Proceedings of the Jangjeon Mathematical Society
21 (2018), No. 3. pp. 413 – 419

www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2018.21.3.413

SOME FAMILIES OF IDENTITIES FOR PADOVAN NUMBERS

TARAS GOY

Abstract. In this paper, we investigate some families of Toeplitz-Hessenberg determinants the entries of which are the Padovan numbers. As a result of these studies, we obtain new identities with multinomial coefficients for Padovan numbers.

2010 Mathematics Subject Classification. 11B37, 15A15.

Keywords and phrases. Padovan number, Padovan sequence, Cordonnier sequence, Toeplitz-Hessenberg matrix, multinomial coefficient.

1. Introduction and preliminaries

The *Padovan* (or *Cordonnier*) *sequence* $\{P_n\}_{n\geqslant 0}$ is the sequence of integers defined by the initial values $P_0 = P_1 = P_2 = 1$ and the recurrence relation

$$P_n = P_{n-2} + P_{n-3}, \qquad n \geqslant 3. \tag{1}$$

The above definition is given by Stewart [16]. Other authors may start the Padovan sequence at a different place, in which case some of the identities in this paper must be corrected with appropriate offsets.

The first few values of P_n are

$$1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, \ldots \text{(sequence A000931 in [17])}.$$

Although the investigation of Padovan numbers started in the beginning of 19th century, the main study was published in 2006 by Shannon et. al. in [15].

The Padovan numbers and their generalizations have been studied by some authors (see, for example, [1, 3, 4, 11, 14, 18, 19] for more details). For example, Ballantine and Merca [1] proved that the nth Padovan number can be expressed as a sum of multinomial coefficients over integer partitions of n into odd parts. Yılmaz and Bozkurt [18] obtained some relations between Padovan sequence and permanents of one type of Hessenberg matrix. Using Hessenberg matrices, Cereceda [3] provided some determinantal representations of the Padovan numbers. Şahin [14] studied the associated polynomials of Padovan numbers and gave some determinantal representations of these polynomials. Yilmaz and Taskara [19] developed the matrix sequence that represent Padovan numbers and examined their properties. Kaygısız and Şahin [11] calculated terms of associated polynomials of Padovan numbers by using determinants and permanents of various Hessenberg matrices. Deveci and Karaduman [4] defined the Padovan p-numbers and obtained their different properties such as the generating function, the Binet formula, the combinatorial and exponential representations.

The purpose of the present paper is to study the Padovan numbers. We investigate some families of Toeplitz-Hessenberg determinants the entries of which are Padovan numbers with sequential, odd and even subscripts. As result, we obtain some new identities with multinomial coefficients for Padovan numbers. Our approach is similar in spirit to [6, 7, 8, 10].

The main results of this paper were announced without proofs in [9].

2. Determinants of Toeplitz-Hessenberg matrices

An $n \times n$ matrix $M_n = (m_{ij})$ is called a *lower Hessenberg matrix* if all entries above the main diagonal are zero but the matrix is not lower triangular, i.e.,

$$M_n = \begin{pmatrix} m_{11} & m_{12} & 0 & \cdots & 0 & 0 \\ m_{21} & m_{22} & m_{23} & \cdots & 0 & 0 \\ m_{31} & m_{32} & m_{33} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\ m_{n-1,1} & m_{n-1,2} & m_{n-1,3} & \cdots & m_{n-1,n-1} & m_{n-1,n} \\ m_{n1} & m_{n2} & m_{n3} & \cdots & m_{n,n-1} & m_{nn} \end{pmatrix},$$

where $m_{i,i+1} \neq 0$ for some $i = 1, 2, \ldots, n-1$.

Hessenberg matrices play an important role in numerical mathematics, because their determinants can be calculated very fast by the recurrence [2]

$$\det M_n = m_{nn} \det M_{n-1} + \sum_{k=1}^{n-1} (-1)^{n-k} m_{nk} \det M_{n-k} \prod_{i=k}^{n-1} m_{i,i+1}. \tag{2}$$

Since there exist fast algorithms, which allows to transform a given quadratic matrix to a Hessenberg matrix, the evaluation of the characteristic polynomial of every quadratic matrix can be reduced to the same problem for an equivalent Hessenberg matrix [5].

For the special choice $m_{ij} = a_{i-j+1}$ for all i and j, i.e., on each diagonal all the elements are the same, we have the *Toeplitz-Hessenberg matrix*

$$A_n \equiv A_n(a_0; a_1, \ldots, a_n) = \begin{pmatrix} a_1 & a_0 & 0 & \cdots & 0 & 0 \\ a_2 & a_1 & a_0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_1 & a_0 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & a_1 \end{pmatrix}.$$

Let us consider the sequence of determinants $\{\det A_n\}_{n \geqslant 1}$ and define $\det A_0 = 1$. Then from (2) we obtain

$$\det A_n = \sum_{k=1}^{n} (-a_0)^{k-1} a_k \det A_{n-k}. \tag{3}$$

The next result gives the multinomial extension of $\det A_n$ [13]:

$$\det A_n = \sum_{s_1+2s_2+\cdots+ns_n=n} (-a_0)^{n-(s_1+\cdots+s_n)} p_n(s) a_1^{s_1} a_2^{s_2} \cdots a_n^{s_n}, \tag{4}$$

where

$$p_n(s) = \frac{(s_1 + \cdots + s_n)!}{s_1! \cdots s_n!}$$

is the multinomial coefficient and $n = s_1 + 2s_2 + \cdots + ns_n$ is a partition of the integer $n \geqslant 0$ where each integers $i \geqslant 0$ appears s_i times (see also [12]).

For simplicity and clarity of notation, we write $\det(a_1, a_2, \ldots, a_n)$ instead of $\det A_n(1; a_1, a_2, \ldots, a_n)$.

In the next two sections, we evaluate $\det(a_1, a_2, \ldots, a_n)$ with special entries given by $a_i = P_i$.

3. Determinants of Toeplitz-Hessenberg matrices whose entries are Padovan numbers with sequential subscripts

The next proposition gives the value of some determinant whose entries are Padovan numbers with sequential subscripts.

Proposition 3.1. *For $n \geqslant 1$, the following identities hold:*

$$\begin{aligned}
\det(P_0, P_1, \ldots, P_{n-1}) &= \sum_{i=1}^{\lfloor \frac{n-1}{3} \rfloor} (-1)^i \binom{n-2-2i}{i-1} + \delta_{n1},\\
\det(P_1, P_2, \ldots, P_n) &= 1 - \delta_{n2},\\
\det(P_2, P_3, \ldots, P_{n+1}) &= \frac{1}{2}\left((-1)^{\lfloor \frac{n+1}{3} \rfloor} + (-1)^{\lfloor \frac{n+2}{3} \rfloor}\right) + \delta_{n1}, \qquad (5)\\
\det(P_3, P_4, \ldots, P_{n+2}) &= n + \delta_{n1},\\
\det(P_4, P_5, \ldots, P_{n+3}) &= 1 + \frac{1}{2}\left((-1)^{\lfloor \frac{n}{3} \rfloor} + (-1)^{\lfloor \frac{n+1}{3} \rfloor}\right),\\
\det(P_5, P_6, \ldots, P_{n+4}) &= \frac{n^2+n+4}{2},
\end{aligned}$$

where δ_{nk} is the Kronecker delta and $\lfloor x \rfloor$ stands of the greatest integer less than or equal to x (the floor of x).

Proof. We will prove formula (5) using induction on n. The other proofs follow similarly, so we omit them for brevity. For simplicity of notation, we denote

$$D_n = \det(P_2, P_3, \ldots, P_{n+1}).$$

Clearly, formula (5) works, when $n = 1$ and $n = 2$. Suppose it is true for all positive integers $k \leqslant n-1$, where $n \geqslant 2$.

Using recurrence (3) and Padovan relation (1), we have

$$\begin{aligned}
D_n &= \sum_{i=1}^{n} (-1)^{i-1} P_{i+1} D_{n-i}\\
&= P_2 D_{n-1} - P_3 D_{n-2} + \sum_{i=3}^{n} (-1)^{i-1} \big(P_{i-1} + P_{i-2}\big) D_{n-i}\\
&= D_{n-1} - 2D_{n-2} + \sum_{i=1}^{n-2} (-1)^{i-1} P_{i+1} D_{n-i-2} + \sum_{i=0}^{n-3} (-1)^i P_{i+1} D_{n-i-3}\\
&= D_{n-1} - 2D_{n-2} + D_{n-2} + \left(\sum_{i=1}^{n-3} (-1)^i P_{i+1} D_{n-i-3} + P_1 D_{n-3}\right)
\end{aligned}$$

$$= D_{n-1} - D_{n-2} - D_{n-3} + D_{n-3}$$
$$= D_{n-1} - D_{n-2}.$$

Using the induction hypothesis, we obtain

$$D_n = \frac{1}{2}\left((-1)^{\lfloor \frac{n}{3} \rfloor} + (-1)^{\lfloor \frac{n+1}{3} \rfloor} - (-1)^{\lfloor \frac{n-1}{3} \rfloor} - (-1)^{\lfloor \frac{n}{3} \rfloor}\right)$$
$$= \frac{1}{2}\left((-1)^{\lfloor \frac{n+1}{3} \rfloor} + (-1)^{\lfloor \frac{n+2}{3} \rfloor}\right).$$

Consequently, the formula (5) is true for n. Therefore, by induction, the formula works for all positive integers n. □

4. Determinants of Toeplitz-Hessenberg matrices whose entries are Padovan numbers with even (odd) subscripts

The next proposition gives the value of some Toeplitz-Hessenberg determinants whose entries are Padovan numbers with even or odd subscripts.

Proposition 4.1. *For $n \geq 1$, the following identities hold:*

$$\det(P_0, P_2, \ldots, P_{2n-2}) = (-1)^{n-1} + \delta_{n2},$$
$$\det(P_1, P_3, \ldots, P_{2n-1}) = \frac{1}{2}\left((-1)^{\frac{2n-1+(-1)^n}{4}} - (-1)^n\right),$$
$$\det(P_2, P_4, \ldots, P_{2n}) = \frac{1}{2}\left((-1)^{\lfloor \frac{2n}{3} \rfloor} + (-1)^{\lfloor \frac{2n+1}{3} \rfloor}\right) + \delta_{n1},$$
$$\det(P_3, P_5, \ldots, P_{2n+1}) = 2\delta_{n1} + \delta_{n2} + \delta_{n3},$$
$$\det(P_4, P_6, \ldots, P_{2n+2}) = \frac{1}{2}\left((-1)^{\lfloor \frac{n}{2} \rfloor} - (-1)^{\lfloor \frac{3n}{2} \rfloor}\right) + \delta_{n2},$$
$$\det(P_5, P_7, \ldots, P_{2n+3}) = \sum_{i=0}^{\lfloor \frac{n+3}{3} \rfloor} \binom{n+3-2i}{i}, \tag{6}$$

where δ_{nk} is the Kronecker delta, $\lfloor x \rfloor$ is the floor of x.

Proof. We will prove formula (6) using induction on n; the others can be confirmed similarly.

The result clearly holds for $n = 1$ and $n = 2$. Suppose it is true for all positive integers $k \leqslant n-1$, where $n \geqslant 2$.

It may be note that the Padovan numbers satisfy the recurrence

$$P_n = 2P_{n-2} - P_{n-4} + P_{n-6}, \tag{7}$$

for $n \geqslant 6$.

Let

$$D_n = \det(P_5, P_7, \ldots, P_{2n+3}).$$

Using recurrences (3) and (7), we then have

$$D_n = \sum_{i=1}^{n} (-1)^{i-1} P_{2i+3} D_{n-i}$$
$$= P_5 D_{n-1} + \sum_{i=2}^{n} (-1)^{i-1} \big(2P_{2i+1} - P_{2i-1} + P_{2i-3}\big) D_{n-i}$$

$$
\begin{aligned}
&= 3D_{n-1} + 2\sum_{i=1}^{n-1}(-1)^i P_{2i+3} D_{n-1-i} \\
&\quad - \left(-P_3 D_{n-2} + \sum_{i=1}^{n-2}(-1)^{i-1} P_{2i+3} D_{n-2-i}\right) \\
&\quad + \left(\sum_{i=3}^{n}(-1)^{i-1} P_{2i-3} D_{n-i} - P_1 D_{n-2}\right) \\
&= 3D_{n-1} - 2D_{n-1} + 2D_{n-2} - 2D_{n-2} \\
&\quad + \sum_{i=1}^{n-3}(-1)^i P_{2i+3} D_{n-3} + P_2 D_{n-3} - D_{n-2} \\
&= D_{n-1} + D_{n-3}.
\end{aligned}
$$

Using the induction hypothesis, from above formula we obtain

$$
D_n = \sum_{i=0}^{\lfloor \frac{n+2}{3} \rfloor} \binom{n+2-2i}{i} + \sum_{i=0}^{\lfloor \frac{n+1}{3} \rfloor} \binom{n+1-2i}{i}. \tag{8}
$$

Let $n = 3k$. Then $\lfloor \frac{n+2}{3} \rfloor = \lfloor \frac{n}{3} \rfloor$. From (8), using well-known formula

$$
\binom{n-1}{m} + \binom{n-1}{m-1} = \binom{n}{m}, \tag{9}
$$

we have

$$
\begin{aligned}
D_n &= \sum_{i=0}^{\lfloor \frac{n}{3} \rfloor} \binom{n+2-2i}{i} + \sum_{i=0}^{\lfloor \frac{n}{3} \rfloor} \binom{n-2i}{i} \\
&= 1 + \sum_{i=1}^{\lfloor \frac{n}{3} \rfloor + 1} \binom{n+2-2i}{i} - \binom{k}{k+1} + \sum_{i=1}^{\lfloor \frac{n}{3} \rfloor + 1} \binom{n+2-2i}{i-1} \\
&= 1 + \sum_{i=1}^{\lfloor \frac{n}{3} \rfloor + 1} \left(\binom{n+2-2i}{i} + \binom{n+2-2i}{i-1}\right) \\
&= \sum_{i=0}^{\lfloor \frac{n+3}{3} \rfloor} \binom{n+3-2i}{i}.
\end{aligned}
$$

Let $n \neq 3k$. Then $\lfloor \frac{n+2}{3} \rfloor = \lfloor \frac{n}{3} \rfloor + 1$. From (8) we have

$$
\begin{aligned}
D_n &= \sum_{i=0}^{\lfloor \frac{n}{3} \rfloor + 1} \binom{n+2-2i}{i} + \sum_{i=0}^{\lfloor \frac{n}{3} \rfloor} \binom{n-2i}{i} \\
&= 1 + \sum_{i=1}^{\lfloor \frac{n}{3} \rfloor + 1} \left(\binom{n+2-2i}{i} + \binom{n+2-2i}{i-1}\right) \\
&= \sum_{i=0}^{\lfloor \frac{n+3}{3} \rfloor} \binom{n+3-2i}{i}.
\end{aligned}
$$

Consequently, the formula (6) is true for n. Therefore, by induction, the formula works for all positive integers n. $\square$

5. Multinomial extension of Toeplitz-Hessenberg determinants

Formula (4), coupled with Propositions 3.1 and 4.1, yields the following identities for Padovan numbers with sequential, even, and odd subscripts.

Proposition 5.1. *Let $n \geqslant 1$, except when noted otherwise. Then*

$$\sum_{\sigma_n=n}(-1)^{|s|}p_n(s)P_0^{s_1}\cdots P_{n-1}^{s_n}=\sum_{i=1}^{\lfloor\frac{n-1}{3}\rfloor}(-1)^{n+i}\binom{n-2-2i}{i-1},\qquad n\geqslant 2,$$

$$\sum_{\sigma_n=n}(-1)^{|s|}p_n(s)P_0^{s_1}\cdots P_{2n-2}^{s_n}=-1,\qquad n\geqslant 3,$$

$$\sum_{\sigma_n=n}(-1)^{|s|}p_n(s)P_1^{s_1}\cdots P_{n}^{s_n}=(-1)^n,\qquad n\geqslant 3,\tag{10}$$

$$\sum_{\sigma_n=n}(-1)^{|s|}p_n(s)P_1^{s_1}\cdots P_{2n-1}^{s_n}=\frac{(-1)^{\frac{2n+1-(-1)^n}{4}}-1}{2},$$

$$\sum_{\sigma_n=n}(-1)^{|s|}p_n(s)P_2^{s_1}\cdots P_{n+1}^{s_n}=\frac{(-1)^{\lfloor\frac{4n+1}{3}\rfloor}+(-1)^{\lfloor\frac{4n+2}{3}\rfloor}}{2},\qquad n\geqslant 2,$$

$$\sum_{\sigma_n=n}(-1)^{|s|}p_n(s)P_2^{s_1}\cdots P_{2n}^{s_n}=\frac{(-1)^{\lfloor\frac{5n}{3}\rfloor}+(-1)^{\lfloor\frac{5n+1}{3}\rfloor}}{2},\qquad n\geqslant 2,$$

$$\sum_{\sigma_n=n}(-1)^{|s|}p_n(s)P_3^{s_1}\cdots P_{n+2}^{s_n}=(-1)^n n,\qquad n\geqslant 2,$$

$$\sum_{\sigma_n=n}(-1)^{|s|}p_n(s)P_3^{s_1}\cdots P_{2n+1}^{s_n}=0,\qquad n\geqslant 4,\tag{11}$$

$$\sum_{\sigma_n=n}(-1)^{|s|}p_n(s)P_4^{s_1}\cdots P_{n+3}^{s_n}=(-1)^n+\frac{(-1)^{\lfloor\frac{4n}{3}\rfloor}+(-1)^{\lfloor\frac{4n+1}{3}\rfloor}}{2},$$

$$\sum_{\sigma_n=n}(-1)^{|s|}p_n(s)P_4^{s_1}\cdots P_{2n+2}^{s_n}=\frac{(-1)^{\lfloor\frac{3n}{2}\rfloor}-(-1)^{\lfloor\frac{n}{2}\rfloor}}{2},\qquad n\geqslant 2,\tag{12}$$

$$\sum_{\sigma_n=n}(-1)^{|s|}p_n(s)P_5^{s_1}\cdots P_{n+4}^{s_n}=\frac{(-1)^n(n^2+n+4)}{2},\tag{13}$$

$$\sum_{\sigma_n=n}(-1)^{|s|}p_n(s)P_5^{s_1}\cdots P_{2n+3}^{s_n}=(-1)^n\sum_{i=0}^{\lfloor\frac{n+3}{3}\rfloor}\binom{n+3-2i}{i},$$

where $\sigma_n=s_1+2s_2+\cdots+ns_n$, $|s|=s_1+\cdots+s_n$, $p_n(s)=\frac{(s_1+\cdots+s_n)!}{s_1!\cdots s_n!}$ is the multinomial coefficient, and the summation is over integers $s_j\geqslant 0$ satisfying $s_1+2s_2+\cdots+ns_n=n$.

For example, it follows from (13), (11), (10), and (12) that

$$P_5^3 - 2P_5P_6 + P_7 = 8,$$

$$P_3^4 - 3P_3^2P_5 + 2P_3P_7 + P_5^2 - P_9 = 0,$$

$$P_1^5 - 4P_1^3P_2 + 3P_1^2P_3 + 3P_1P_2^2 - 2P_1P_4 - 2P_2P_3 + P_5 = 1,$$

$$P_4^6 - 5P_4^4P_6 + 4P_4^3P_8 + 6P_4^2P_6^2 - 3P_4^2P_{10} - 6P_4P_6P_8 + 2P_4P_{12} - P_6^3 + 2P_6P_{10} + P_8^2 - P_{14} = 0,$$

respectively.

References

[1] C. Ballantine and M. Merca, *Padovan numbers as sums over partitions into odd parts*, J. Inequal. Appl. (2016), 2016:1.

[2] N. D. Cahill, J. R. D'Errico, D. A. Narayan and J. Y. Narayan, *Fibonacci determinants*, College Math. J. 3(3) (2002), 221-225.

[3] J. L. Cereceda, *Determinantal represantations for generalized Fibonacci and tribonacci numbers*, Int. J. Contemp. Math. Sci. 9(6) (2014), 269-285.

[4] Ö. Deveci and E. Karaduman, *On the Padovan p-numbers*, Hacet. J. Math. Stat. 40(4) (2017), 579-592.

[5] G. H. Golub and C. F. Van Loan, *Matrix Computations*, Johns Hopkins University Press, Baltimore, 2012.

[6] T. Goy, *On Jacobsthal and Jacobsthal-Lucas identities with multinomial coefficients*, International Conference "Actual Problems of Pure and Applied Mathematics", Almaty, Kazakhstan (2017), 61-64.

[7] T. Goy, *On Pell identities with multinomial coefficients*, International Conference "Numbers, Forms and Geometry", Sochi, Russia (2017), 23-24.

[8] T. Goy, *Some combinatorial identities for two-periodic Fibonacci sequence*, 12th International Scientific Conference "Fundamental and Applied Problems of Mathematics and Informatics", Mahachkala, Russia (2017), 107-109.

[9] T. Goy, *Some identities for Padovan numbers via the determinants of Toeplitz-Hessenberg matrices*, 30th International Conference of The Jangjeon Mathematical Society "Pure and Applied Mathematics", Bab-Ezzouar, Algeria (2017), 242-244.

[10] T. Goy, *Some tribonacci identities using Toeplitz-Hessenberg determinants*, 18th International Scientific M. Kravchuk Conference, Kyiv-Lutsk, Ukraine 1 (2017), 159-161.

[11] K. Kaygısız and A. Şahin, *Calculating terms of associated polynomials of Perrin and Cordonnier numbers*, Notes Number Theory Discrete Math. 20(1) (2014), 10-18.

[12] M. Merca, *A note on the determinant of a Toeplitz-Hessenberg matrix*, Spec. Matrices, 1 (2013), 10-16.

[13] T. Muir, *The Theory of Determinants in the Historical Order of Development*. Vol. 3, New York, Dover Publications, 1960.

[14] A. Şahin, *On the generalized Perrin and Cordonnier matrices*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. Tech. 66(1) (2017), 242-253.

[15] A. G. Shannon, P. G. Anderson and A. F. Horadam, *Properties of Cordonnier, Perrin and Van der Laan numbers*, Int. J. Math. Ed. Sci. Tech. 37(17) (2006), 825-831.

[16] I. Stewart, *Tales of a neglected number*, Sci. Am. 274 (1996), 102-103.

[17] N. J. A. Sloane, editor, *The On-Line Encyclopedia of Integer Sequences*, published electornically at https://oeis.org.

[18] F. Yılmaz and D. Bozkurt, *Some properties of Padovan sequence by matrix methods*, Ars Comb. 104 (2012), 149-160.

[19] F. Yılmaz and N. Taskara, *Matrix sequences in terms of Padovan and Perrin numbers*, J. Appl. Math. (2013), Article ID 941673.

Vasyl Stefanyk Precarpathian National University, 76018 Ivano-Frankivsk, Ukraine

E-mail address: tarasgoy@gmail.com