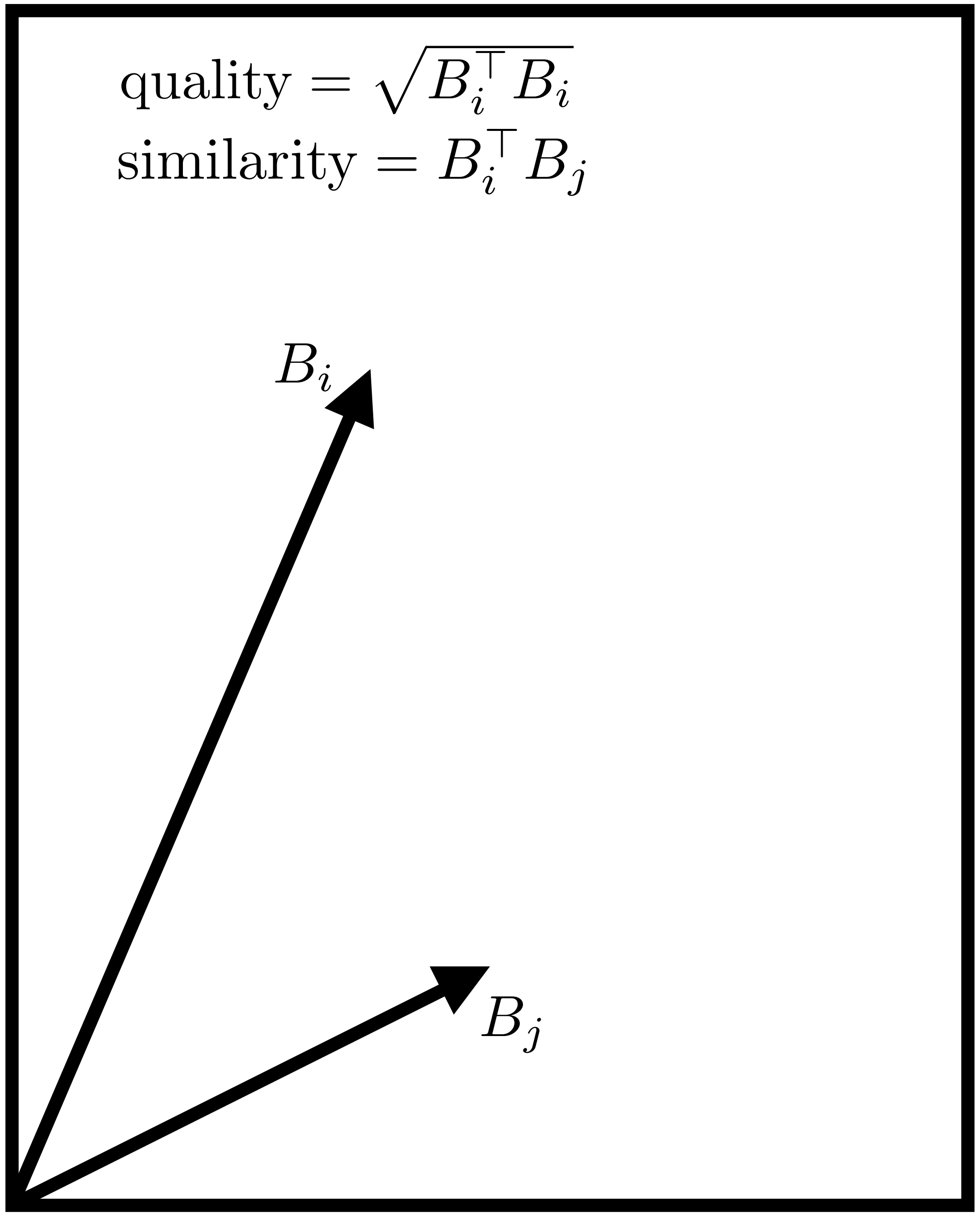


AREA AS SET-GOODNESS

feature space



quality $= \sqrt{B_i^\top B_i}$

similarity $= B_i^\top B_j$

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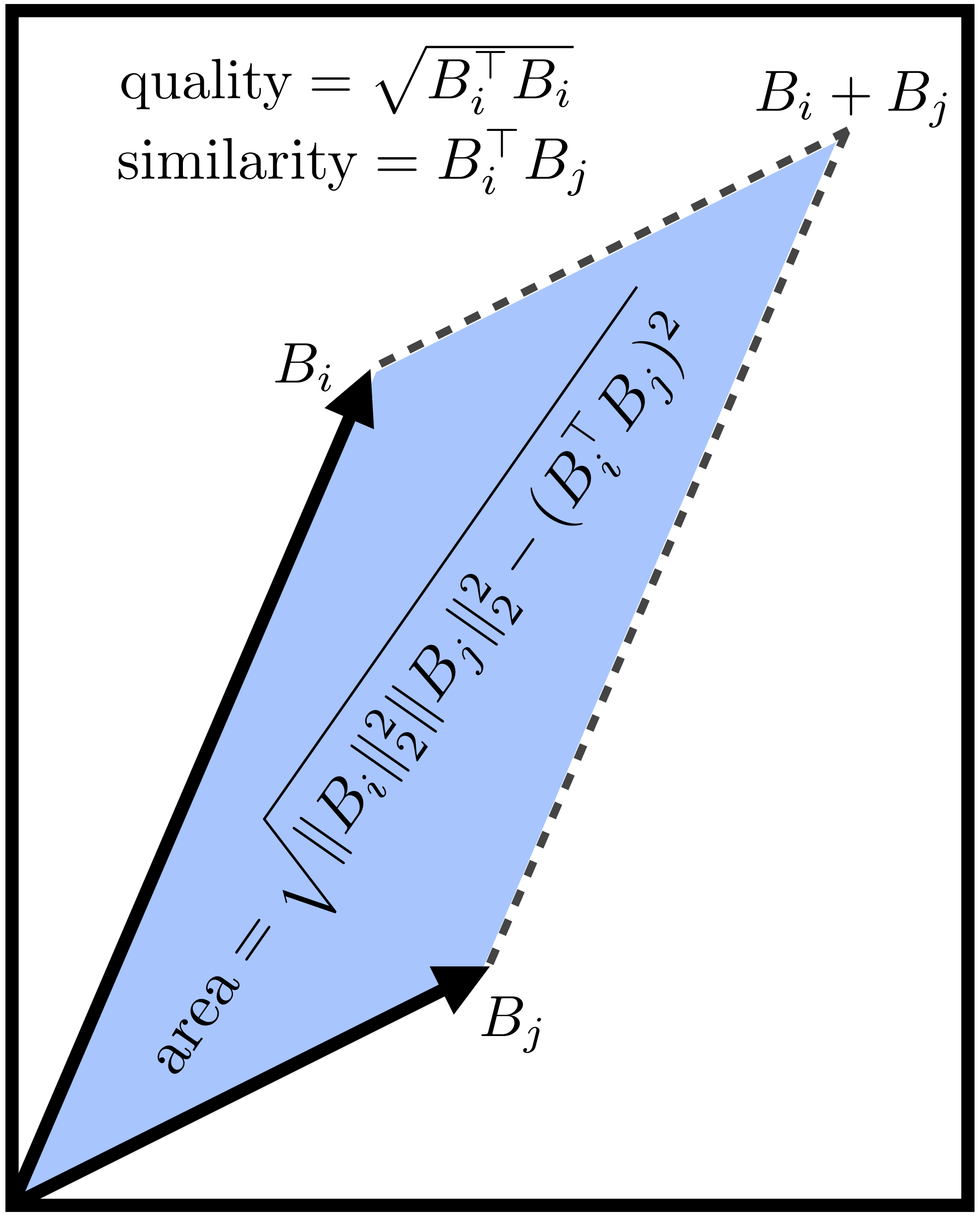
feature space



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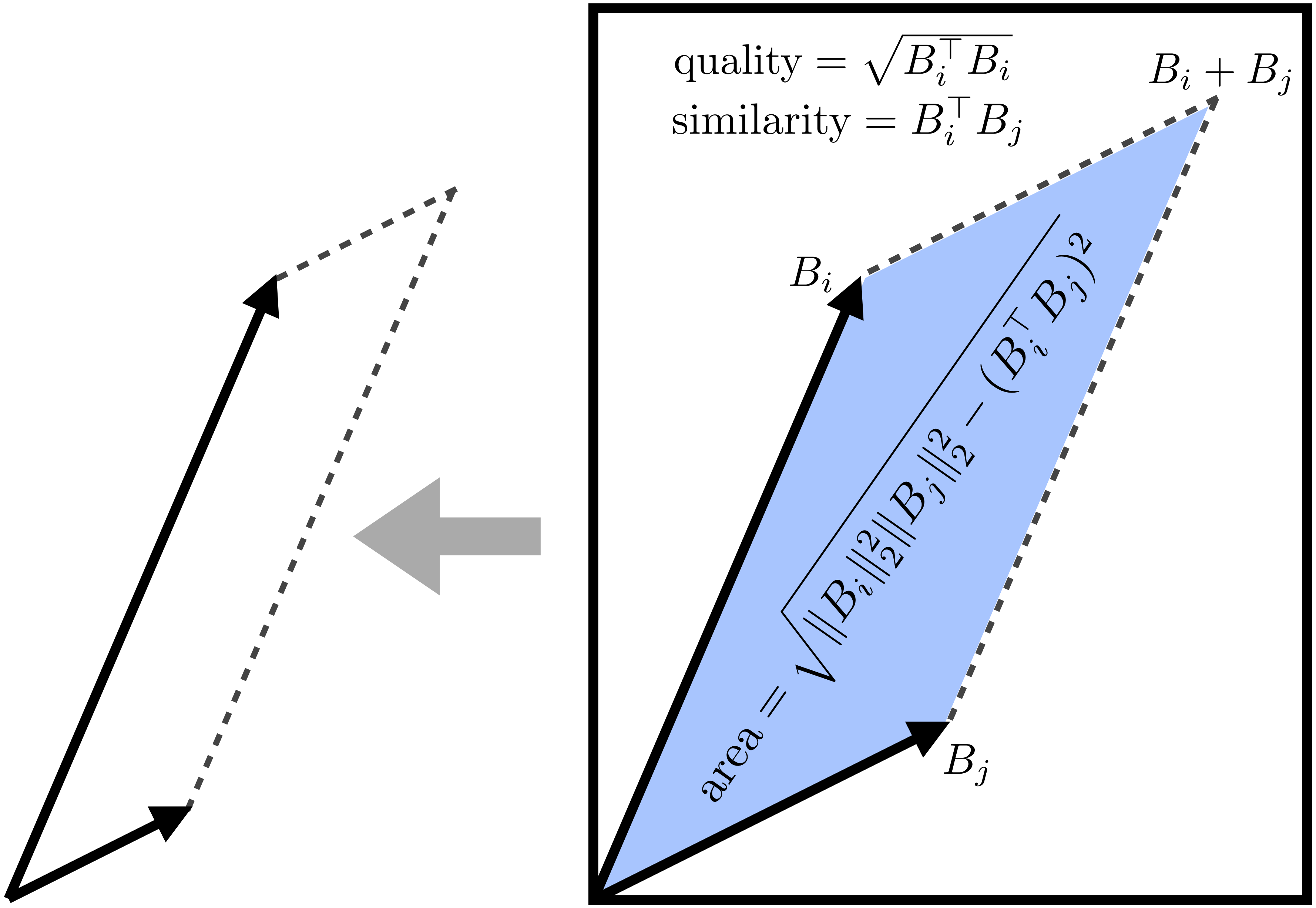

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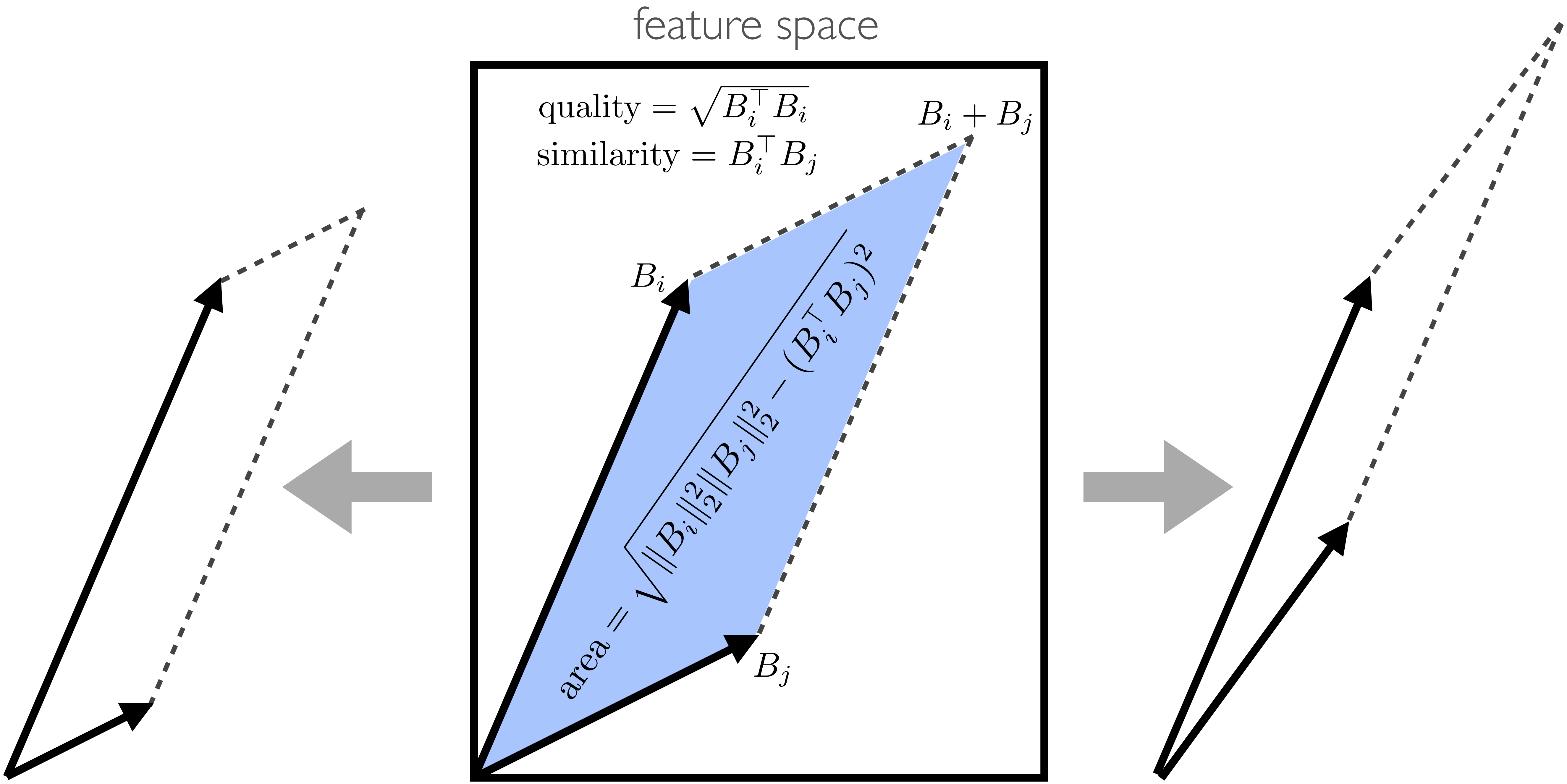

$$\text{quality} = \sqrt{B_i^\top B_i}$$

$$\text{similarity} = B_i^\top B_j$$

$$\text{area} = \sqrt{\|B_i\|_2^2 \|B_j\|_2^2 - (B_i^\top B_j)^2}$$

VOLUME AS SET-GOODNESS

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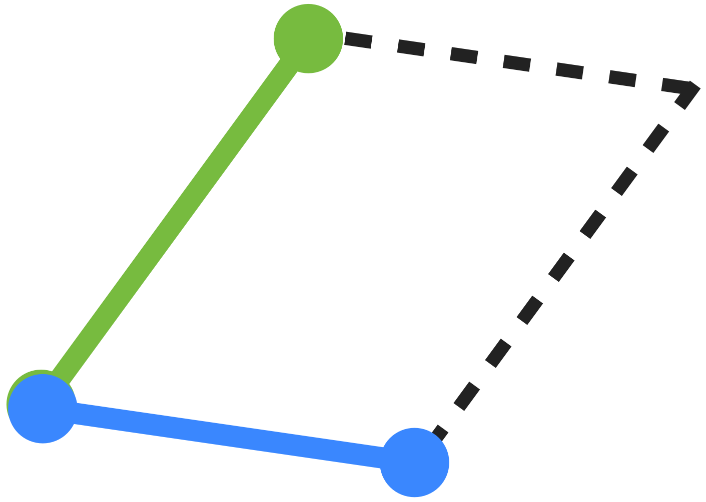

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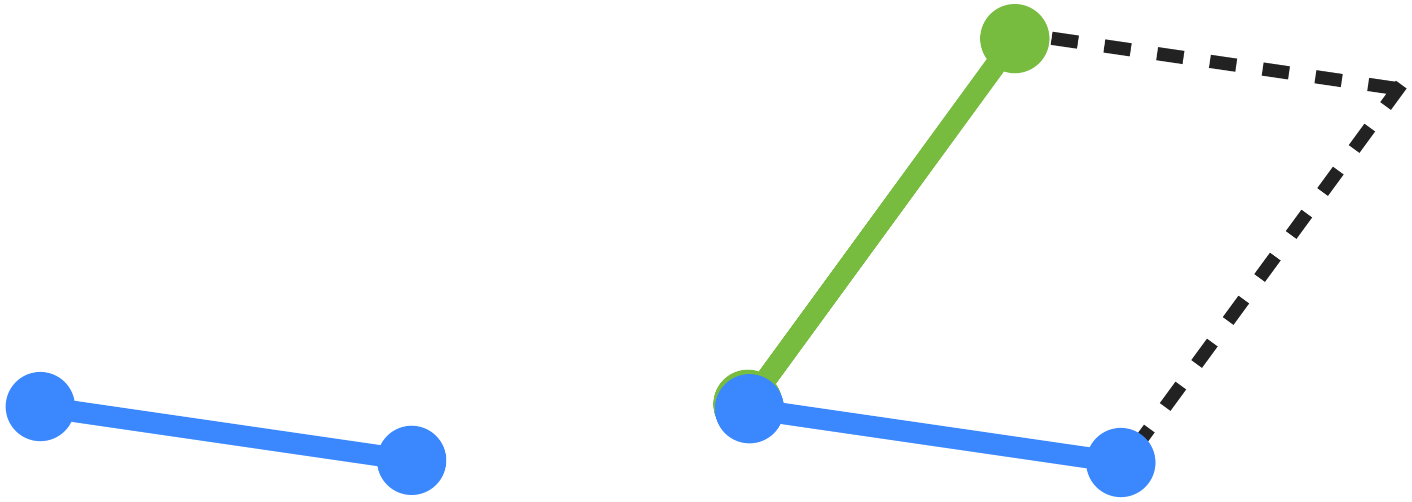

$$\text{length} = \|B_i\|_2$$

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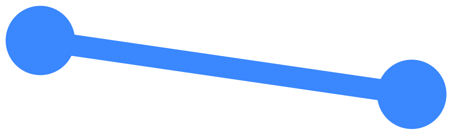
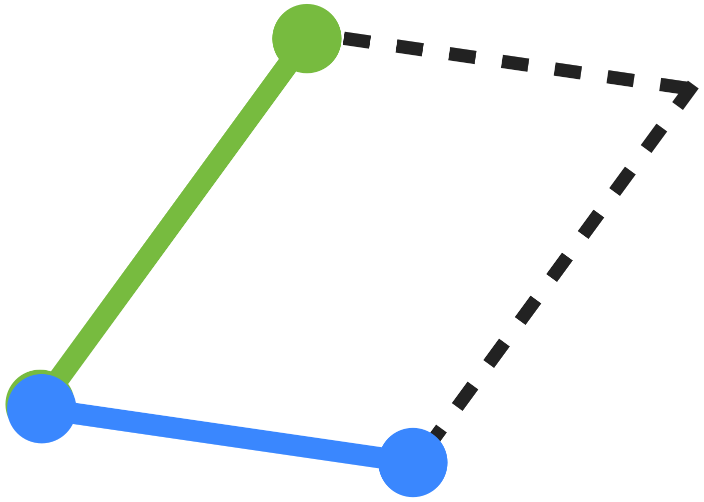
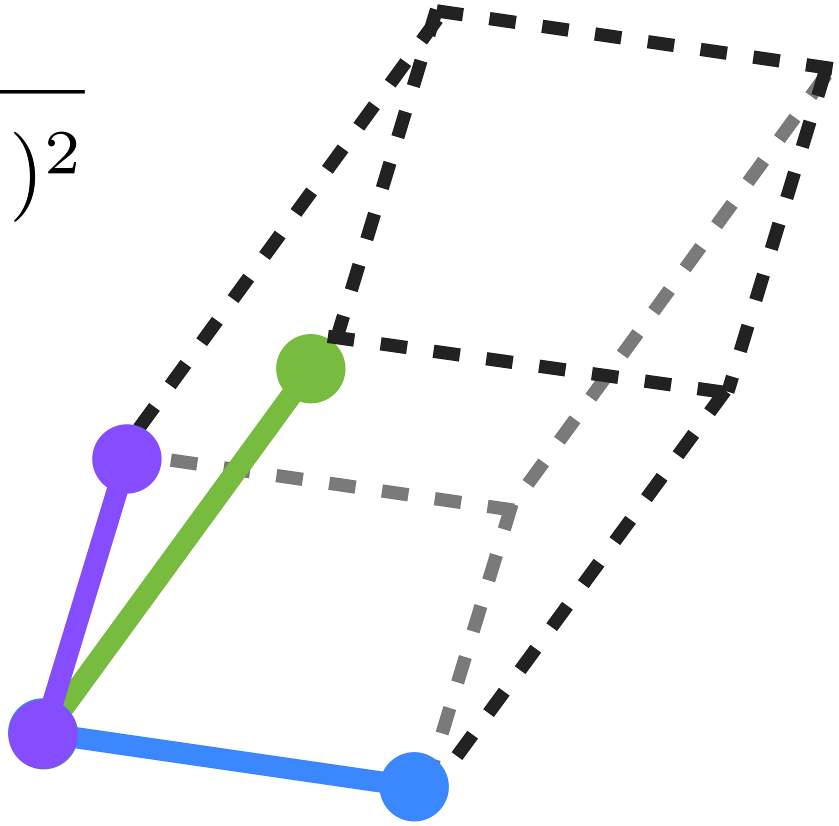


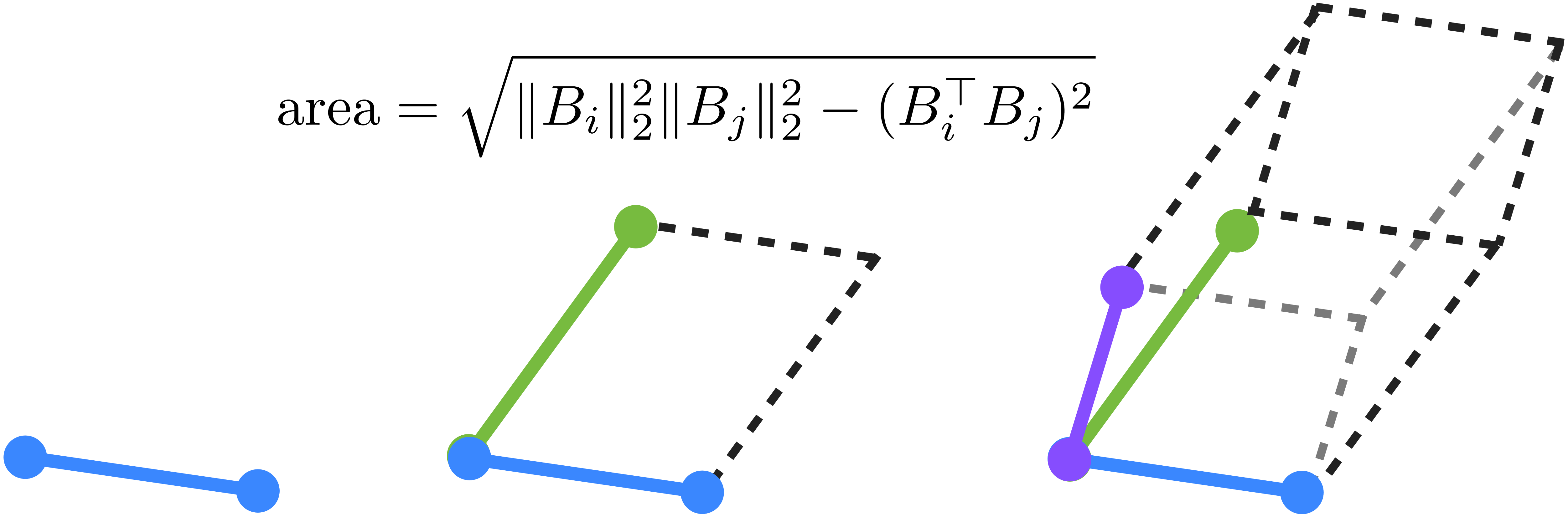




$$\text{length} = \|B_i\|_2$$

$$\text{volume} = \text{base} \times \text{height}$$

VOLUME AS SET-GOODNESS



$\text{area} = \sqrt{\|B_i\|_2^2 \|B_j\|_2^2 - (B_i^\top B_j)^2}$

$\text{length} = \|B_i\|_2$

$\text{volume} = \text{base} \times \text{height}$

$$\begin{aligned}\text{vol}(B) &= \text{height} \times \text{base} \\ &= ||B_1||_2 \text{vol}(\text{proj}_{\perp B_1}(B_{2:N}))\end{aligned}$$

AREA AS A DET

$$\text{area} = \sqrt{\|B_i\|_2^2\|B_j\|_2^2 - (B_i^\top B_j)^2}$$

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$$= \det \left(\begin{bmatrix} \text{---} \; B_i \; \text{---} \\ \text{---} \; B_j \; \text{---} \end{bmatrix} \begin{bmatrix} | & | \\ B_i & B_j \\ | & | \end{bmatrix} \right)^{\frac{1}{2}}$$

VOLUME AS A DET

$$\mathrm{vol}(B_{\{i,j\}}) = \det\left(\begin{pmatrix} - & B_i & - \\ - & B_j & - \end{pmatrix} \begin{pmatrix} | & | \\ B_i & B_j \\ | & | \end{pmatrix} \right)^{\frac{1}{2}}$$

VOLUME AS A DET

$$\mathrm{vol}(B_{\{i,j\}}) = \det\left(\begin{bmatrix} \text{—}\; B_i \;\text{—} \\ \text{—}\; B_j \;\text{—} \end{bmatrix}\begin{bmatrix} | & | \\ B_i & B_j \\ | & | \end{bmatrix}\right)^{\frac{1}{2}}$$

$$\mathrm{vol}(B) = \det\left(\begin{bmatrix} \text{—}\; B_1 \;\text{—} \\ \vdots \\ \text{—}\; B_N \;\text{—} \end{bmatrix}\begin{bmatrix} | & & | \\ B_1 & \cdots & B_N \\ | & & | \end{bmatrix}\right)^{\frac{1}{2}}$$

$$\mathrm{vol}(B)^2 = \det(B^\top B) = \det(L)$$

COMPLEX STATISTICS

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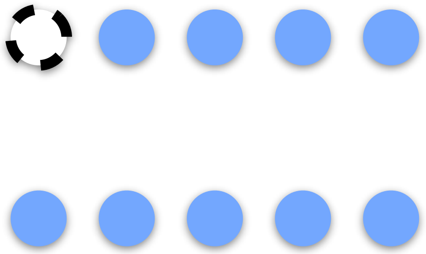
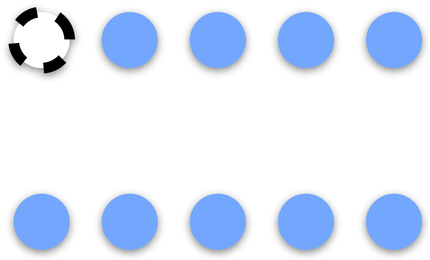
COMPLEX STATISTICS



COMPLEX STATISTICS

$$\Sigma$$



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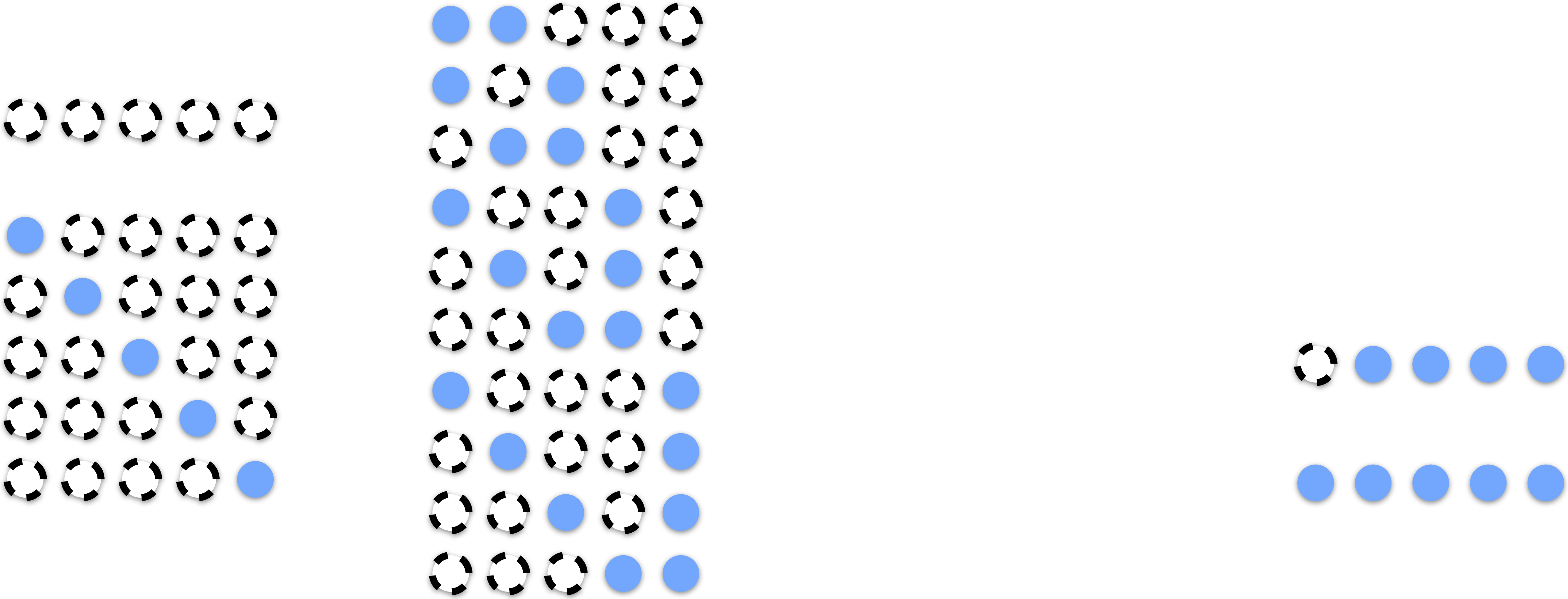

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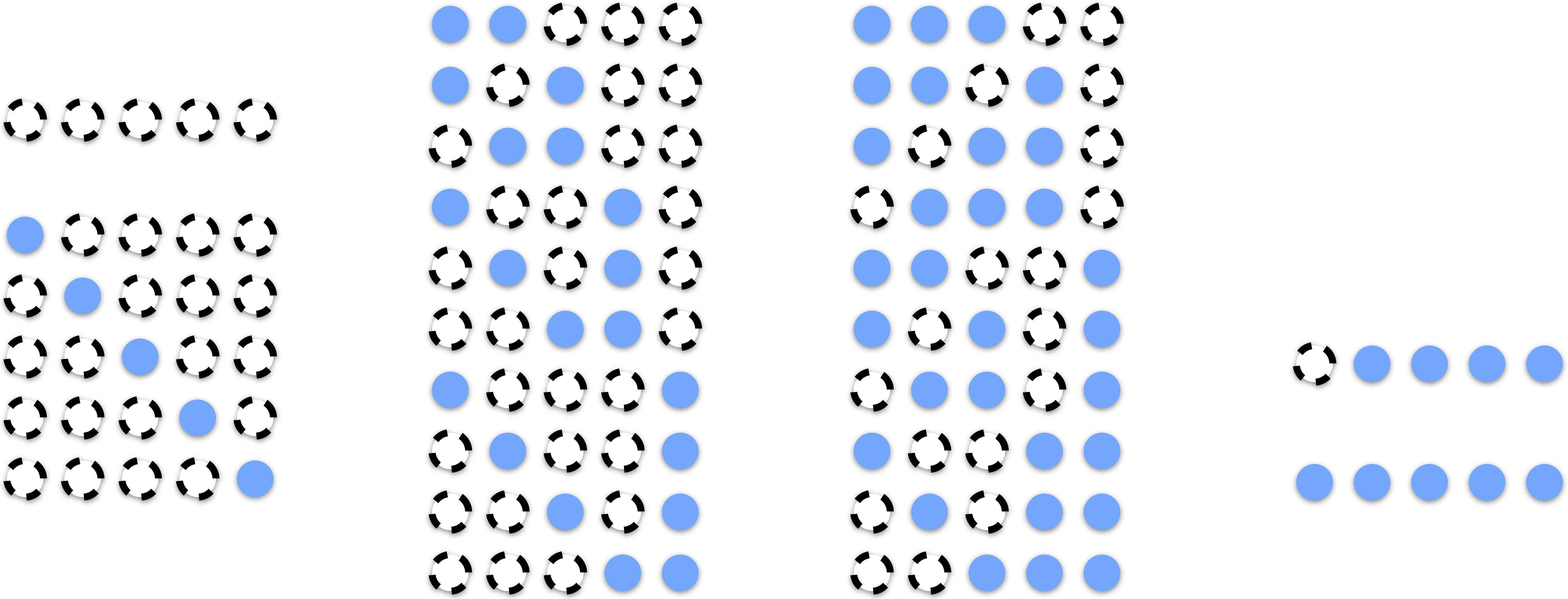

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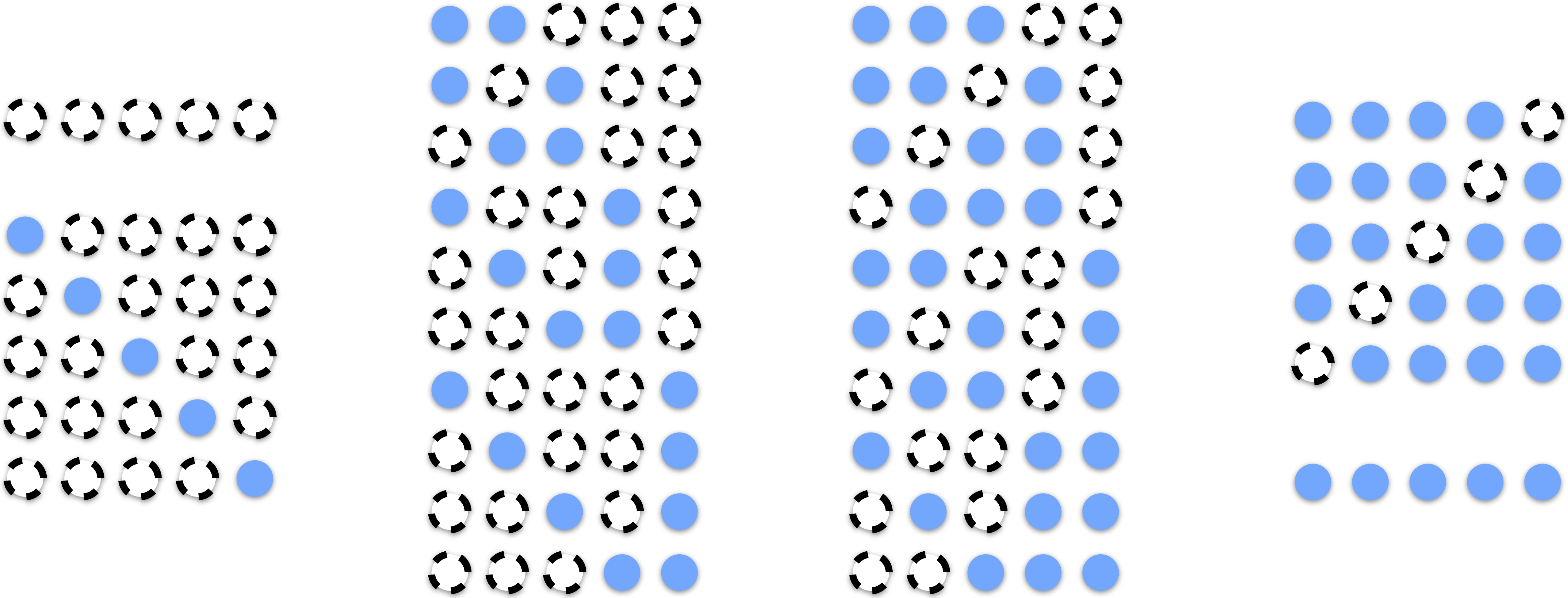

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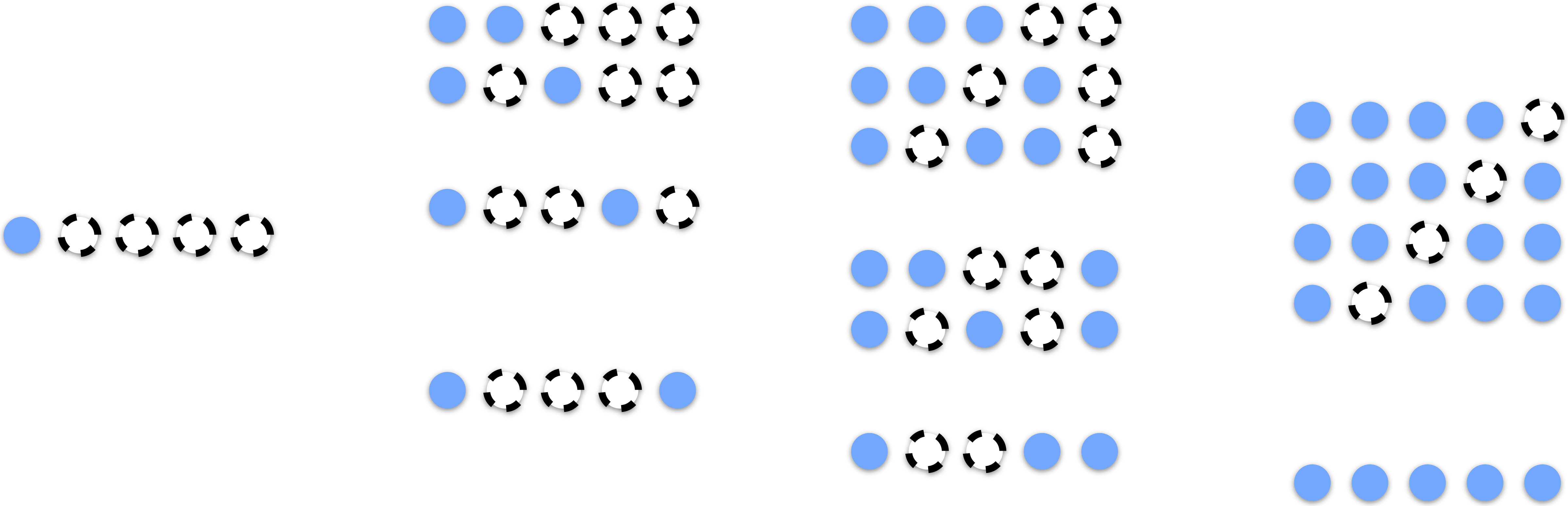

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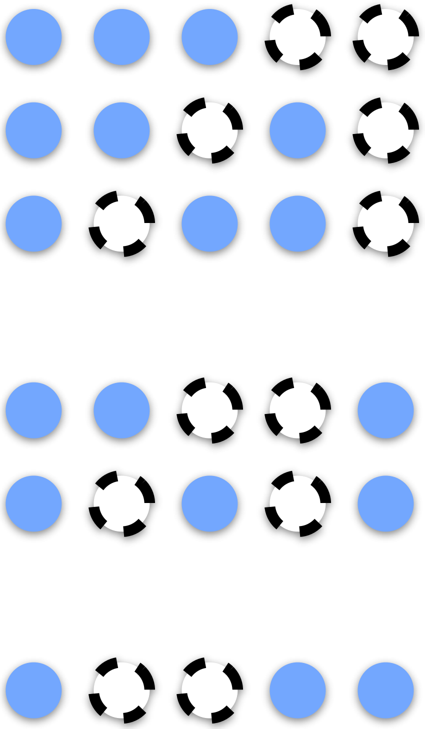
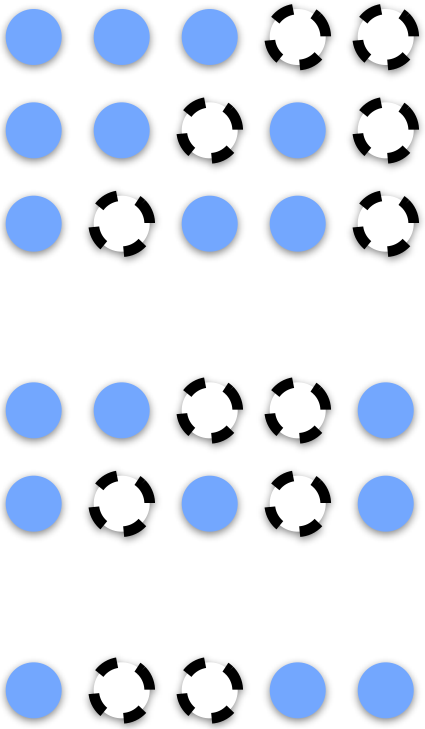


COMPLEX STATISTICS

$$N \text{ items} \implies 2^N \text{ sets}$$



EFFICIENT COMPUTATION

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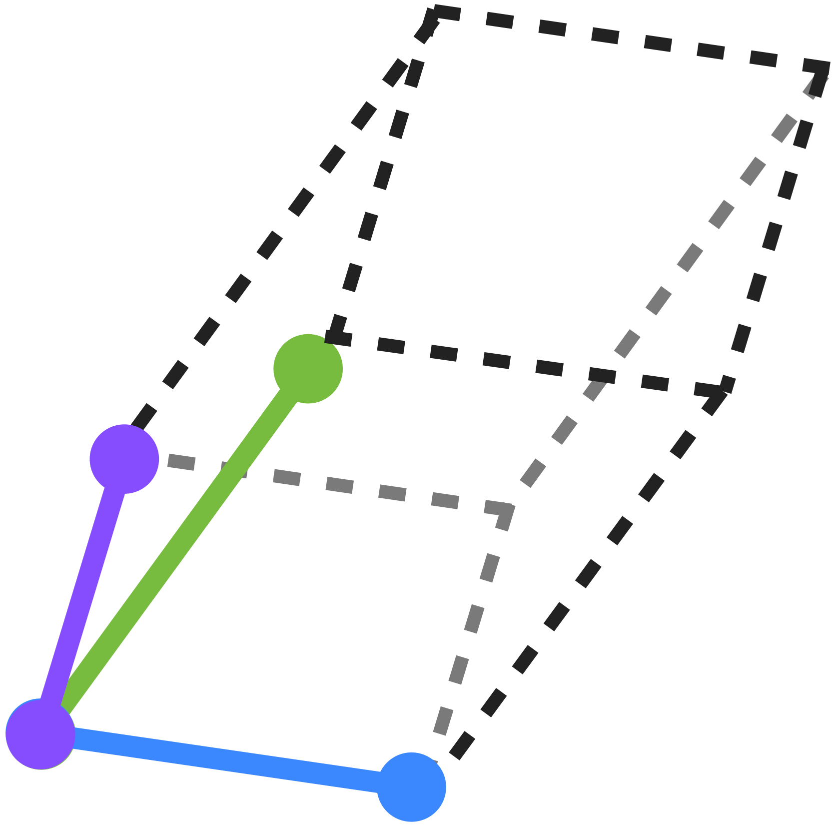
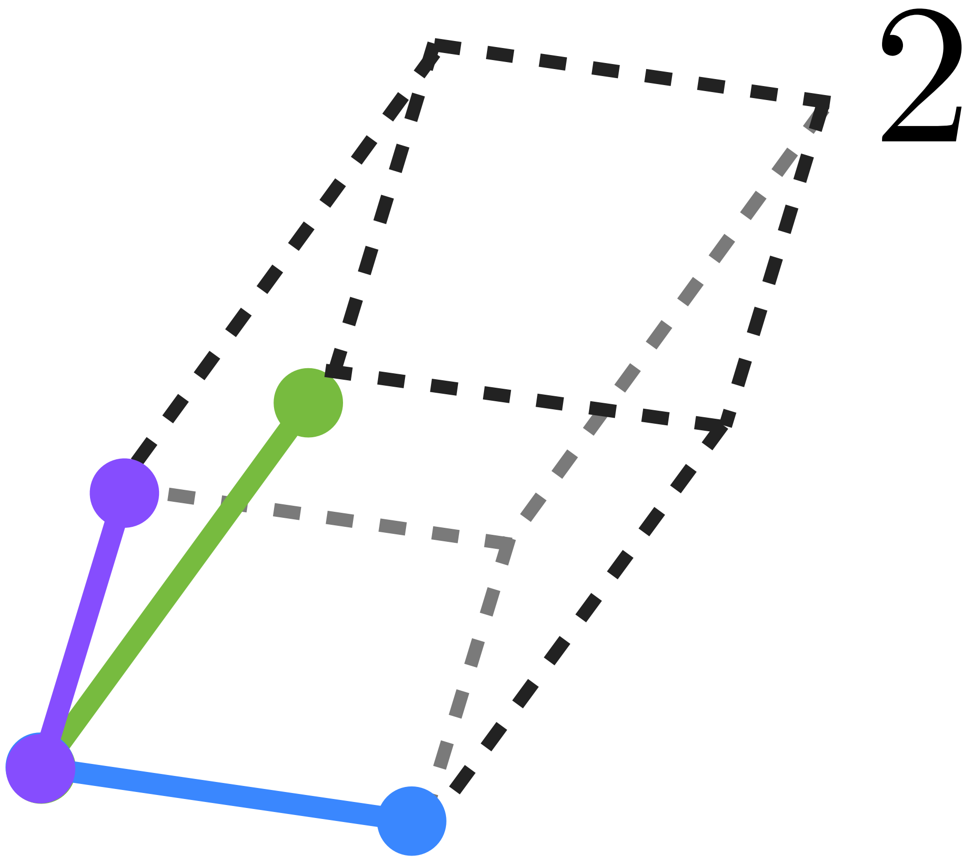


$$\det^{\frac{1}{2}}$$

EFFICIENT COMPUTATION



$$\det$$

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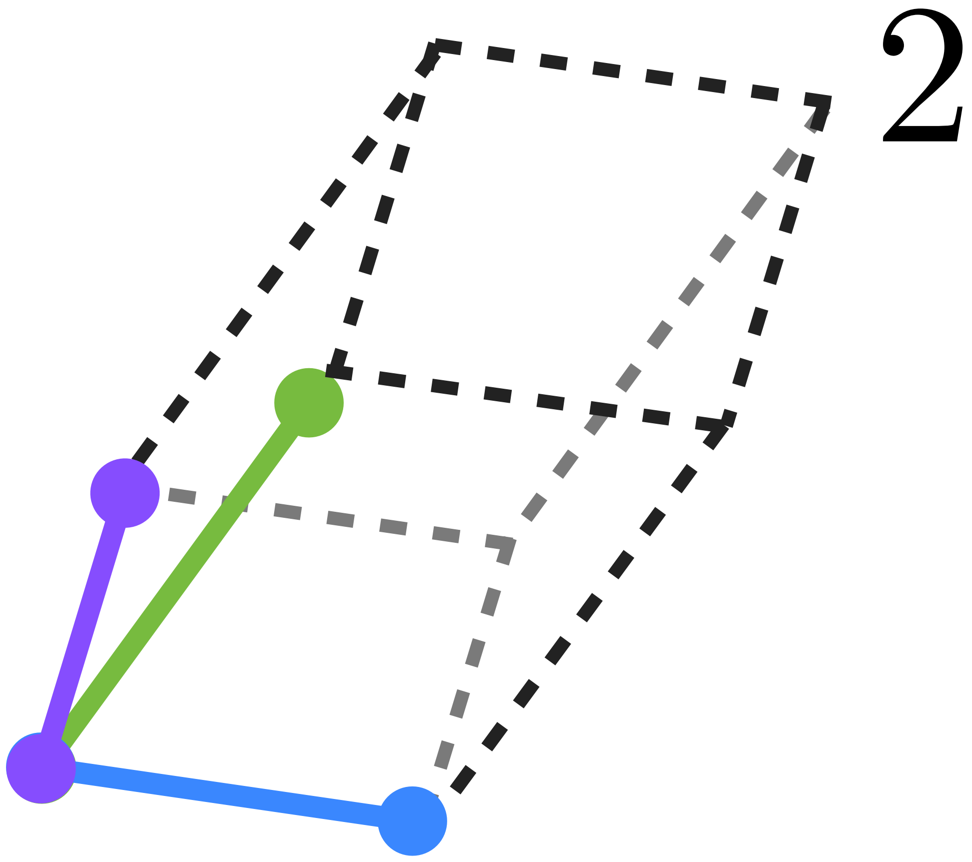
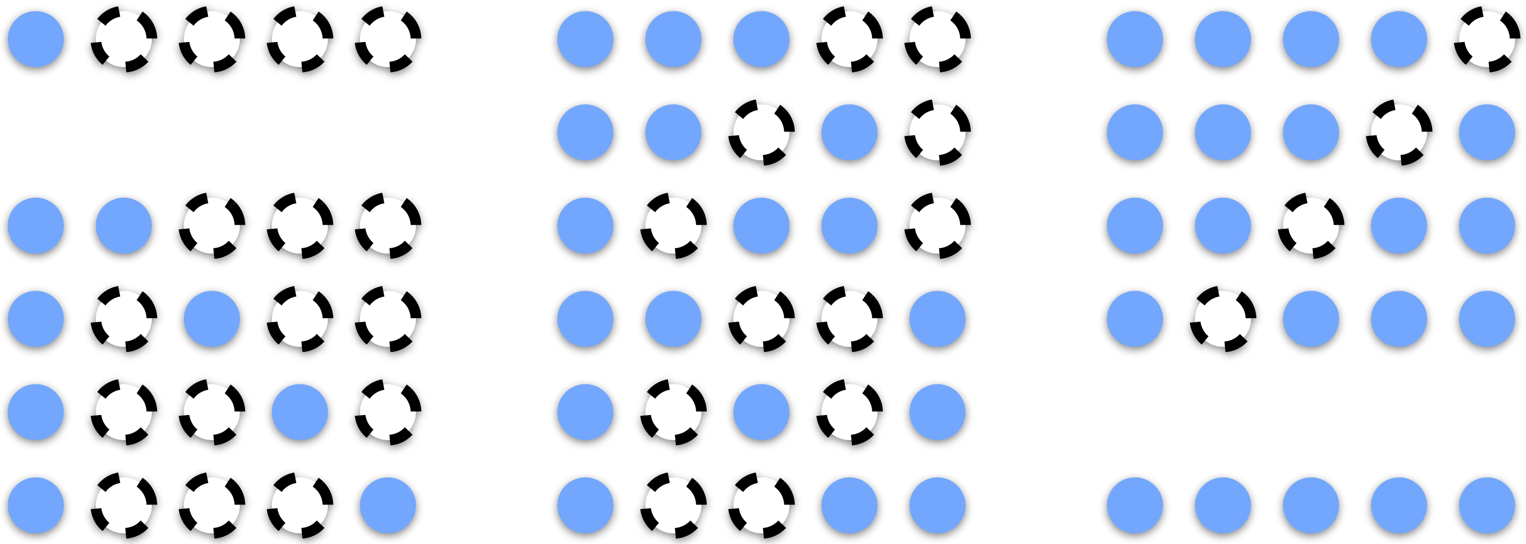


det

$$\Sigma$$



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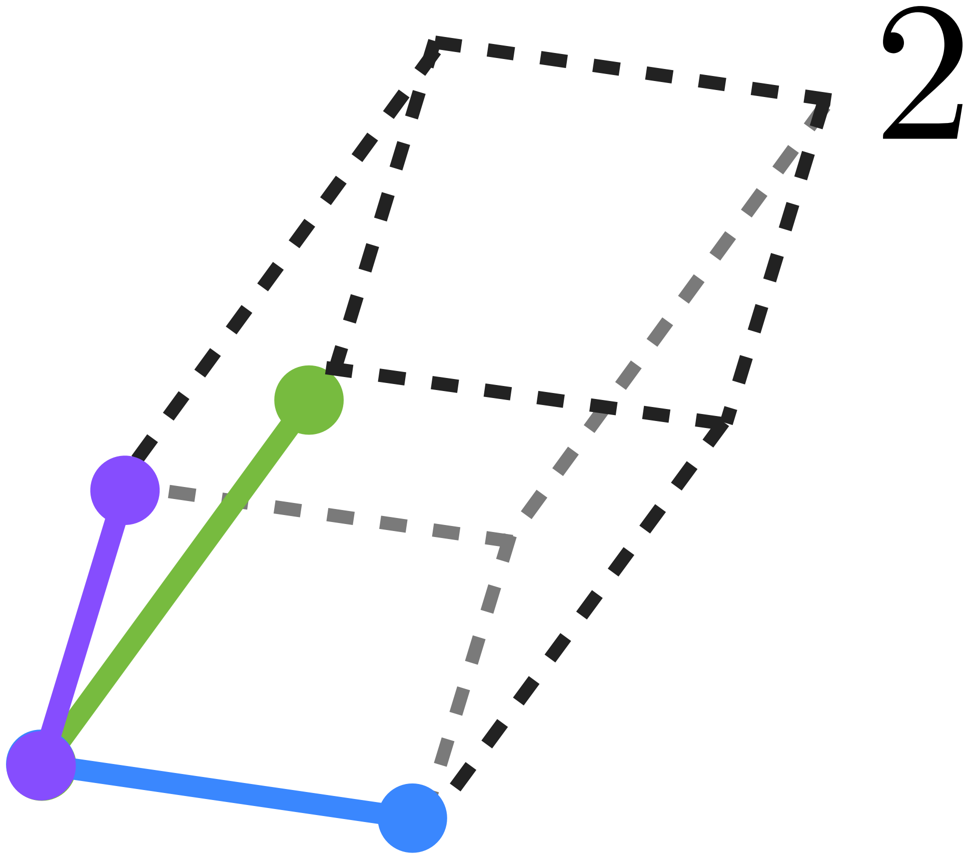
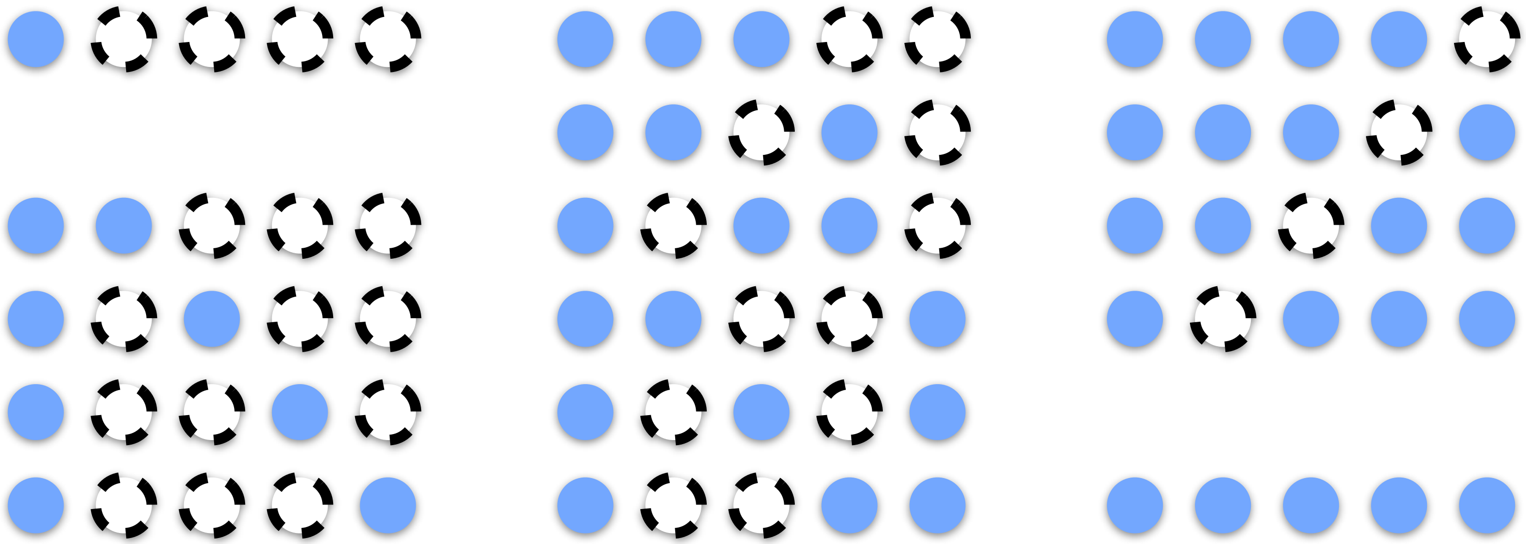


det

$O(N^3)$



POINT PROCESSES

$$\mathcal{Y} = \{1, \ldots, N\}$$

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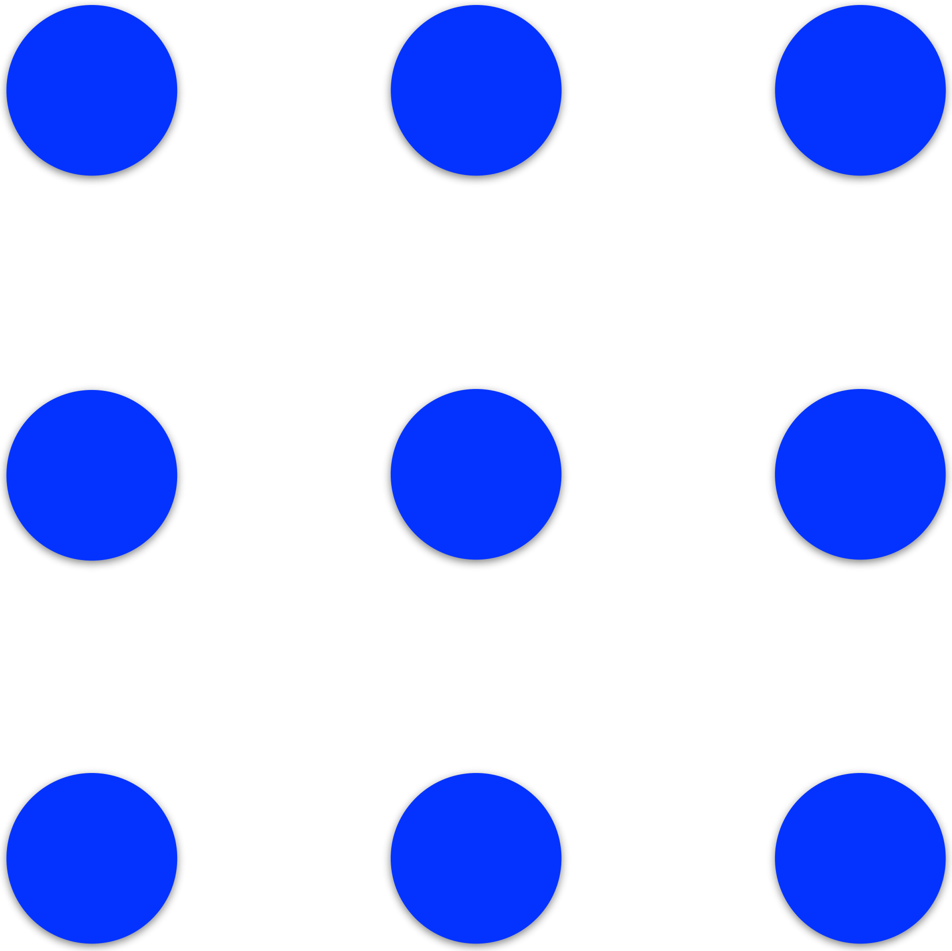

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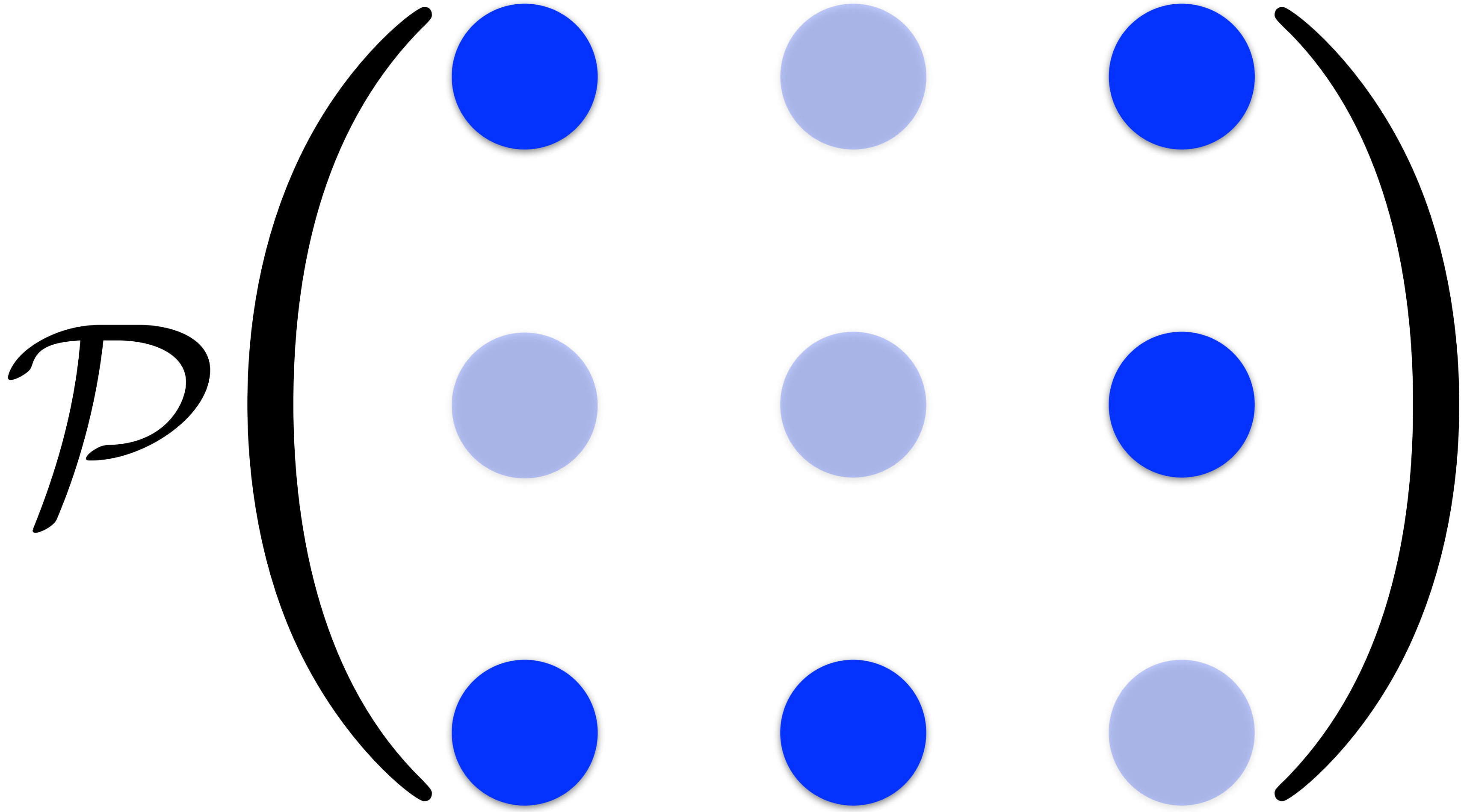

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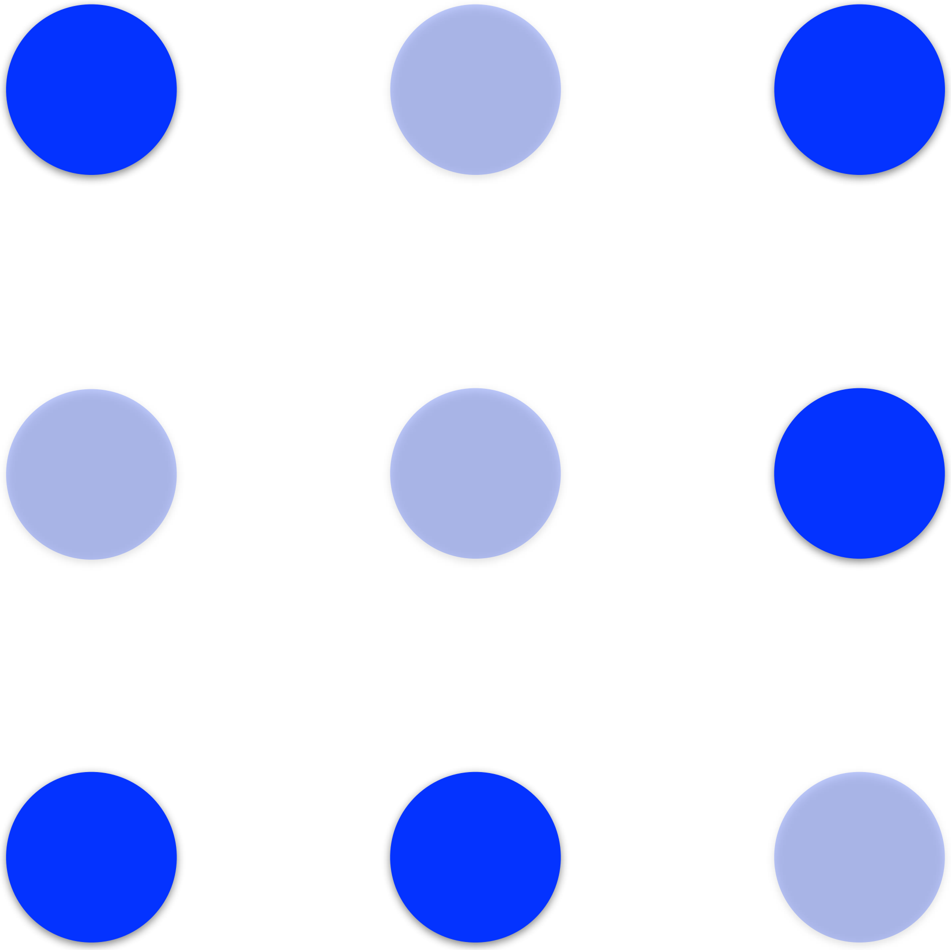
$$\mathcal{Y} = \{1, \ldots, N\}$$

$\mathcal{P}\Bigg($



$\Bigg) = 0.2$

DETERMINANTAL

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$$\mathcal{P}(\{2,3,5\}) \propto$$

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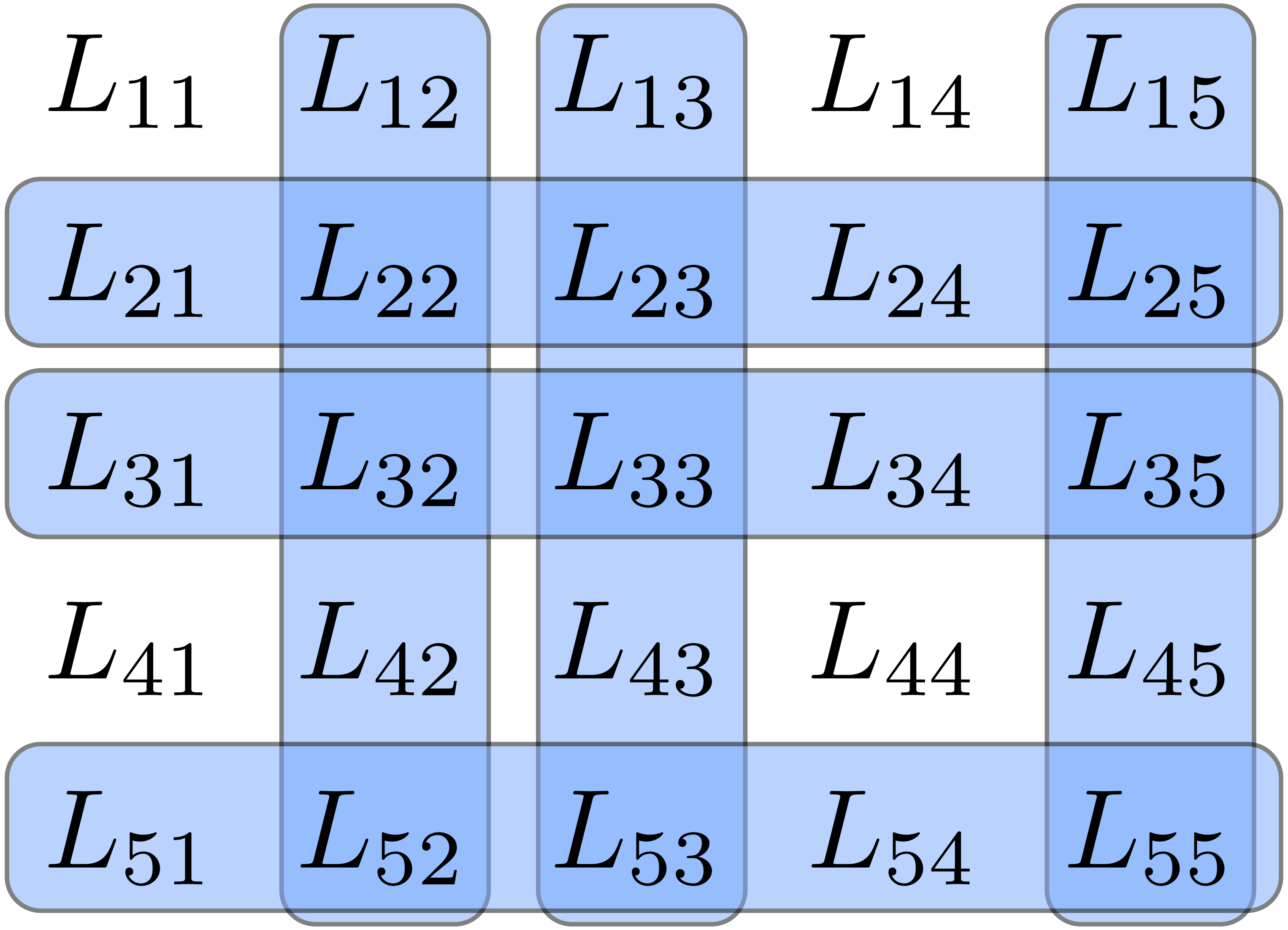
$$\mathcal{P}(\{2, 3, 5\}) \propto$$

$$\begin{matrix} L_{11} & L_{12} & L_{13} & L_{14} & L_{15} \\ L_{21} & L_{22} & L_{23} & L_{24} & L_{25} \\ L_{31} & L_{32} & L_{33} & L_{34} & L_{35} \\ L_{41} & L_{42} & L_{43} & L_{44} & L_{45} \\ L_{51} & L_{52} & L_{53} & L_{54} & L_{55} \end{matrix}$$

DETERMINANTAL

$$\mathcal{P}(\{2, 3, 5\}) \propto$$



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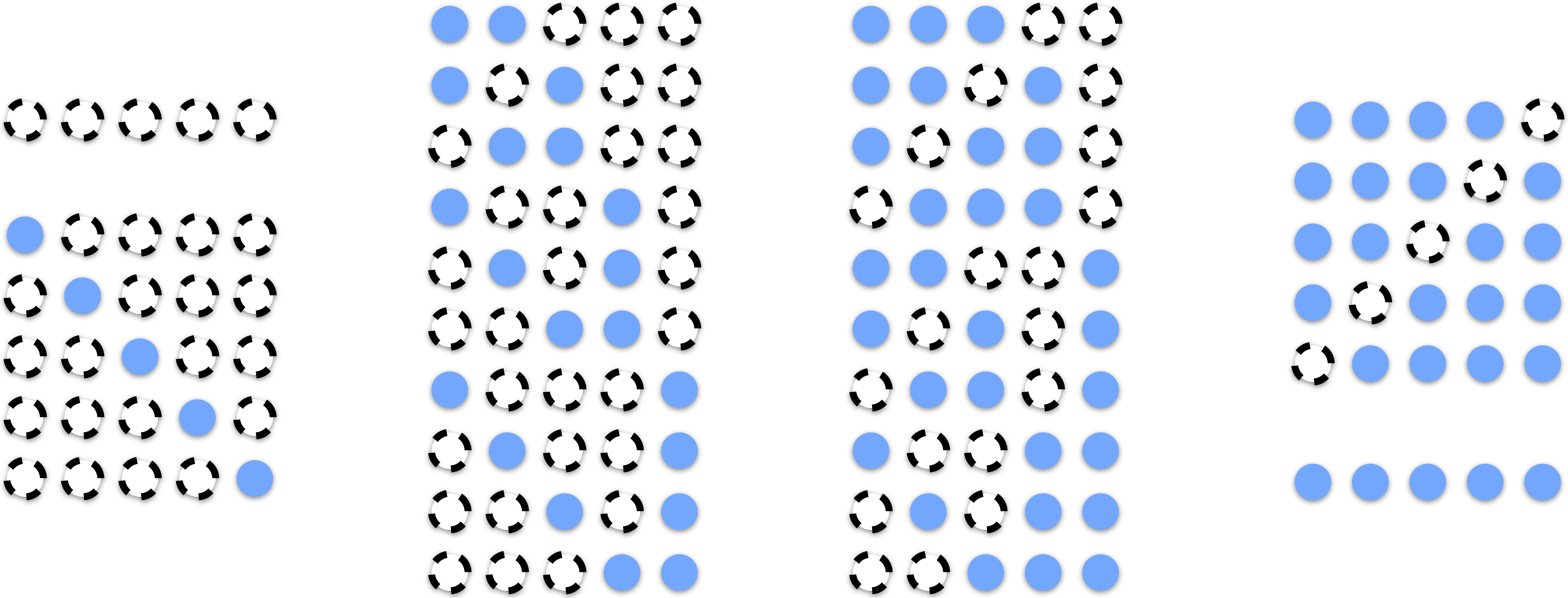
DETERMINANTAL

$$\mathcal{P}(\{2,3,5\}) = \det\begin{pmatrix} \boxed{\begin{matrix} L_{22} & L_{23} & L_{25} \\ L_{32} & L_{33} & L_{35} \\ L_{52} & L_{53} & L_{55} \end{matrix}} \end{pmatrix}$$



DETERMINANTAL

$$\mathcal{P}(\{2,3,5\}) = \frac{\det\begin{pmatrix} L_{22} & L_{23} & L_{25} \\ L_{32} & L_{33} & L_{35} \\ L_{52} & L_{53} & L_{55} \end{pmatrix}}{\det(L+I)}$$

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Normalizing: $\mathcal{P}_L(\mathbf{Y} = Y)$

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Sampling: $\mathbf{Y} \sim \mathcal{P}_L$

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$O(N^3)$

1. DIMENSIONALITY REDUCTION

DUAL KERNEL

KULESZA AND TASKAR (NIPS 2010)

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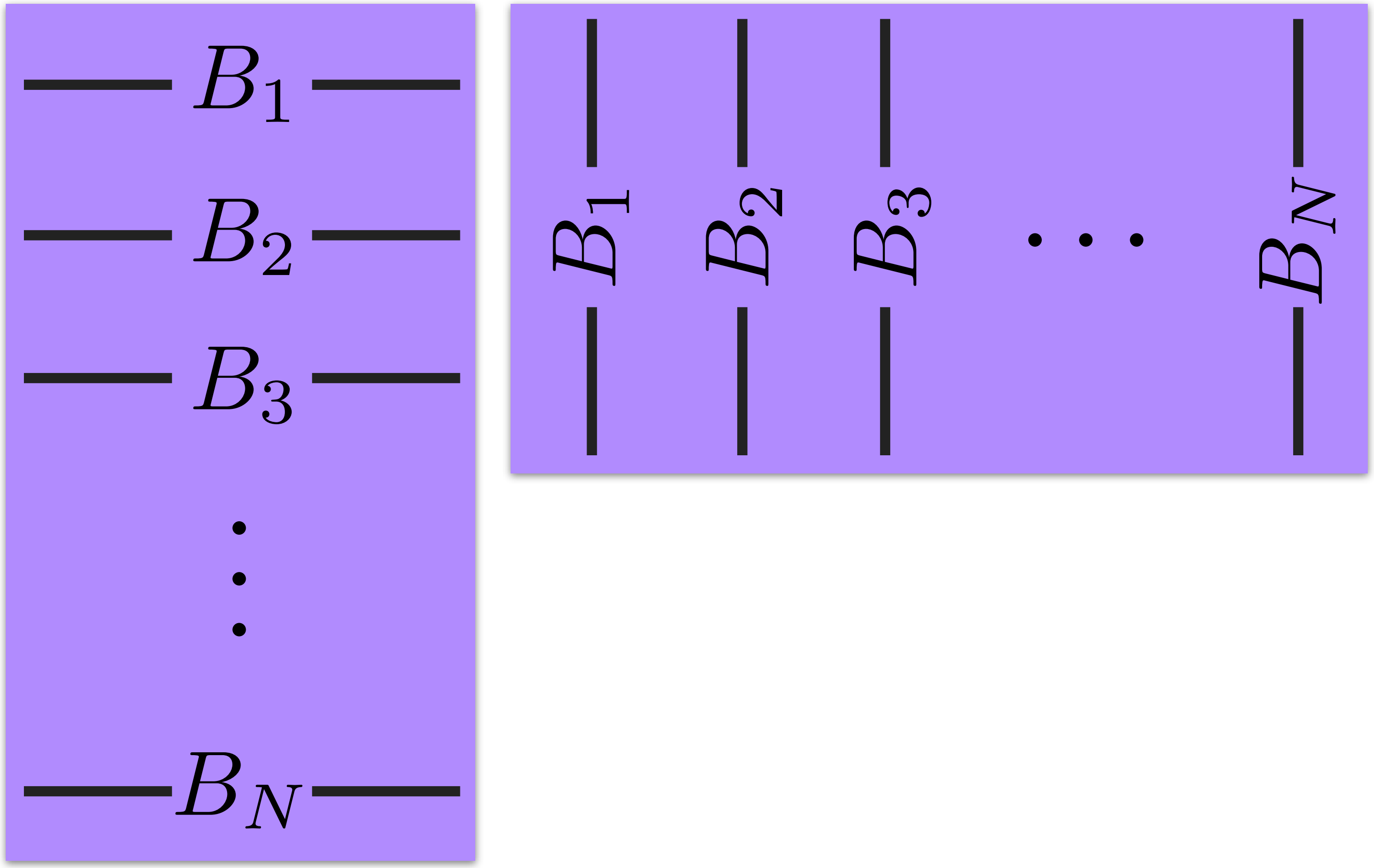
$$L$$



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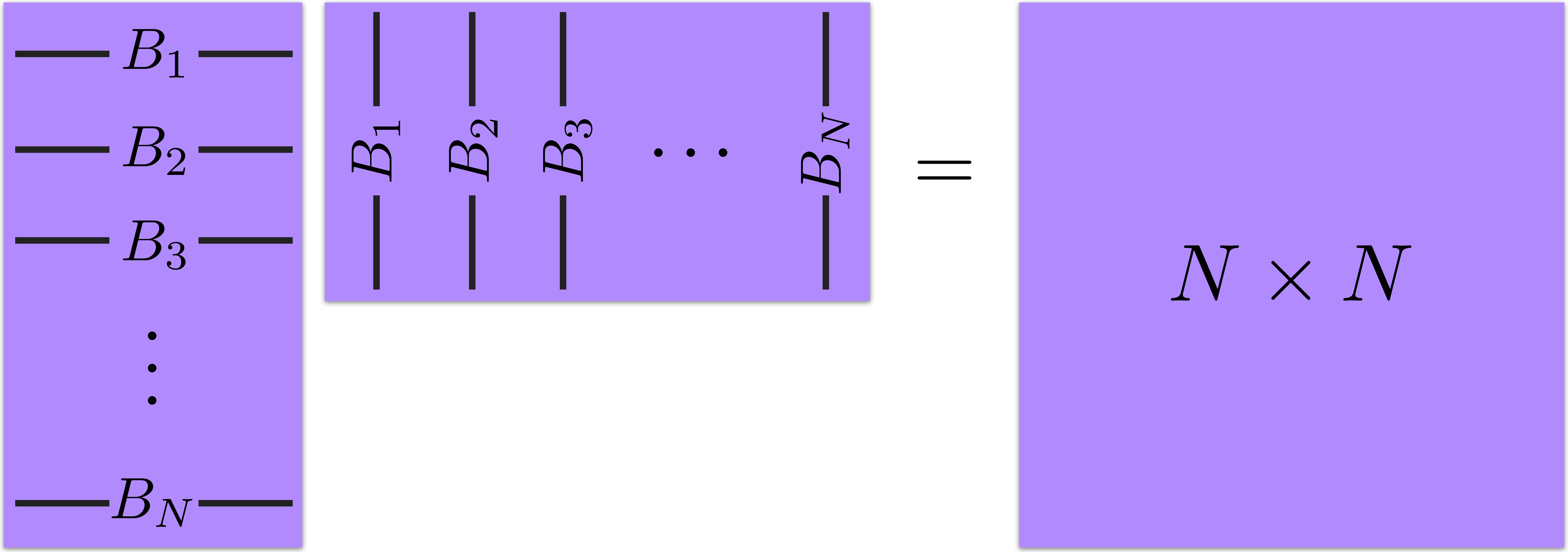
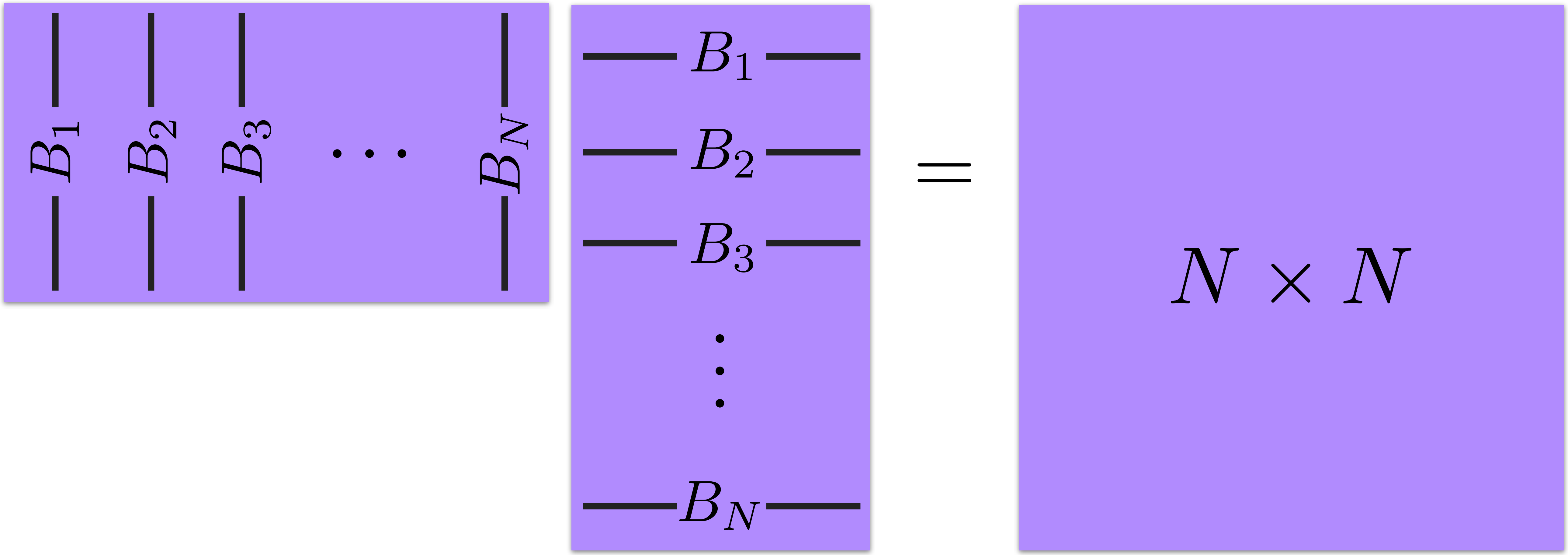


DUAL KERNEL

KULESZA AND TASKAR (NIPS 2010)

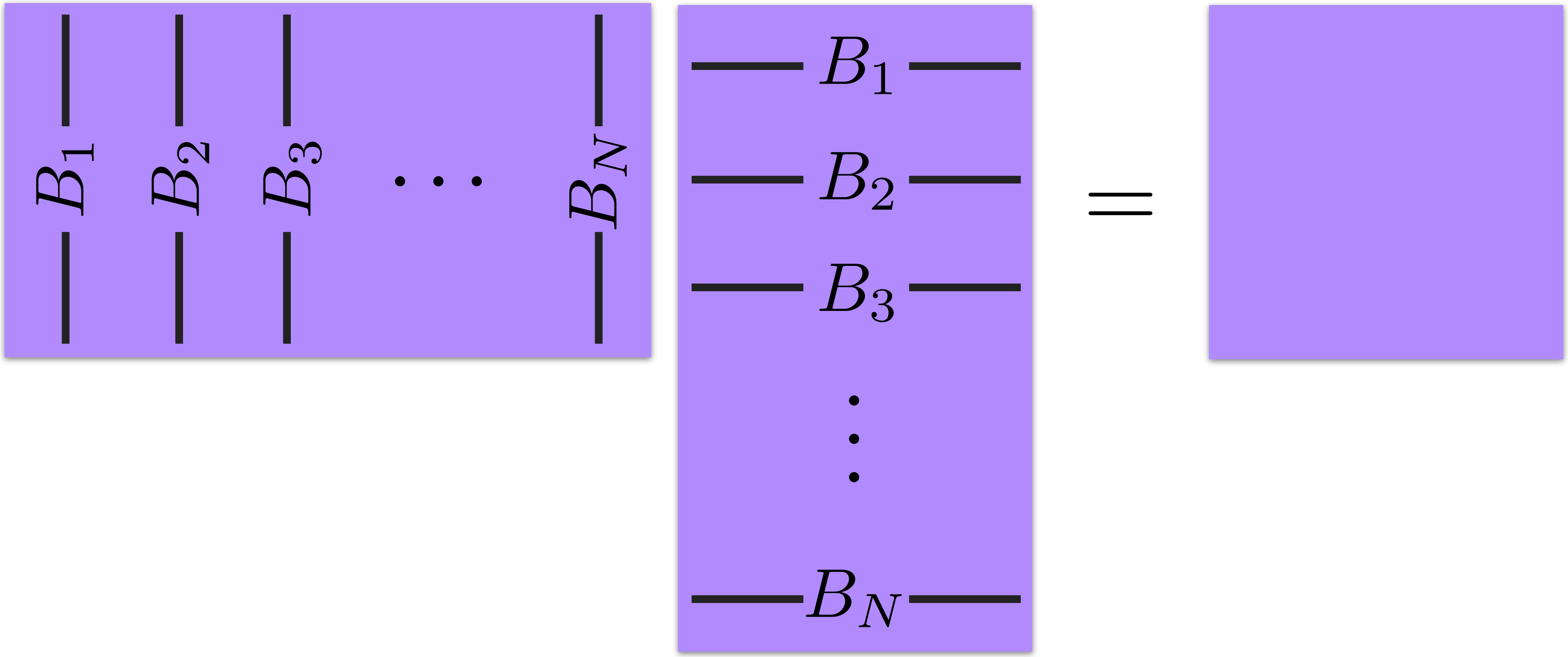
$$C$$



DUAL KERNEL

KULESZA AND TASKAR (NIPS 2010)

$$C$$



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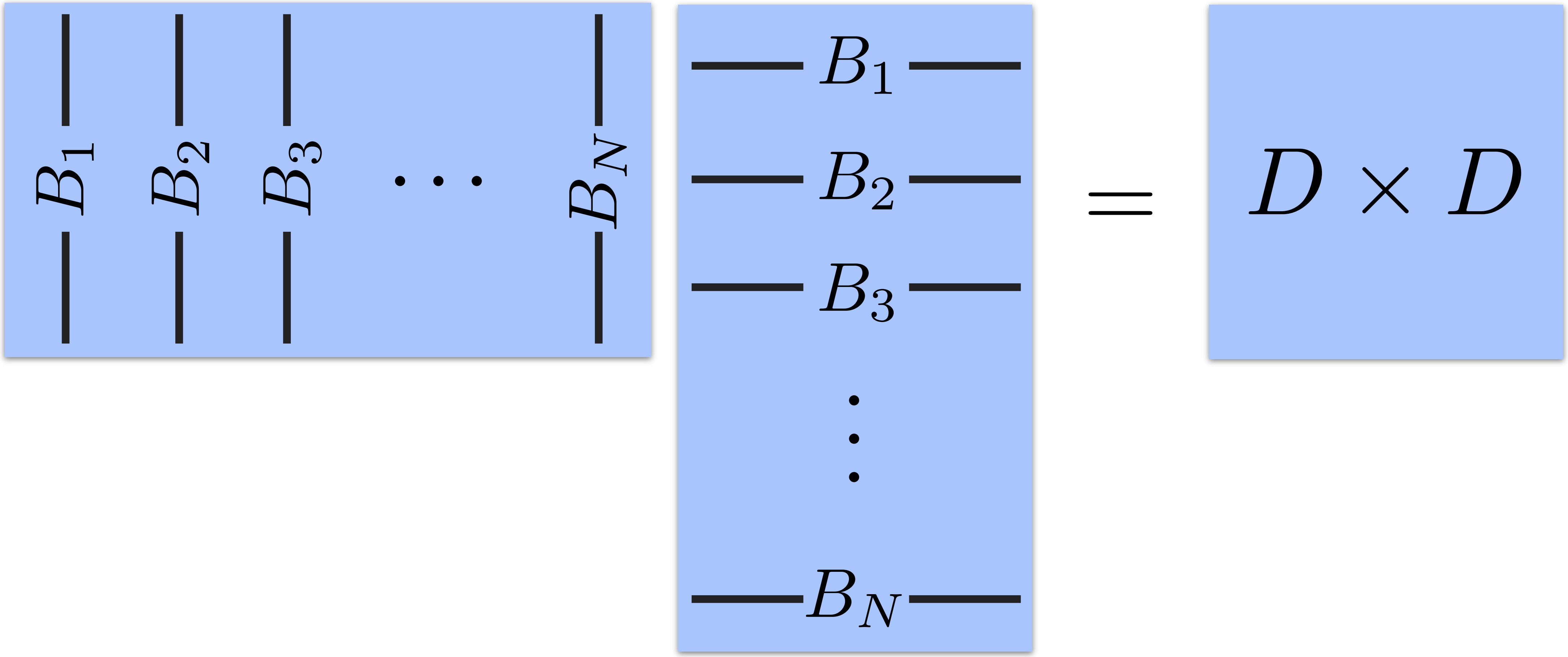
KULESZA AND TASKAR (NIPS 2010)

$$C$$



DUAL INFERENCE

DUAL INFERENCE

$L = V \Lambda V^\top$

$C = \hat{V} \Lambda \hat{V}^\top$

DUAL INFERENCE

$$L = V \Lambda V^\top \quad \overset{V = B^\top \hat{V} \Lambda^{-\frac{1}{2}}}{\Longleftrightarrow} \quad C = \hat{V} \Lambda \hat{V}^\top$$

DUAL INFERENCE

$L = V \Lambda V^\top$ $\quad V = B^\top \hat{V} \Lambda^{-\frac{1}{2}} \quad$ $C = \hat{V} \Lambda \hat{V}^\top$

Normalizing $\sum_Y \det(L_Y)$ $\quad O(D^3)$

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$$L = V \Lambda V^\top \quad \xLeftrightarrow{\;V = B^\top \hat{V} \Lambda^{-\frac{1}{2}}\;} \quad C = \hat{V} \Lambda \hat{V}^\top$$

Normalizing $\sum_Y \det(L_Y)$ $O(D^3)$

Marginalizing & Conditioning $O(D^3 + D^2 k^2)$

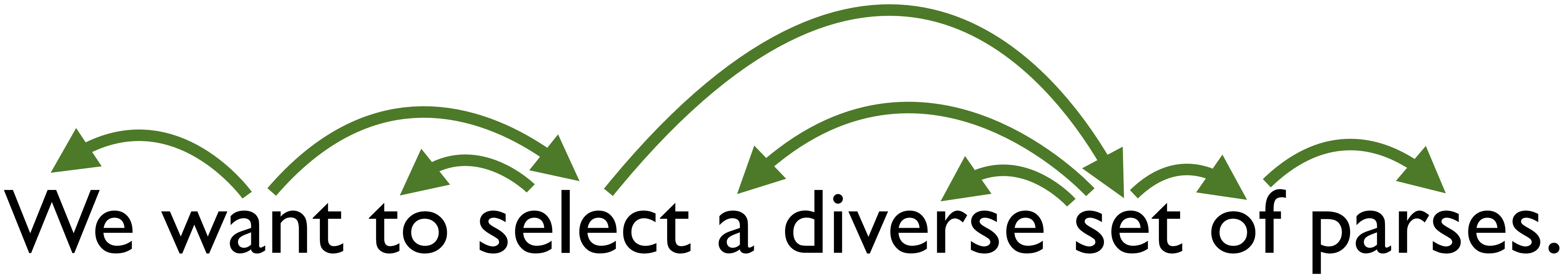
DUAL INFERENCE

$$L = V\Lambda V^\top \quad \xLeftrightarrow{\; V = B^\top \hat{V}\Lambda^{-\frac{1}{2}} \;} \quad C = \hat{V}\Lambda\hat{V}^\top$$

Normalizing $\sum_Y \det(L_Y)$	$O(D^3)$
Marginalizing & Conditioning	$O(D^3 + D^2k^2)$
Sampling $\mathbf{Y} \sim \mathcal{P}_L$	$O(ND^2k)$

EXPONENTIAL N

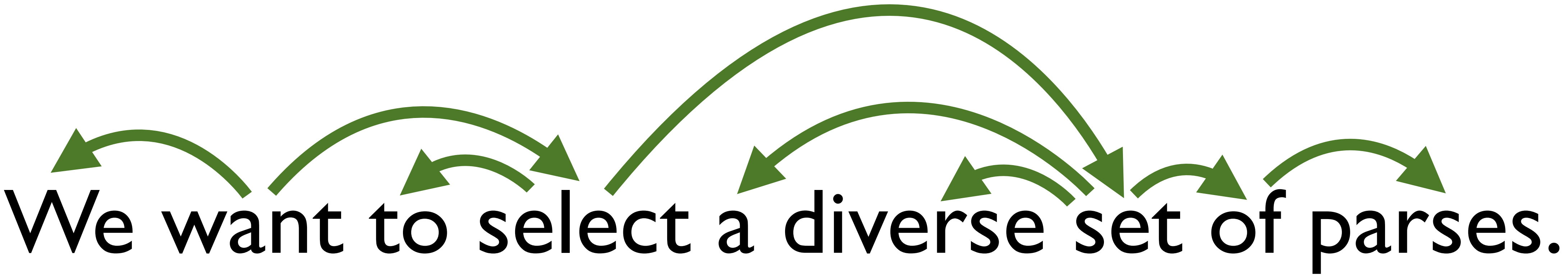
EXPONENTIAL N



We want to select a diverse set of parses.

$$N = O(\{\text{sentence length}\}^{\{\text{sentence length}\}})$$

EXPONENTIAL N



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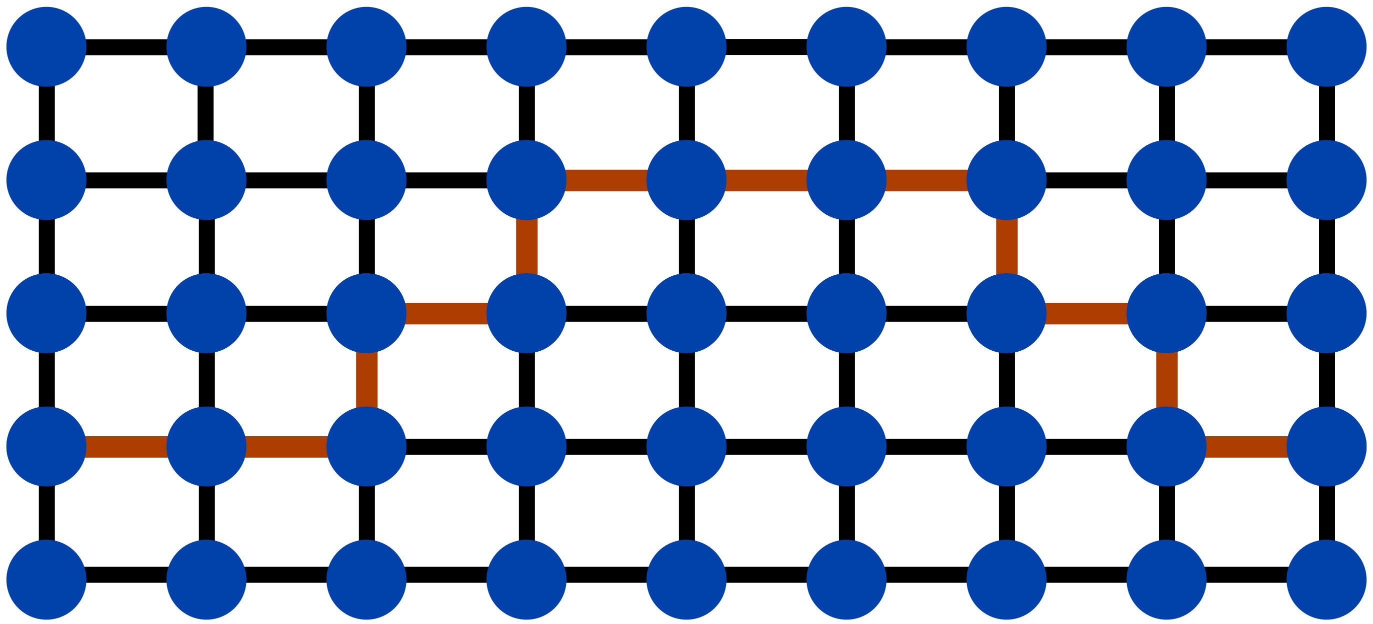
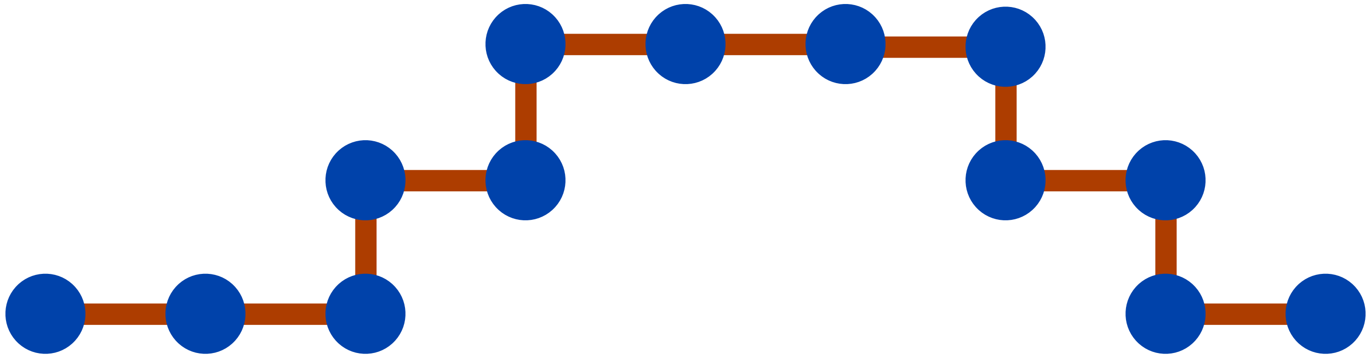


$$N = O(\{\text{node degree}\}^{\{\text{path length}\}})$$

STRUCTURE FACTORIZATION

KULESZA AND TASKAR (NIPS 2010)

$\mathbf{i} =$ 

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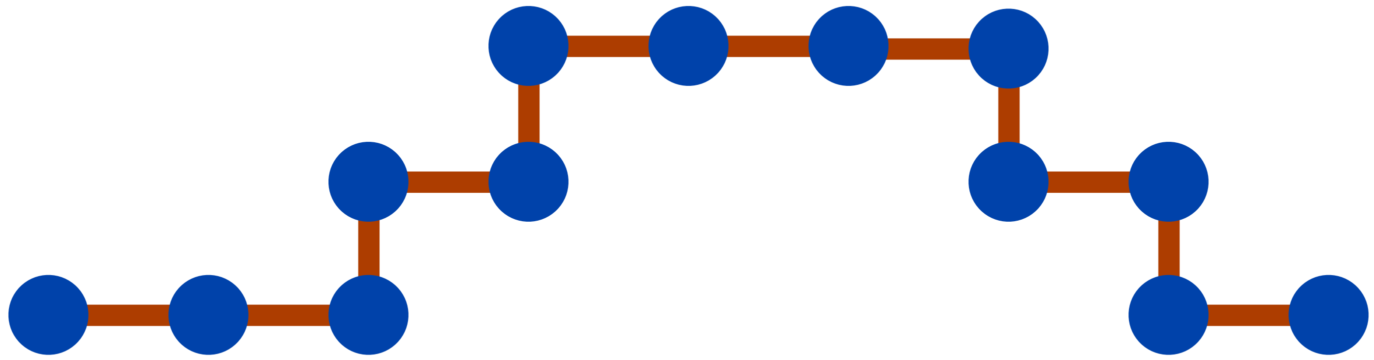
$\mathbf{i} =$ 

$$B_{\mathbf{i}} = q(\mathbf{i})\phi(\mathbf{i})$$

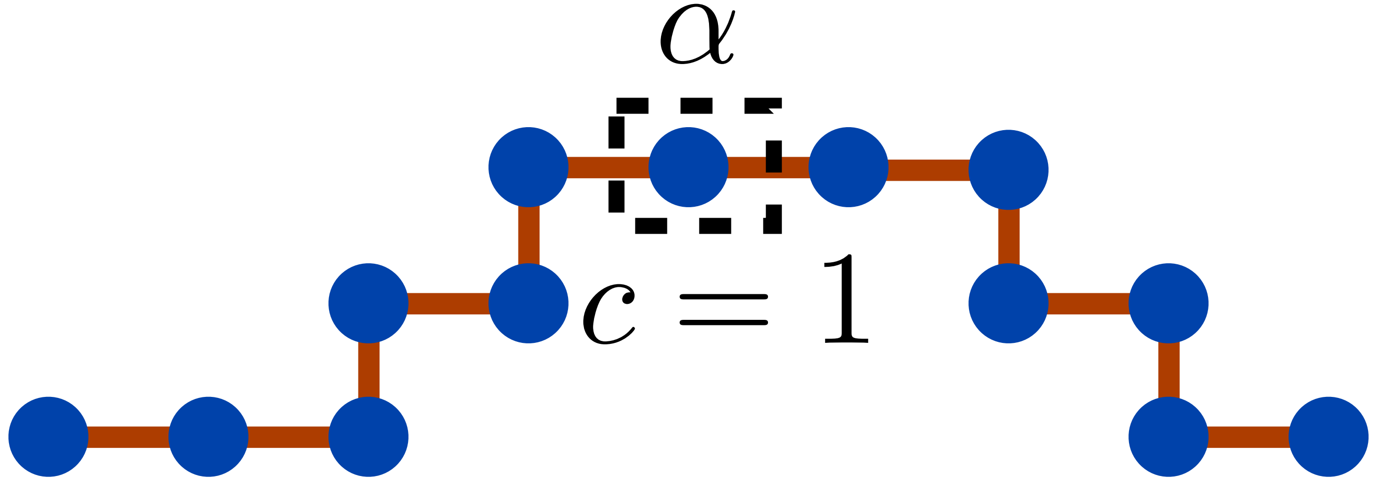
quality similarity

STRUCTURE FACTORIZATION

KULESZA AND TASKAR (NIPS 2010)

$$\mathbf{i} = \{i_\alpha\}_{\alpha \in F}$$

$$\mathbf{i} =$$



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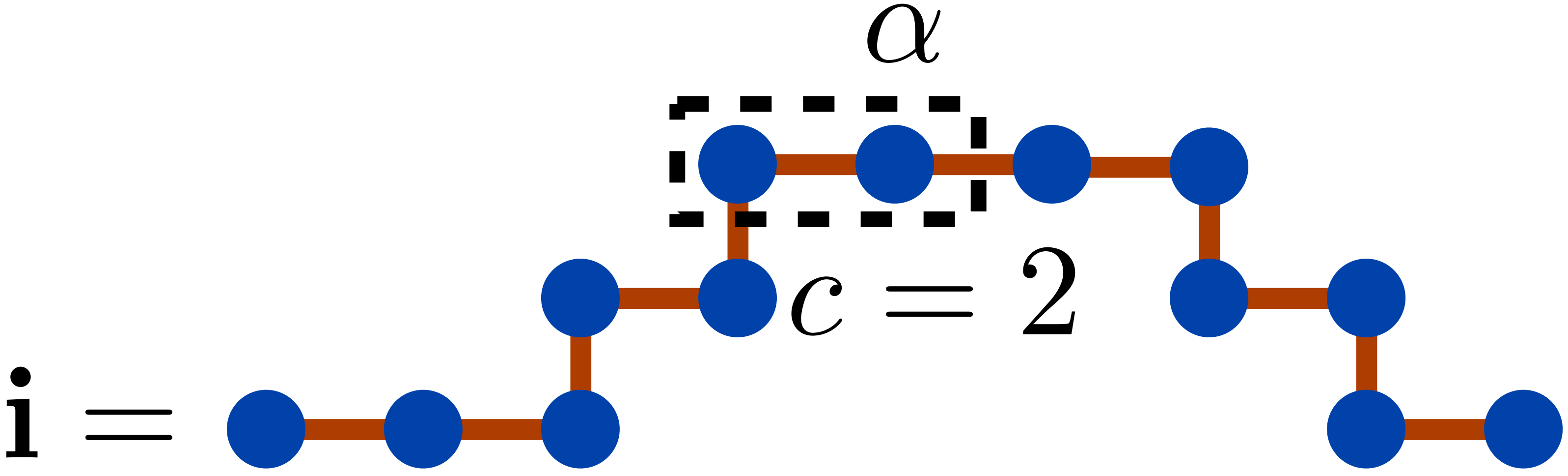

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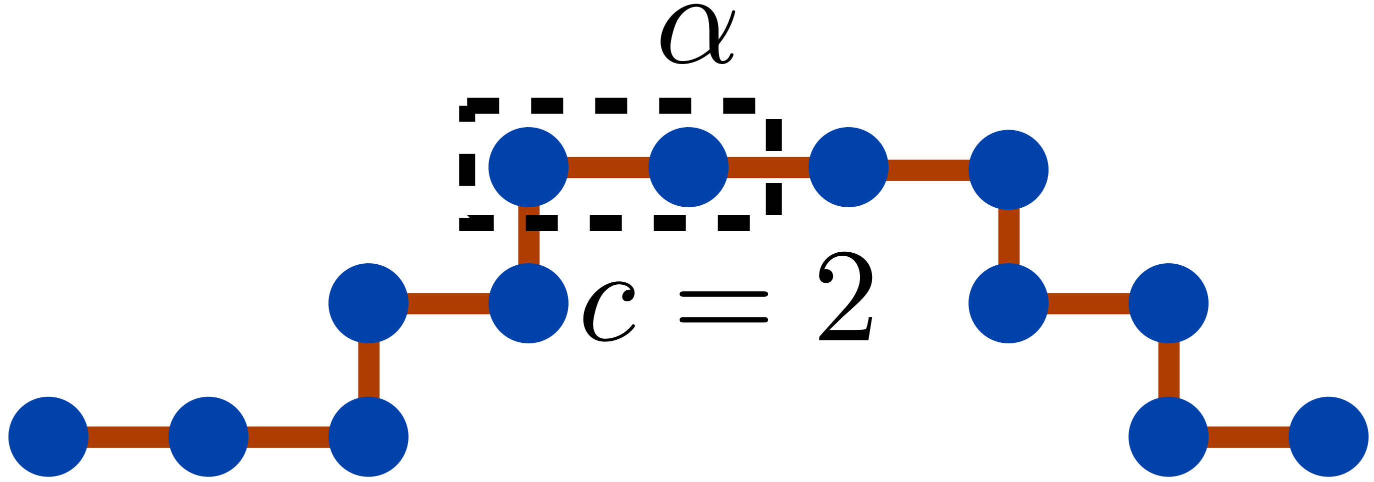


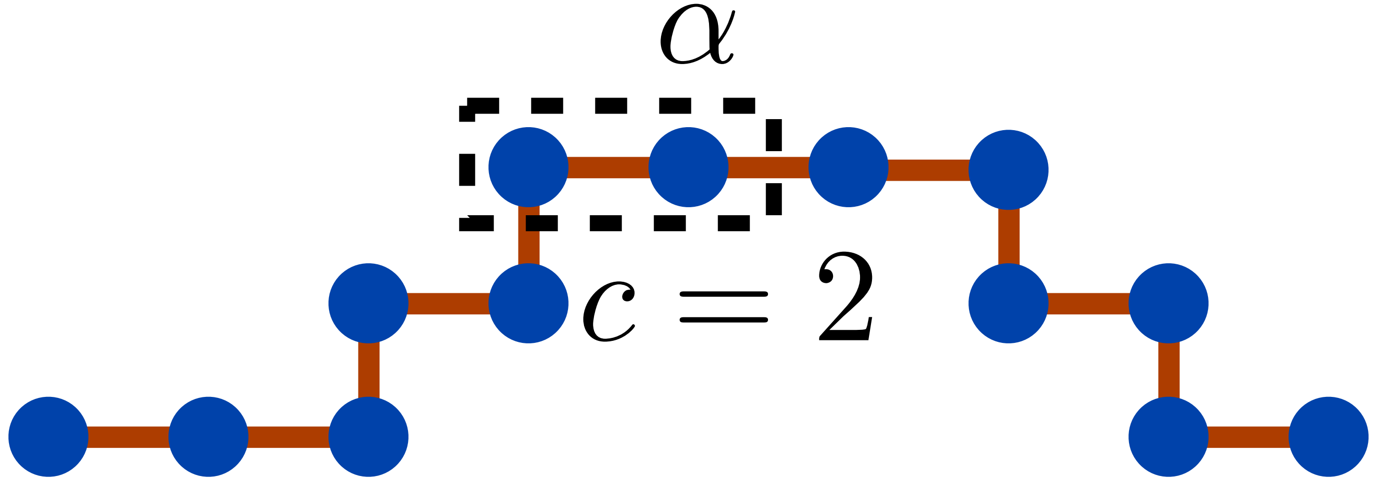
$$B_{\mathbf{i}} = \left[\prod_{\alpha \in F} q(i_\alpha)\right] \phi(\mathbf{i})$$

STRUCTURE FACTORIZATION

KULESZA AND TASKAR (NIPS 2010)

$$\mathbf{i} = \{i_\alpha\}_{\alpha \in F}$$

$\mathbf{i} =$

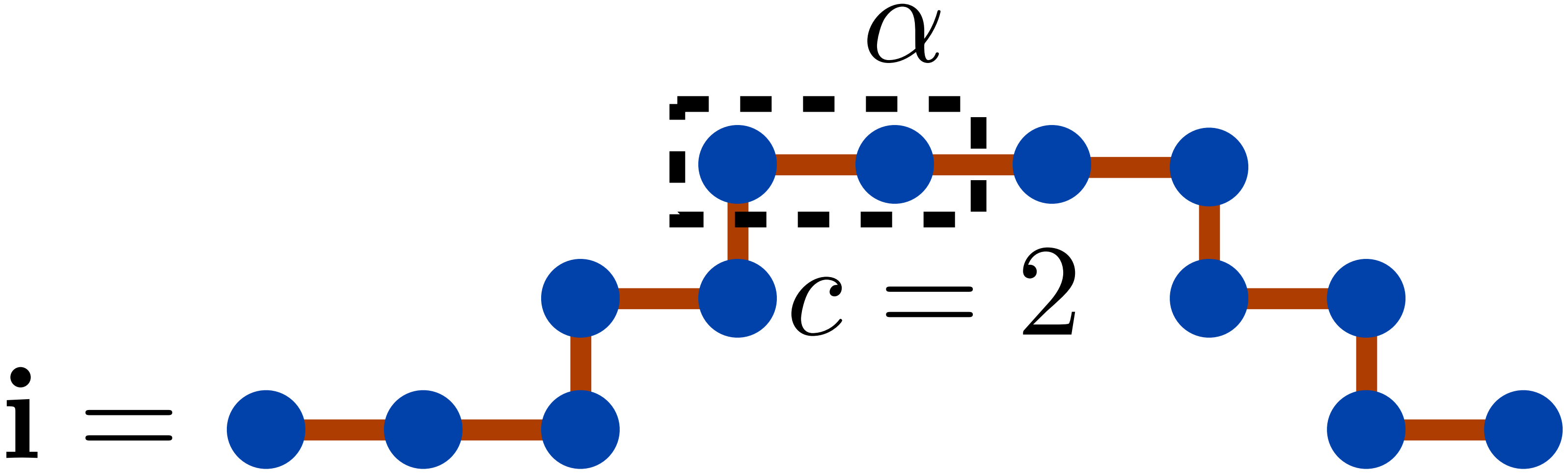


$$B_{\mathbf{i}} = \left[\prod_{\alpha \in F} q(i_\alpha)\right]\left[\sum_{\alpha \in F} \phi(i_\alpha)\right]$$

STRUCTURE FACTORIZATION

KULESZA AND TASKAR (NIPS 2010)

$$\mathbf{i} = \{i_\alpha\}_{\alpha \in F}$$



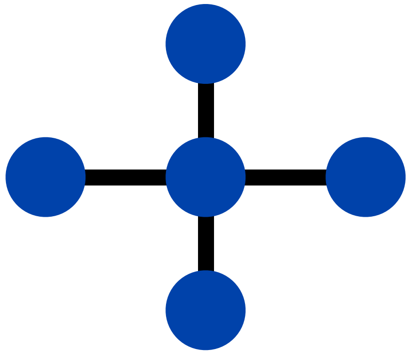
$$B_{\mathbf{i}} = \left[\prod_{\alpha \in F} q(i_\alpha)\right]\left[\sum_{\alpha \in F} \phi(i_\alpha)\right]$$

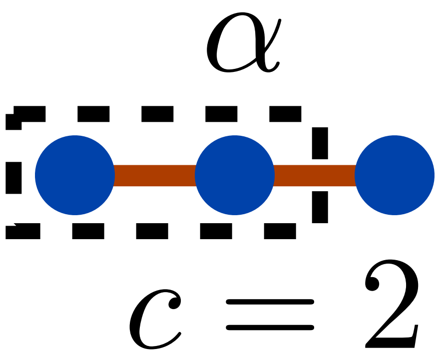
$$\mathbf{Y} \sim \mathcal{P}_L \qquad O(ND^2k)$$

STRUCTURE FACTORIZATION

KULESZA AND TASKAR (NIPS 2010)

$M =$





$R =$

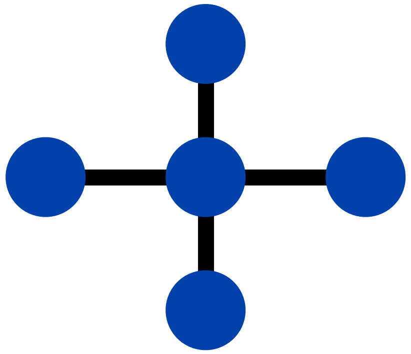
$$B_{\mathbf{i}} = \left[\prod_{\alpha\in F} q(i_\alpha)\right]\left[\sum_{\alpha\in F}\phi(i_\alpha)\right]$$

$$\mathbf{Y}\sim\mathcal{P}_L \quad O(D^2k^3 + Dk^2M^cR)$$

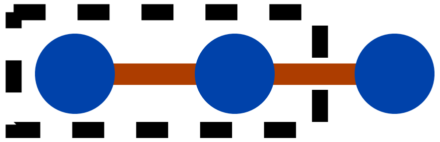
STRUCTURE FACTORIZATION

KULESZA AND TASKAR (NIPS 2010)

$M =$



α



$c = 2$

$R =$

$$B_{\mathbf{i}} = \left[\prod_{\alpha\in F} q(i_\alpha)\right]\left[\sum_{\alpha\in F} \phi(i_\alpha)\right]$$

$$\mathbf{Y} \sim \mathcal{P}_L \quad O(D^2k^3 + Dk^2M^cR)$$

$$M^cR = 4^2 * 12 = 192 \ll N = 4^{12} = 16{,}777{,}216$$

LARGE FEATURE SETS

LARGE FEATURE SETS

$N = \#\text{ of items}$

$D = \#\text{ of features}$

	Large	**Exponential**
Small	dual	dual + structure

LARGE FEATURE SETS

$N = \#$ of items

$D = \#$ of features

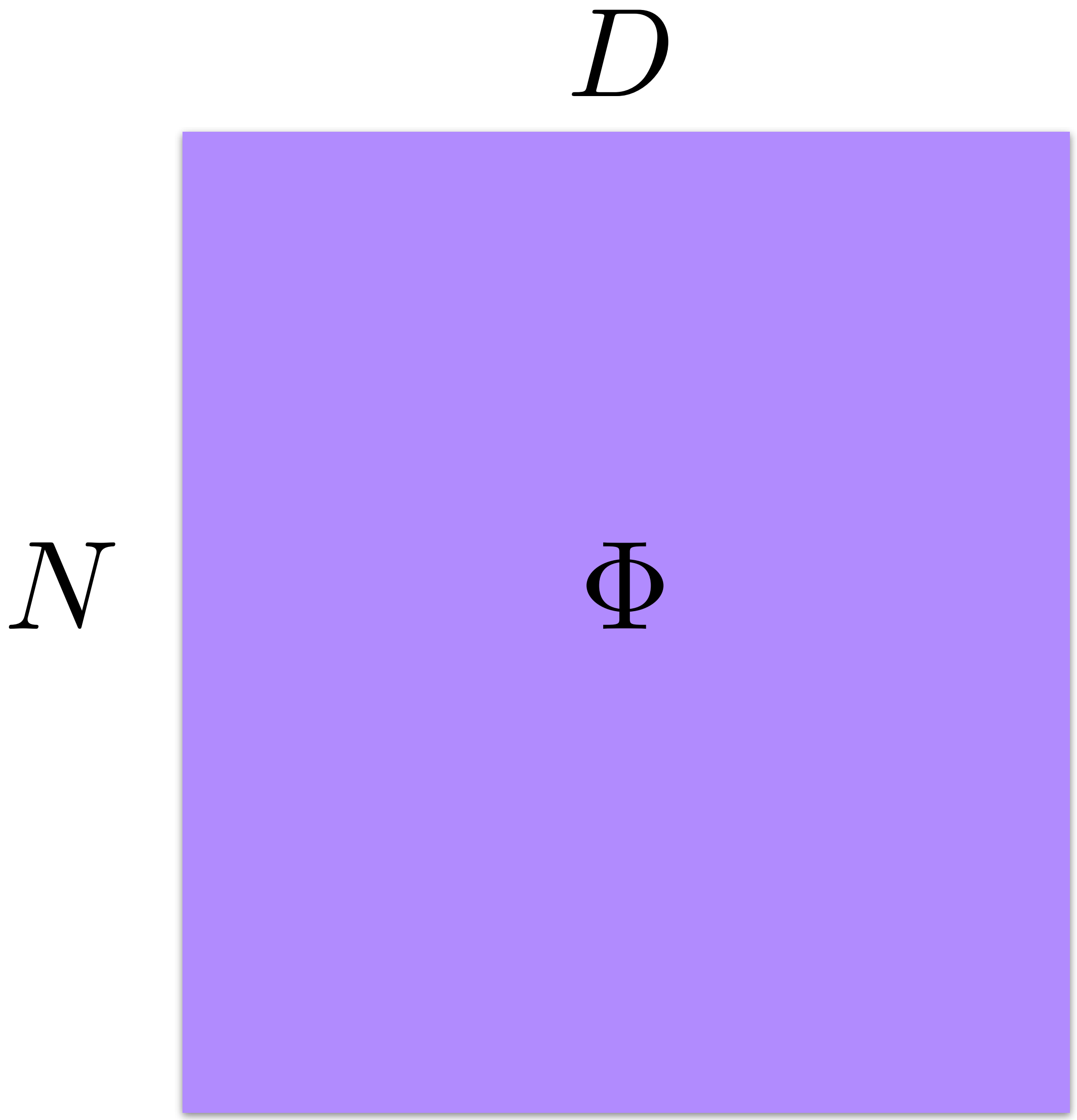
	Large	**Exponential**
Small	dual	dual + structure
Large	?	?

RANDOM PROJECTIONS

GILLENWATER, KULESZA, AND TASKAR (EMNLP 2012)

RANDOM PROJECTIONS

GILLENWATER, KULESZA, AND TASKAR (EMNLP 2012)



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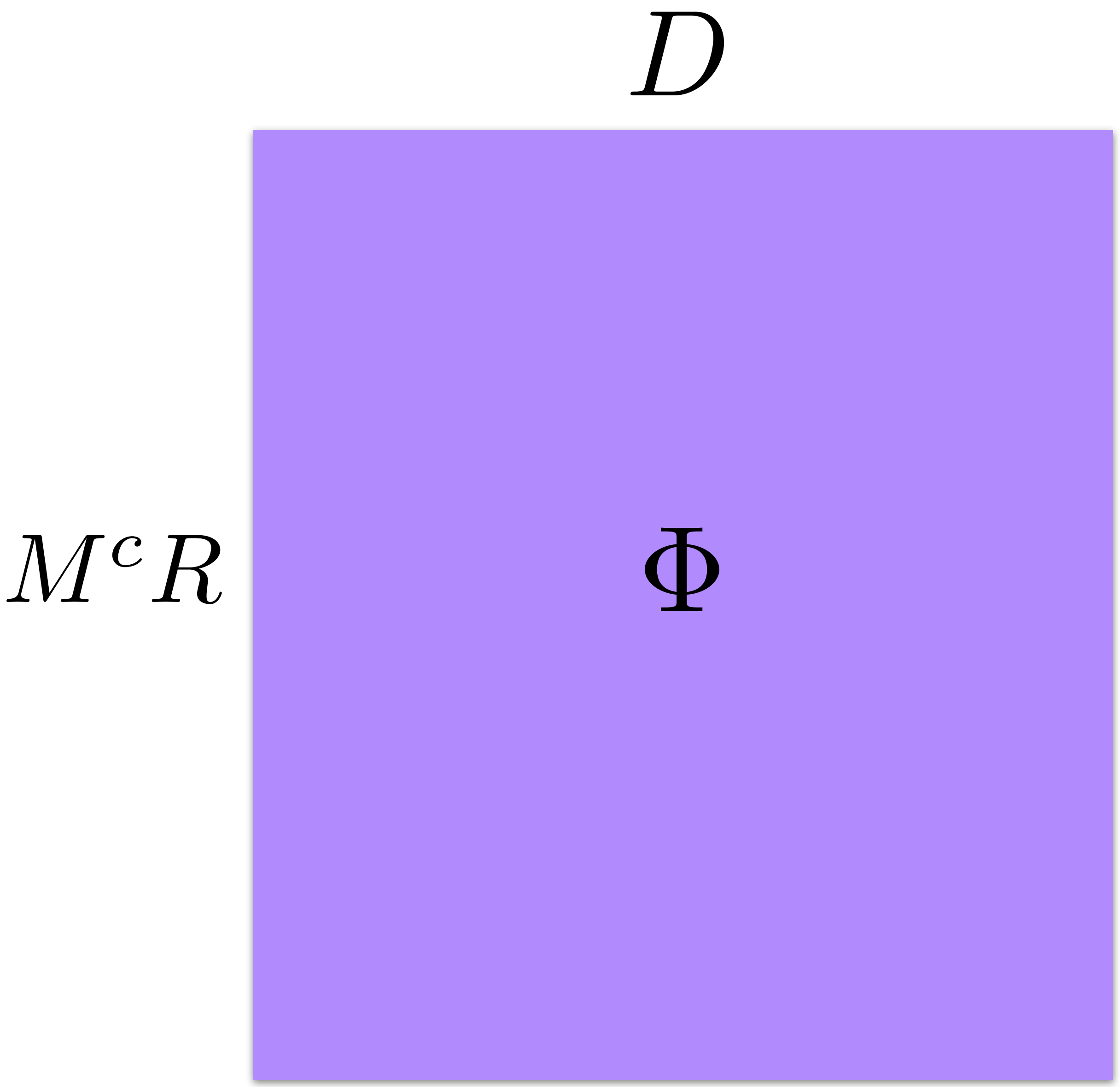

D

$M^c R$

Φ

RANDOM PROJECTIONS

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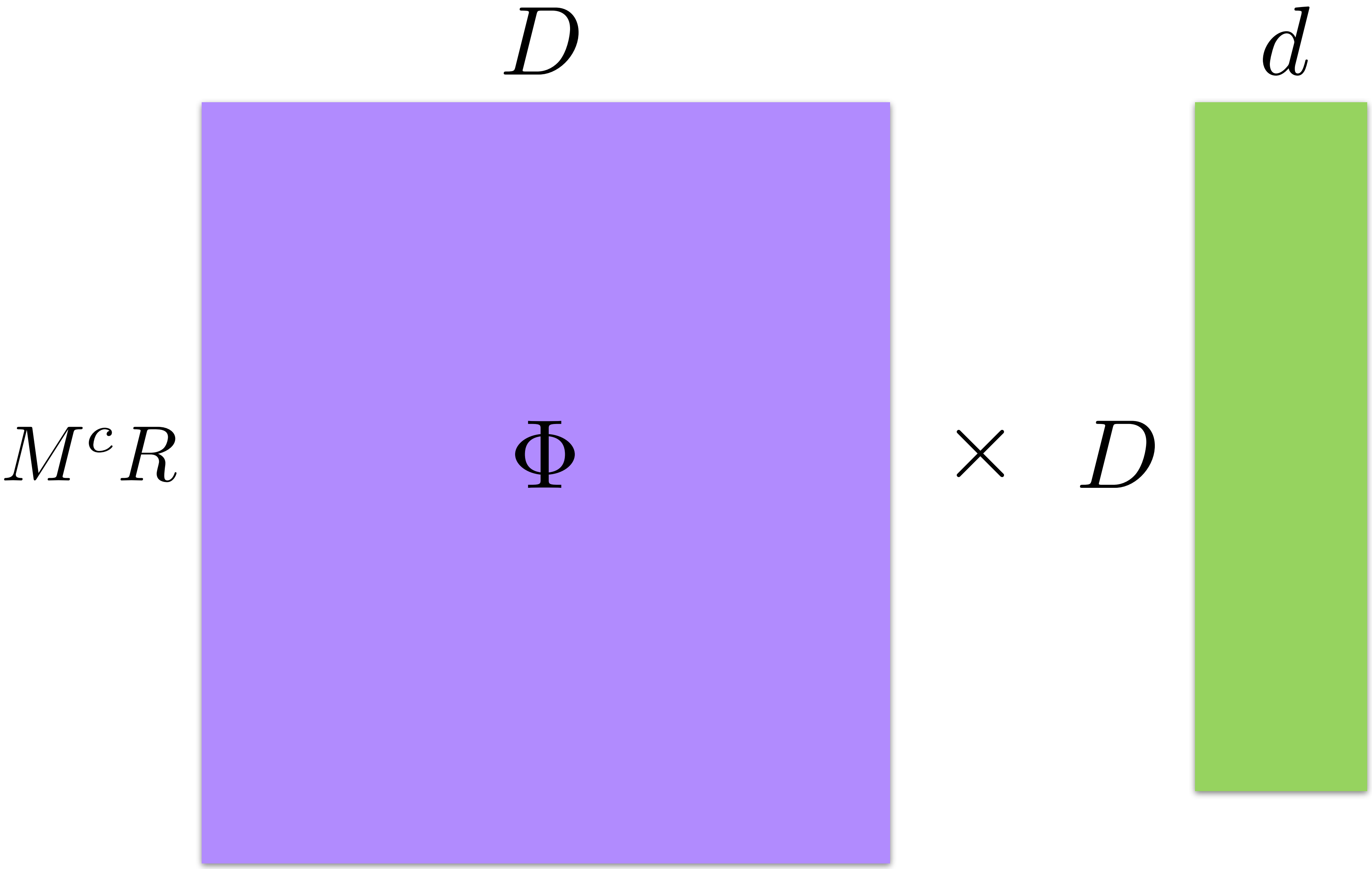

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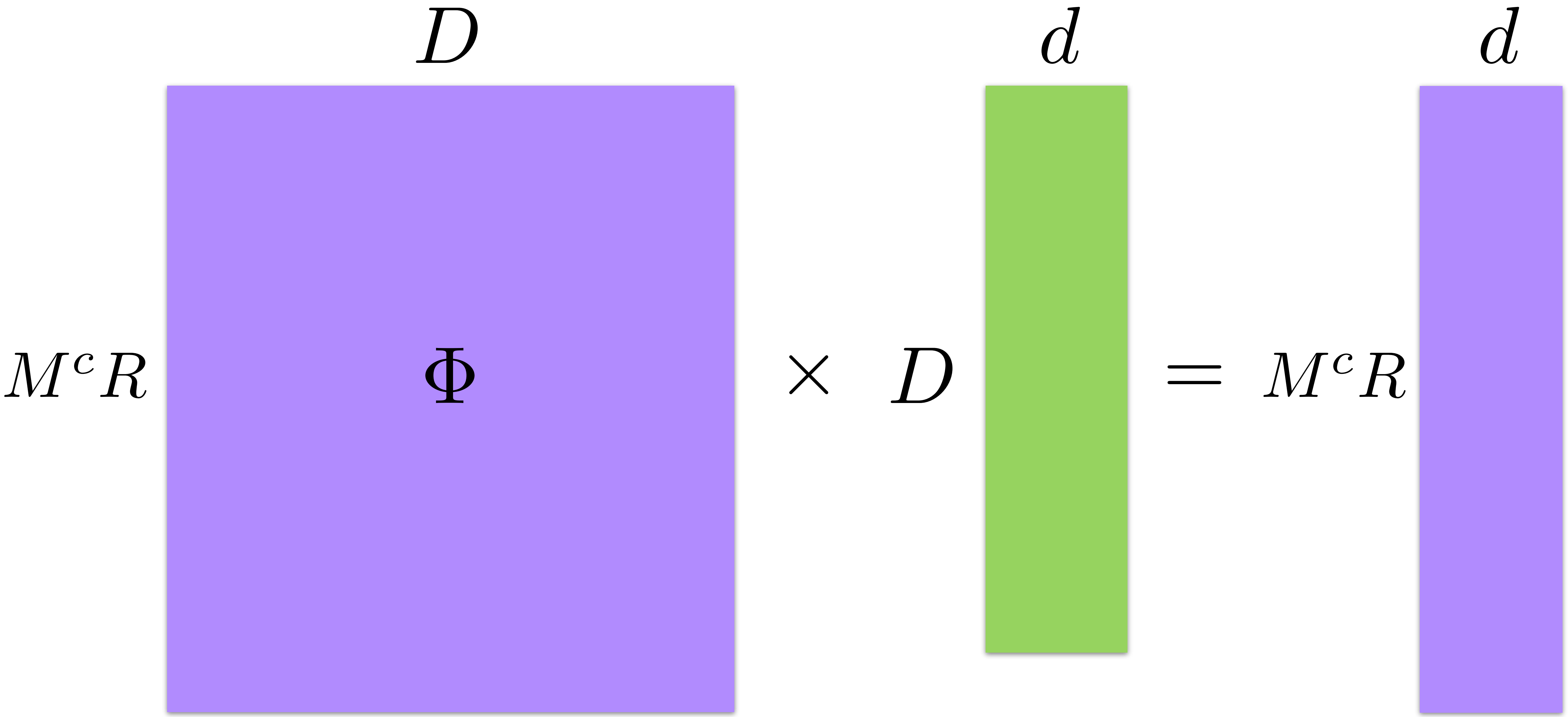

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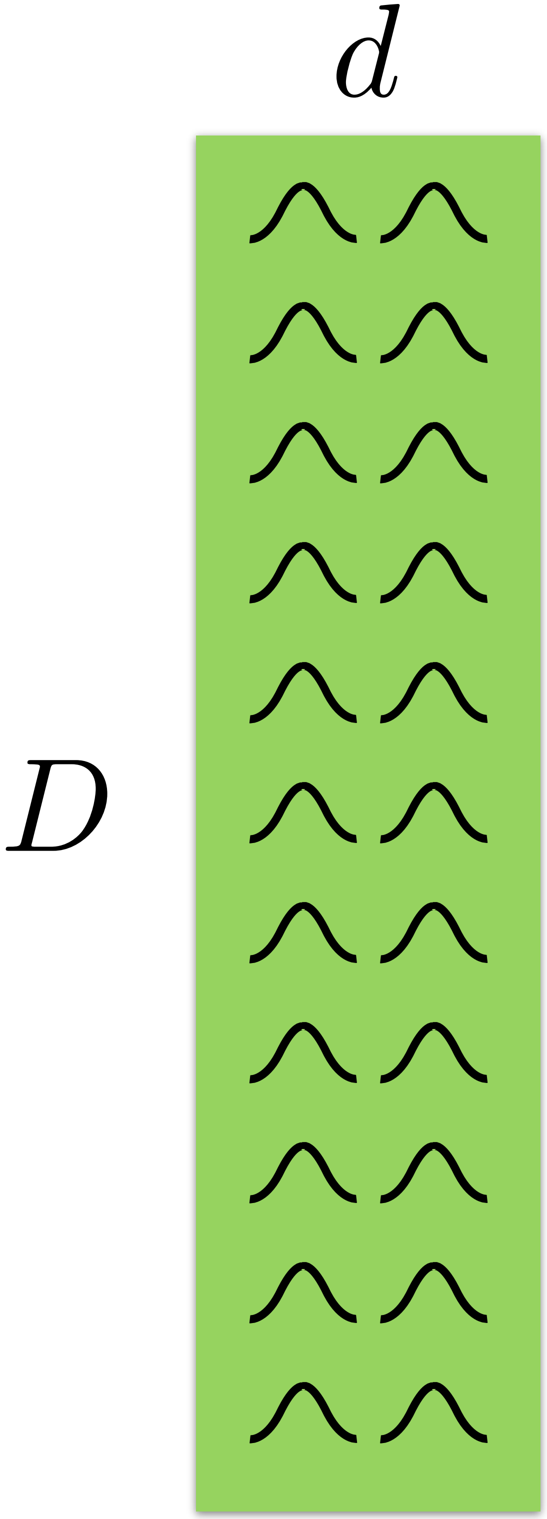

VOLUME PRESERVATION

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JOHNSON AND LINDENSTRAUSS (1984)

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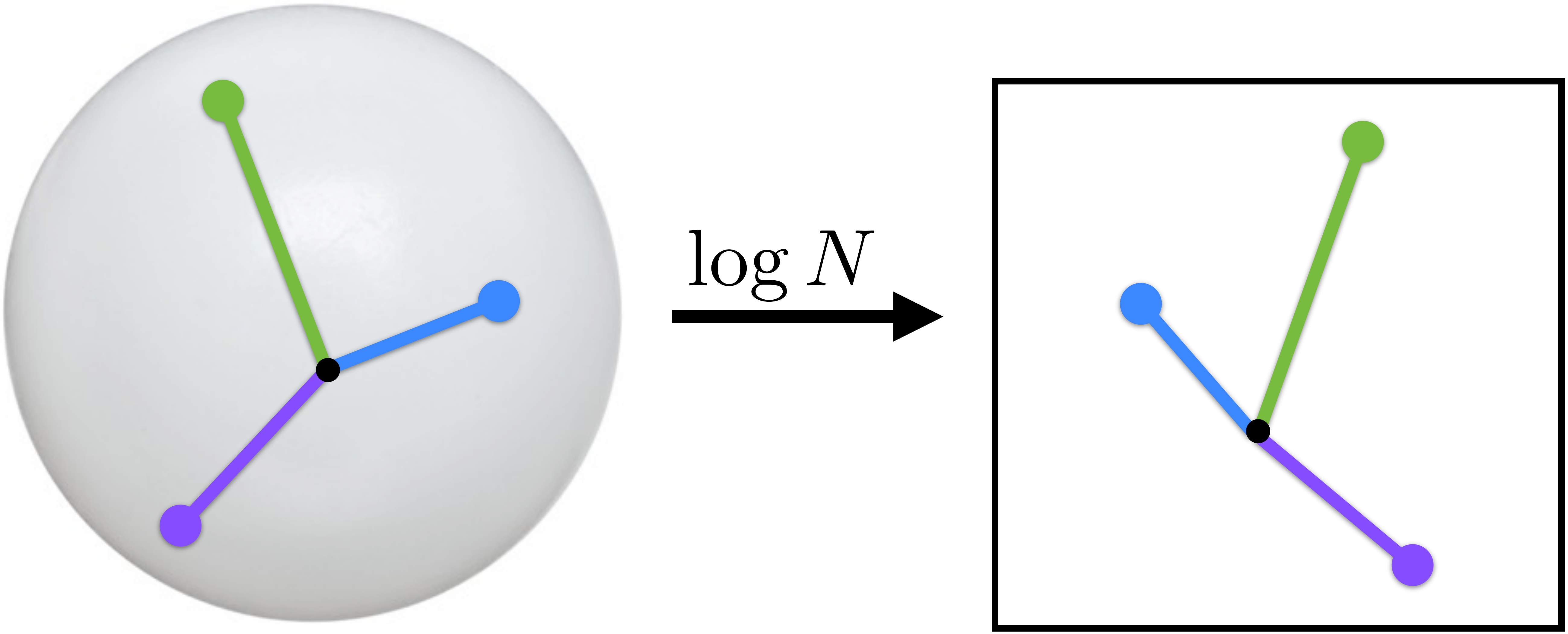
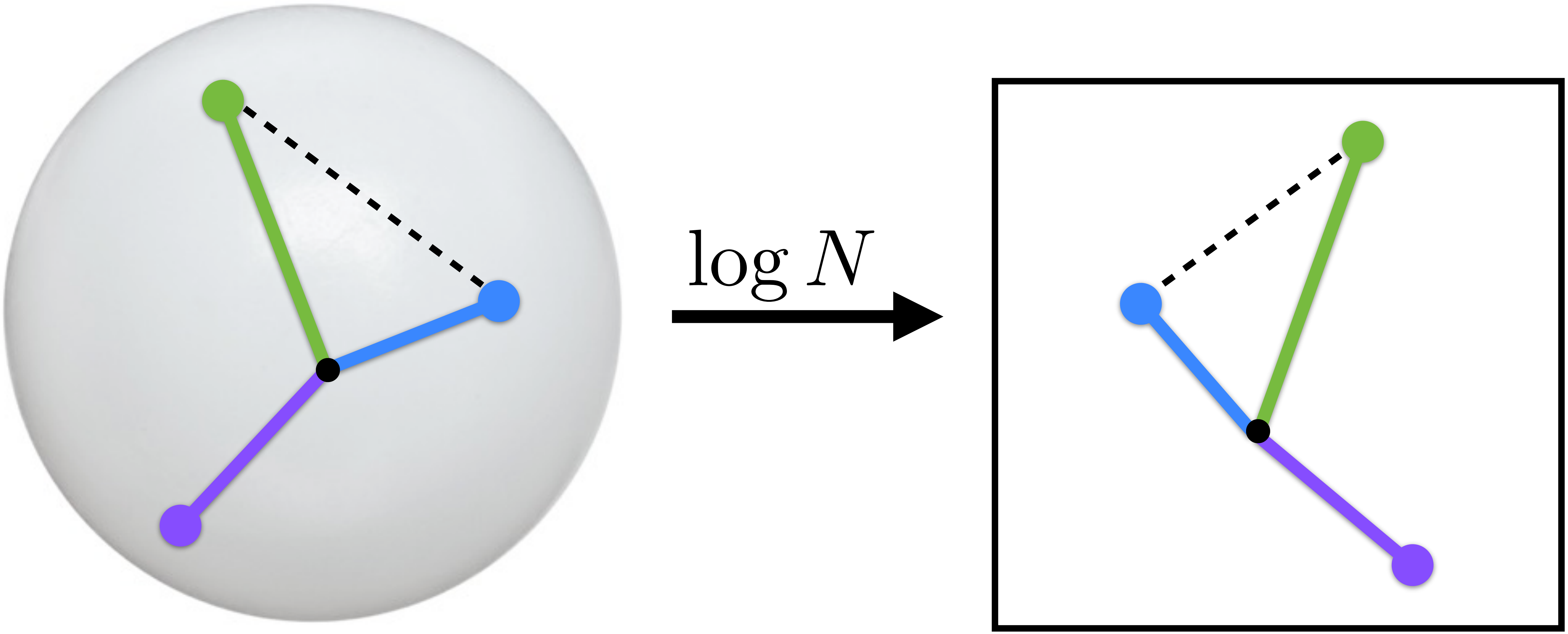


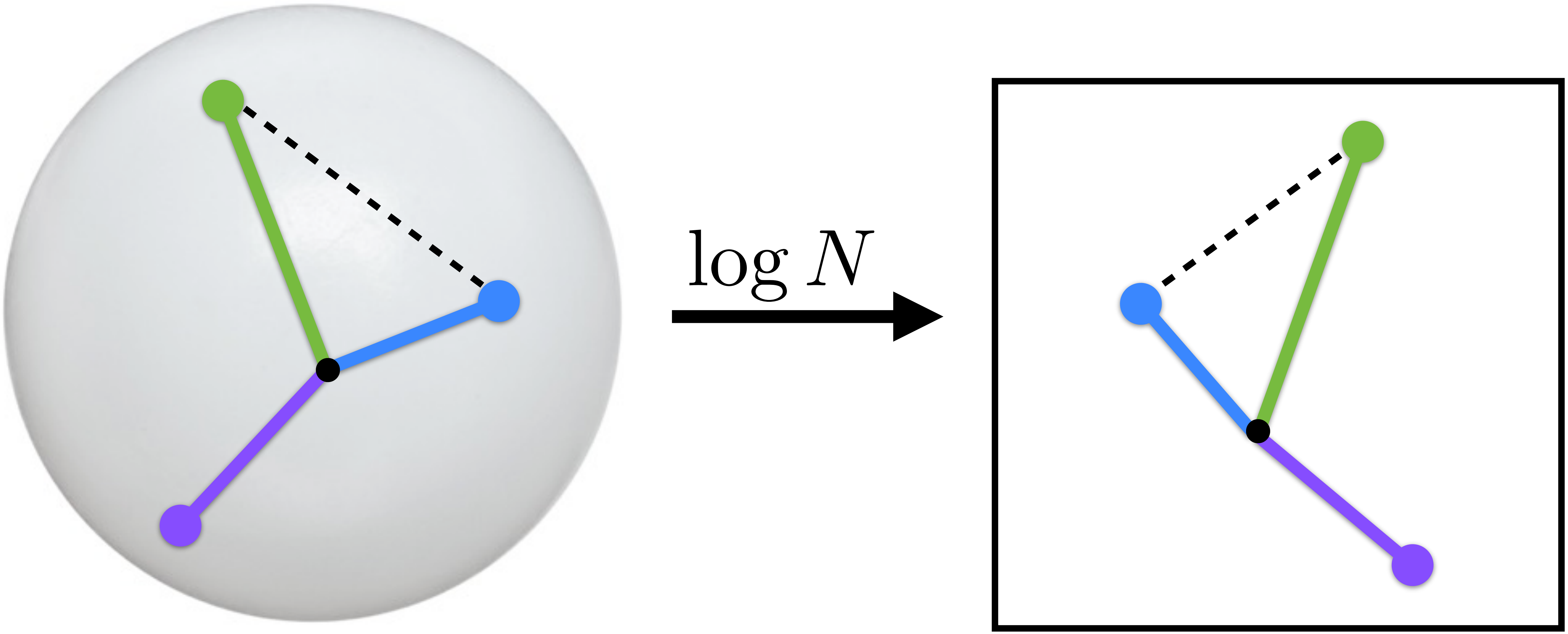
VOLUME PRESERVATION

JOHNSON AND LINDENSTRAUSS (1984)



VOLUME PRESERVATION

MAGEN AND ZOUZIAS (2008)



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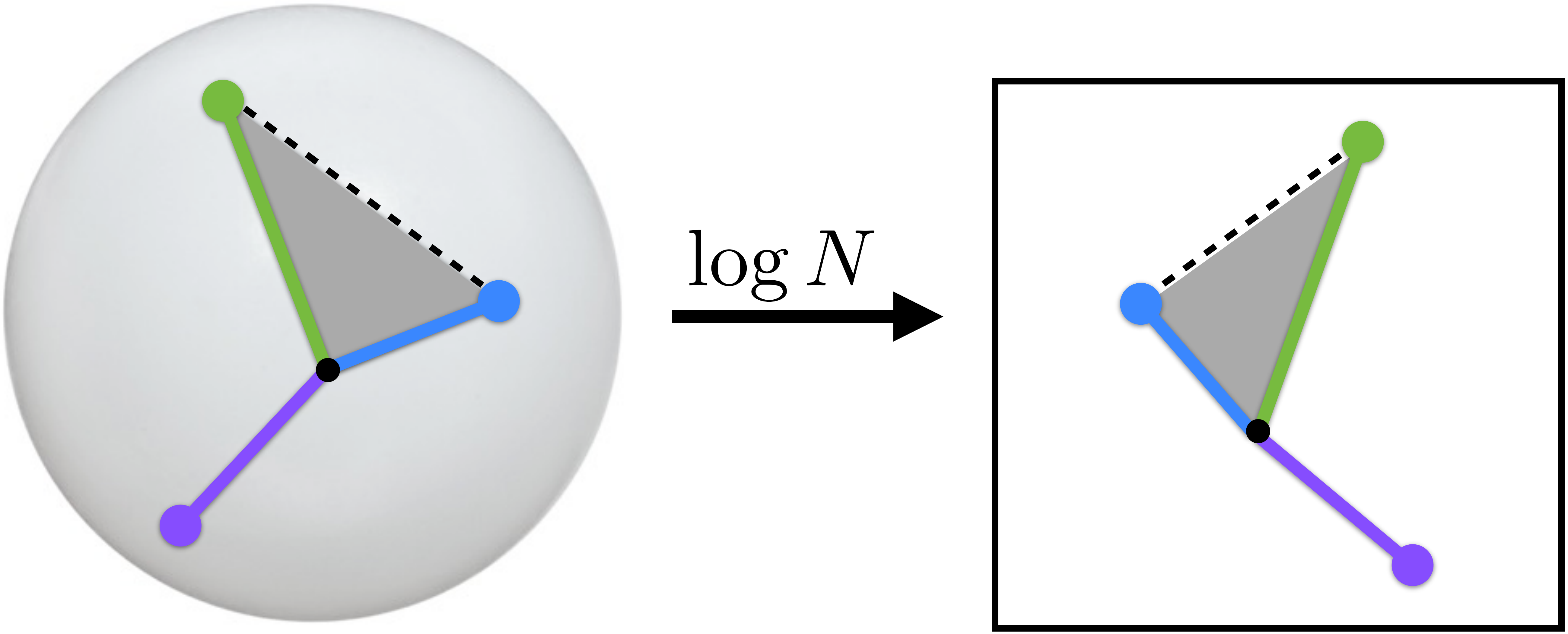


DPP PRESERVATION

GILLENWATER, KULESZA, AND TASKAR (EMNLP 2012)

$$\mathrm{vol}^2 = \det$$

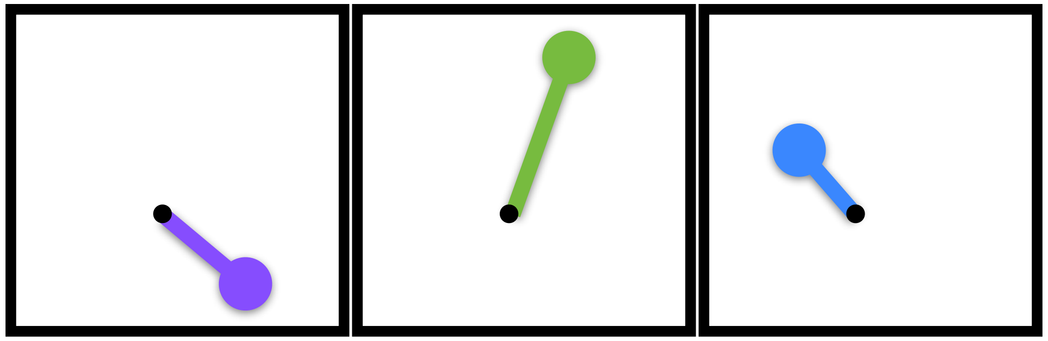
DPP PRESERVATION

GILLENWATER, KULESZA, AND TASKAR (EMNLP 2012)

$$\mathrm{vol}^2 = \det$$

$$k = 1$$

$$\approx$$



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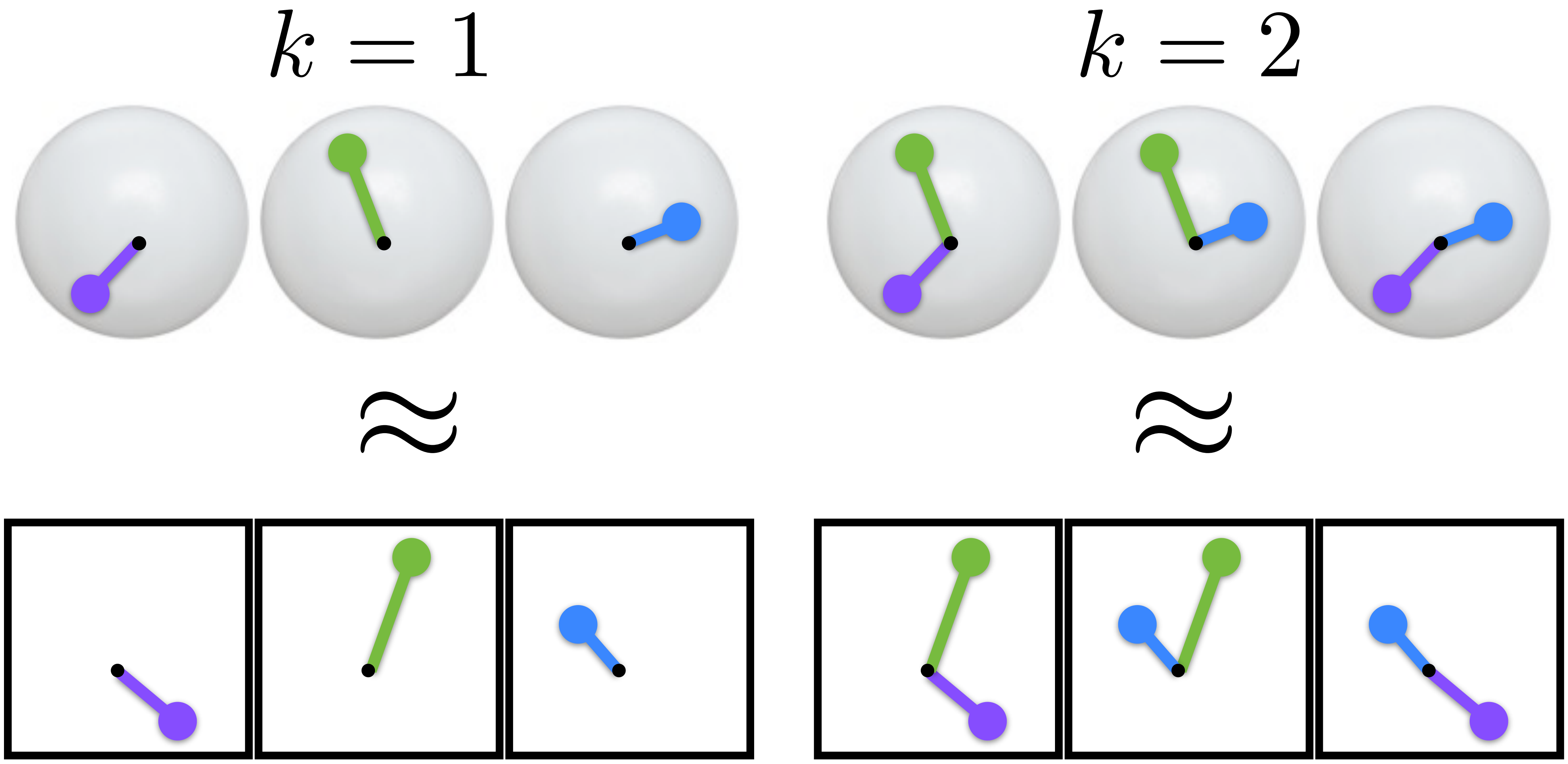

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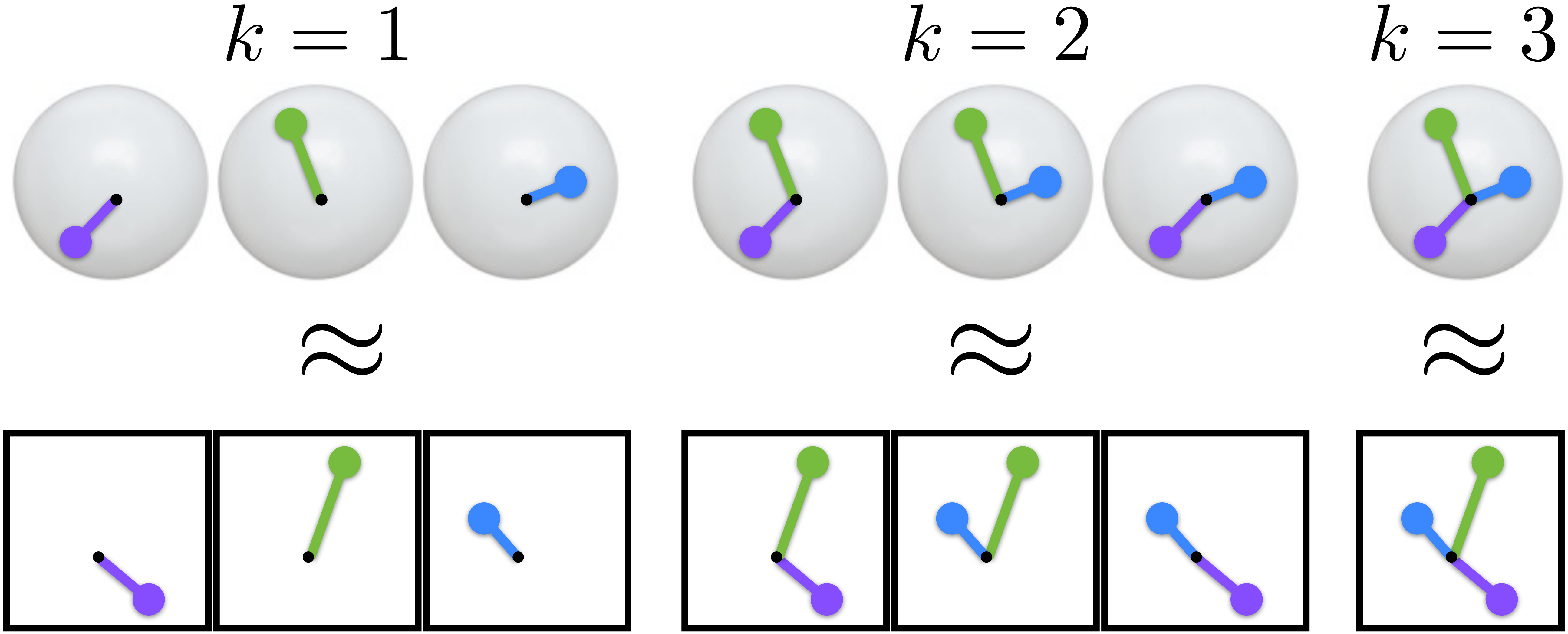

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$$d = O\left(\max\left\{\frac{k}{\epsilon}, \frac{\log(1/\delta)+\log(N)}{\epsilon^2} + k\right\}\right)$$

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$$\text{w.p. } 1-\delta: \quad \|\mathcal{P}^k - \tilde{\mathcal{P}}^k\|_1 \le e^{6k\epsilon} - 1$$

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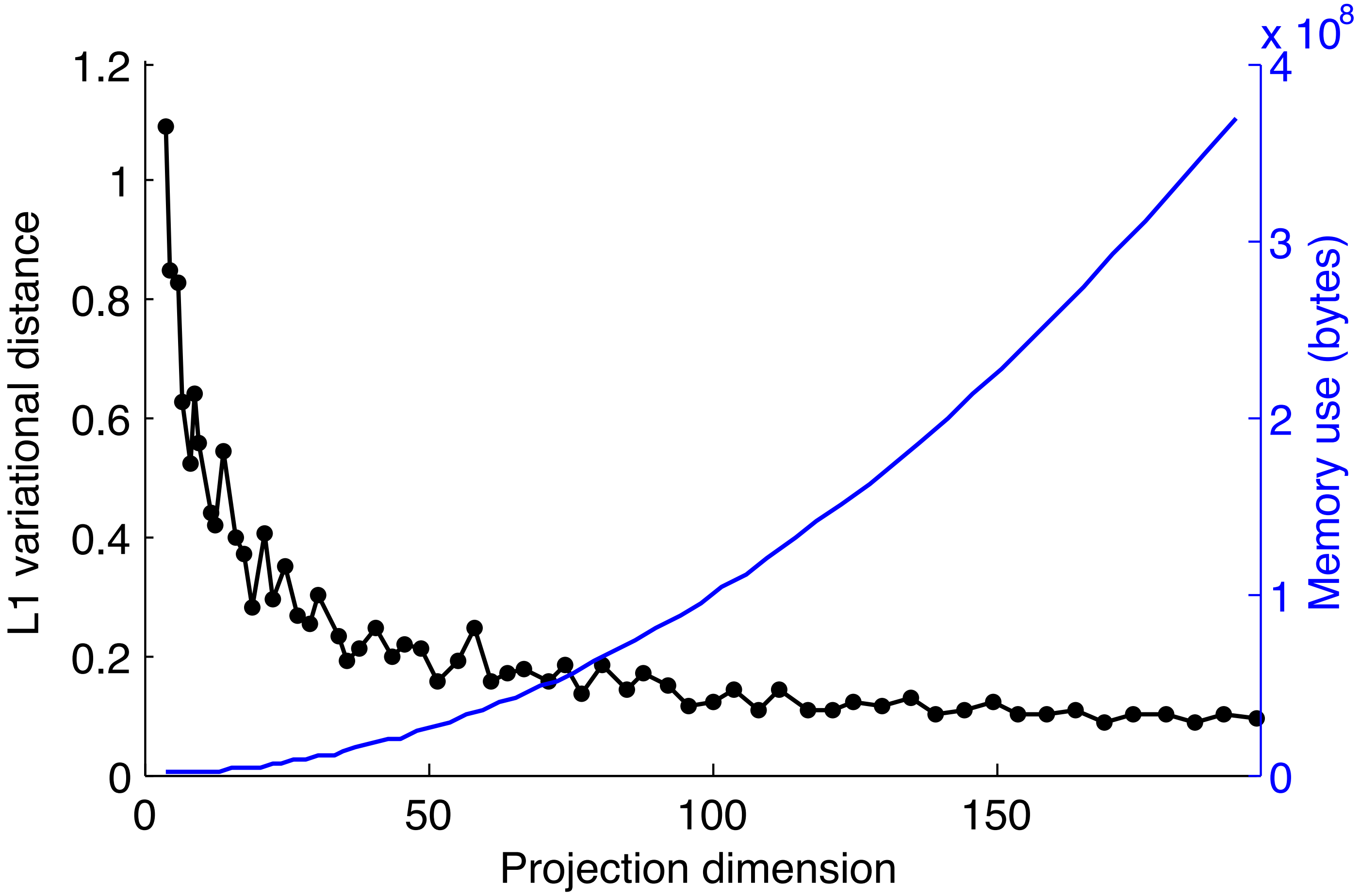

NEWS THREADING

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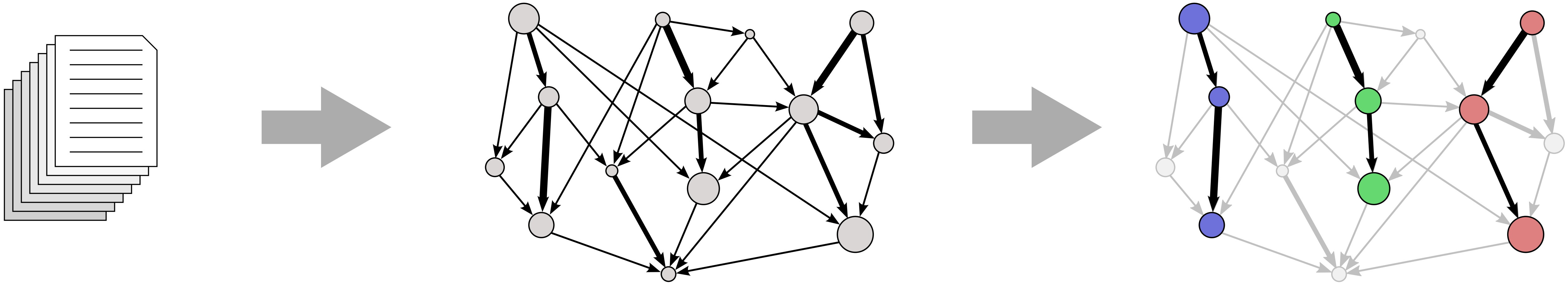

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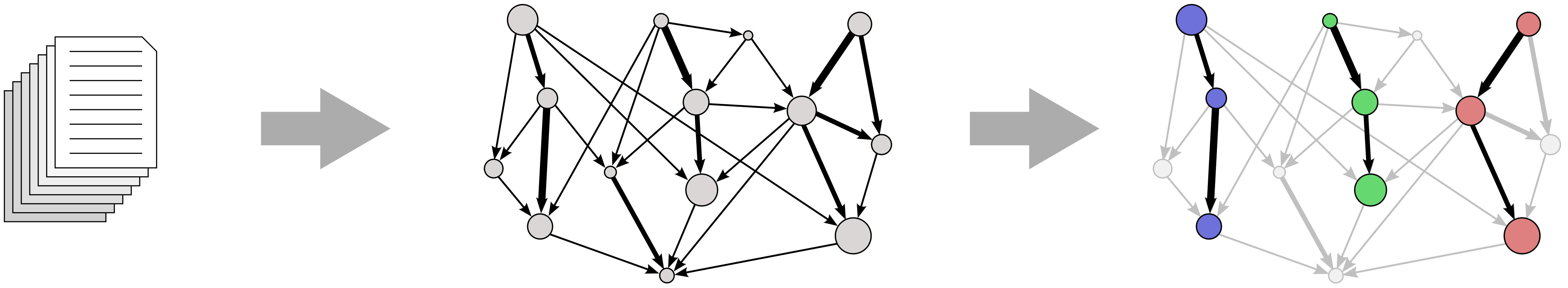
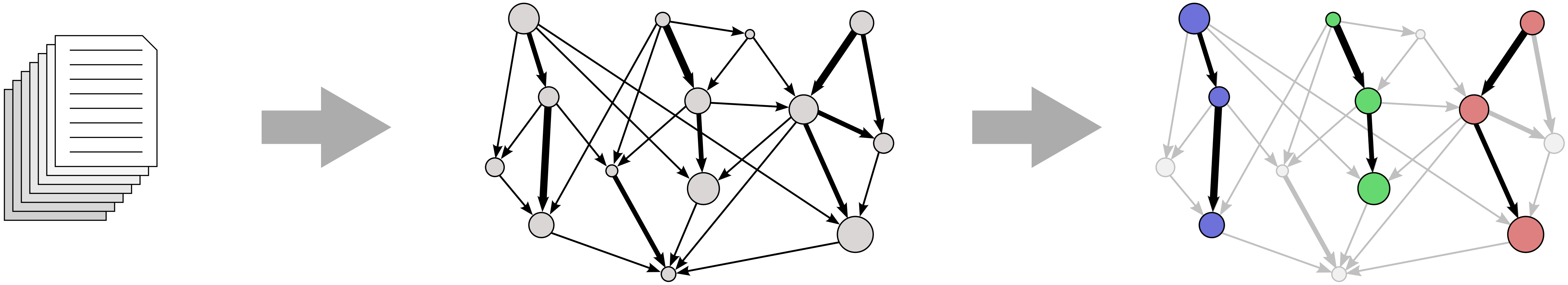
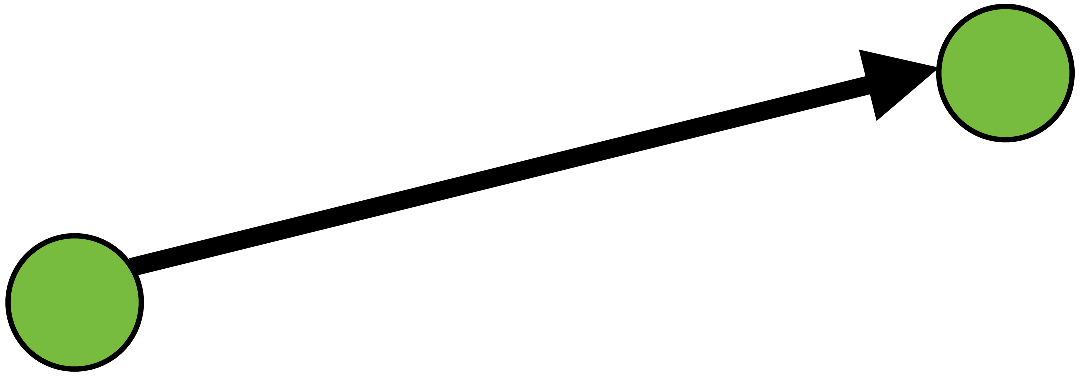


March 28: Health officials confirm Ebola outbreak in Guinea's capital

NEWS THREADING





August 8: World Health Organization declares Ebola epidemic an international health emergency

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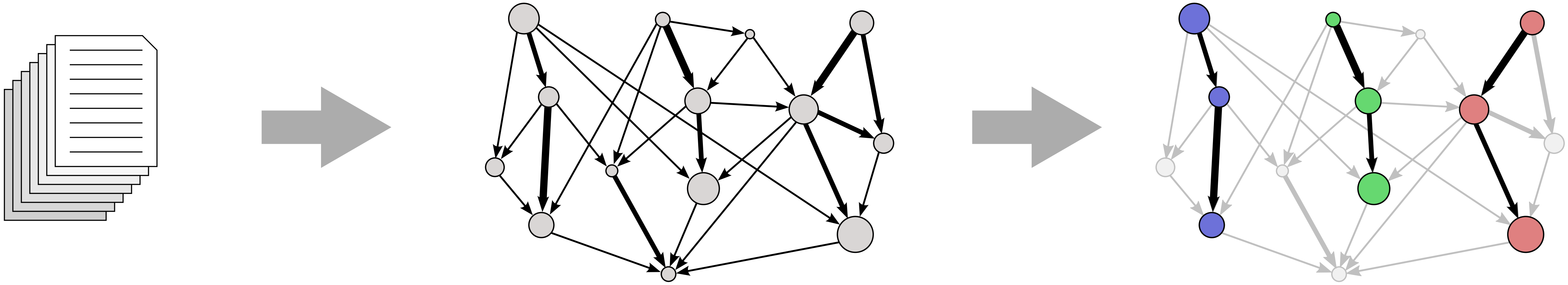

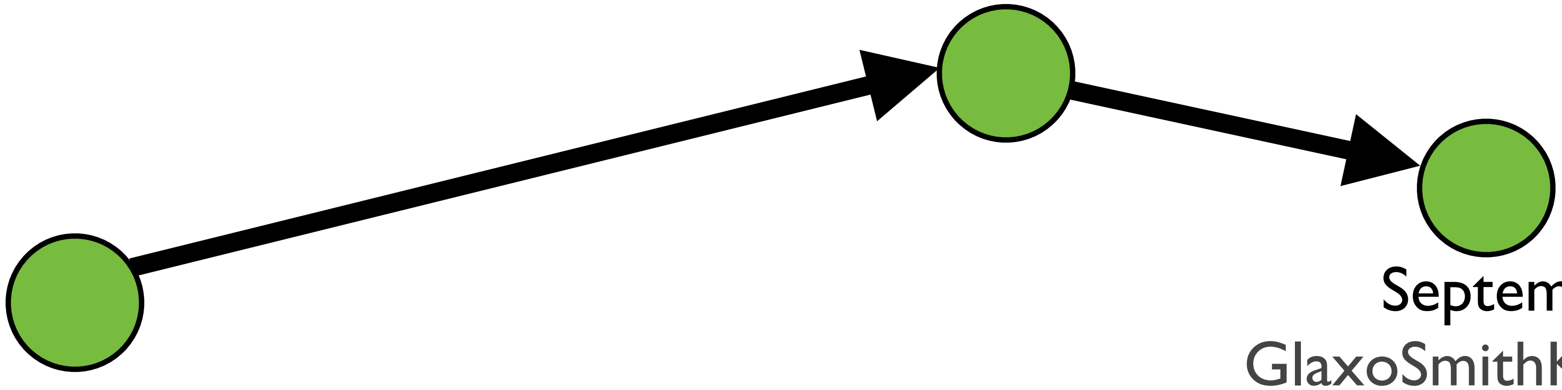

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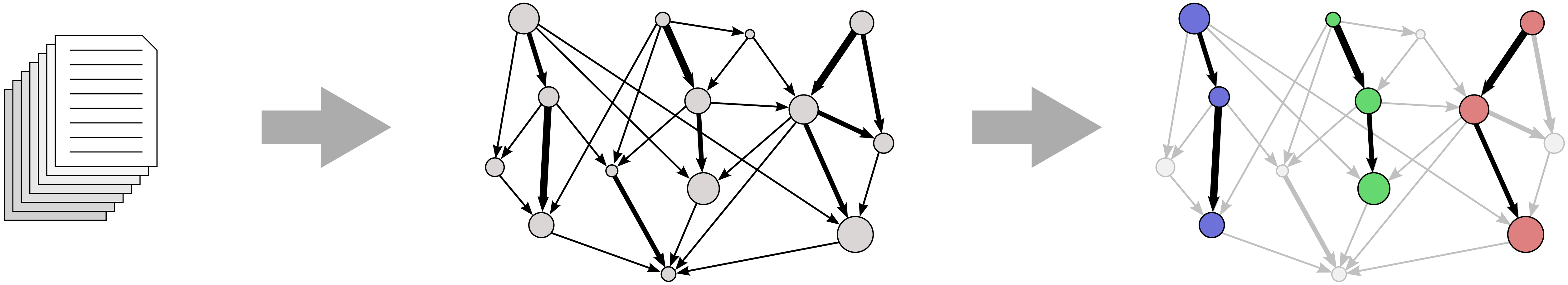
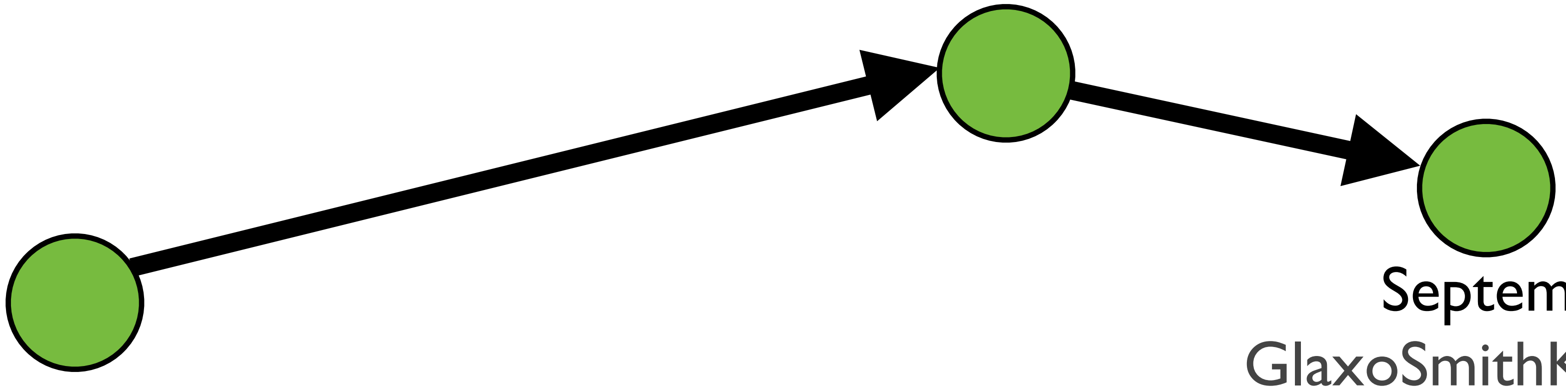


$M \approx 35{,}000$

10^{360}



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PROJECTING NEWS FEATURES

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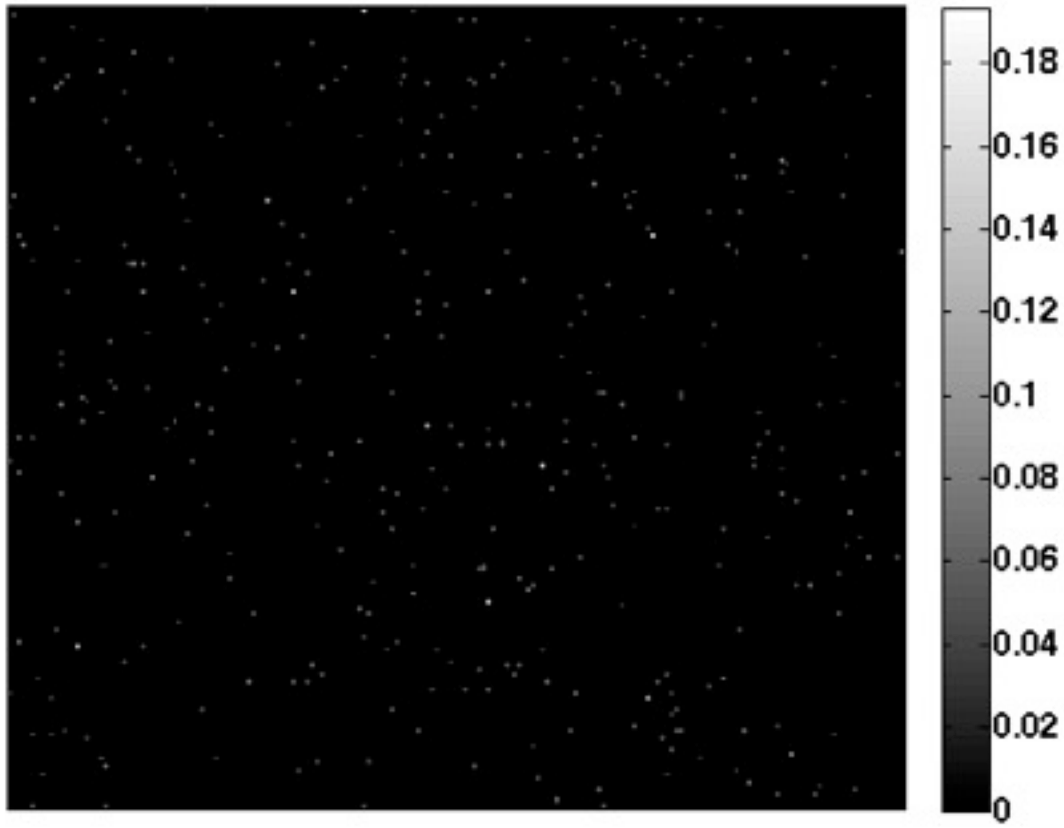
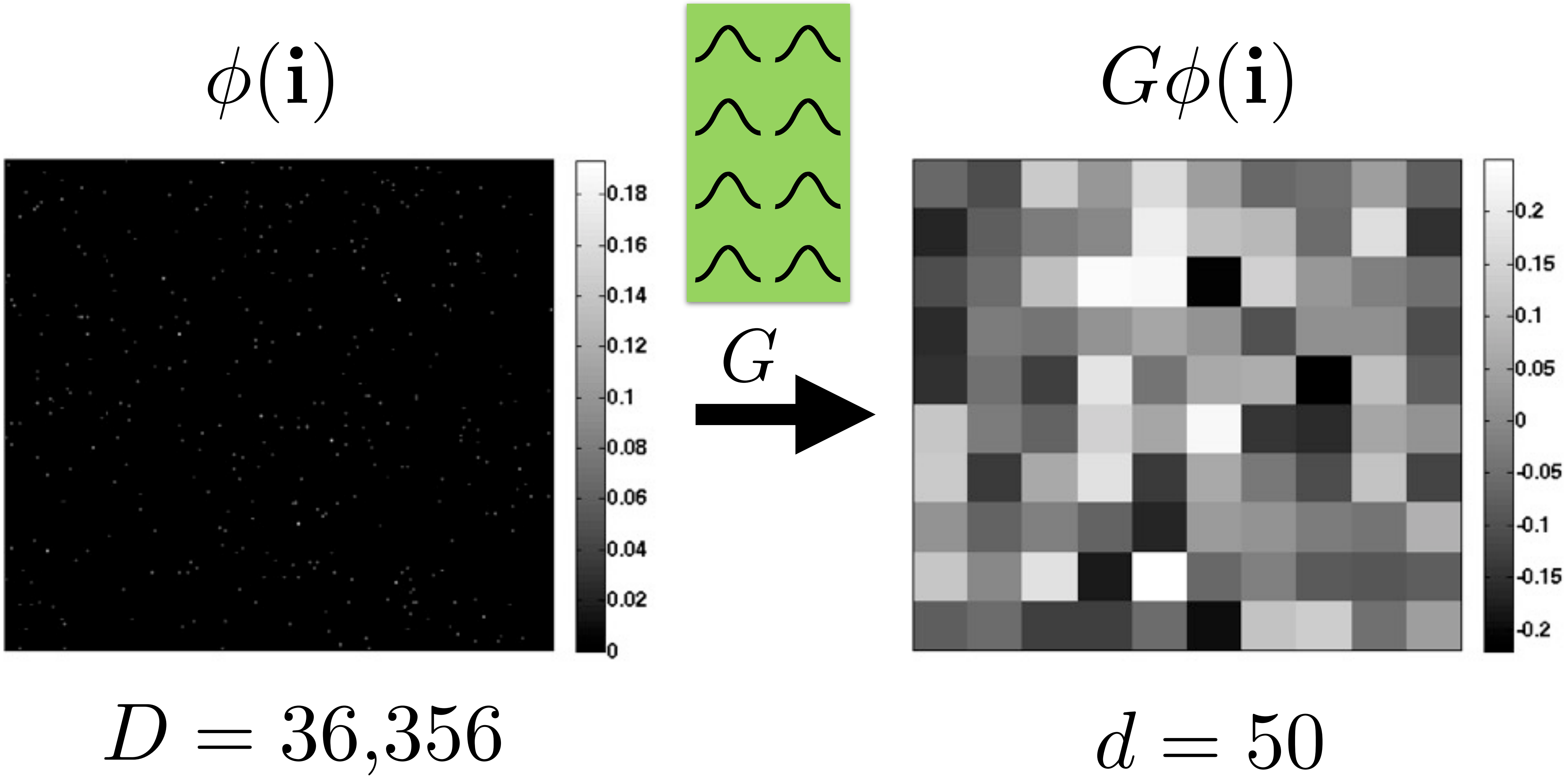
$\phi(\mathbf{i})$



$D = 36{,}356$

PROJECTING NEWS FEATURES



$\phi(\mathbf{i})$

$G\phi(\mathbf{i})$

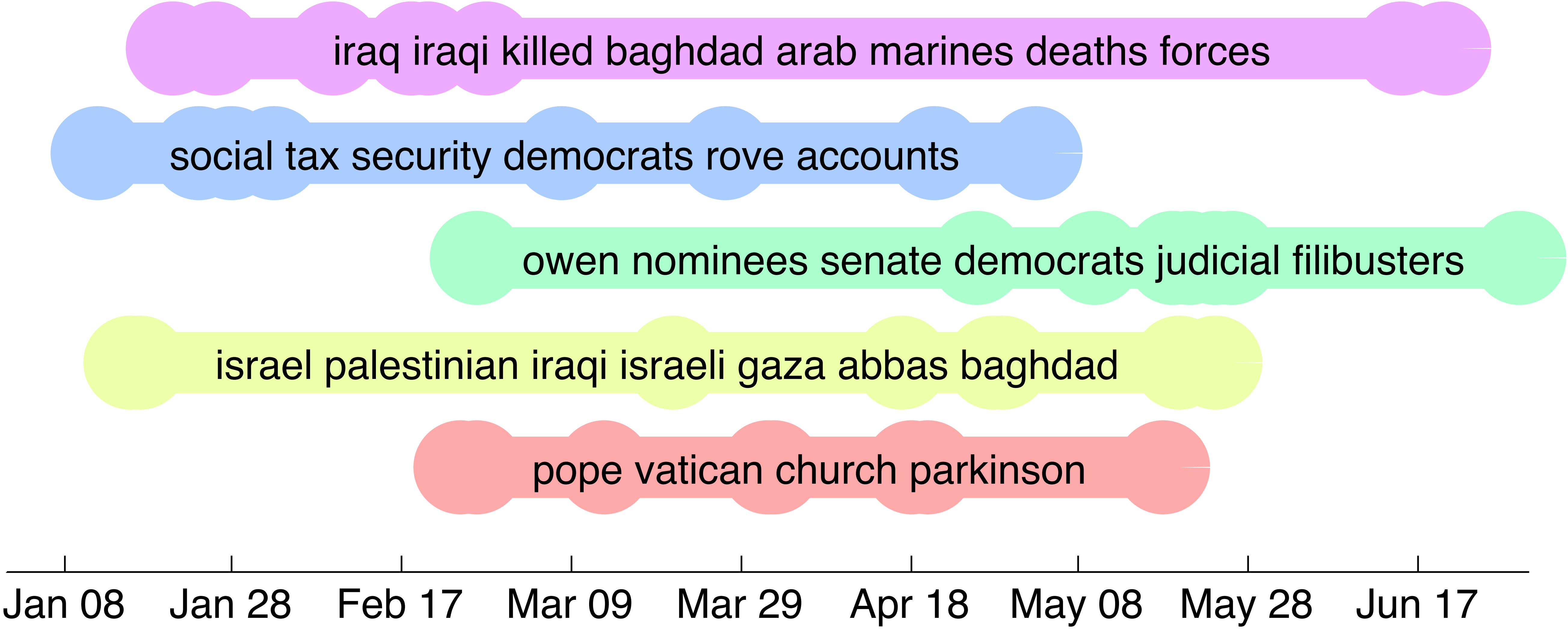
$\xrightarrow{G}$

$D = 36{,}356$

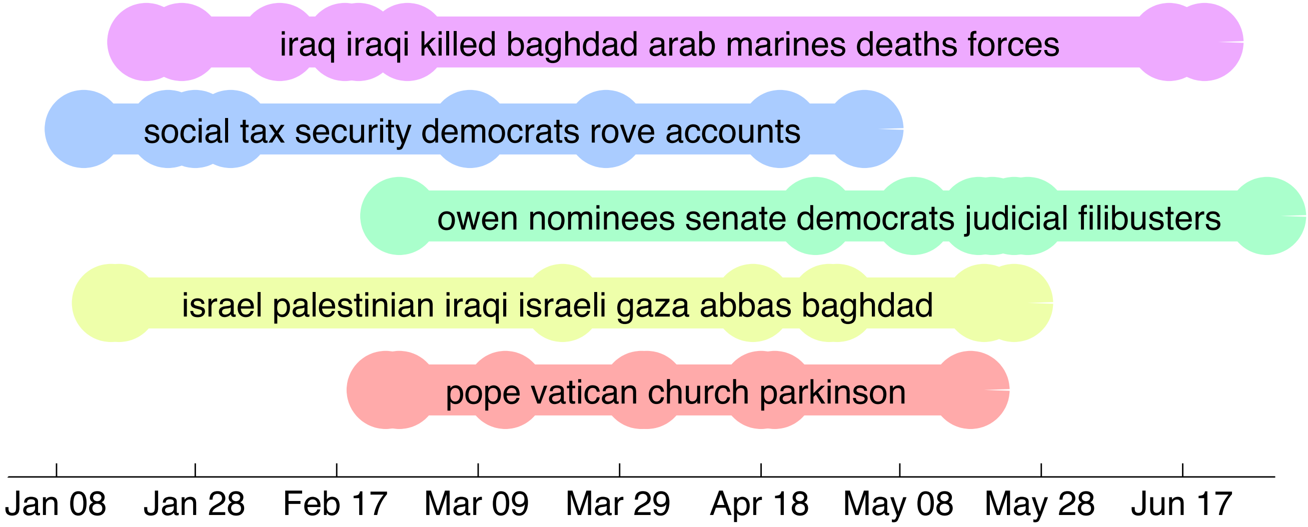
$d = 50$

DPP THREADS

DPP THREADS


iraq iraqi killed baghdad arab marines deaths forces
social tax security democrats rove accounts
owen nominees senate democrats judicial filibusters
israel palestinian iraqi israeli gaza abbas baghdad
pope vatican church parkinson
Jan 08
Jan 28
Feb 17
Mar 09
Mar 29
Apr 18
May 08
May 28
Jun 17

DPP THREADS



Feb 24: Parkinson's Disease Increases Risks to Pope
Feb 26: Pope's Health Raises Questions About His Ability to Lead
Mar 13: Pope Returns Home After 18 Days at Hospital
Apr 01: Pope's Condition Worsens as World Prepares for End of Papacy
Apr 02: Pope, Though Gravely Ill, Utters Thanks for Prayers
Apr 18: Europeans Fast Falling Away from Church
Apr 20: In Developing World, Choice [of Pope] Met with Skepticism
May 18: Pope Sends Message with Choice of Name

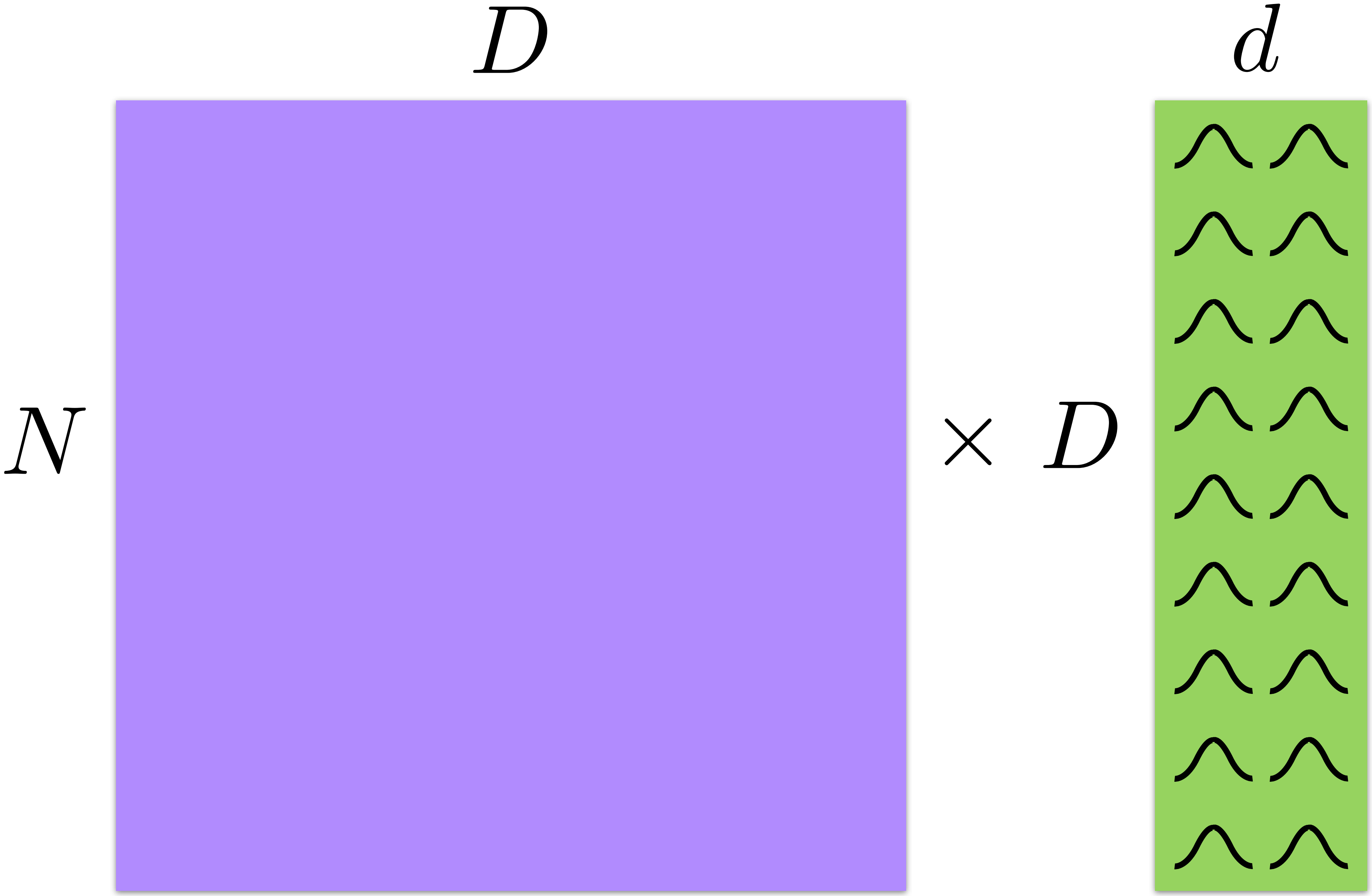
PROPOSED WORK

PROPOSED WORK

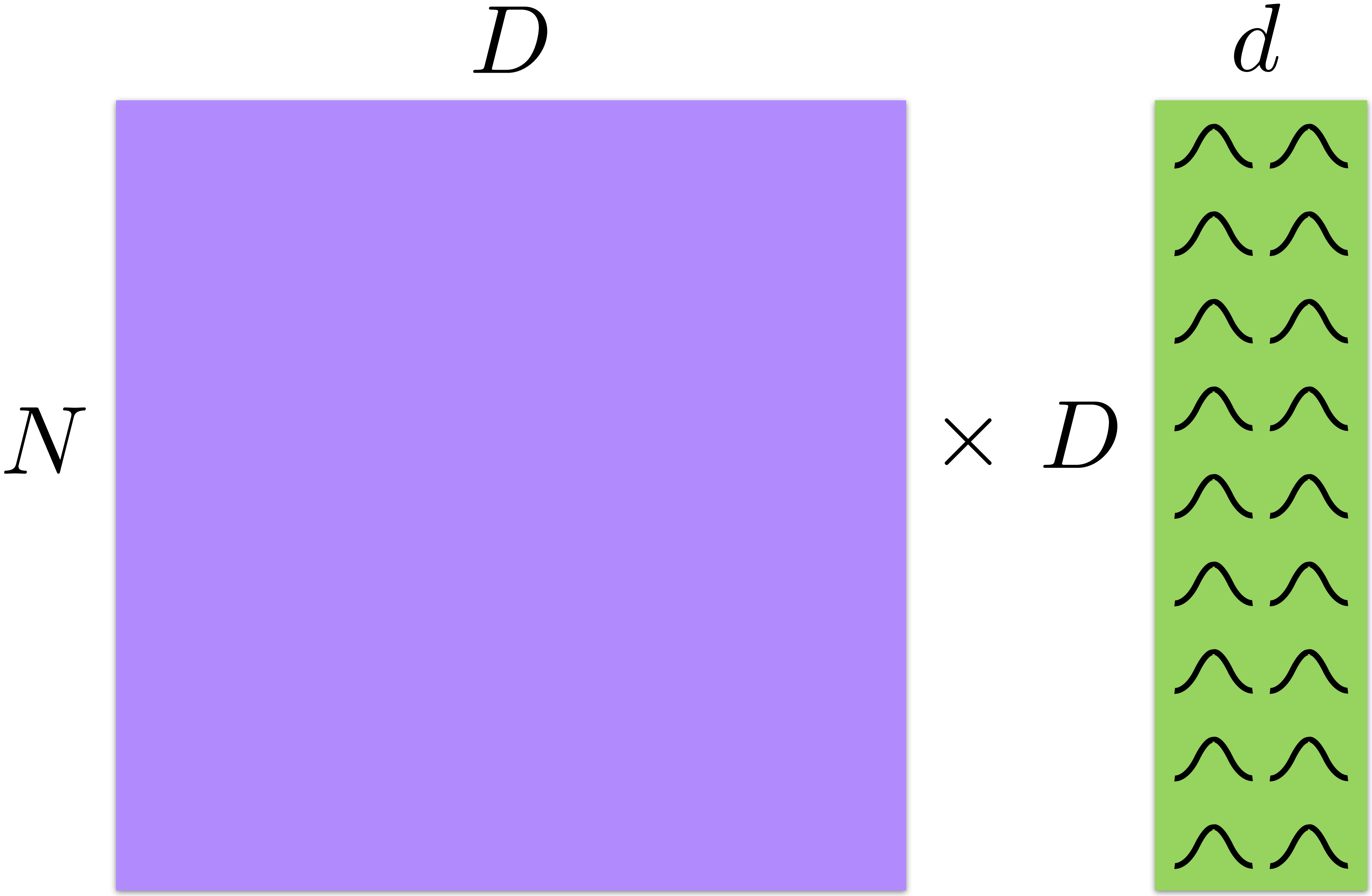
- Survey algorithms for large-scale eigendecomps

LARGE-SCALE EIGENDECOMP

LARGE-SCALE EIGENDECOMP



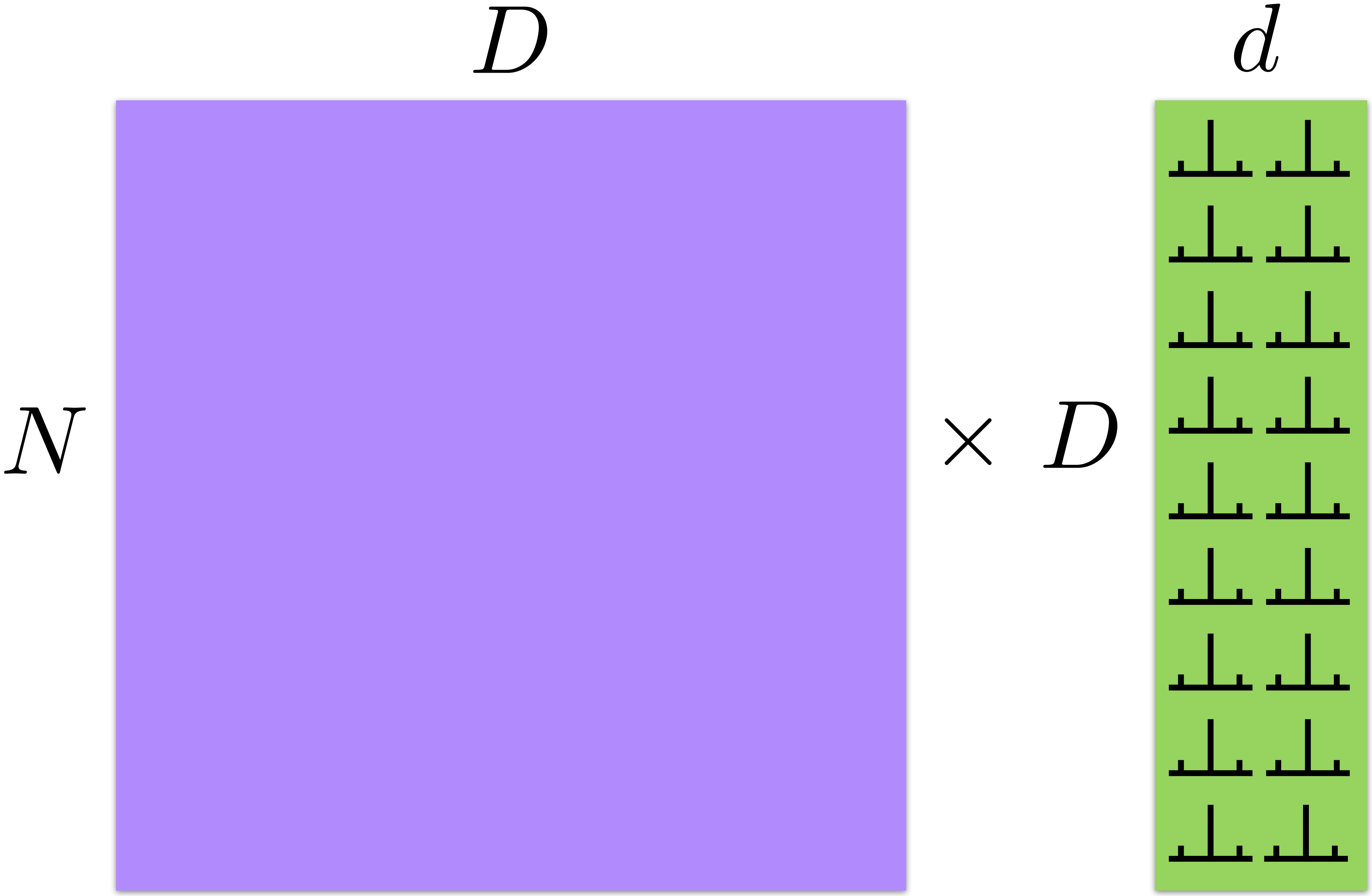
LARGE-SCALE EIGENDECOMP



$O(NDd)$

LARGE-SCALE EIGENDECOMP

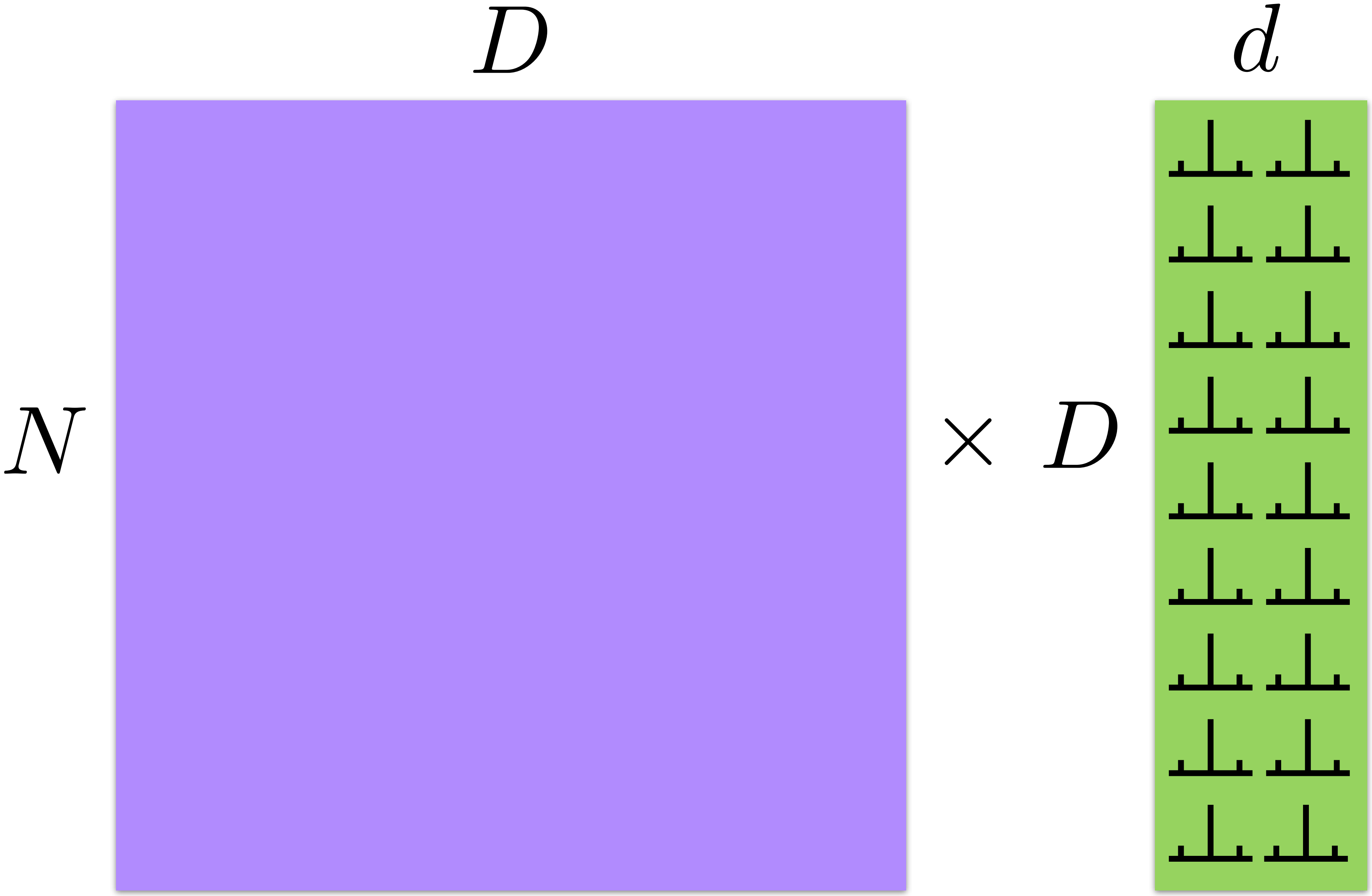
ACHLIOPTAS (2002)



$O(NDd)$

LARGE-SCALE EIGENDECOMP

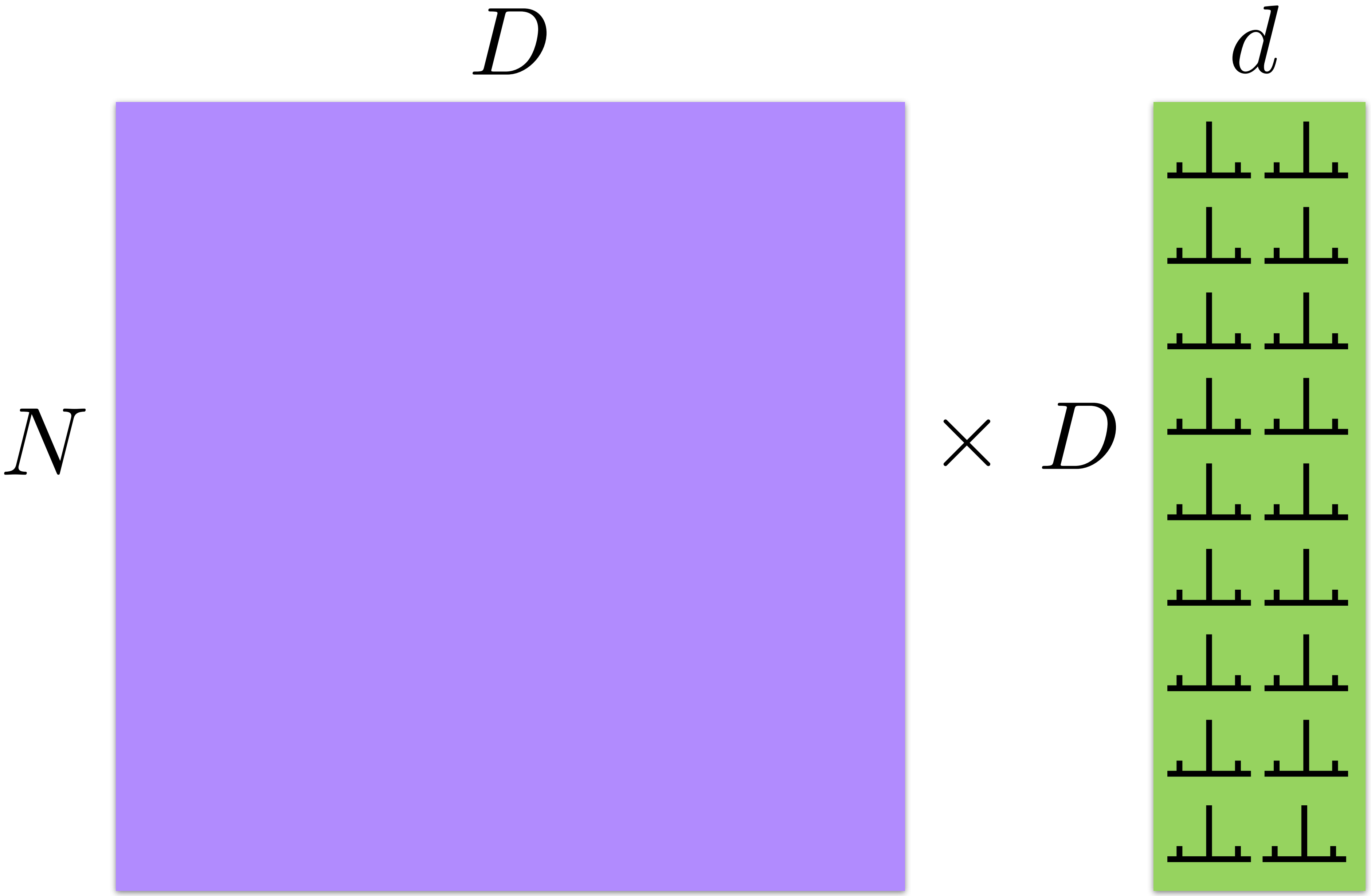
ACHLIOPTAS (2002)



$$\frac{1}{3}O(NDd)$$

LARGE-SCALE EIGENDECOMP

AILON AND CHAZELLE (2009)



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LARGE-SCALE EIGENDECOMP

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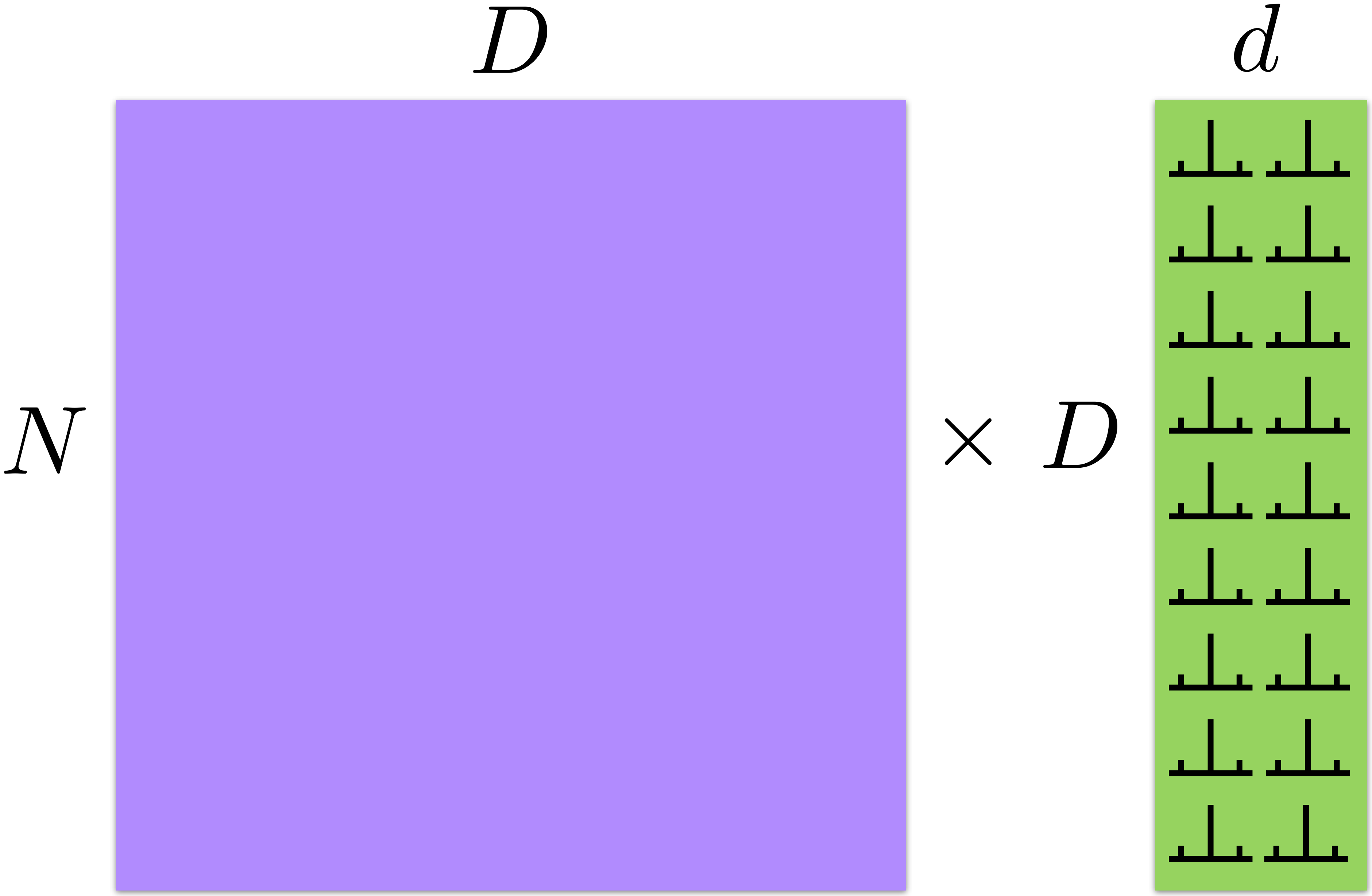

$O(ND(< d))$

LARGE-SCALE EIGENDECOMP

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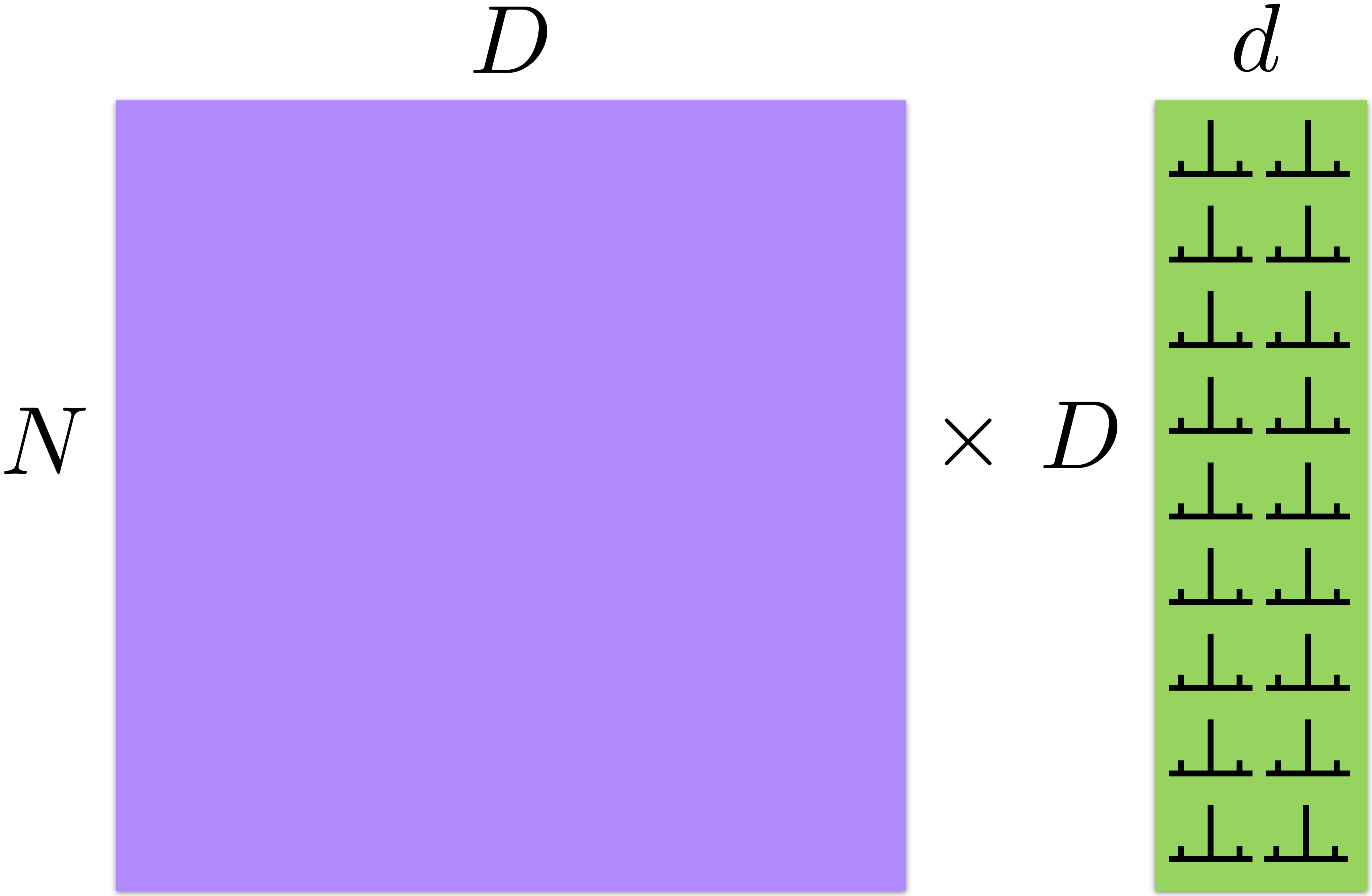

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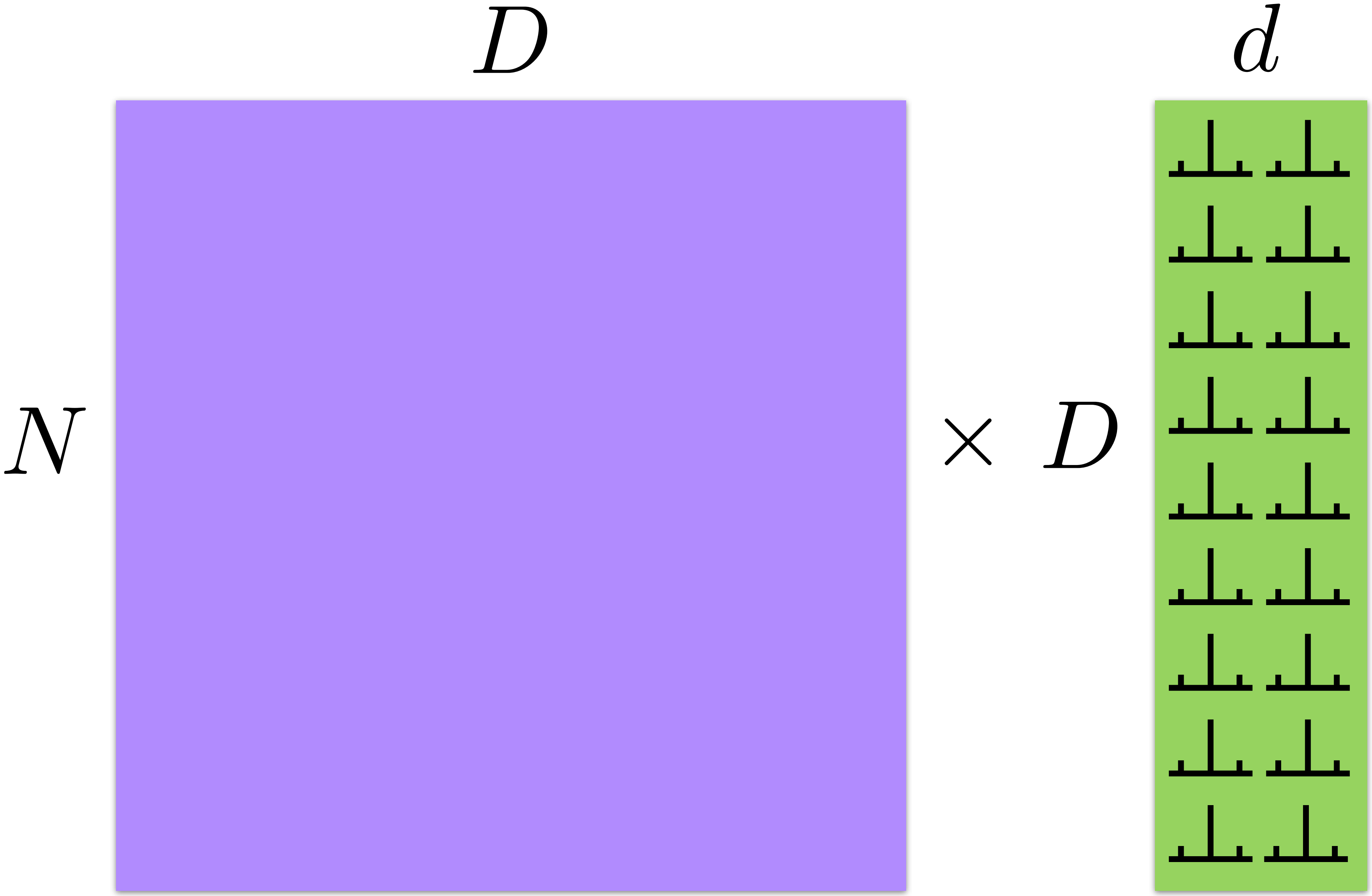
LARGE-SCALE EIGENDECOMP

MAGEN AND ZOUZIAS (2008)



$O(ND(< d))$

Take-away message for these other random projection methods:
1) no volume-preservation guarantees, and 2) runtime will be dominated by DPP sampling anyway.

PROPOSED WORK

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- Survey algorithms for large-scale eigendecomps

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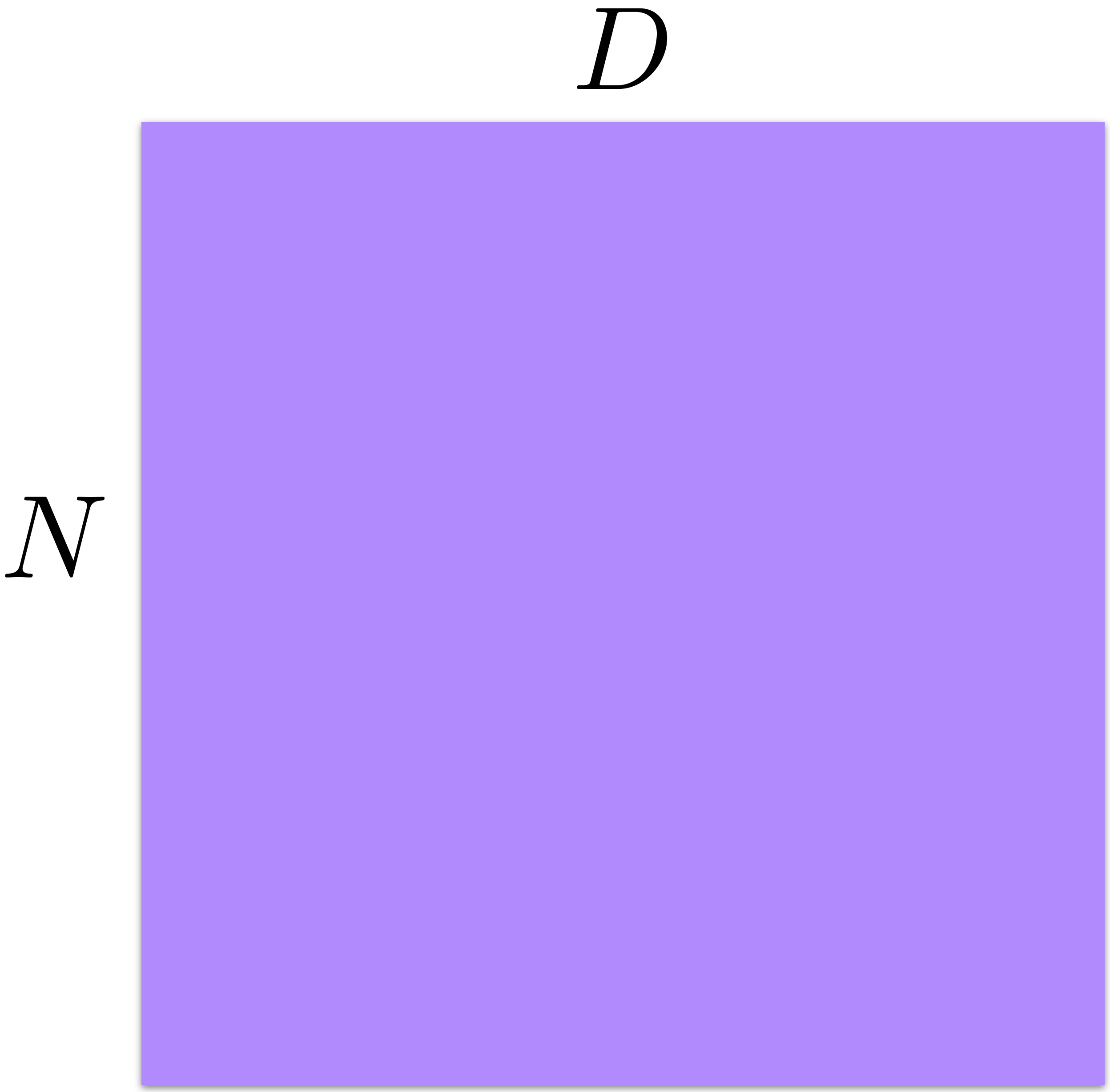
NYSTRÖM APPROX



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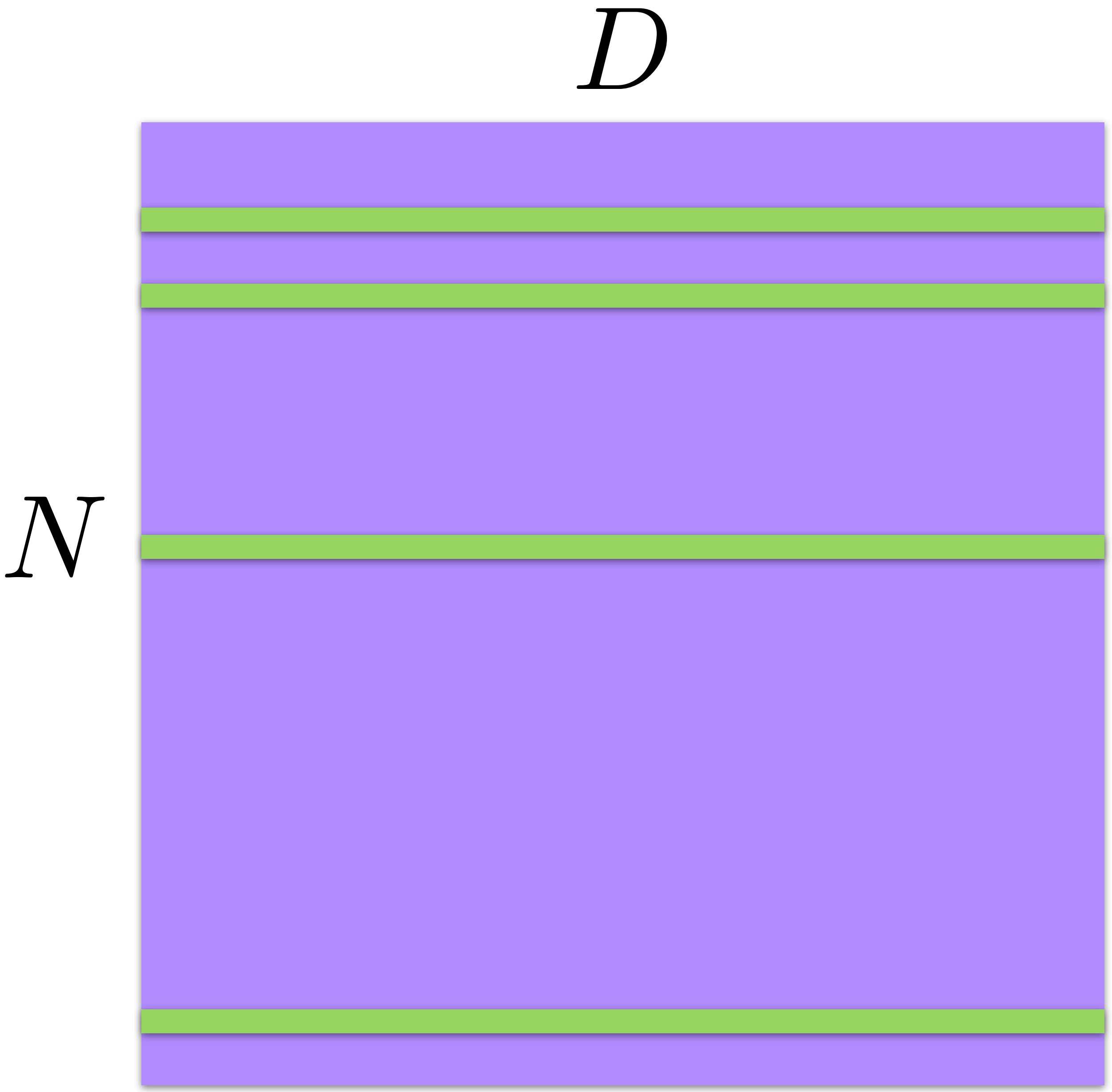

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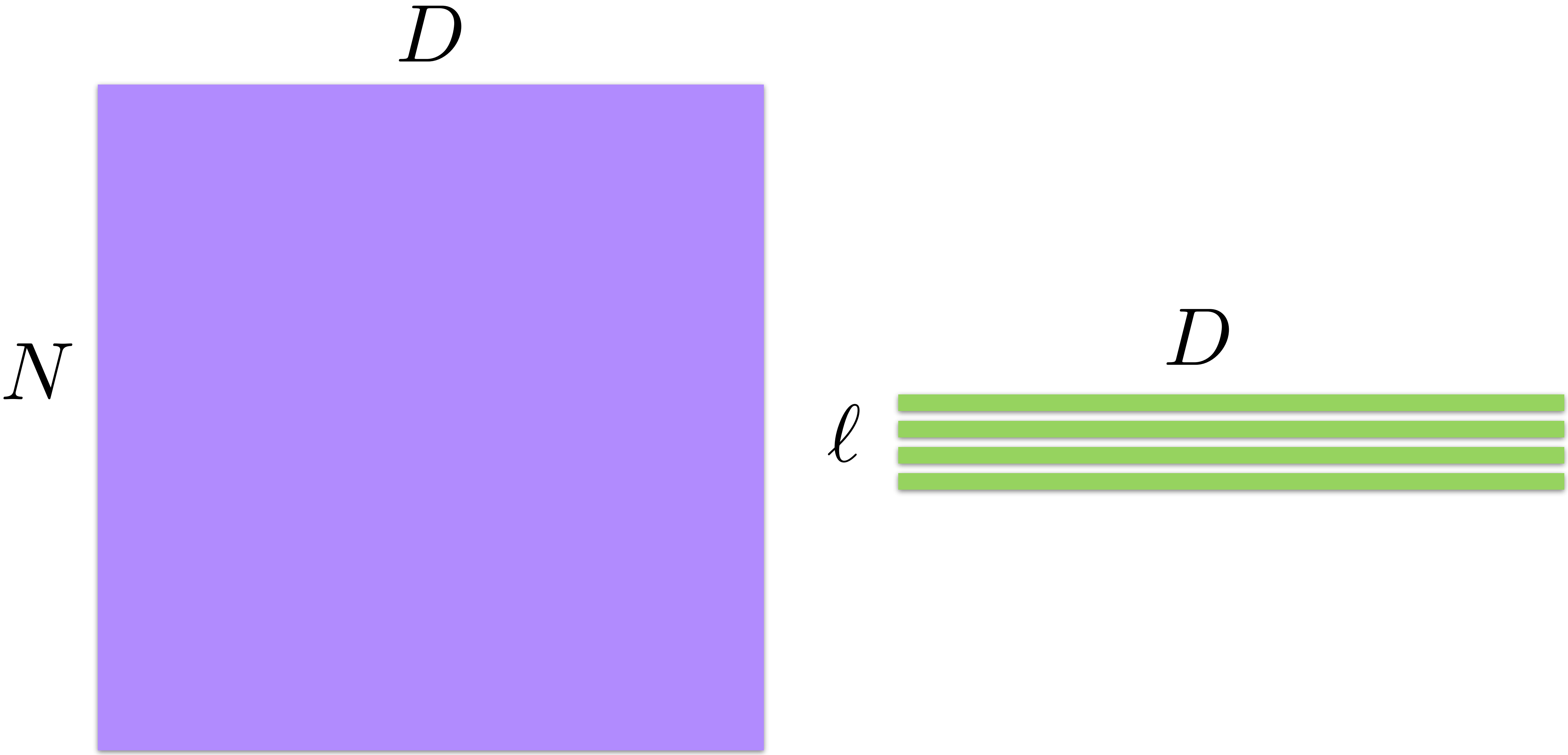
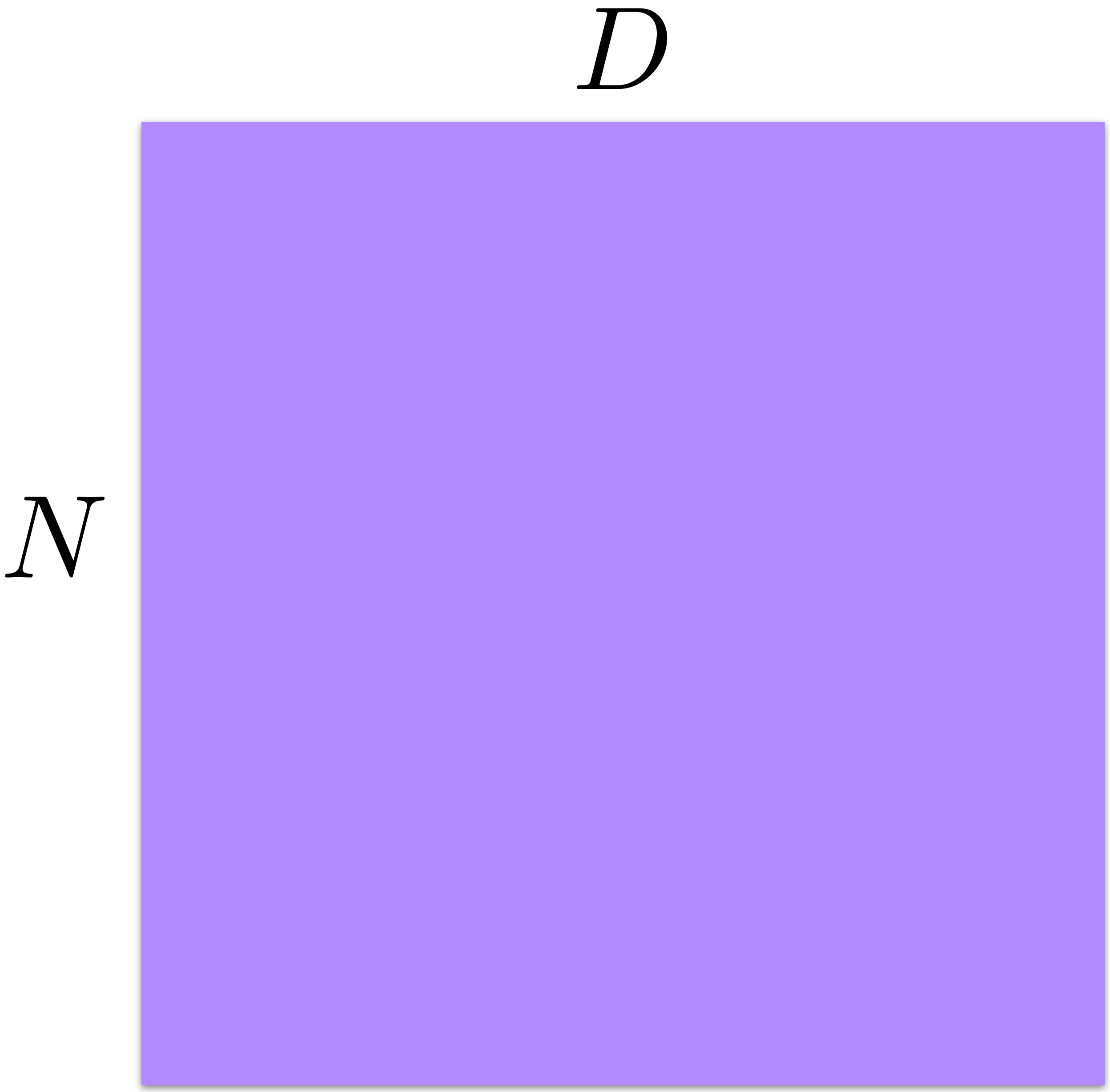
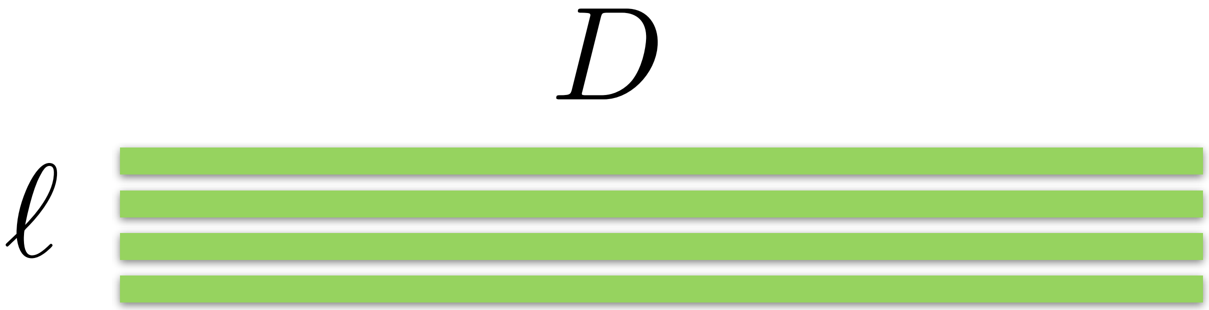


NYSTRÖM APPROX

AFFANDI, KULESZA, FOX, AND TASKAR (AISTATS 2013)



$|\mathcal{P}(Y) - \tilde{\mathcal{P}}(Y)|$ bounded



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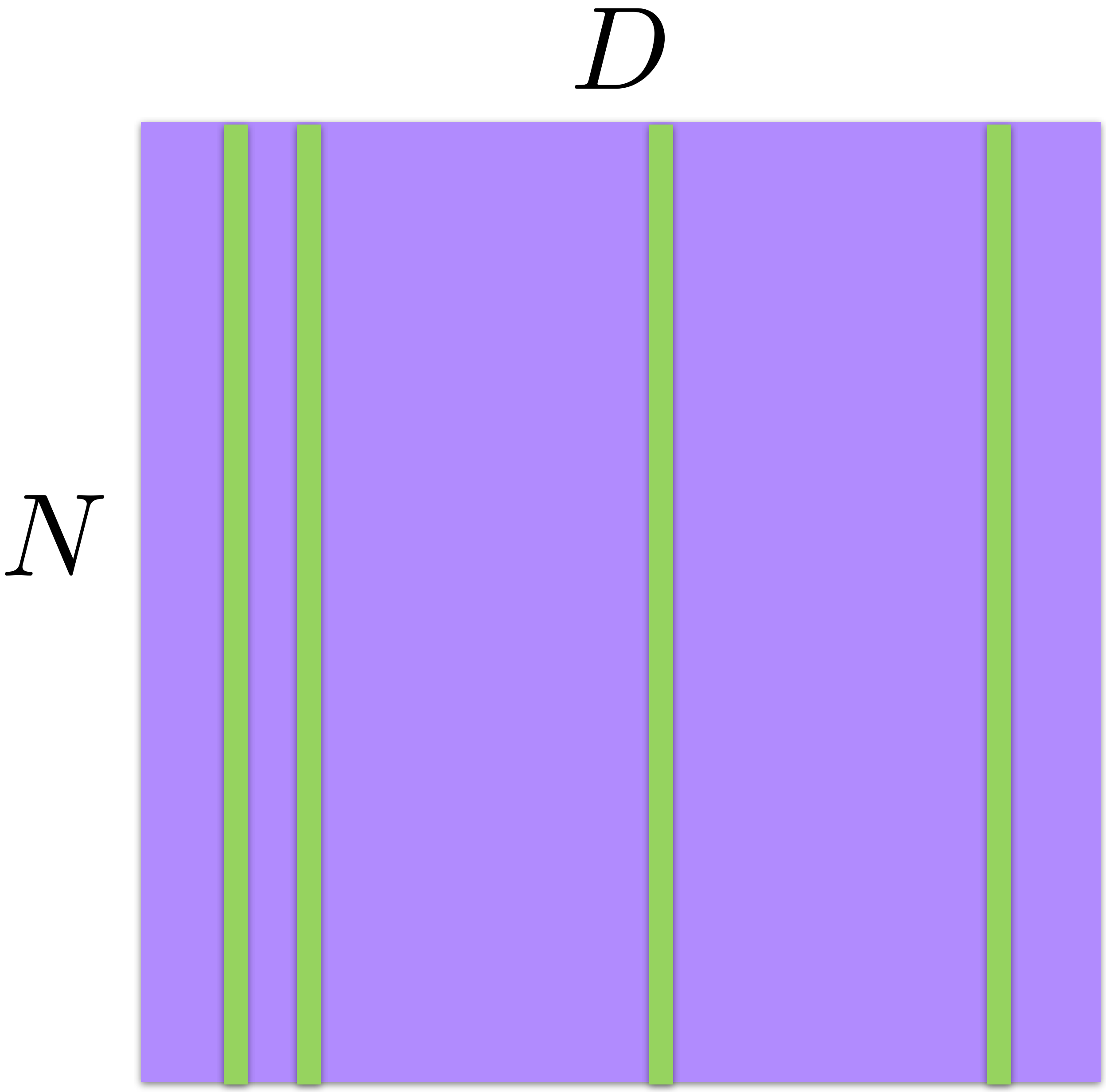

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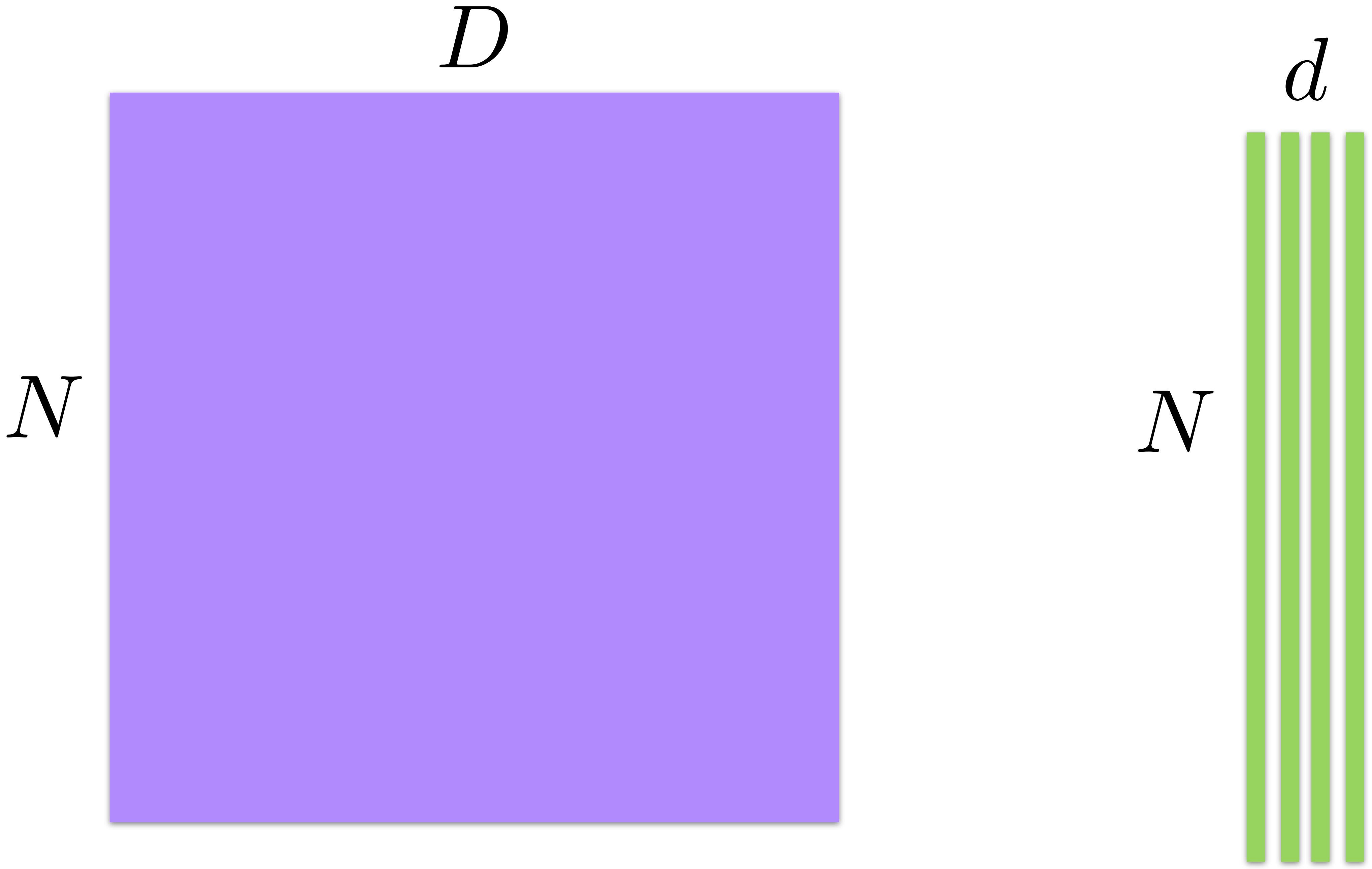

NYSTRÖM APPROX

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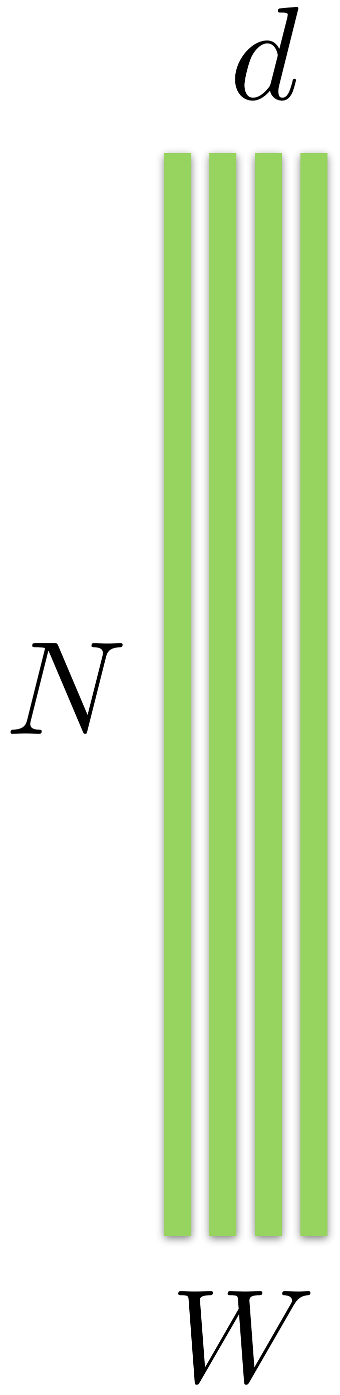

$$\tilde{C} = \begin{pmatrix} C_W & C_{W,\overline{W}} \\ C_{\overline{W},W} & C_{\overline{W},W} C_W^{+} C_{W,\overline{W}} \end{pmatrix}$$

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$$B_{\mathbf{i}} = \left[\prod_{\alpha\in F} q(i_\alpha)\right]\left[\sum_{\alpha\in F}\phi(i_\alpha)\right]$$

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$$\tilde{C} = \tilde{V} \tilde{\Lambda} \tilde{V}^\top$$

$$L \approx B^\top \tilde{V} \tilde{V}^\top B$$

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$$O(k^3 \log(k/\epsilon))$$

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$$\text{mixing time} \to \infty \text{ as eigenvalues of } L \to 0$$

$$\text{condition number bounded: } \frac{\lambda_{\max}}{\lambda_{\min}} \leq c$$

$$\implies \text{mixing time bounded?}$$

2. MAP ESTIMATION

HARDNESS OF MAP

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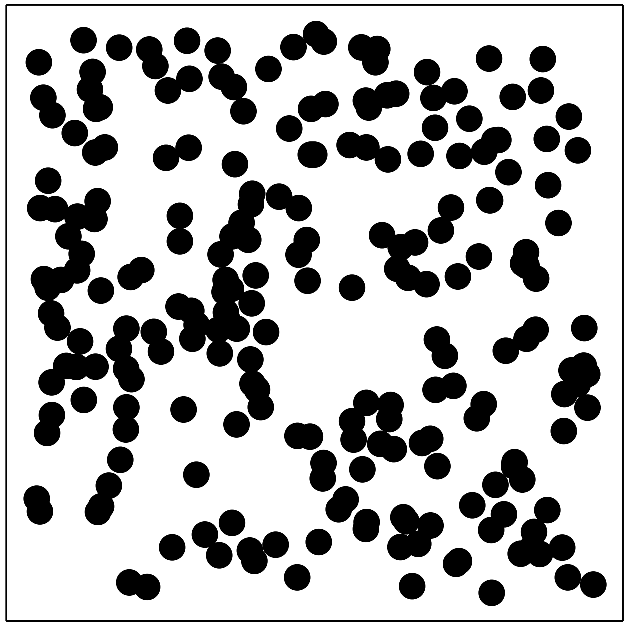
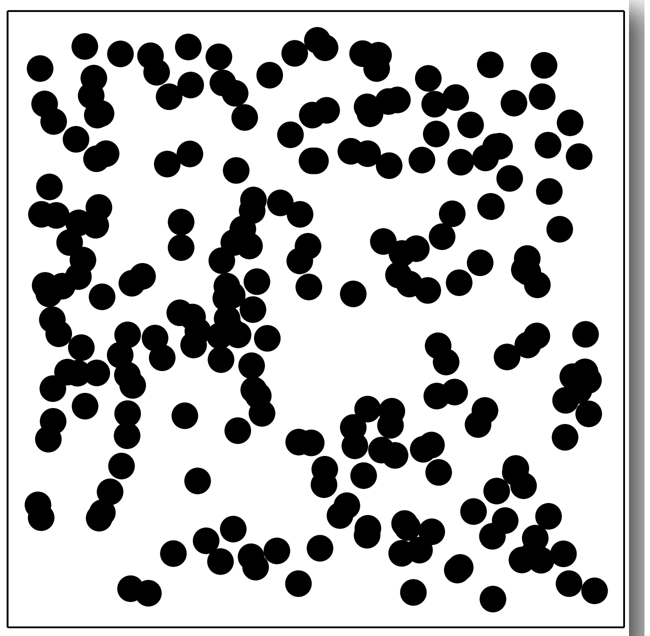


All points

HARDNESS OF MAP



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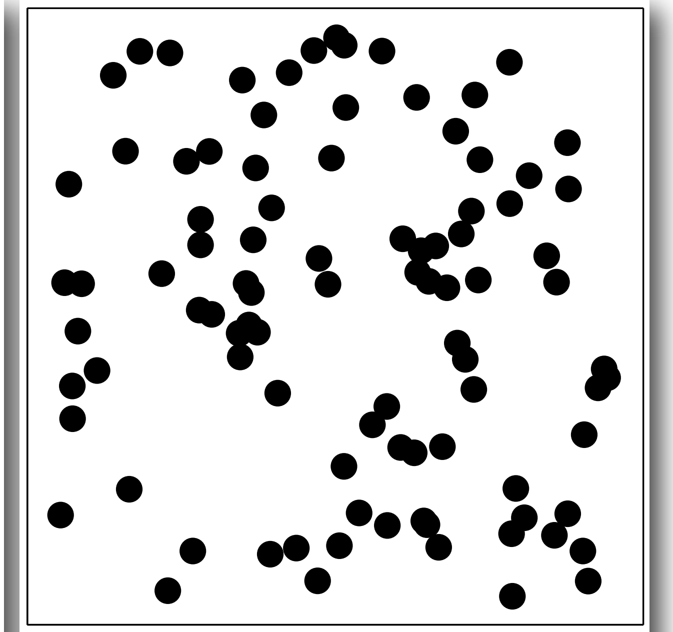
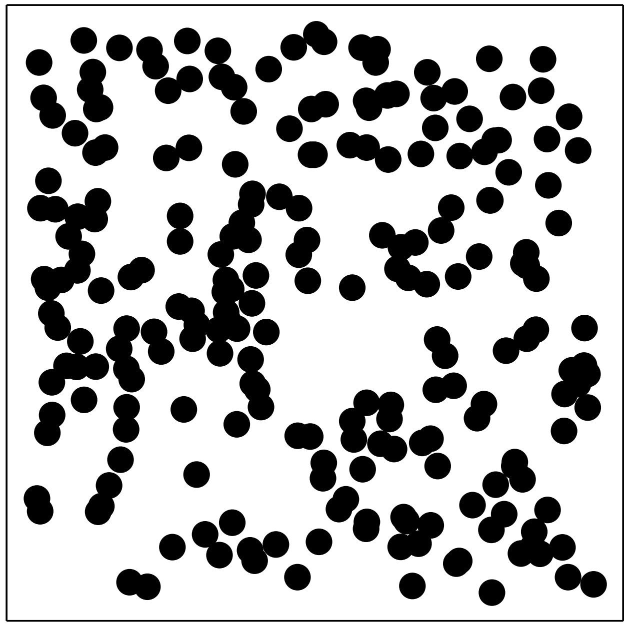
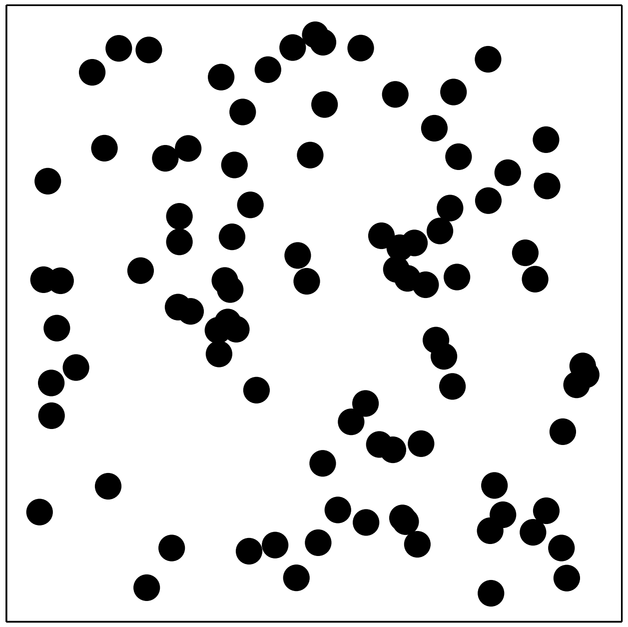
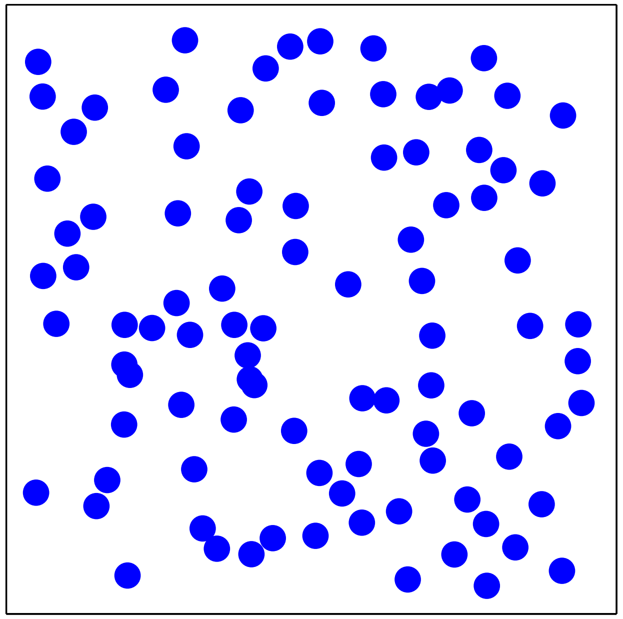


Independent sample

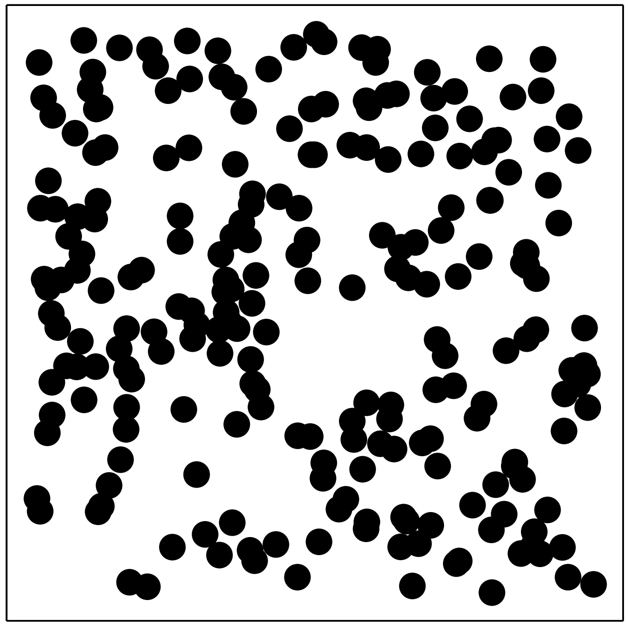
HARDNESS OF MAP


All points

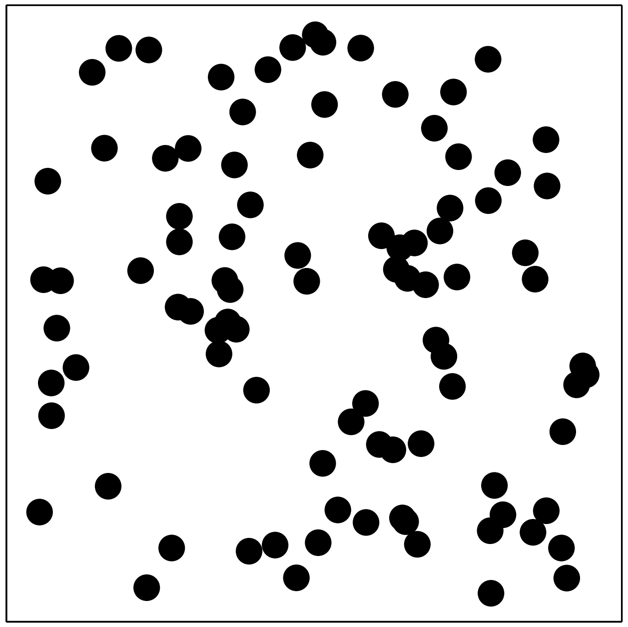

Independent sample


DPP sample

HARDNESS OF MAP



All points



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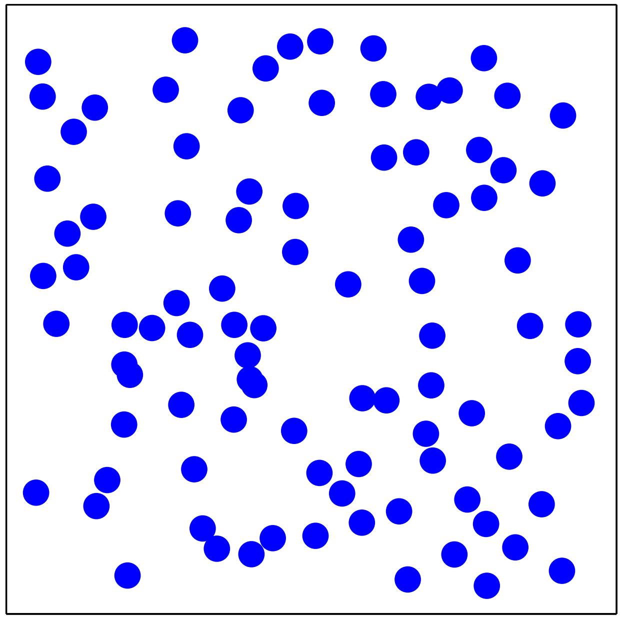

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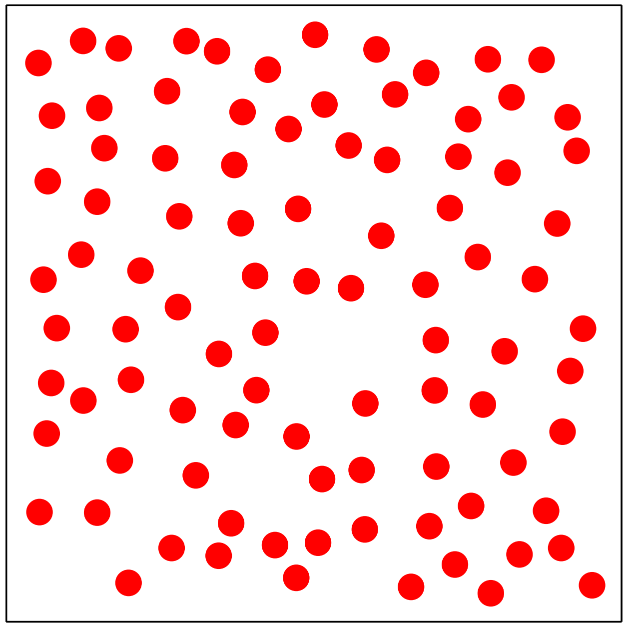
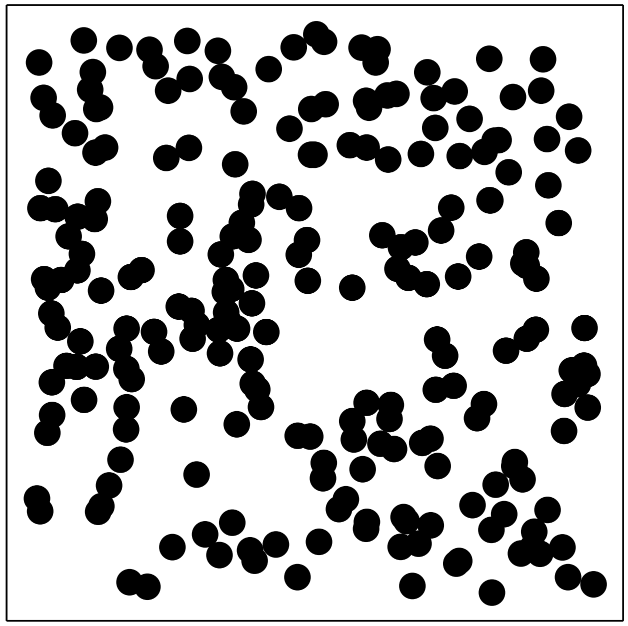
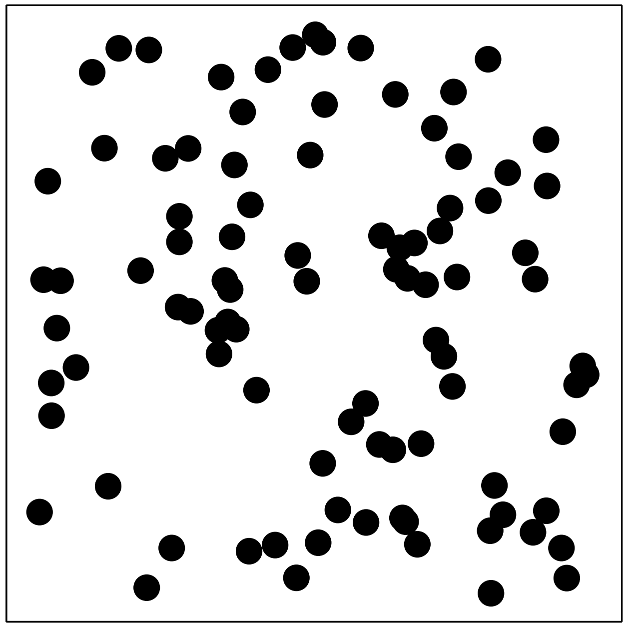


DPP (approx) MAP

HARDNESS OF MAP



All points



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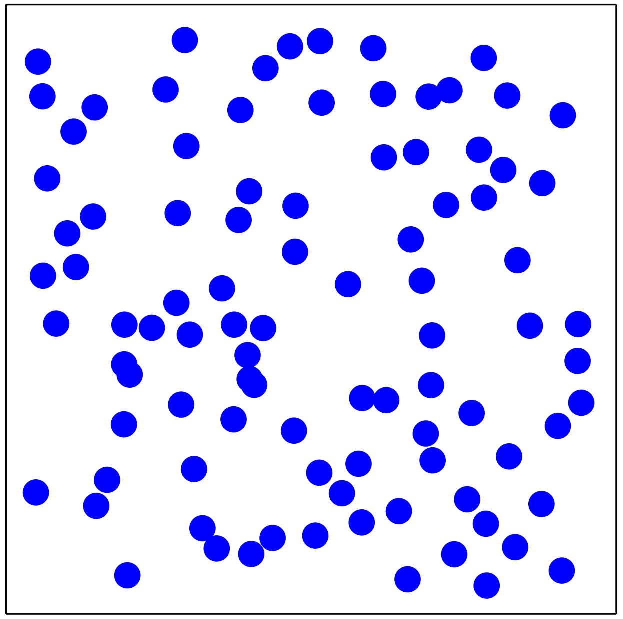

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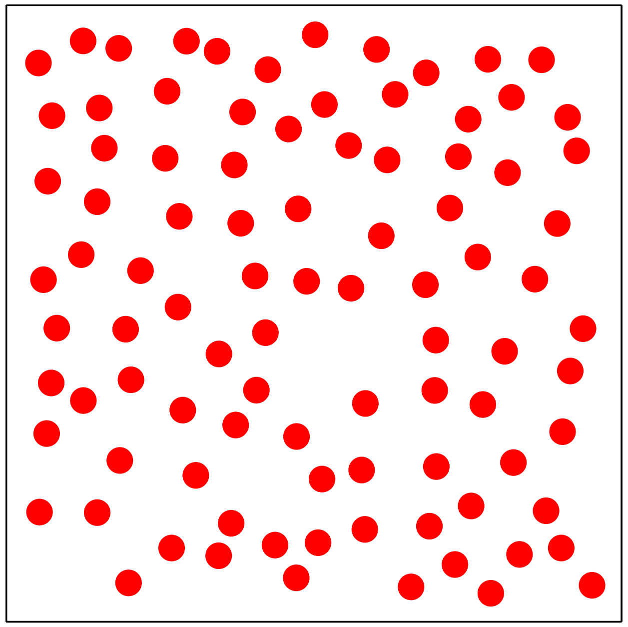
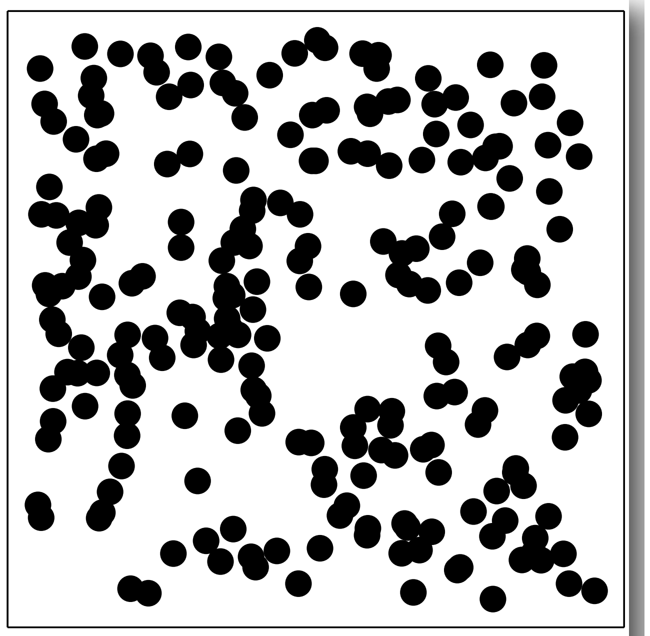


DPP (approx) MAP

$$Y = \underset{Y':Y'\subseteq\mathcal{Y}}{\arg\max}\ \det(L_{Y'})$$

HARDNESS OF MAP



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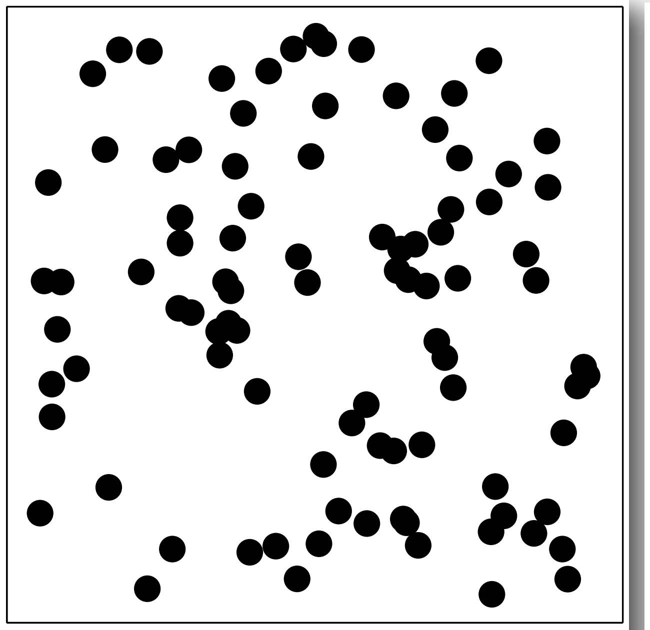

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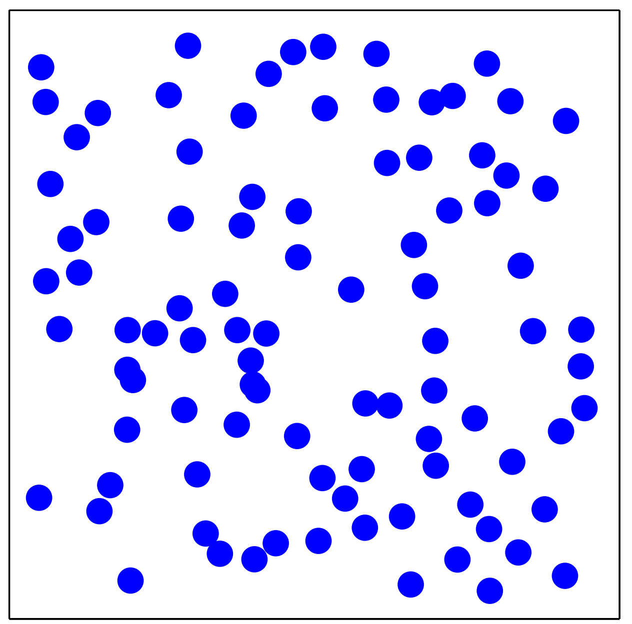

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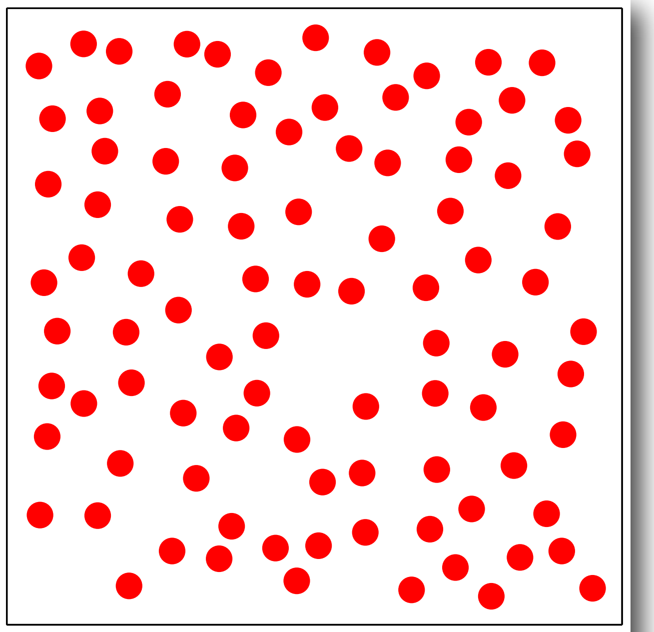

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$$Y = \underset{Y' : Y' \subseteq \mathcal{Y}}{\arg\max} \; \det(L_{Y'})$$

NP-hard, no PTAS: $\det(L_{\hat{Y}}) \geq \left(\frac{8}{9} + \epsilon\right) \det(L_{Y^*})$

SUBMODULARITY

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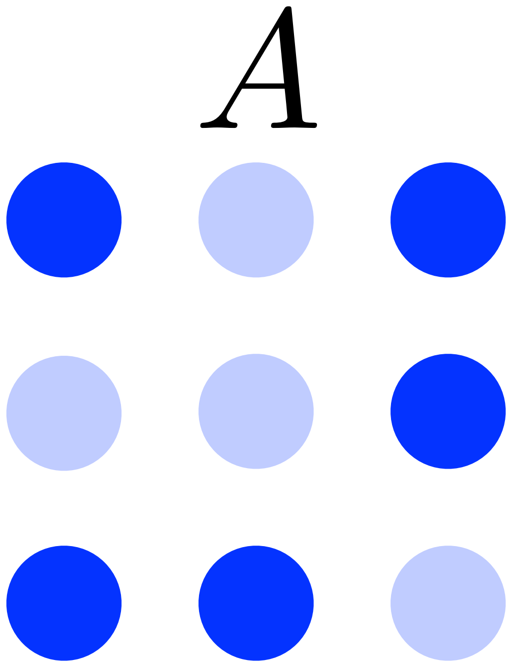

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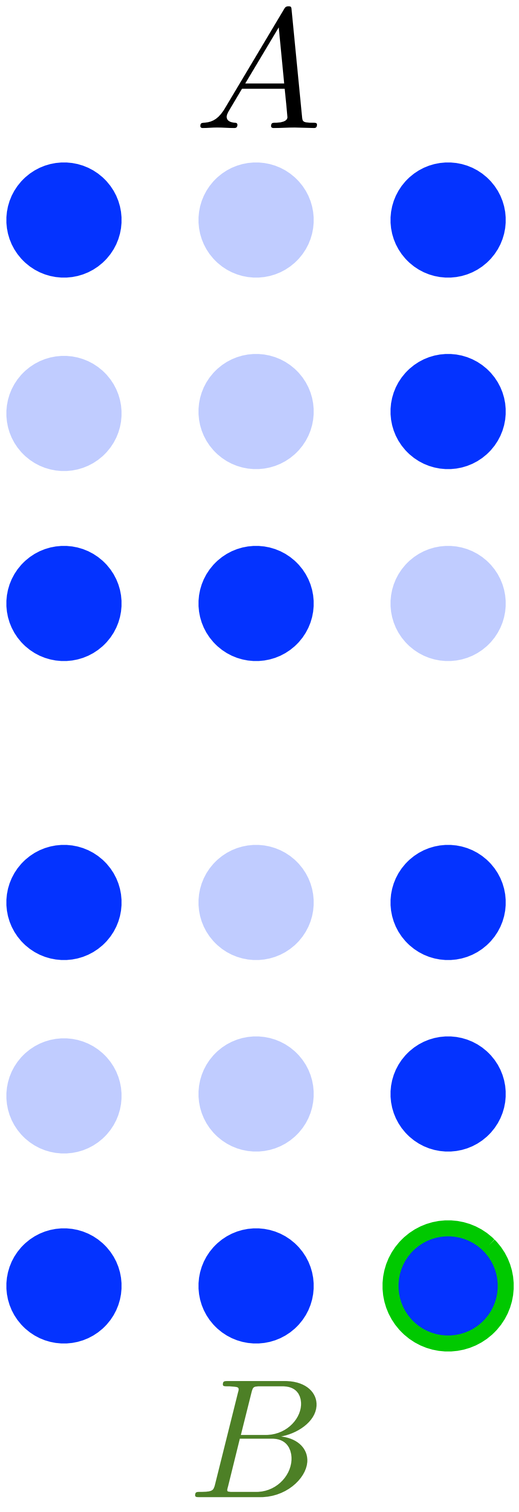

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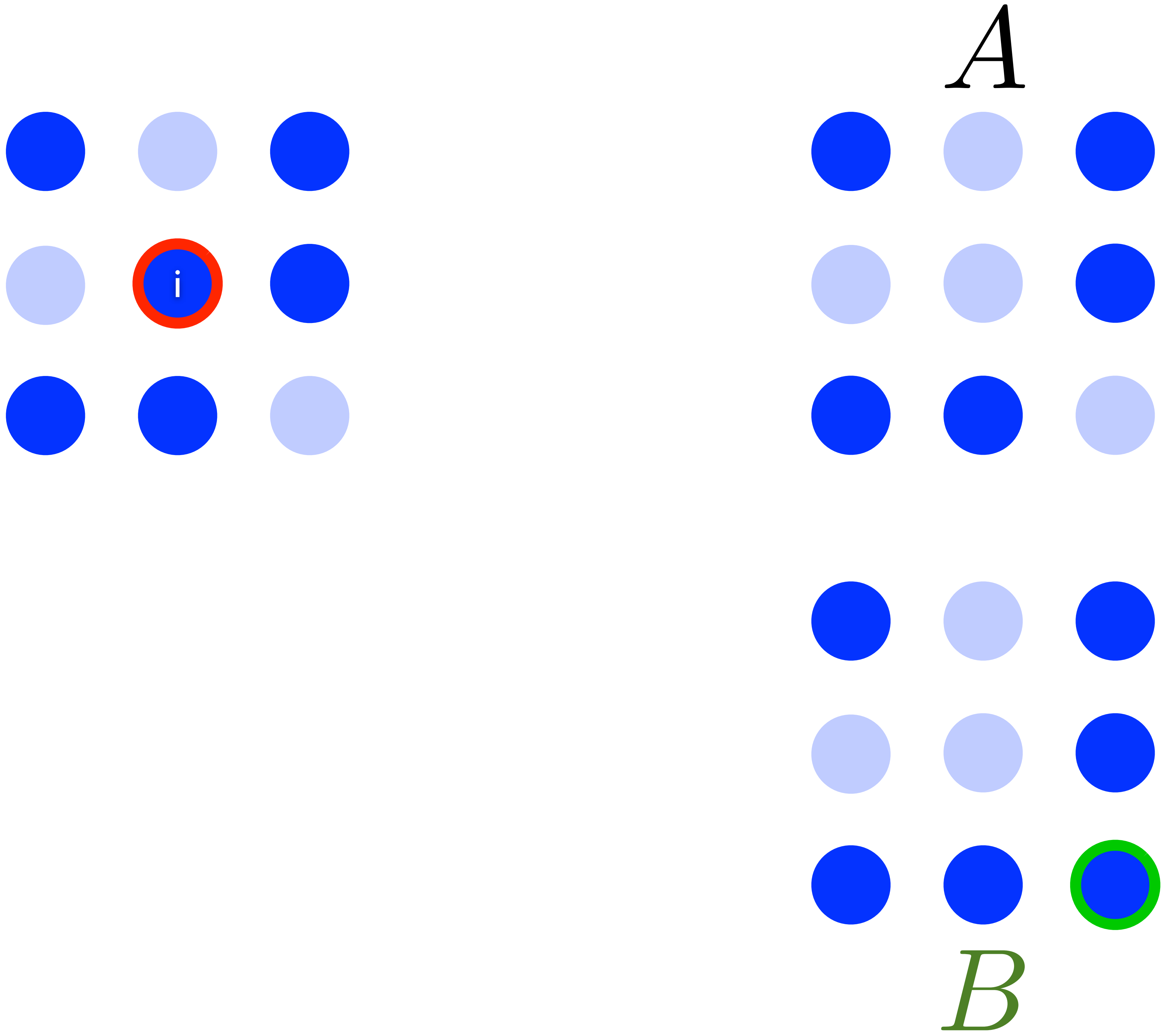

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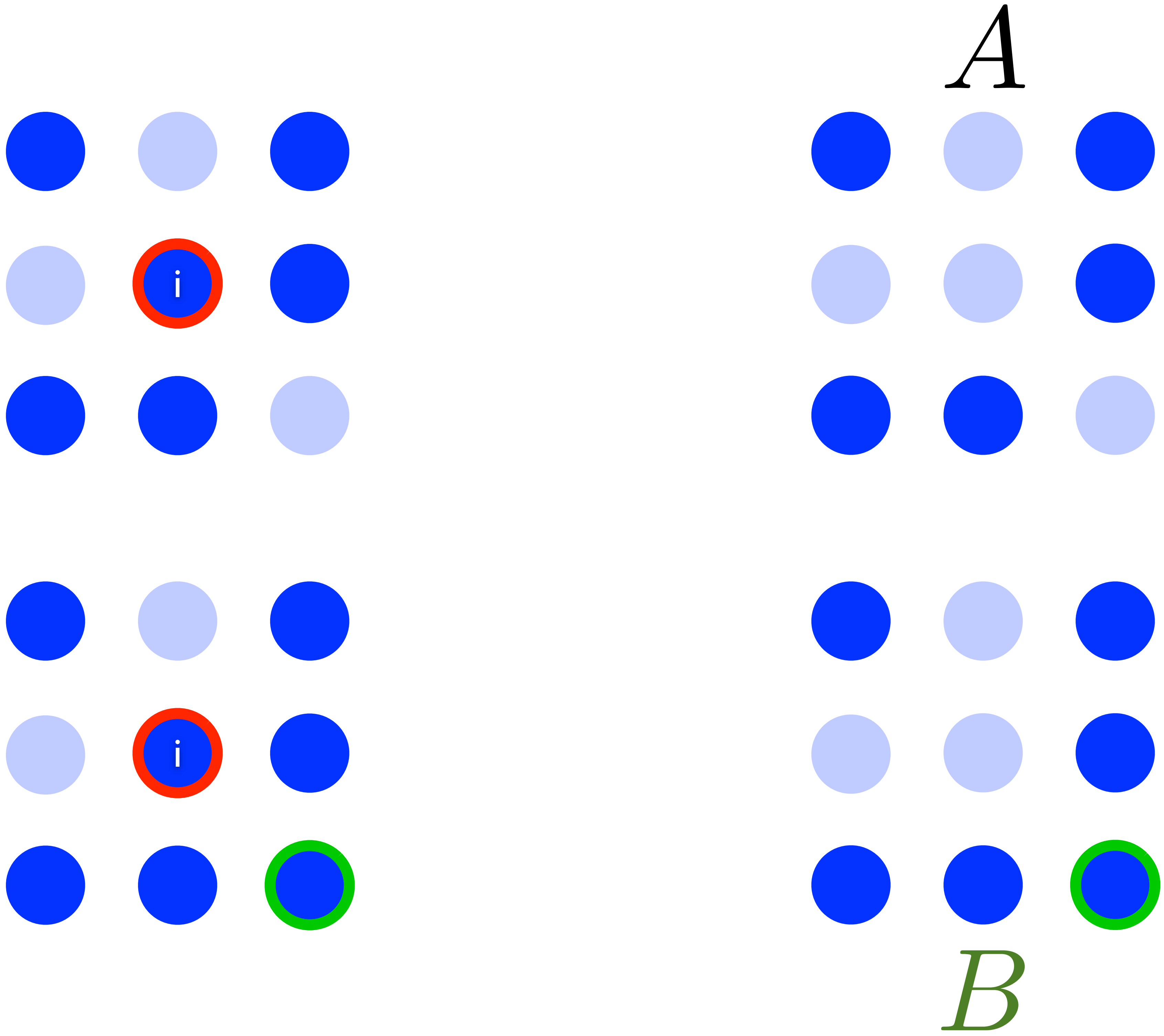

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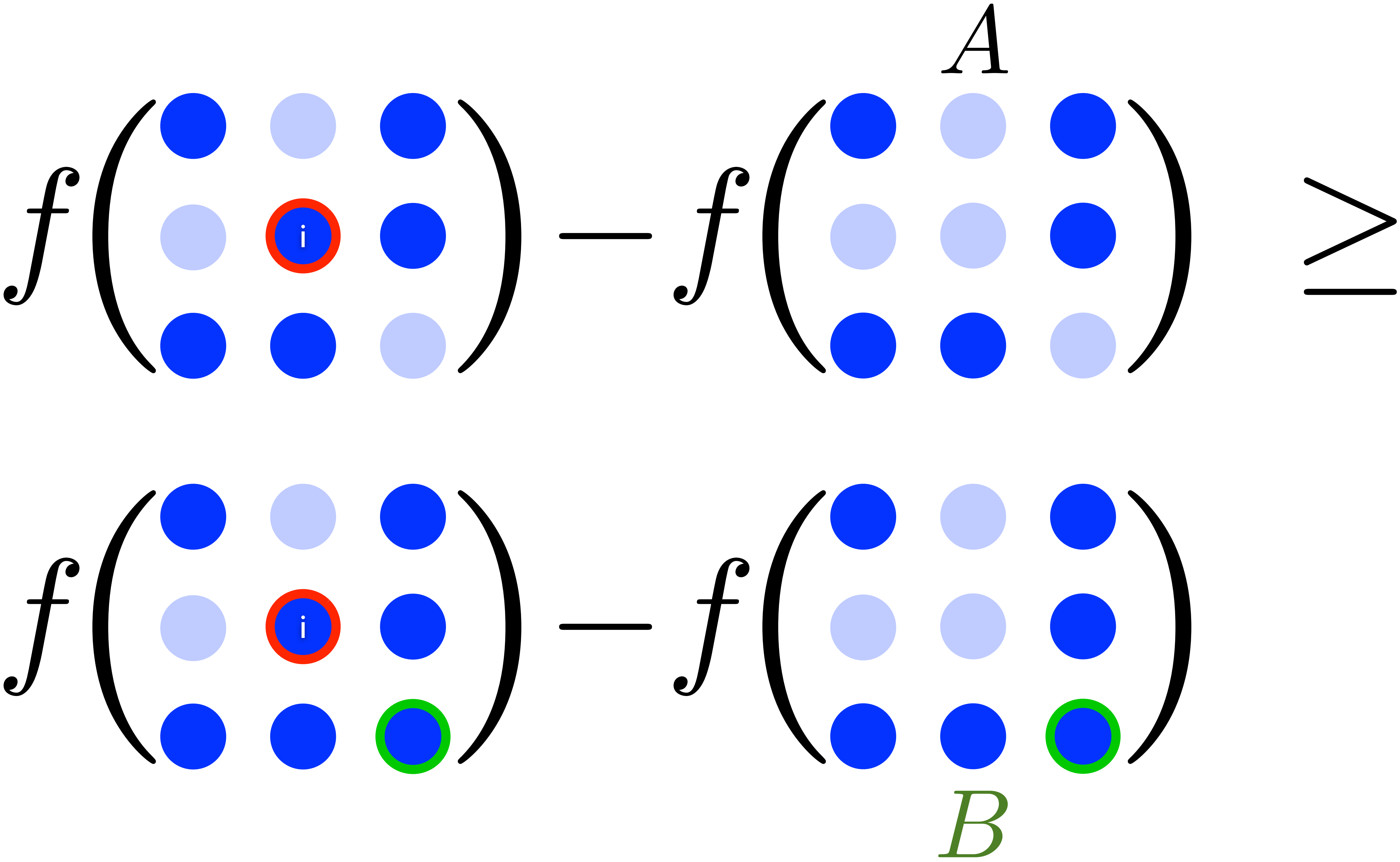
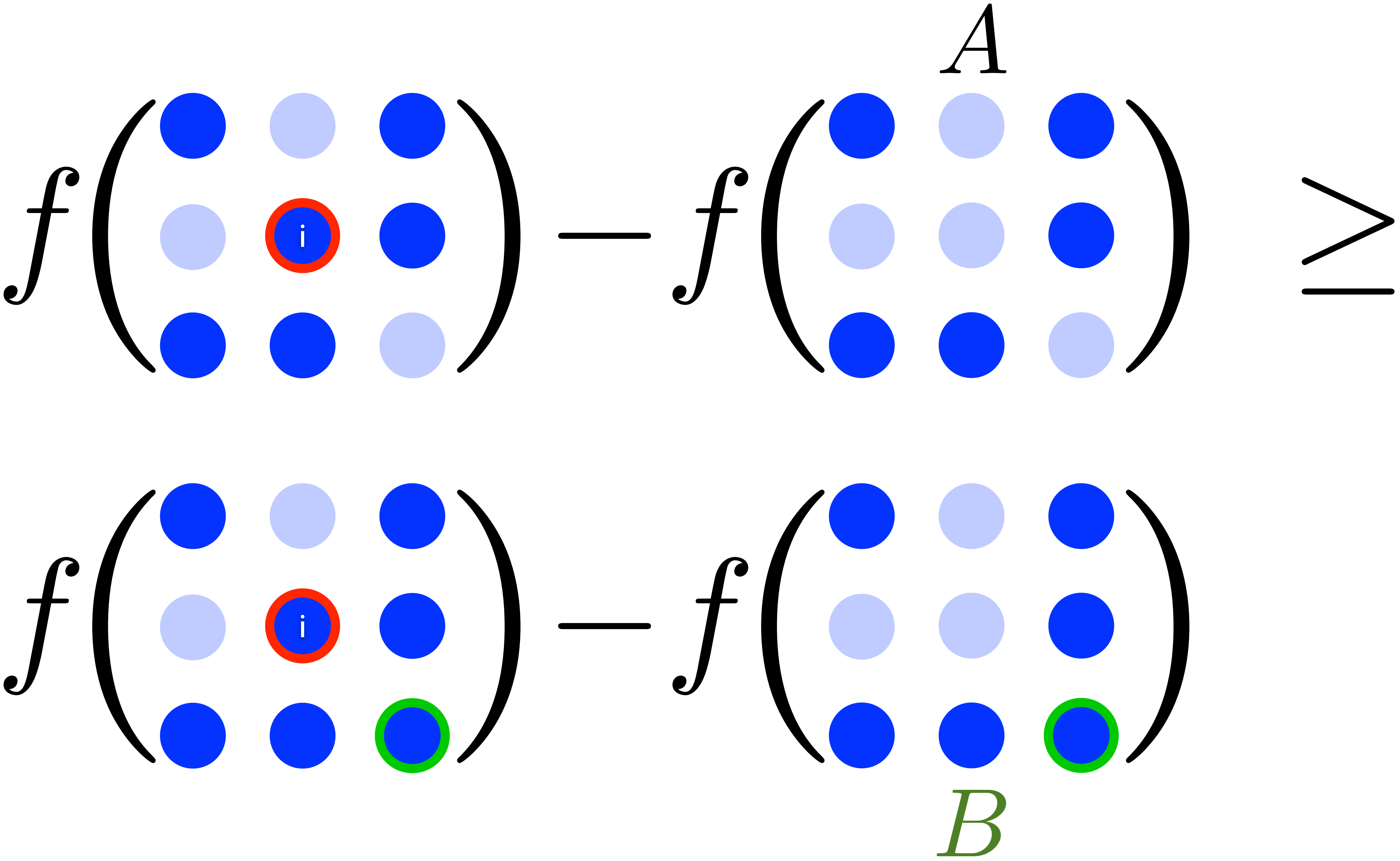


LOG-SUBMODULARITY



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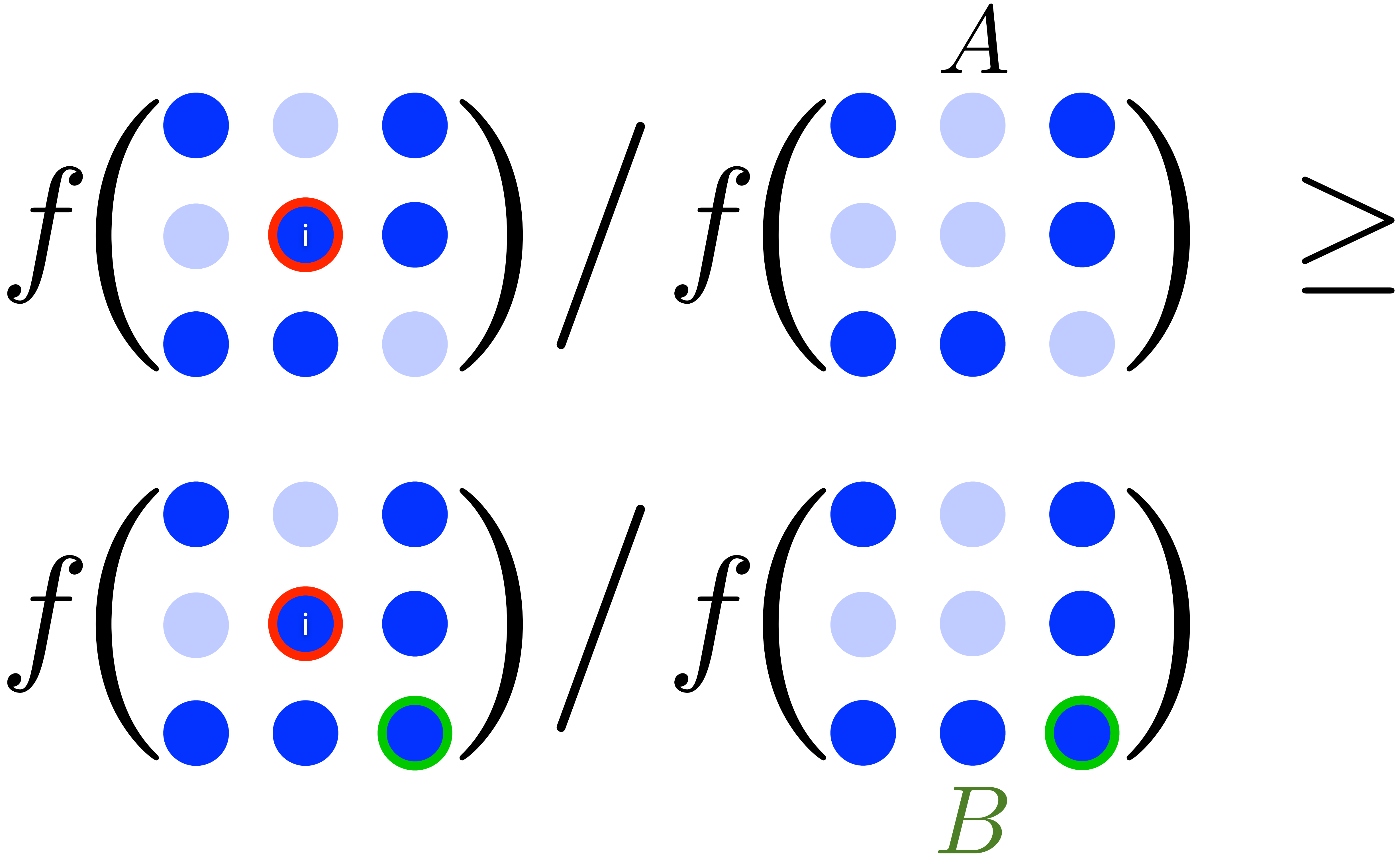
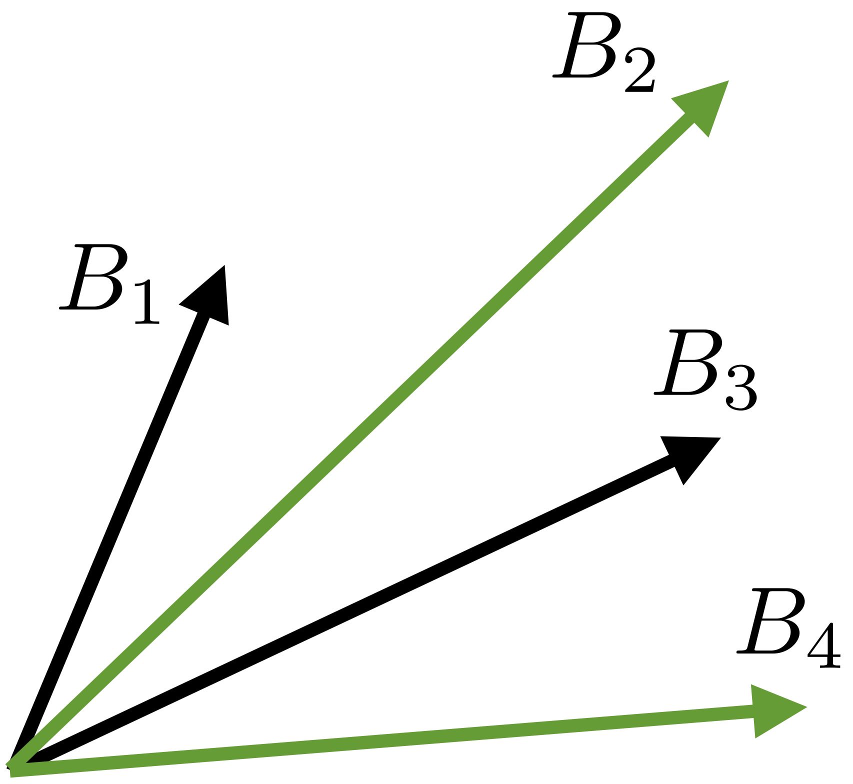


CHEKURI ET AL. 2011

CHEKURI ET AL. 2011

Step 1: Relax inclusion-exclusion



$Y = \{2, 4\}$

CHEKURI ET AL. 2011

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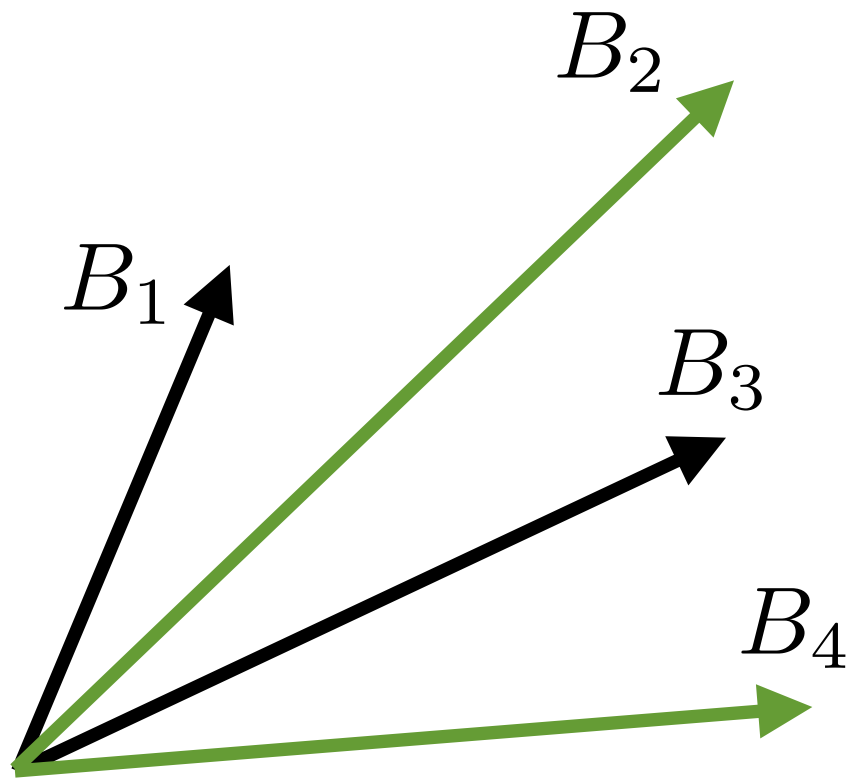

$$Y = \{2, 4\}$$

$$\mathbf{x} = [0, 1, 0, 1]$$

CHEKURI ET AL. 2011

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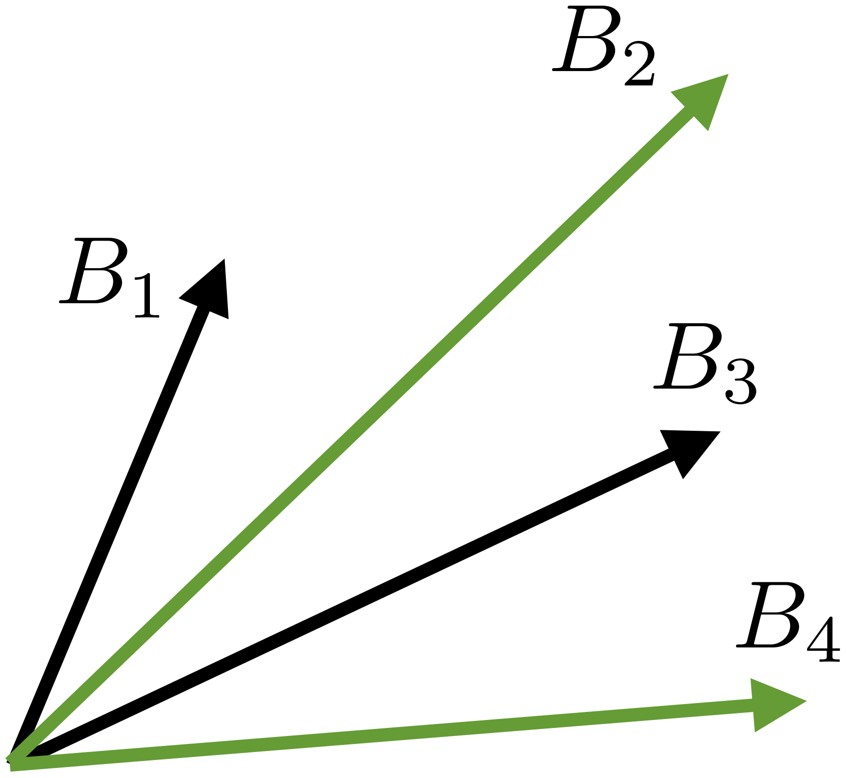

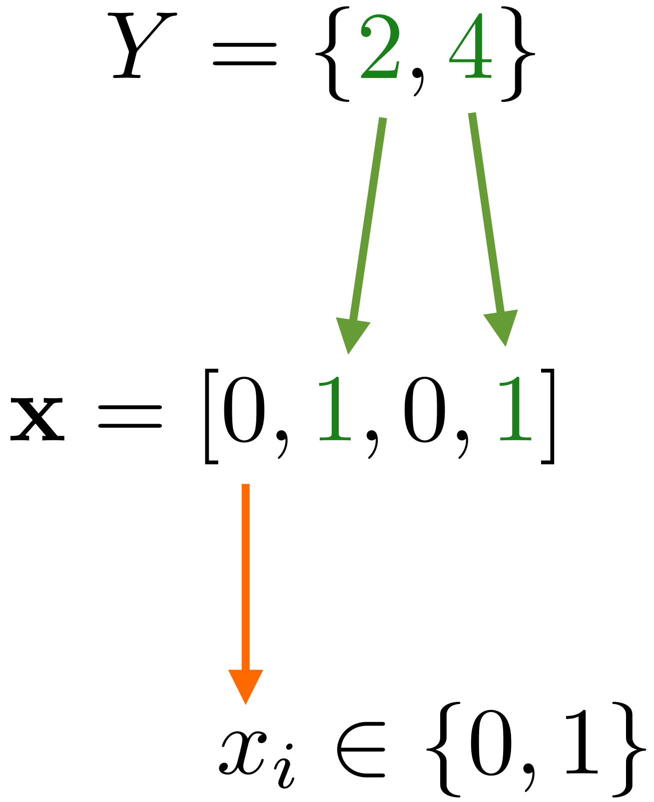

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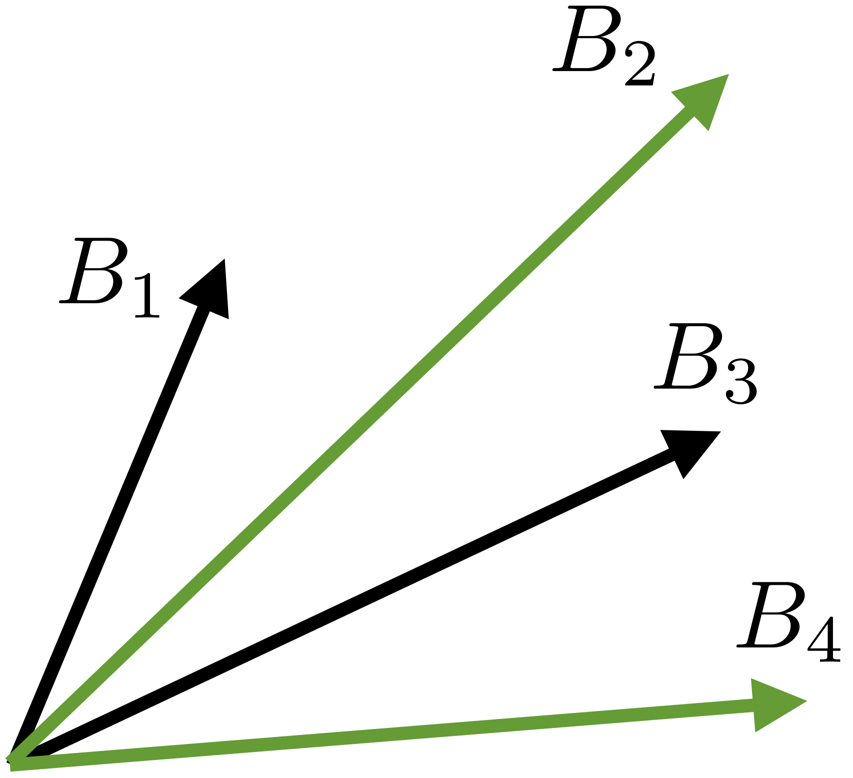

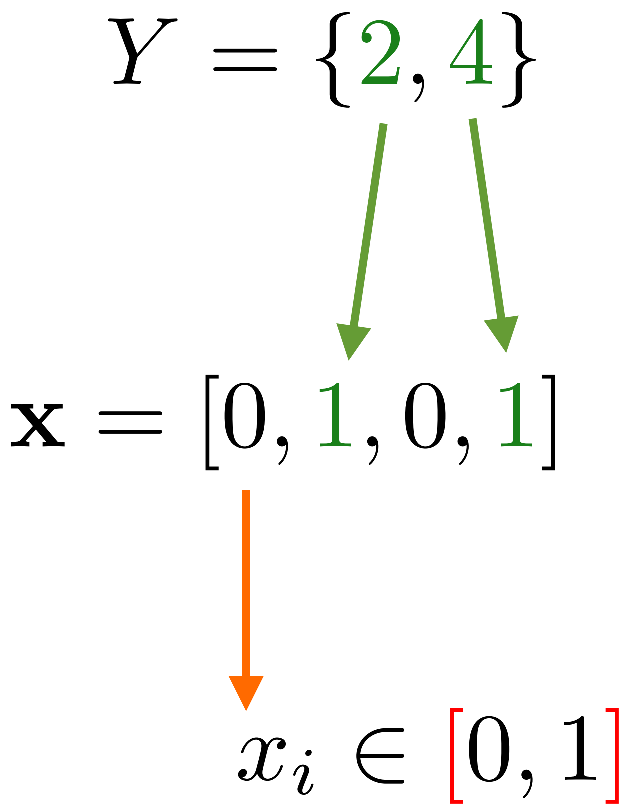

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CHEKURI ET AL. 2011

Step 2: Extend objective

$$F(\mathbf{x}) = E_{\mathbf{x}}[\log \boxed{f(Y)}]$$ multilinear extension

$\downarrow$

log-submodular, like $\det(L_Y)$

CHEKURI ET AL. 2011

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$$= \sum_Y \prod_{i \in Y} x_i \prod_{i \notin Y} (1 - x_i) \log f(Y)$$

CHEKURI ET AL. 2011

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$$F(\mathbf{x}) = E_{\mathbf{x}}[\log f(Y)] \qquad \text{multilinear extension}$$

$$= \sum_Y \underbrace{\boxed{\prod_{i \in Y} x_i \prod_{i \notin Y} (1 - x_i)}}_{p_Y} \log f(Y)$$

CHEKURI ET AL. 2011

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2^N subsets

CHEKURI ET AL. 2011

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$$F(\mathbf{x}) = E_{\mathbf{x}}[\log f(Y)]$$ multilinear extension

$$= \underbrace{\boxed{\sum_{Y}}}_{2^N \ \text{subsets}} \prod_{i \in Y} x_i \prod_{i \notin Y} (1 - x_i) \log f(Y)$$

Step 3: Optimize using gradient-based methods $\frac{\partial F(\mathbf{x})}{\partial \mathbf{x}}$

CHEKURI ET AL. 2011

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CHEKURI ET AL. 2011

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$$2^N \text{ subsets } \implies \text{Monte Carlo required}$$

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SOFTMAX EXTENSION

GILLENWATER, KULESZA, AND TASKAR (NIPS 2012)

Multilinear: $F(\mathbf{x}) = \sum_Y \prod_{i \in Y} x_i \prod_{i \notin Y} (1 - x_i) \log f(Y)$

SOFTMAX EXTENSION

GILLENWATER, KULESZA, AND TASKAR (NIPS 2012)

Multilinear: $F(\mathbf{x}) = \sum_Y \prod_{i \in Y} x_i \prod_{i \notin Y} (1 - x_i) \log f(Y)$

Softmax: $\tilde{F}(\mathbf{x}) = \log \sum_Y \prod_{i \in Y} x_i \prod_{i \notin Y} (1 - x_i) f(Y)$

SOFTMAX EXTENSION

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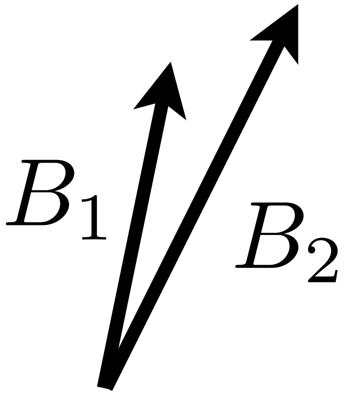
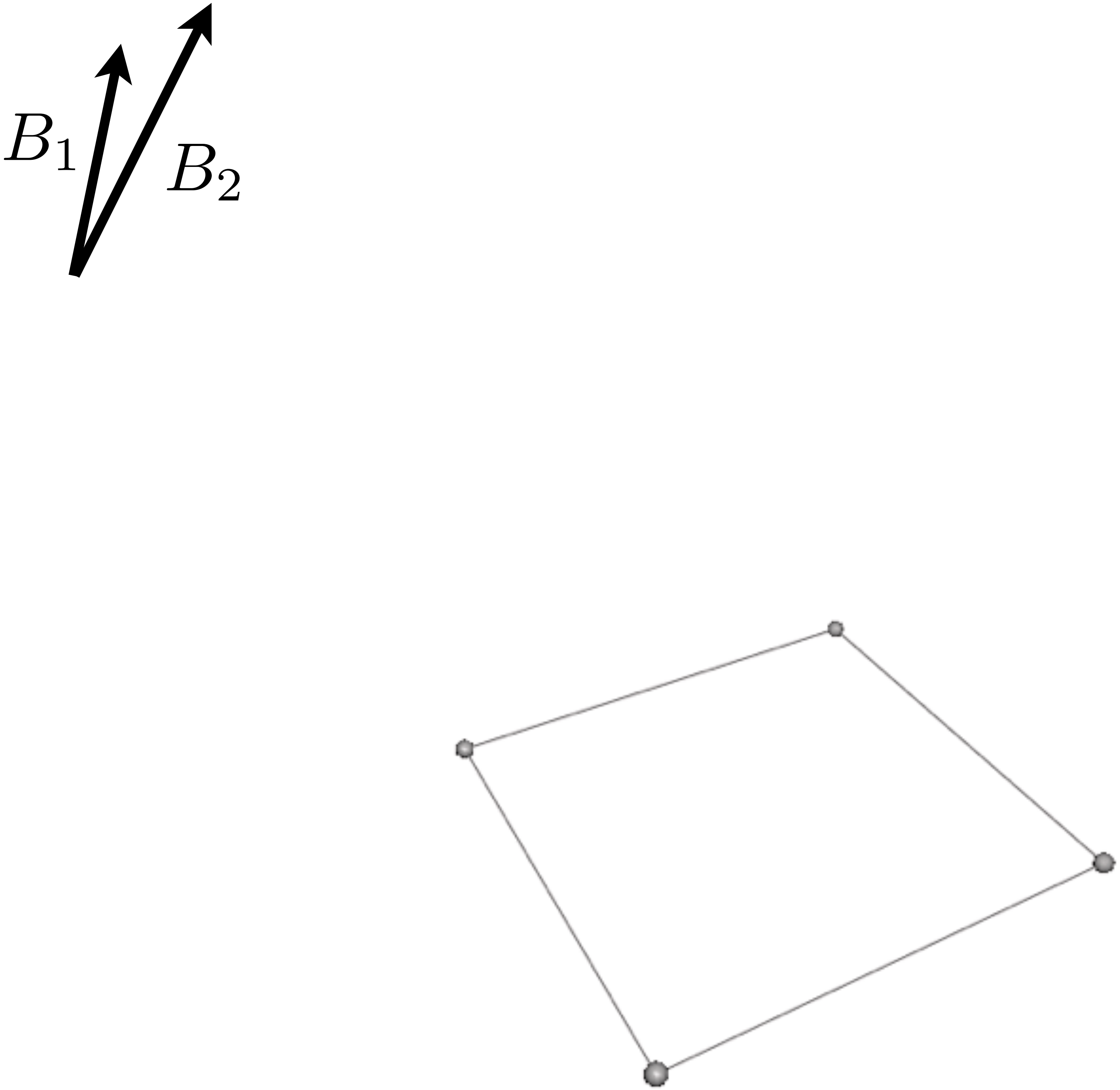
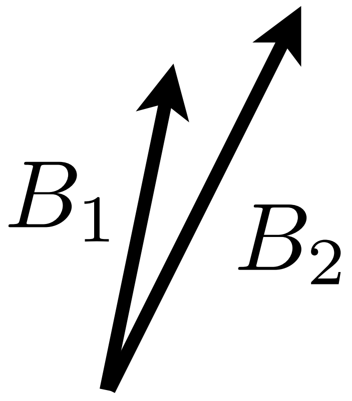
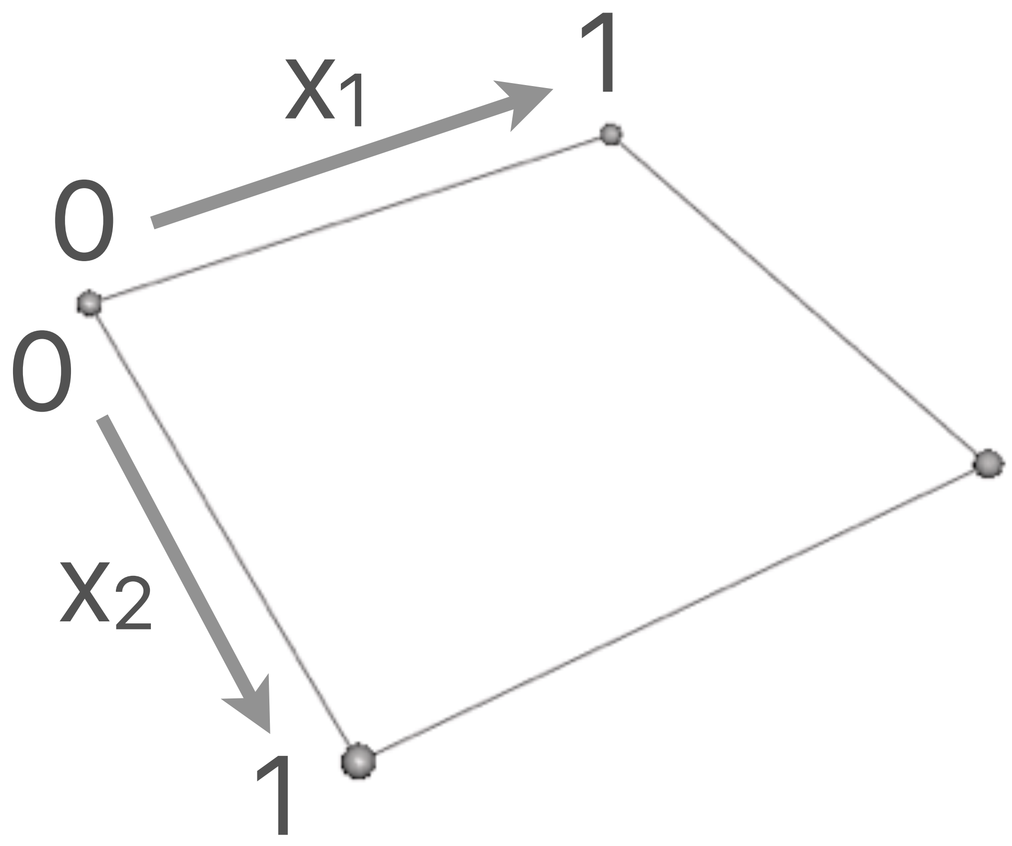
$N = 2$

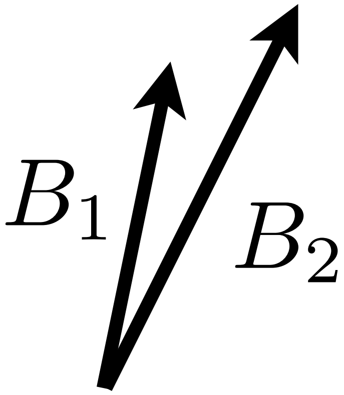



B_1
B_2

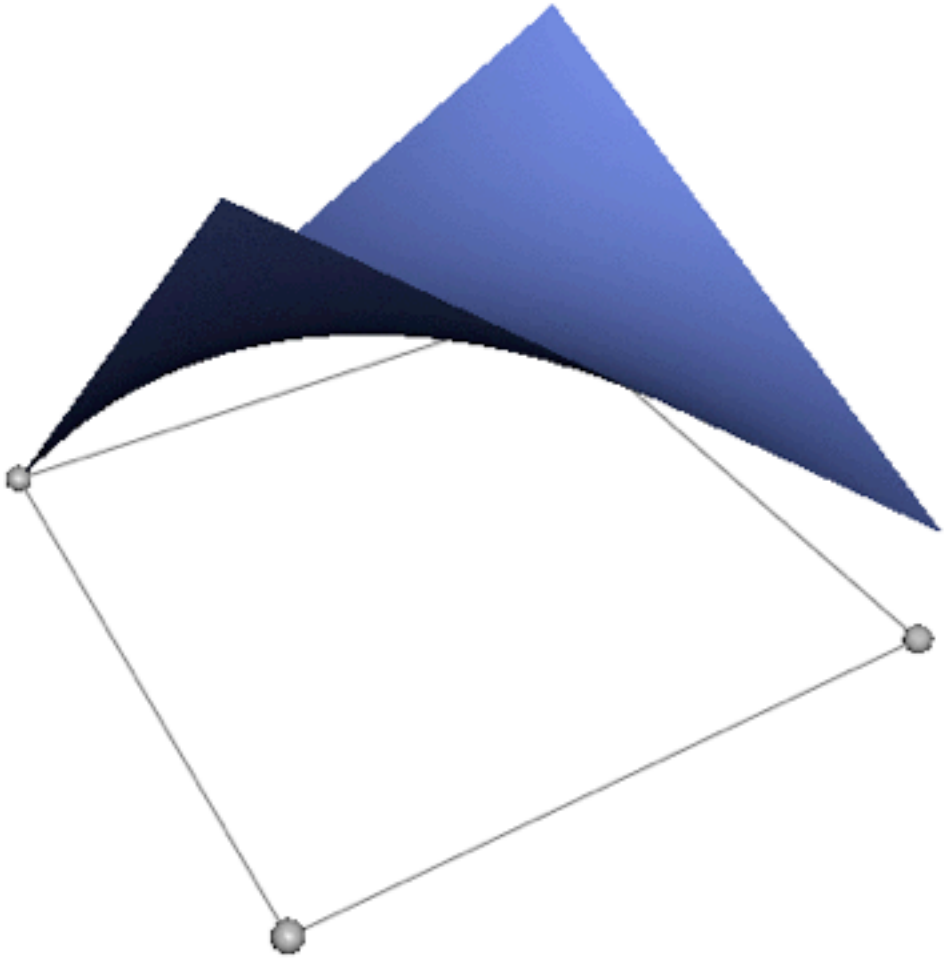


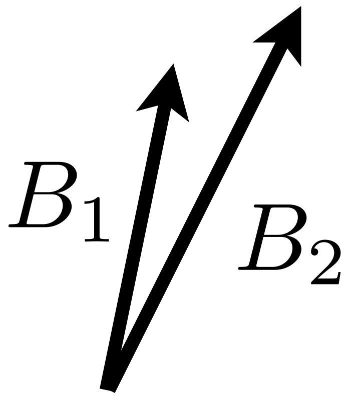
Relaxed domain





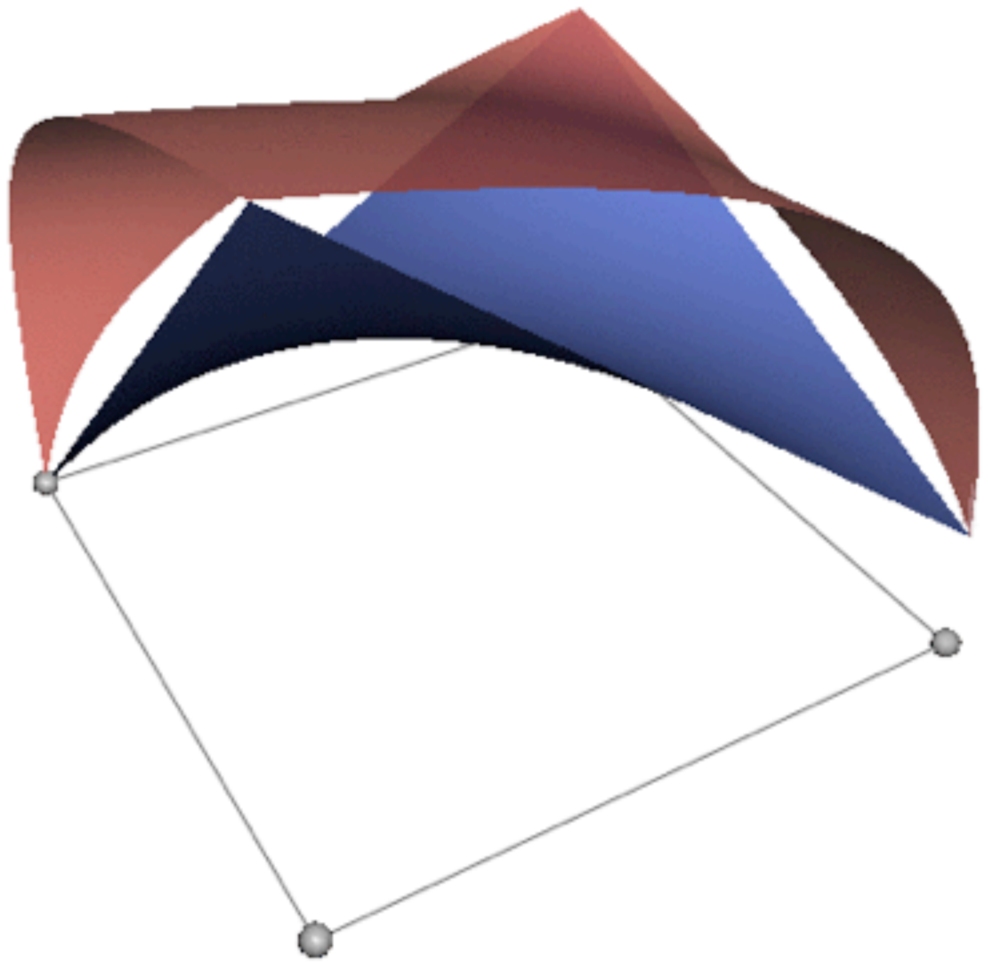
Multilinear extension

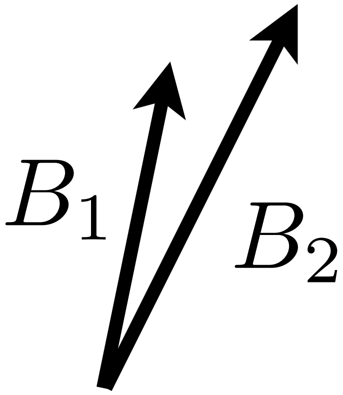




Softmax extension

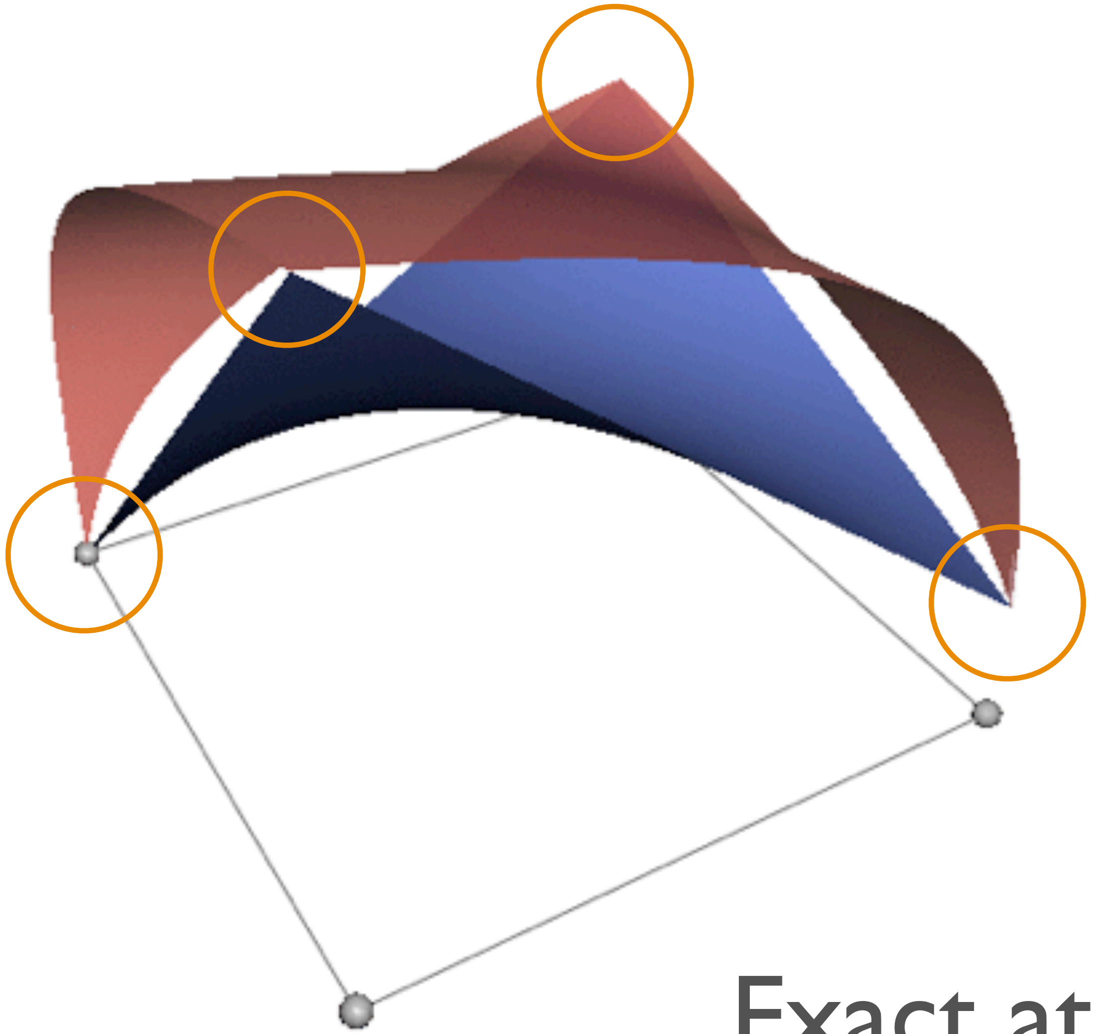
Multilinear extension





Softmax extension

Multilinear extension



Exact at integral points

SOFTMAX EXTENSION

GILLENWATER, KULESZA, AND TASKAR (NIPS 2012)

$$\tilde{F}(\mathbf{x}) = \log \sum_Y \prod_{i \in Y} x_i \prod_{i \notin Y} (1 - x_i) f(Y)$$

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Theorem:

Efficiently computable for $f(Y) = \det(L_Y)$

$$O(N^3)$$

$$\tilde{F}(\mathbf{x}) = \log \det(\operatorname{diag}(\mathbf{x})(L - I) + I)$$

EFFICIENCY PROOF

$$\exp(\tilde{F}(\boldsymbol{x})) = \sum_{Y : Y \subseteq \mathcal{Y}} \det(L_Y) \prod_{i : i \in Y} x_i \prod_{i : i \notin Y} (1 - x_i)$$

EFFICIENCY PROOF

$$\begin{aligned}\exp(\tilde{F}(\boldsymbol{x})) &= \sum_{Y:Y\subseteq\mathcal{Y}} \det(L_Y) \prod_{i:i\in Y} x_i \prod_{i:i\notin Y} (1-x_i) \\ &= \prod_{i=1}^{N}(1-x_i) \sum_{Y:Y\subseteq\mathcal{Y}} \det(L_Y) \prod_{i:i\in Y} \frac{x_i}{1-x_i}\end{aligned}$$

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EFFICIENCY PROOF

$$
\begin{aligned}
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&= \prod_{i=1}^{N}(1-x_i)\det(\mathrm{diag}(\boldsymbol{x})\mathrm{diag}(1-\boldsymbol{x})^{-1}L+I)
\end{aligned}
$$

Concave in all-positive/all-negative directions

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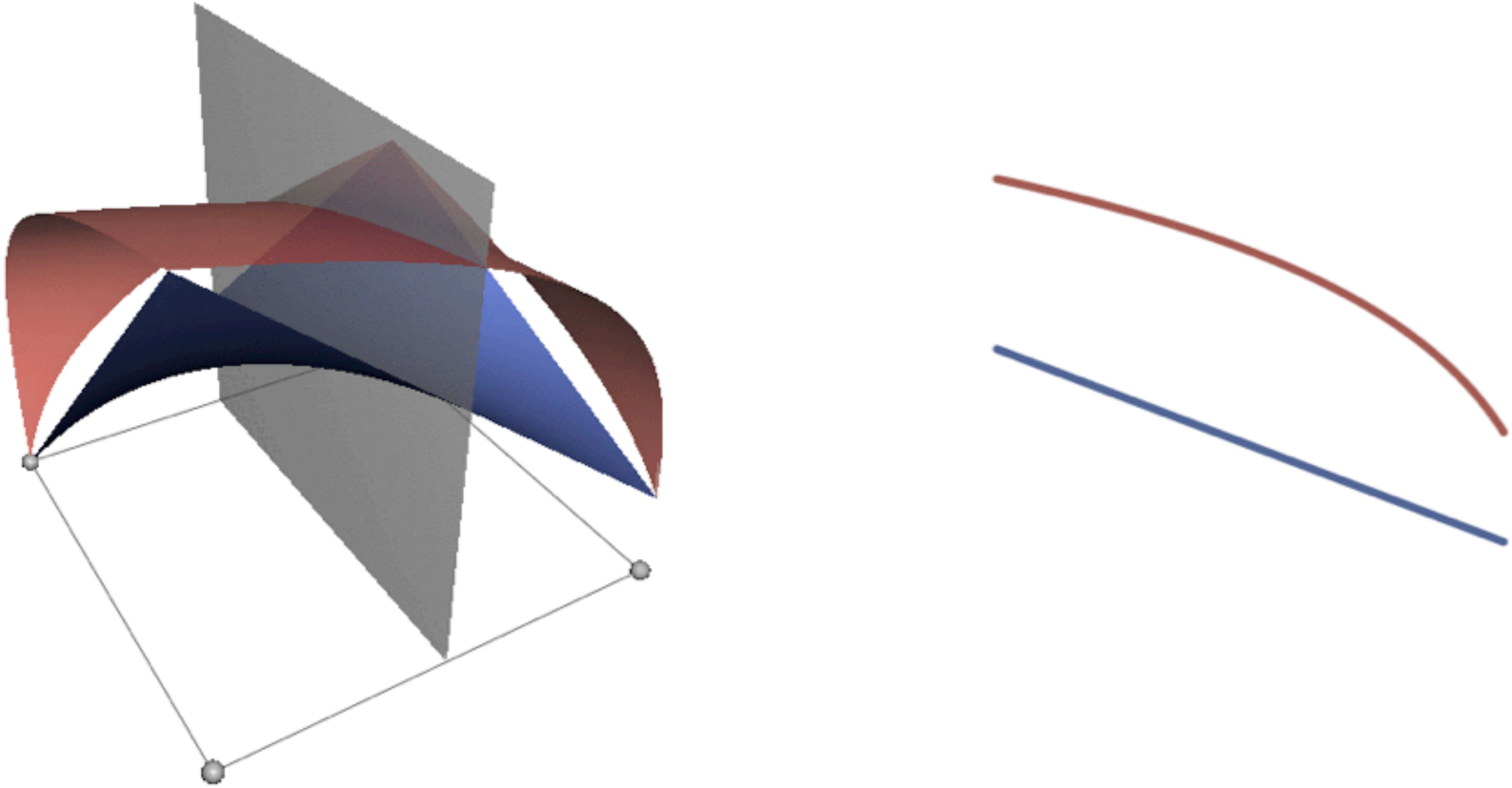

CONCAVITY PROOF

For $\boldsymbol{u} \geq \mathbf{0}$ and $\mathbf{0} < \boldsymbol{x} + s\boldsymbol{u} < \mathbf{1}$: $\frac{\partial^2}{\partial s^2} \tilde{F}(\boldsymbol{x} + s\boldsymbol{u}) \leq 0.$

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$$\frac{\partial^2}{\partial s^2}\tilde{F}(\boldsymbol{x} + s\boldsymbol{u}) = \mathrm{tr}(-D^{-2}\mathrm{diag}(\boldsymbol{u}^2) \\ + 2M^{-1}D^{-3}\mathrm{diag}(\boldsymbol{u}^2) \\ - (M^{-1}D^{-2}\mathrm{diag}(\boldsymbol{u}))^2)$$

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= \; &-\operatorname{tr}(SS^\top) \\
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\leq 0 &
\end{aligned}$$

APPROXIMATION GUARANTEE

Theorem: Concavity in all-positive directions

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+ Submodularity $\Longrightarrow$

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Theorem: In the unconstrained case, LOCAL OPT will be integer (no rounding necessary).

Constrained: No guarantees, but in practice pipage rounding and thresholding work well.

$\max_{Y \in S}$

MATCHED SUMMARIZATION

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20 Republican primary debates

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Average of 179 quotes per candidate

MATCHED SUMMARIZATION

- **R1** (taxes): **No tax on interest, dividends, or capital gains.**

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Matched summary

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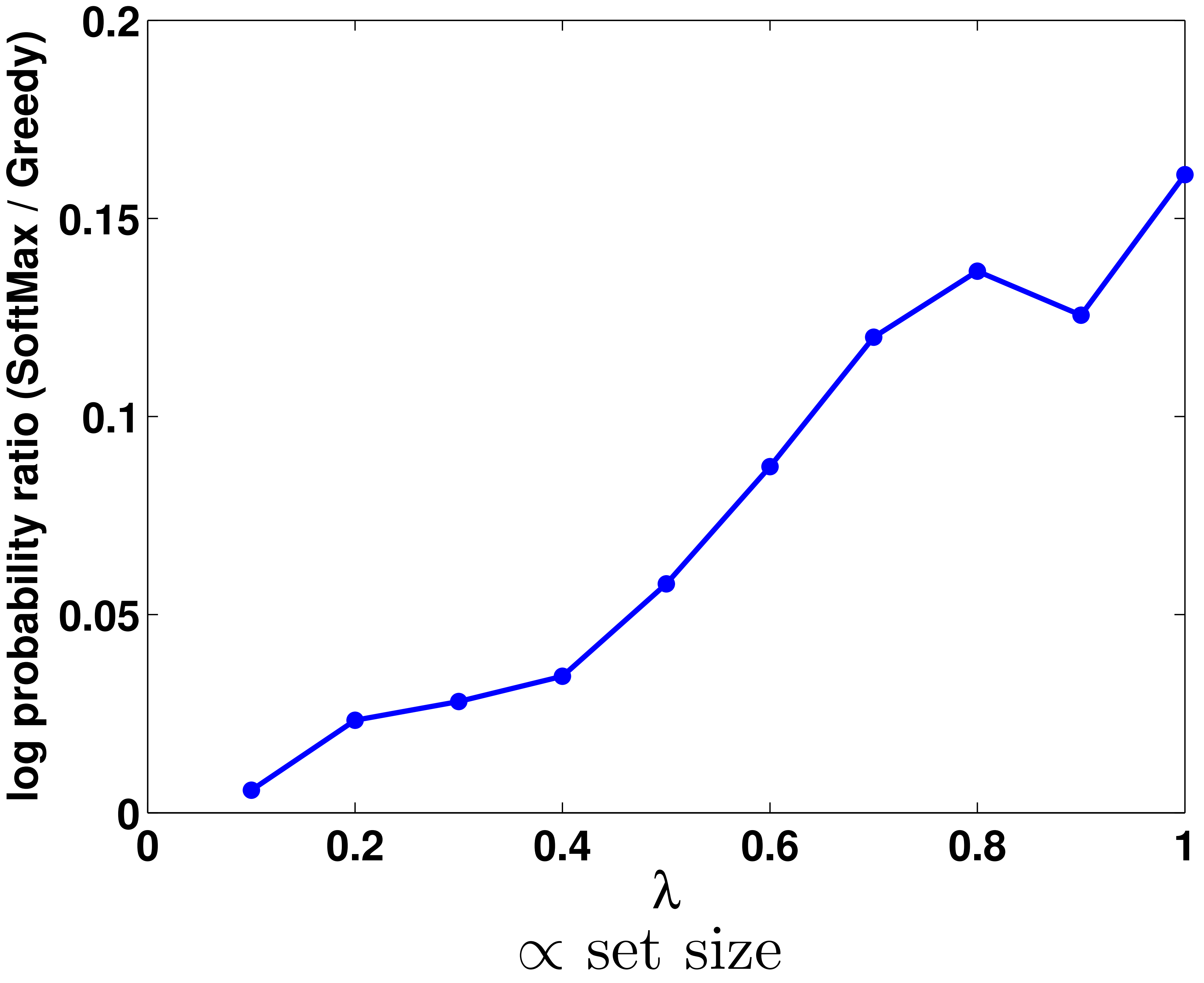
PERFORMANCE



PROPOSED WORK

- Update experiments with faster algorithms and testing with random restarts
- k-DPP MAP estimation algorithm based on marginals, plus extension to structured k-DPPs

FUTURE WORK

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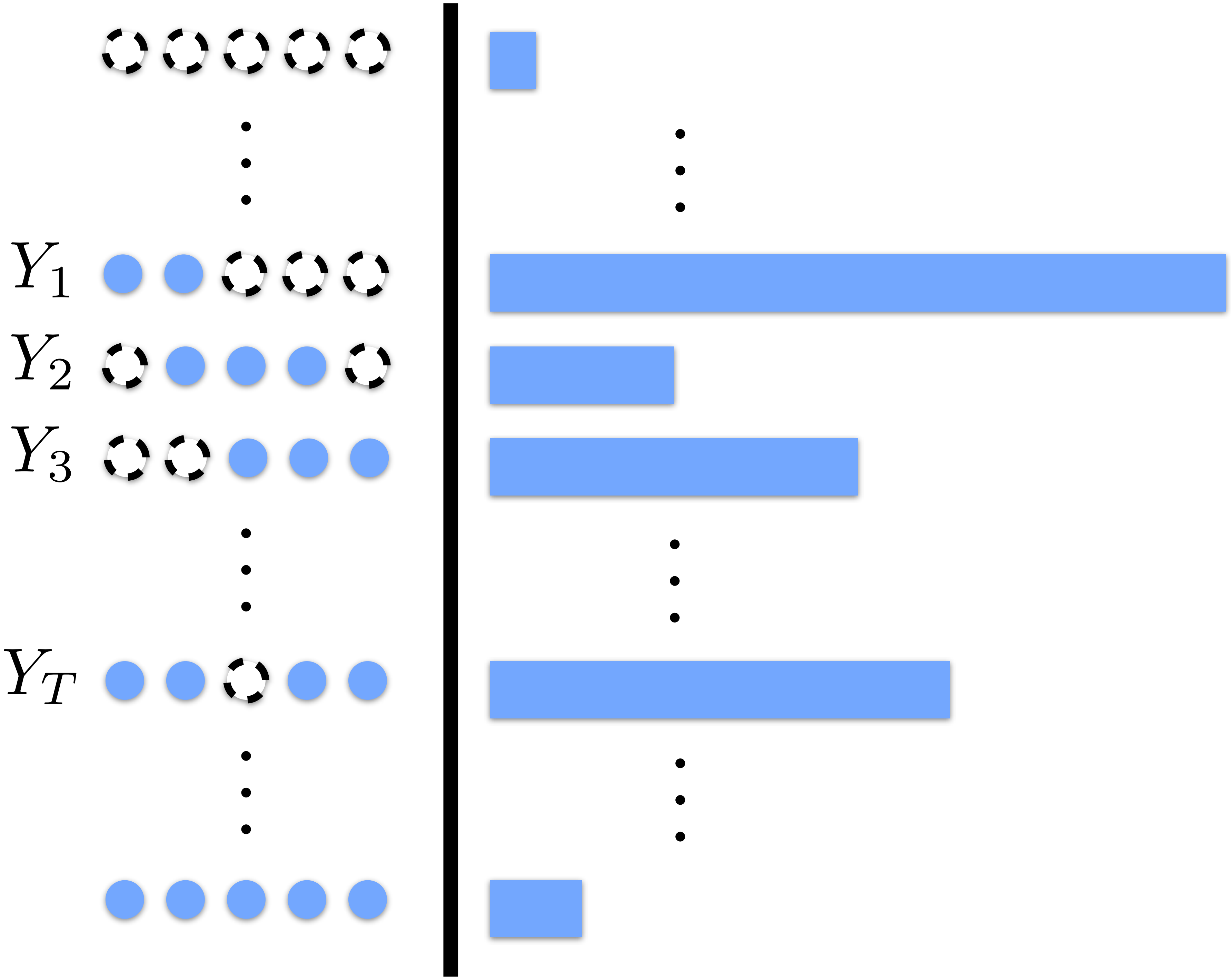
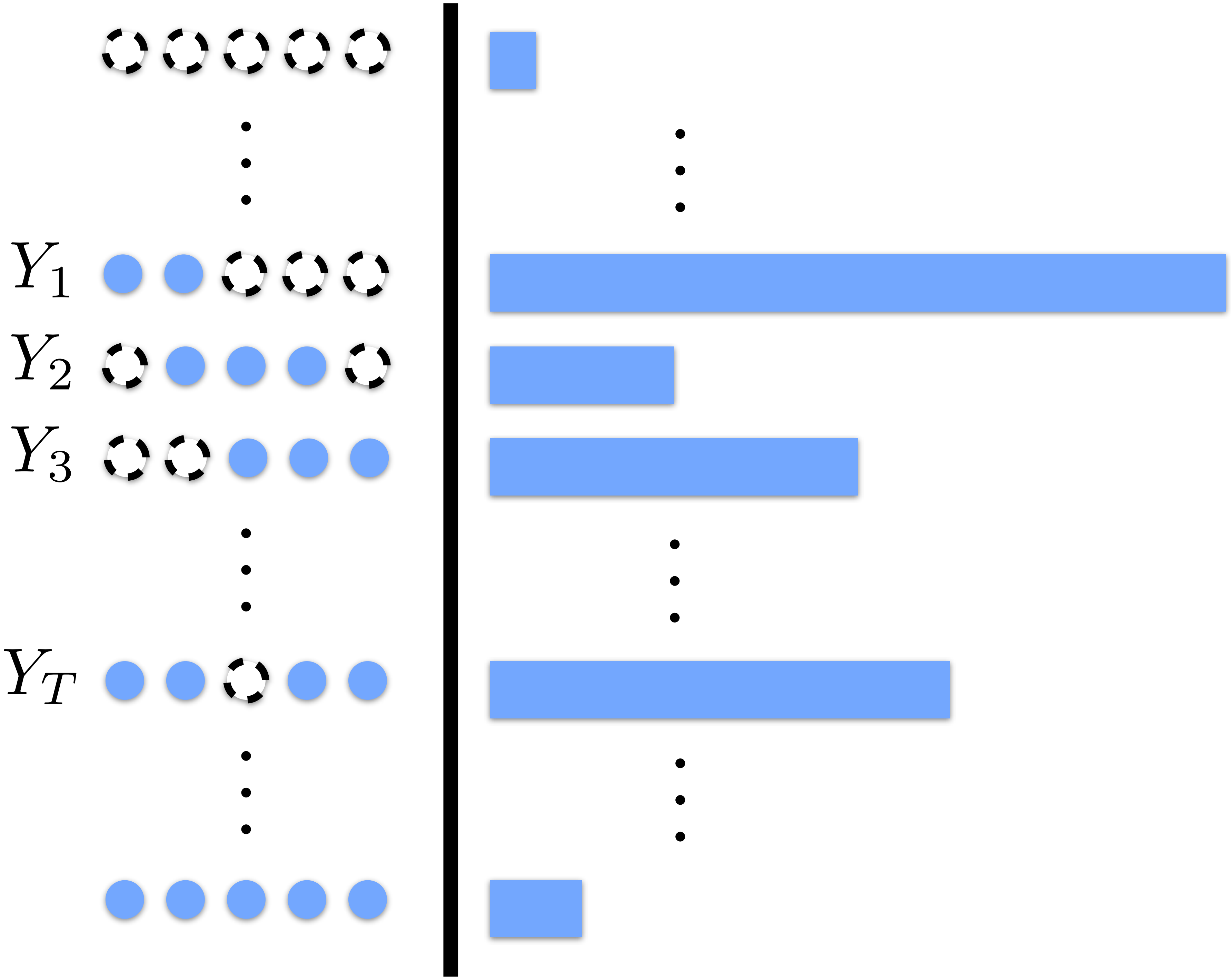
3. LIKELIHOOD MAXIMIZATION

LIKELIHOOD



LIKELIHOOD

$\mathcal{P}$ is a DPP



$$\mathcal{L}(L) = \sum_{t=1}^{T} \log\left(\frac{\det(L_{Y_t})}{\det(L+I)}\right)$$

OR

$$\mathcal{L}(K) = \sum_{t=1}^{T} \log\left(|\det(K - I_{\overline{Y}_t})|\right)$$

NON-CONCAVITY

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$$\mathcal{L}(L) = \sum_{t=1}^{T} \left[\log\det(L_{Y_t}) - \log\det(L + I)\right]$$

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$$\max_{L: L \succeq 0} \mathcal{L}(L)$$

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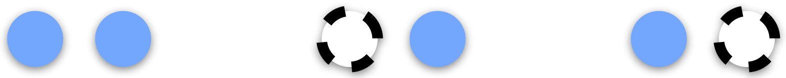

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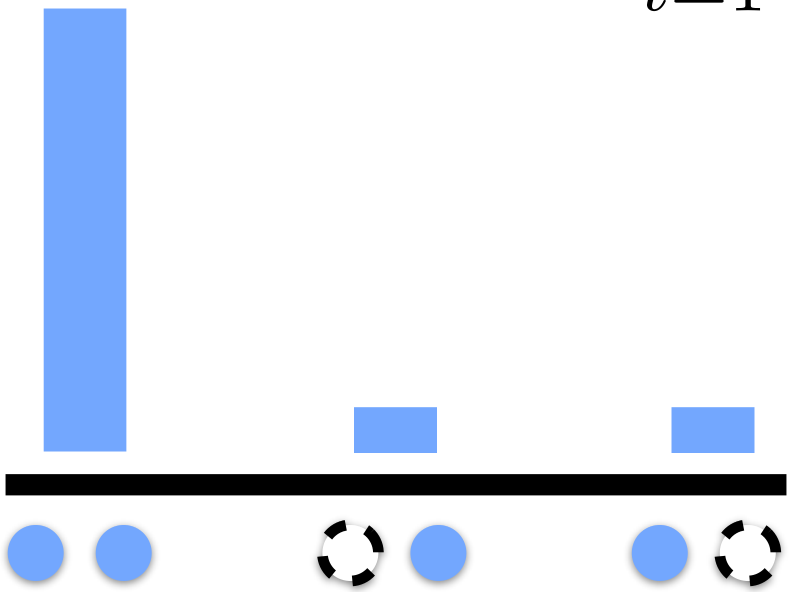

$$L = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$

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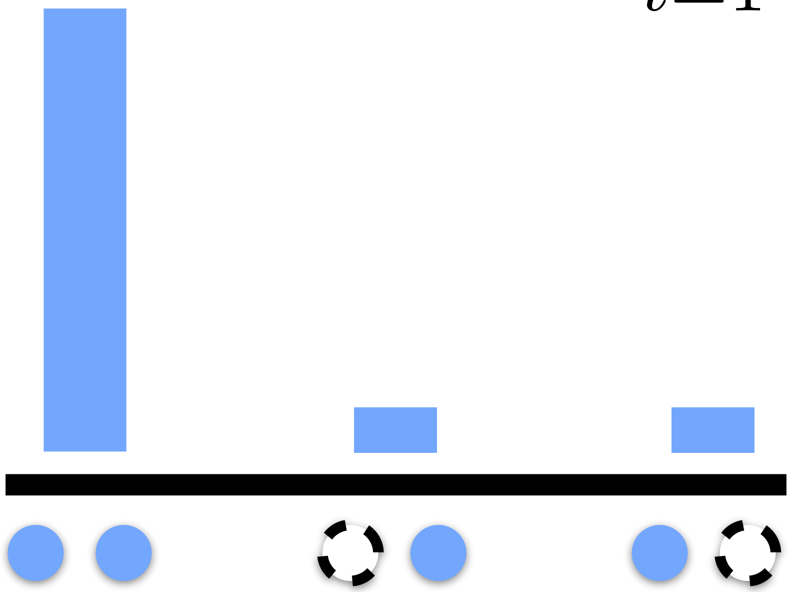

$$L = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$

$$+ \eta \begin{bmatrix} 0.1 & 0.5 \\ 0.5 & 0.1 \end{bmatrix}$$

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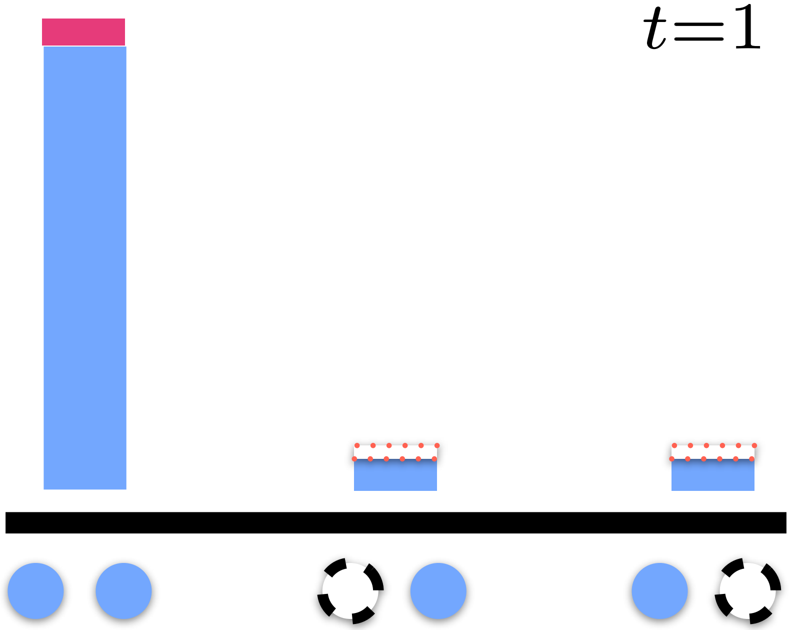

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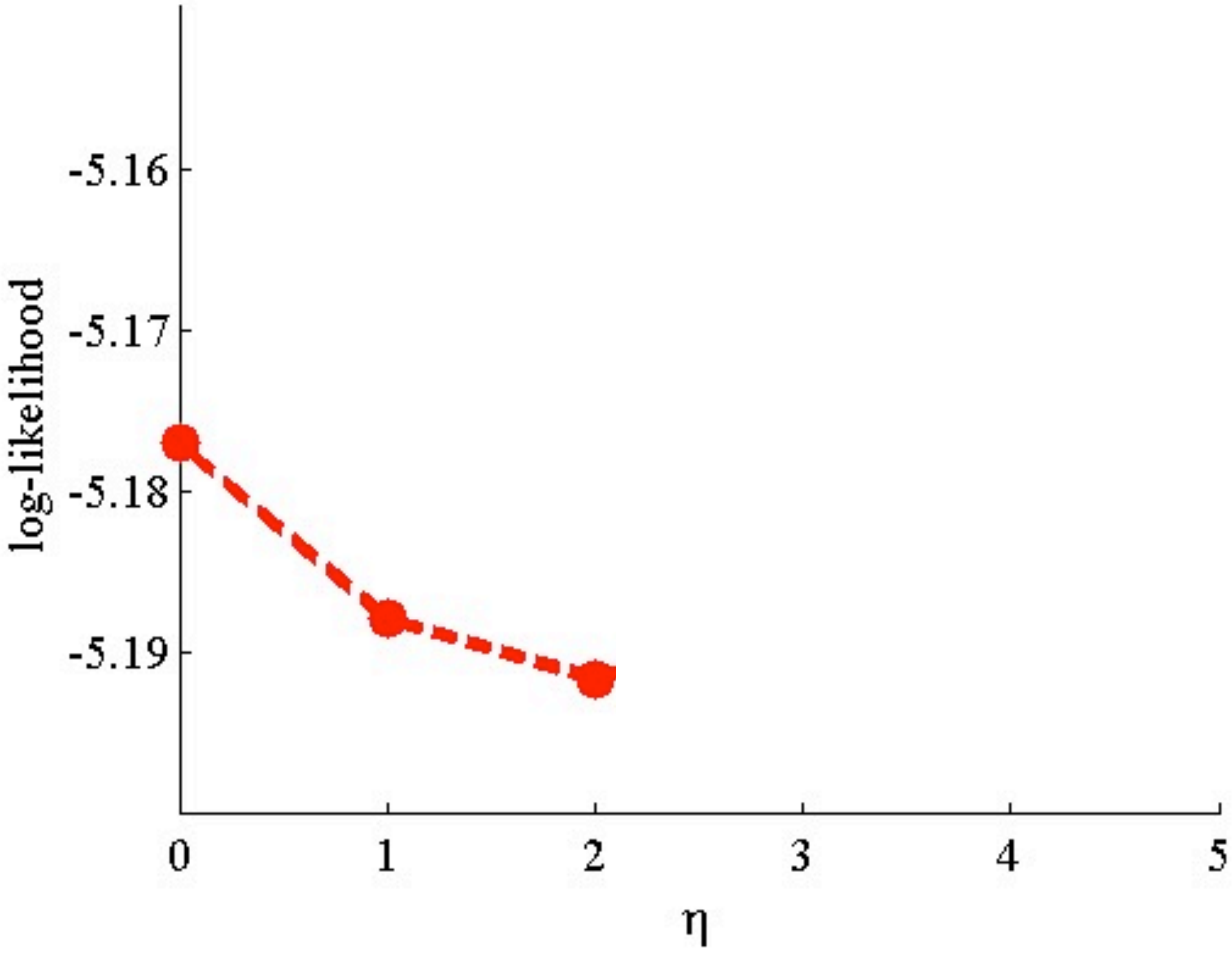

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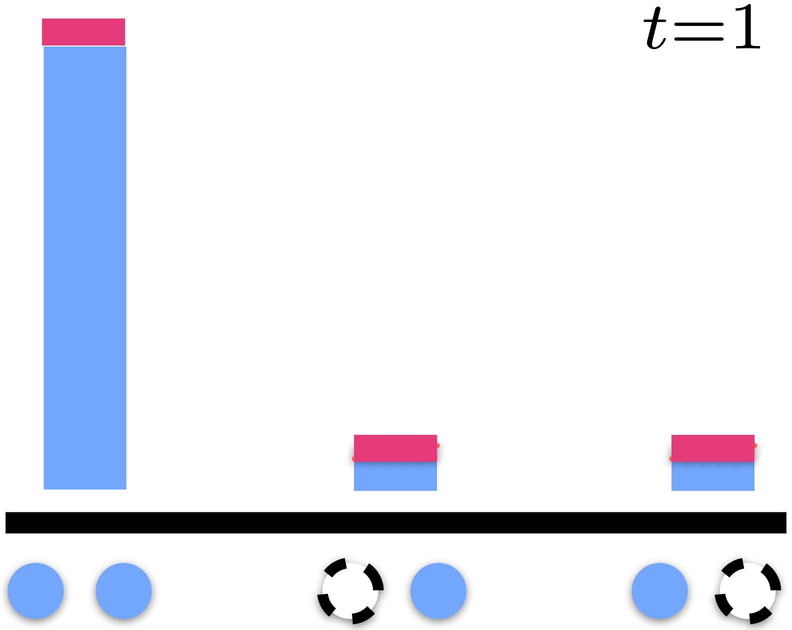

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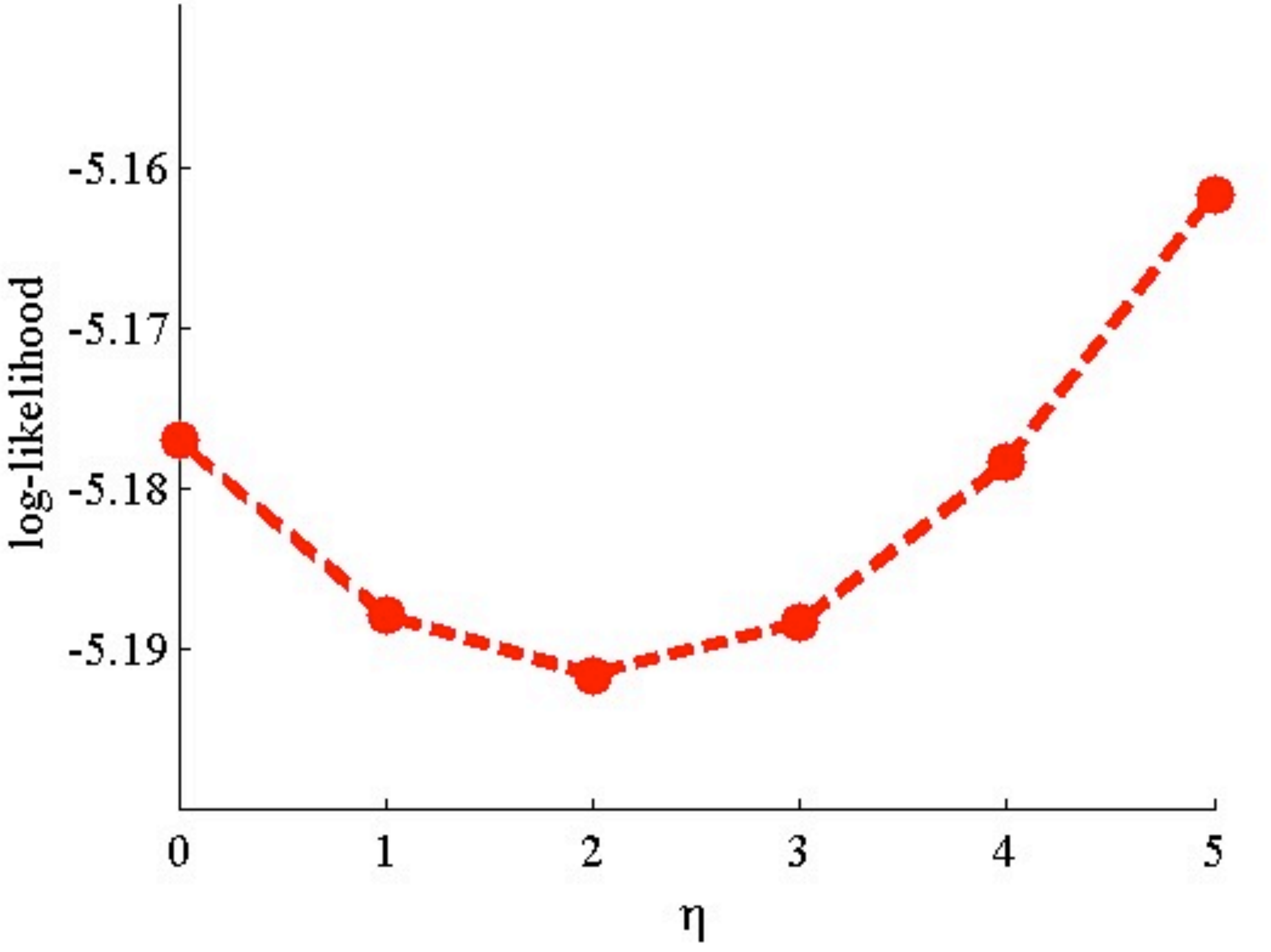

K PARAMETERIZATION

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Change of variables for developing EM algorithm:

$$L = V \operatorname{diag}(\boldsymbol{\lambda}) V^\top$$

$$K = V \operatorname{diag}(\tfrac{\boldsymbol{\lambda}}{1+\boldsymbol{\lambda}}) V^\top$$

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$$\max_{\substack{K:K \succeq 0, \\ I-K \succeq 0}} \mathcal{L}(K)$$

GRADIENT ASCENT

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$$\frac{\partial \mathcal{L}(K)}{\partial K} = \sum_{t=1}^{T} (K - I_{\overline{Y}_t})^{-1}$$

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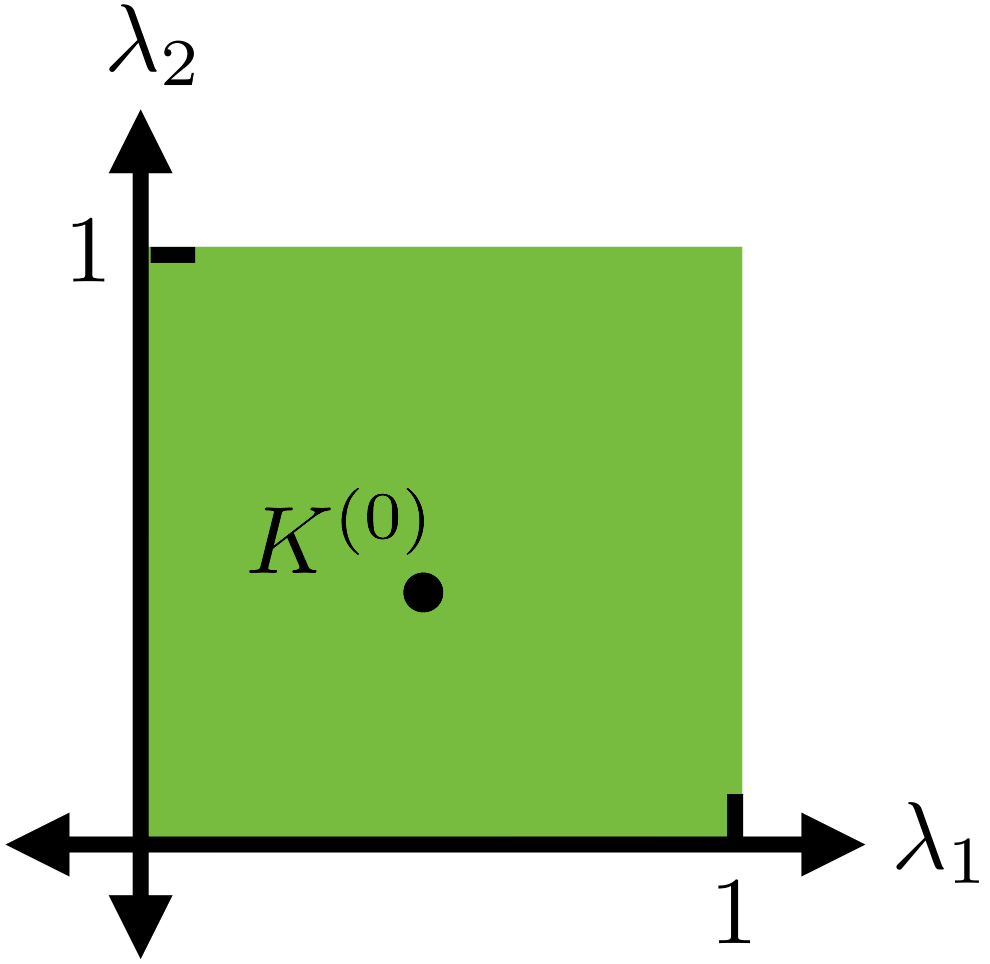

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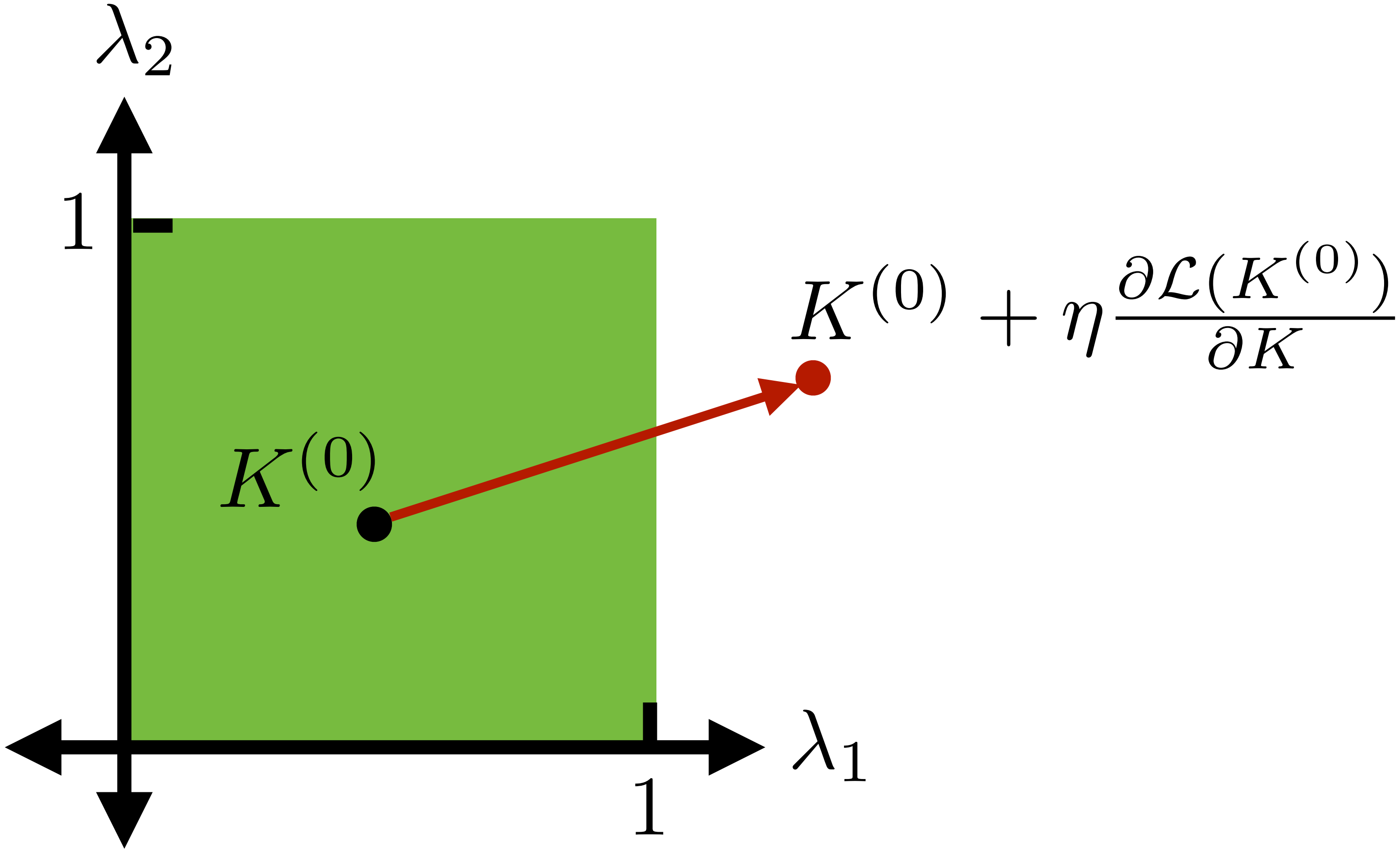

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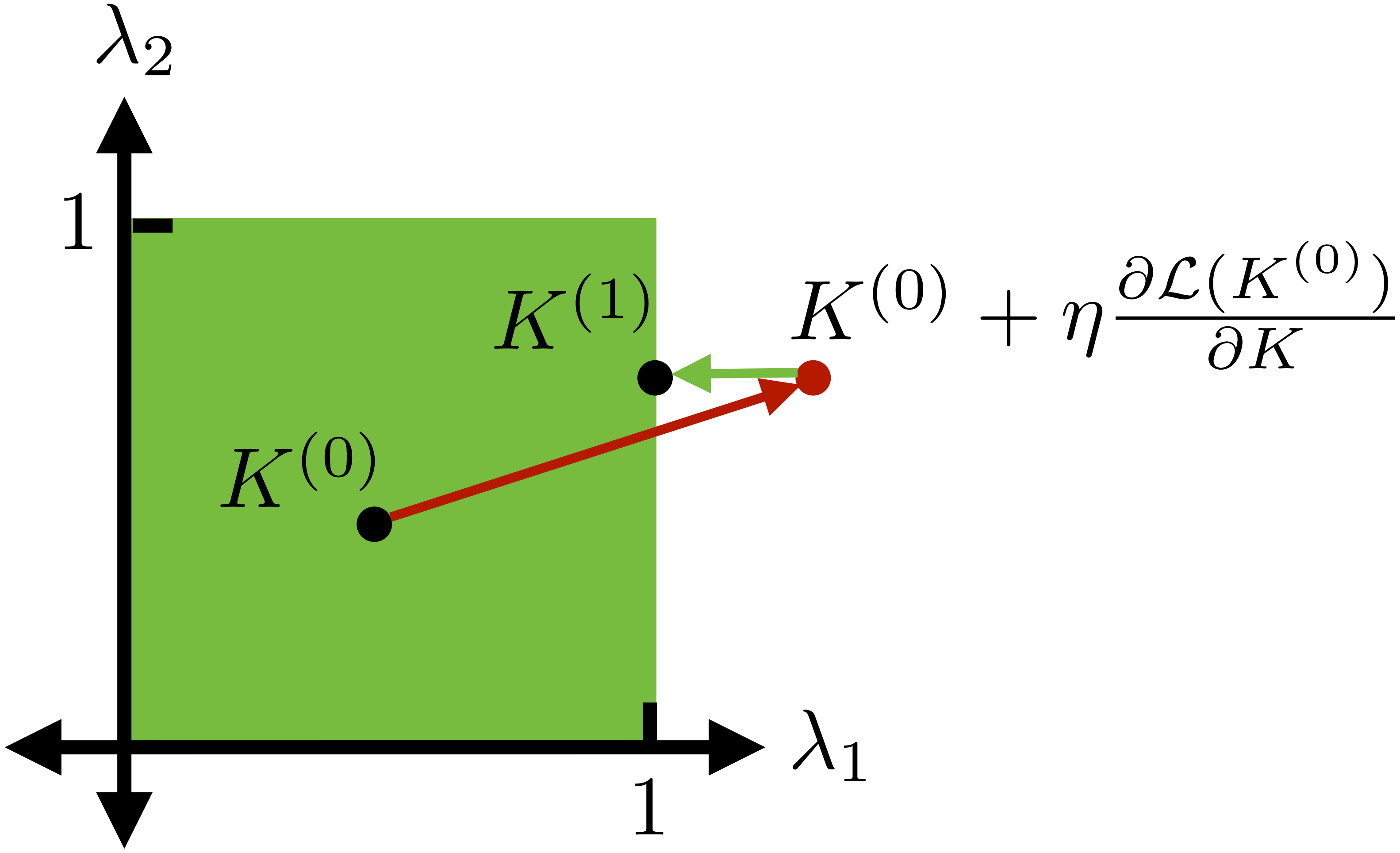

EXPECTATION-MAXIMIZATION

GILLENWATER, KULESZA, FOX, AND TASKAR (NIPS 2014)

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$$= \sum_{J: J \subseteq \{1,\dots,N\}} \Pr_K(Y \mid J) \Pr_K(J)$$

Hidden variable J

EXPECTATION-MAXIMIZATION

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$$\mathcal{L}(K) = \mathcal{L}(V, \Lambda) = \sum_{t=1}^{T} \log p_K(Y_t) = \sum_{t=1}^{T} \log \left(\sum_{J} p_K(J, Y_t) \right)$$

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$$\mathcal{L}(K) = \mathcal{L}(\overset{\substack{\text{eigenvectors}\\ \searrow}}{V}, \overset{\substack{\text{eigenvalues}\\ \swarrow}}{\Lambda}) = \sum_{t=1}^{T} \log p_K(Y_t) = \sum_{t=1}^{T} \log \left(\sum_{J} p_K(J, Y_t) \right)$$

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$$= \sum_{t=1}^{T} \log \left(\sum_{J} q(J \mid Y_t) \frac{p_K(J, Y_t)}{q(J \mid Y_t)} \right)$$

EXPECTATION-MAXIMIZATION

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$$
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&= \sum_{t=1}^{T} \log \left(\sum_{J} q(J \mid Y_t) \frac{p_K(J, Y_t)}{q(J \mid Y_t)} \right) \\
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\end{aligned}
$$

EXPECTATION-MAXIMIZATION

GILLENWATER, KULESZA, FOX, AND TASKAR (NIPS 2014)

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$$\mathcal{P}(Y) = \sum_{J : J \subseteq \{1, \dots, N\}} \mathcal{P}_{K^J}(Y) \prod_{j : j \in J} \lambda_j \prod_{j : j \notin J} (1 - \lambda_j)$$

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EXPECTATION-MAXIMIZATION

GILLENWATER, KULESZA, FOX, AND TASKAR (NIPS 2014)

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V update is more complicated, but still efficient

PRODUCT RECOMMENDATION

furniture

carseats

toys

PRODUCT RECOMMENDATION

$\sim$ 30,000 Amazon baby registries

furniture

carseats

toys

PRODUCT RECOMMENDATION

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13 categories

furniture

carseats

toys

EM VS PROJECTED GRADIENT

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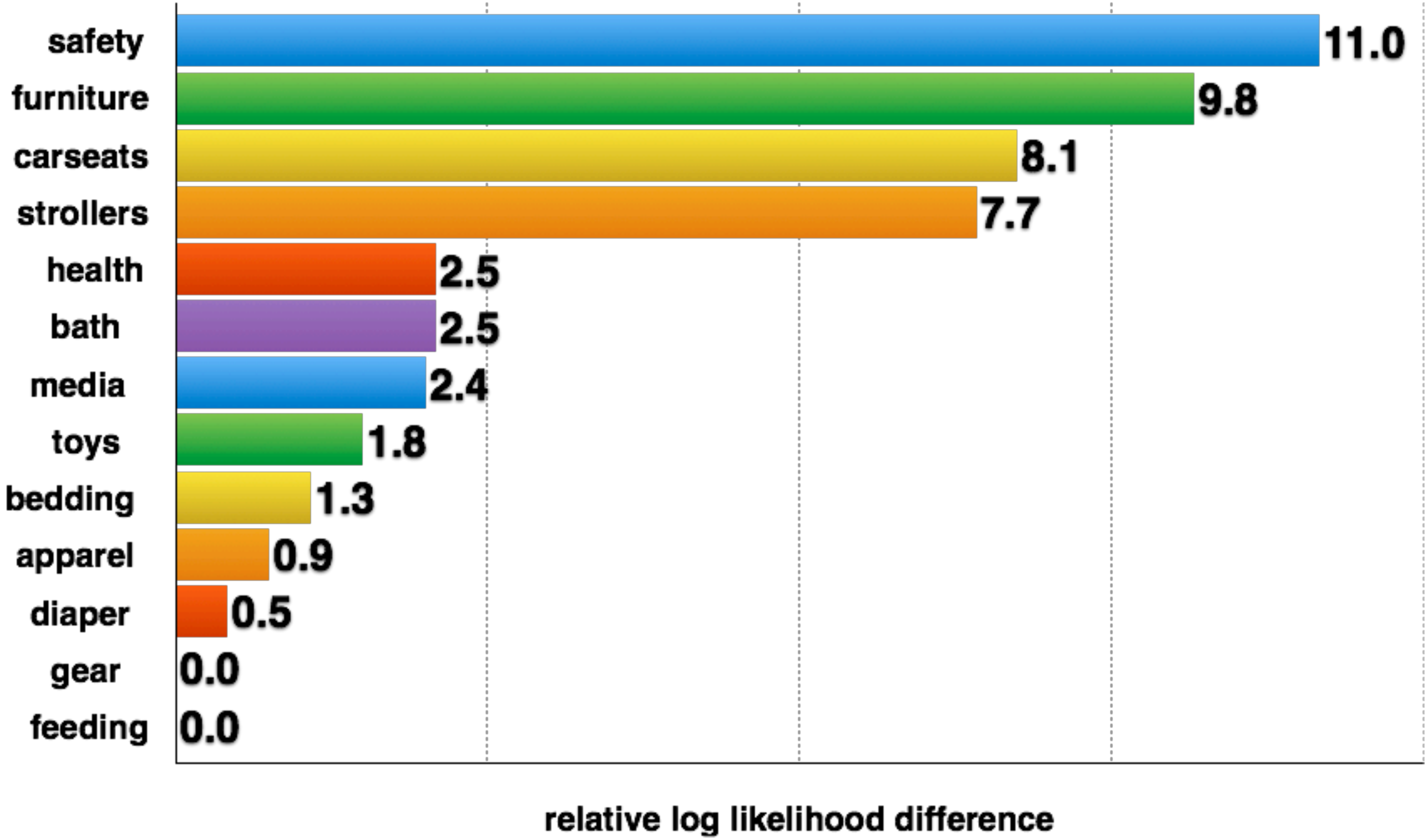

“SAFETY” SELECTION

“SAFETY” SELECTION

Graco Sweet Slumber
Sound Machine

Boppy Noggin Nest
Head Support

Cloud b Twilight
Night Light

Braun ThermoScan
Lens Filters

Aquatopia Bath
Thermometer Alarm

Britax EZ-Cling
Sun Shades

TL Care Organic
Cotton Mittens

Regalo Easy Step
Walk Thru Gate

VTech Comm.
Audio Monitor

Infant Optics
Video Monitor

"SAFETY" SELECTION

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Cloud b Twilight
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Summer Infant
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$$K^{(t+1)} \leftarrow \exp\left(\log(K^{(t)}) + \eta \nabla \mathcal{L}(K^{(t)})\right)$$

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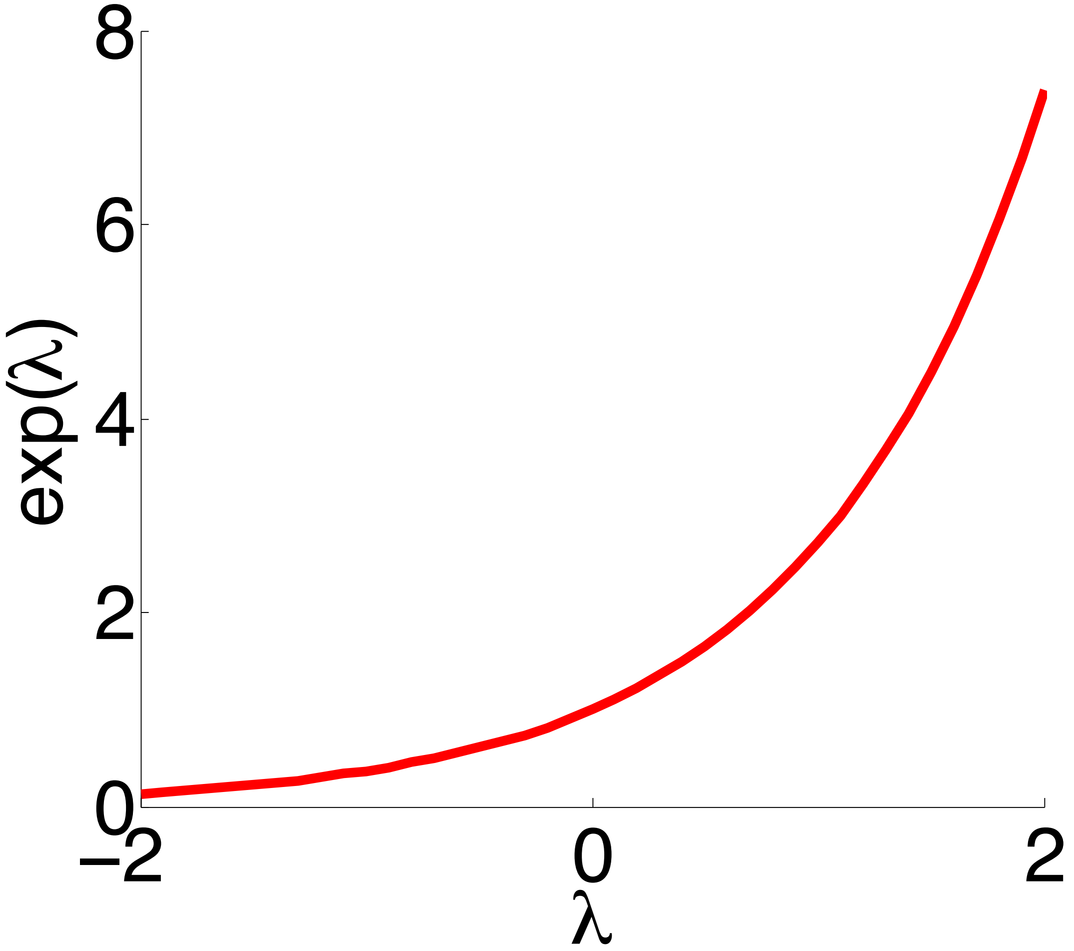

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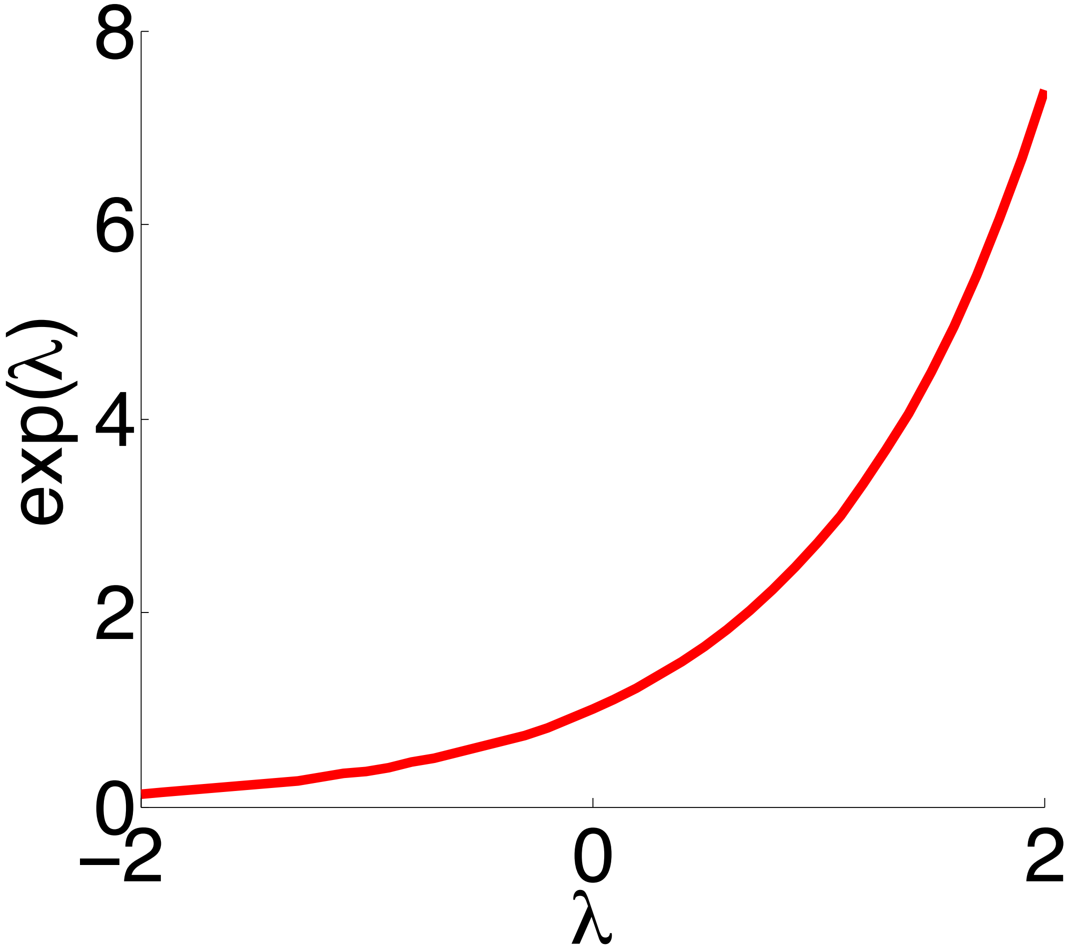

Relative log-likelihood

$\frac{|EM-EG|}{|EG|}$: 0.57%

APPROXIMATE INFERENCE
FOR
DETERMINANTAL POINT PROCESSES

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1. Dimensionality Reduction

GILLENWATER, KULESZA, AND TASKAR (EMNLP 2012)

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1. **Dimensionality Reduction**

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APPROXIMATE INFERENCE
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3. **Likelihood Maximization**

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