International Conference
on the Algebraic and Arithmetic Theory of Quadratic Forms

Lago Lhanquihué 2007

Enumerating perfect forms

Achill Schürmann
(Otto-von-Guericke Universität Magdeburg)

Perfect Forms

Consider the space $\mathcal{S}^n_{>0}$

of positive definite quadratic forms $Q : \mathbb{R}^n \longrightarrow \mathbb{R}$

(of sym. pos. def. matrices in $\mathbb{R}^{n \times n}$)

Perfect Forms

Consider the space $\mathcal{S}^n_{>0}$

of positive definite quadratic forms $Q : \mathbb{R}^n \longrightarrow \mathbb{R}$

(of sym. pos. def. matrices in $\mathbb{R}^{n \times n}$)

DEF: $\lambda(Q) = \min_{x \in \mathbb{Z}^n \setminus \{0\}} Q[x]$ is the arithmetical minimum

Perfect Forms

Consider the space $\mathcal{S}^n_{>0}$

of positive definite quadratic forms $Q : \mathbb{R}^n \longrightarrow \mathbb{R}$

(of sym. pos. def. matrices in $\mathbb{R}^{n\times n}$)

DEF: $\lambda(Q) \ = \ \min_{x\in\mathbb{Z}^n\setminus\{0\}} Q[x]$ is the arithmetical minimum

DEF: $Q \in \mathcal{S}^n_{>0}$ perfect $\Leftrightarrow$ Q is uniquely determined by $\lambda(Q)$ and

$$\operatorname{Min} Q \ = \ \{ \ x \in \mathbb{Z}^n \ : \ Q[x] = \lambda(Q) \ \}$$

Extreme Forms

THM: (Hermite, 1850)

$$\lambda(Q) \quad \leq \quad \left(\frac{4}{3}\right)^{(n-1)/2} (\det Q)^{1/n}$$

(1822–1901)

Extreme Forms

THM: (Hermite, 1850)

$$\lambda(Q) \quad \leq \quad \left(\frac{4}{3}\right)^{(n-1)/2} (\det Q)^{1/n}$$

(1822–1901)

Hermite's constant $\quad \mathcal{H}_n \; = \; \sup_{Q \in \mathcal{S}^n_{>0}} \frac{\lambda(Q)}{(\det Q)^{1/n}}$

Extreme Forms

(1822–1901)

THM: (Hermite, 1850)

$$\lambda(Q) \quad \leq \quad \left(\frac{4}{3}\right)^{(n-1)/2} (\det Q)^{1/n}$$

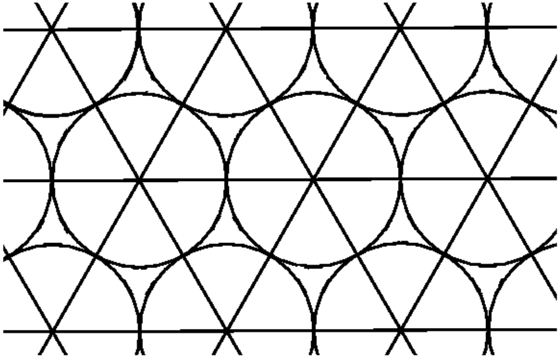
Hermite's constant $\mathcal{H}_n \ = \ \sup_{Q \in \mathcal{S}^n_{>0}} \frac{\lambda(Q)}{(\det Q)^{1/n}}$

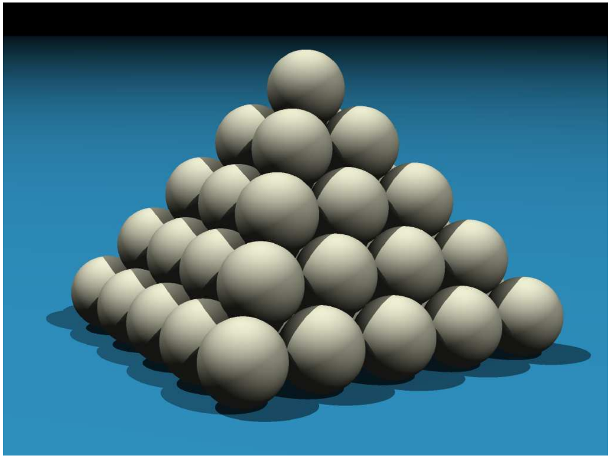
DEF: Q is (geometric) extreme

if it attains a local maximum of $\lambda(Q)/(\det Q)^{1/n}$ on $\mathcal{S}^n_{>0}$

Sphere packings

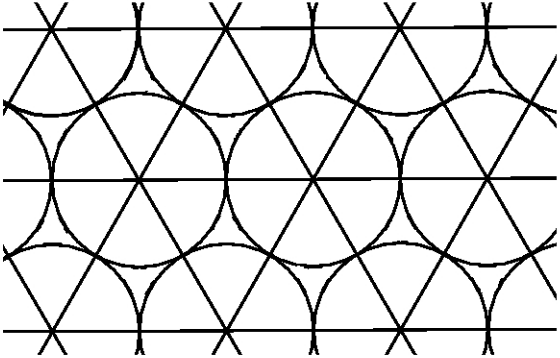
$$\delta_n \;=\; \mathcal{H}_n^{n/2}\,\frac{\mathrm{vol}\,B^n}{2^n}$$ density of densest lattice sphere packing

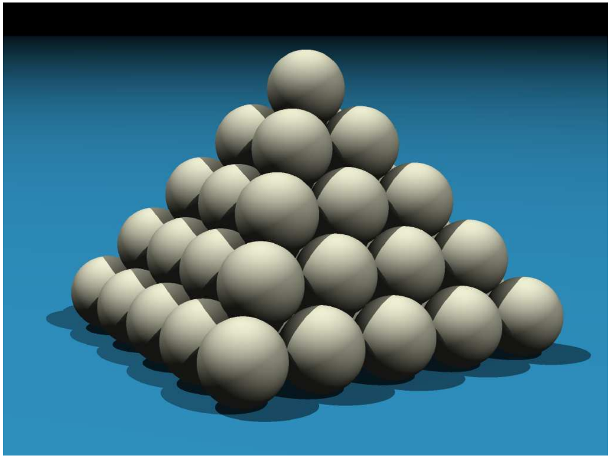




Sphere packings

$$\delta_n = \mathcal{H}_n^{n/2} \frac{\mathrm{vol}\, B^n}{2^n}$$ density of densest lattice sphere packing





- $\lambda(Q)$ — squared length of shortest non-zero lattice vector
- $\det(Q)$ — squared volume of a fundamental cell

Known results

n	PQF/lattice	δ_n	$\mathcal{H}_n$	author(s)
2	A_2	$0.9069\ldots$	$\left(\frac{4}{3}\right)^{1/2}$	Lagrange, 1773
3	$\mathsf{A}_3 = \mathsf{D}_3$	$0.7404\ldots$	$2^{1/3}$	Gauß, 1840
4	D_4	$0.6168\ldots$	$4^{1/4}$	Korkine & Zolotarev 1877
5	D_5	$0.4652\ldots$	$8^{1/5}$	Korkine & Zolotarev 1877
6	E_6	$0.3729\ldots$	$\left(\frac{64}{3}\right)^{1/6}$	Blichfeldt, 1935
7	E_7	$0.2953\ldots$	$64^{1/7}$	Blichfeldt, 1935
8	E_8	$0.2536\ldots$	2	Blichfeldt, 1935

Known results

n	PQF/lattice	δ_n	$\mathcal{H}_n$	author(s)
2	A_2	$0.9069\ldots$	$\left(\frac{4}{3}\right)^{1/2}$	Lagrange, 1773
3	$\mathsf{A}_3 = \mathsf{D}_3$	$0.7404\ldots$	$2^{1/3}$	Gauß, 1840
4	D_4	$0.6168\ldots$	$4^{1/4}$	Korkine & Zolotarev 1877
5	D_5	$0.4652\ldots$	$8^{1/5}$	Korkine & Zolotarev 1877
6	E_6	$0.3729\ldots$	$\left(\frac{64}{3}\right)^{1/6}$	Blichfeldt, 1935
7	E_7	$0.2953\ldots$	$64^{1/7}$	Blichfeldt, 1935
8	E_8	$0.2536\ldots$	2	Blichfeldt, 1935
24	Λ_{24}	$0.0019\ldots$	4	Cohn & Kumar, 2004

Densest lattice sphere packings known

Known results

n	PQF/lattice	δ_n	$\mathcal{H}_n$	author(s)
2	A_2	$0.9069\ldots$	$\left(\frac{4}{3}\right)^{1/2}$	Lagrange, 1773
3	$\mathsf{A}_3 = \mathsf{D}_3$	$0.7404\ldots$	$2^{1/3}$	Gauß, 1840
4	D_4	$0.6168\ldots$	$4^{1/4}$	Korkine & Zolotarev 1877
5	D_5	$0.4652\ldots$	$8^{1/5}$	Korkine & Zolotarev 1877
6	E_6	$0.3729\ldots$	$\left(\frac{64}{3}\right)^{1/6}$	Blichfeldt, 1935
7	E_7	$0.2953\ldots$	$64^{1/7}$	Blichfeldt, 1935
8	E_8	$0.2536\ldots$	2	Blichfeldt, 1935
24	Λ_{24}	$0.0019\ldots$	4	Cohn & Kumar, 2004

Densest lattice sphere packings known

OPEN: What are the densest sphere packings for $n \geq 4$?

Voronoi's characterization

THM: (Voronoi, 1907)

$$Q \text{ extreme} \quad \Leftrightarrow \quad Q \text{ perfect and eutactic}$$

(1868–1908)

Voronoi's characterization

THM: (Voronoi, 1907)

Q extreme $\Leftrightarrow$ Q perfect and eutactic


(1868–1908)

DEF: $Q \in \mathcal{S}^n_{>0}$ is eutactic, if $Q^{-1} = \sum_{v \in \operatorname{Min} Q} \underbrace{\alpha_v}_{>0} v v^t$

Determinant minimization

Extreme forms are local minima of $(\det Q)^{\frac{1}{n}}$

on $\mathcal{R} \;=\; \{\, Q \in \mathcal{S}^n_{>0} \;:\; \lambda(Q) \geq 1 \,\}$

Determinant minimization

Extreme forms are local minima of $(\det Q)^{\frac{1}{n}}$

on $\mathcal{R} = \{ Q \in \mathcal{S}^n_{>0} \ : \ \lambda(Q) \geq 1 \}$

$= \{ Q \in \mathcal{S}^n_{>0} \ : \ Q[x] \geq 1$ for all $x \in \mathbb{Z}^n \setminus \{0\} \}$

Determinant minimization

Extreme forms are local minima of $(\det Q)^{\frac{1}{n}}$

on $\mathcal{R} = \{ Q \in \mathcal{S}^n_{>0} : \lambda(Q) \geq 1 \}$

$= \{ Q \in \mathcal{S}^n_{>0} : Q[x] \geq 1$ for all $x \in \mathbb{Z}^n \setminus \{0\} \}$

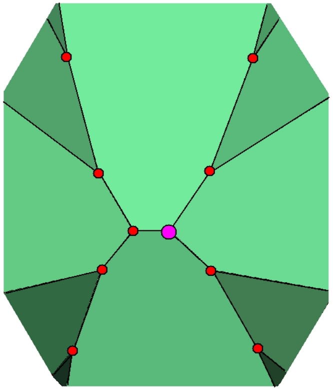
$$Q[x] = \langle Q, xx^t \rangle = \mathrm{trace}(Q\, xx^t)$$

is for fixed $x \in \mathbb{R}^n$

linear in the $\binom{n+1}{2}$ parameters q_{ij} of Q

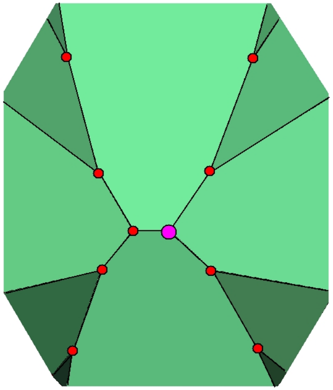
Ryshkov Polyhedra

- $\mathcal{R}$ is a locally finite polyhedron



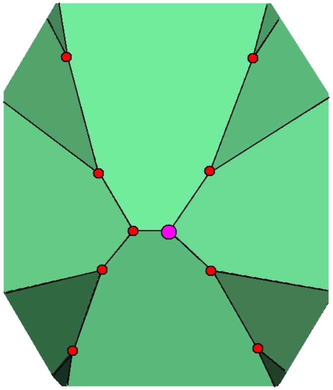
Ryshkov Polyhedra

- $\mathcal{R}$ is a locally finite polyhedron
- Vertices of $\mathcal{R}$ are perfect forms



Ryshkov Polyhedra

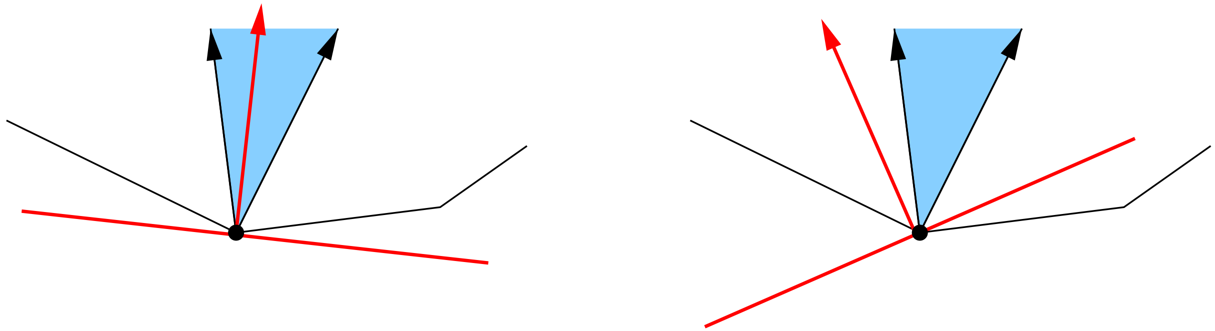
- $\mathcal{R}$ is a locally finite polyhedron
- Vertices of $\mathcal{R}$ are perfect forms
- $\alpha \mapsto (\det(Q + \alpha Q'))^{\frac{1}{n}}$ is strictly concave on $\mathcal{S}^n_{>0}$



Voronoi Cones

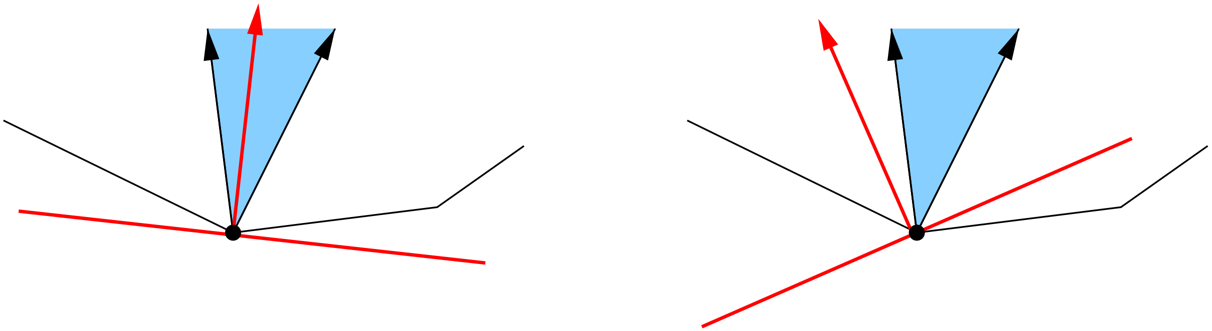
Voronoi Cones

- $\operatorname{grad}\det Q \;=\; (\det Q)Q^{-1}$ for $Q \in \mathcal{S}^n_{>0}$



Voronoi Cones

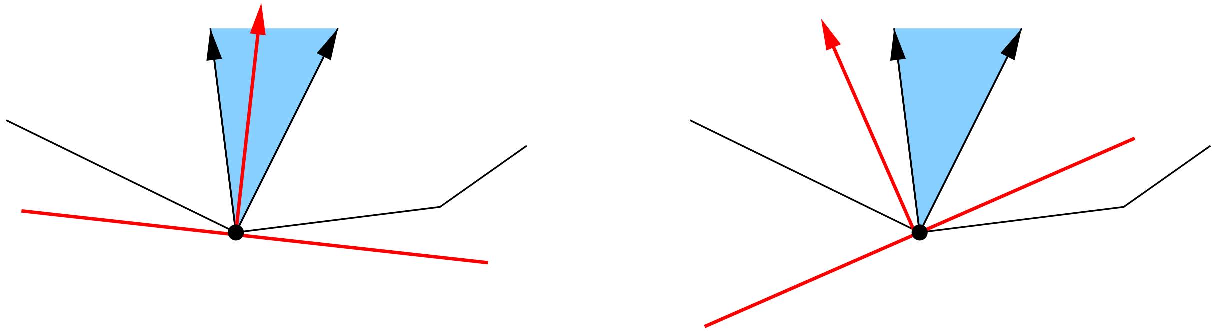
- $\operatorname{grad} \det Q \ = \ (\det Q) Q^{-1}$ for $Q \in \mathcal{S}^n_{>0}$



$$\mathcal{V}(Q) = \operatorname{cone}\{vv^t : v \in \operatorname{Min} Q\}$$

Voronoi Cones

- $\operatorname{grad} \det Q \;=\; (\det Q) Q^{-1}$ for $Q \in \mathcal{S}^n_{>0}$

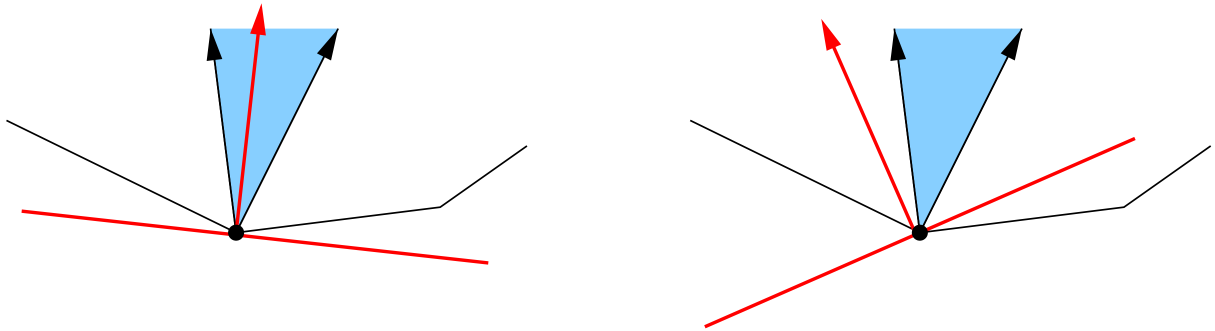


$$\mathcal{V}(Q) = \operatorname{cone}\{vv^t : v \in \operatorname{Min} Q\}$$

- Q eutactic $\Leftrightarrow$ $Q^{-1} \in \operatorname{relint} \mathcal{V}(Q)$

Voronoi Cones

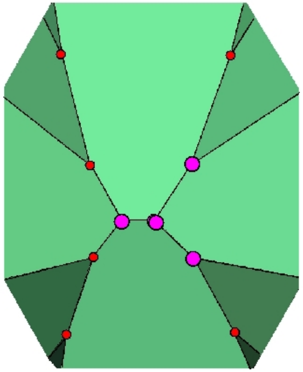
- $\operatorname{grad}\det Q \;=\; (\det Q)Q^{-1}$ for $Q \in \mathcal{S}^n_{>0}$



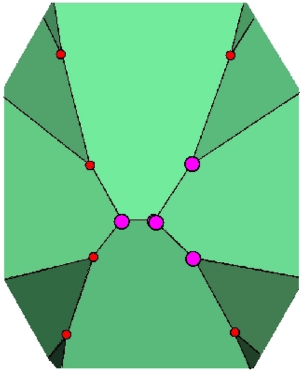
$$\mathcal{V}(Q) = \operatorname{cone}\{vv^t : v \in \operatorname{Min} Q\}$$

- Q eutactic $\quad\Leftrightarrow\quad Q^{-1} \in \operatorname{relint}\mathcal{V}(Q)$
- Q perfect $\quad\Leftrightarrow\quad \mathcal{V}(Q)$ is $\binom{n+1}{2}$-dimensional

Arithmetic equivalence



Arithmetic equivalence

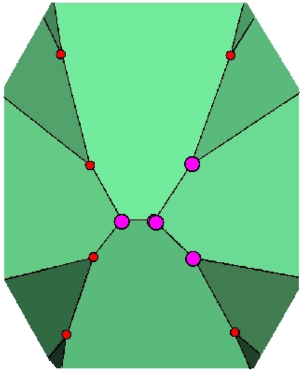


Q and U^tQU with $U \in \mathsf{GL}_n(\mathbb{Z})$ are arithmetical equivalent

$\mathsf{GL}_n(\mathbb{Z})$ operates on $\mathcal{R}$ and its vertices and edges by

$$Q \mapsto U^tQU$$

Arithmetic equivalence



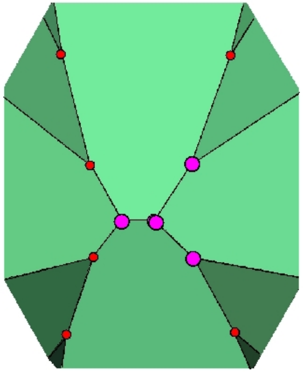
Q and U^tQU with $U \in \mathrm{GL}_n(\mathbb{Z})$ are arithmetical equivalent

$\mathrm{GL}_n(\mathbb{Z})$ operates on $\mathcal{R}$ and its vertices and edges by

$$Q \mapsto U^tQU$$

THM (Voronoi, 1907): $\{ \, Q \in \mathcal{S}^n_{>0}$ perfect with $\lambda(Q) = 1 \, \} \; / \; \sim$ finite

Arithmetic equivalence



Q and U^tQU with $U \in \mathrm{GL}_n(\mathbb{Z})$ are arithmetical equivalent

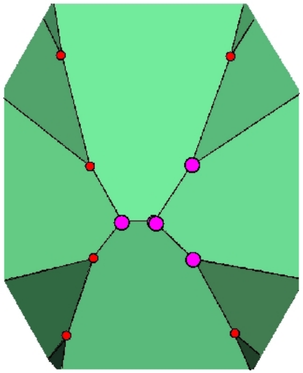
$\mathrm{GL}_n(\mathbb{Z})$ operates on $\mathcal{R}$ and its vertices and edges by

$$Q \mapsto U^tQU$$

THM (Voronoi, 1907): $\{ Q \in \mathcal{S}^n_{>0}$ perfect with $\lambda(Q) = 1 \} / \sim$ finite

$\Rightarrow$ Enumeration of perfect and extreme forms is possible

Arithmetic equivalence



Q and $U^t Q U$ with $U \in \mathrm{GL}_n(\mathbb{Z})$ are arithmetical equivalent

$\mathrm{GL}_n(\mathbb{Z})$ operates on $\mathcal{R}$ and its vertices and edges by

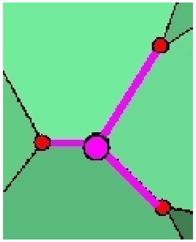
$$Q \mapsto U^t Q U$$

THM (Voronoi, 1907): $\{ \, Q \in \mathcal{S}^n_{>0}$ perfect with $\lambda(Q) = 1 \, \} \; / \; \sim$ finite

$\Rightarrow$ Enumeration of perfect and extreme forms is possible

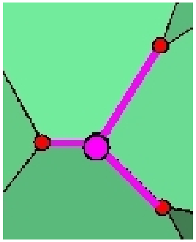
Voronoi's algorithm : Vertex enumeration up to arithmetical equivalence

Voronoi's algorithm



Start with a perfect form Q

Voronoi's algorithm

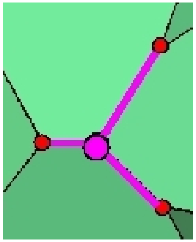


Start with a perfect form Q

1. SVP: Compute $\operatorname{Min} Q$ and describing inequalities of the polyhedral cone

$$\mathcal{P}(Q) \;=\; \{\; Q' \in \mathcal{S}^n \;:\; Q'[x] \geq 1 \text{ for all } x \in \operatorname{Min} Q \;\}$$

Voronoi's algorithm



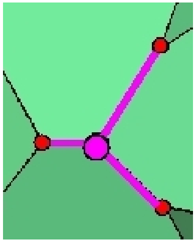
Start with a perfect form Q

1. SVP: Compute $\mathrm{Min}\, Q$ and describing inequalities of the polyhedral cone

$$\mathcal{P}(Q) \; = \; \{ \; Q' \in \mathcal{S}^n \; : \; Q'[x] \geq 1 \text{ for all } x \in \mathrm{Min}\, Q \; \}$$

2. PolyRepConv: Enumerate extreme rays $R_1, \ldots, R_k$ of $\mathcal{P}(Q)$

Voronoi's algorithm



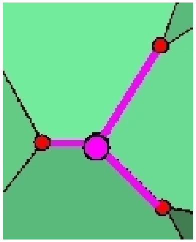
Start with a perfect form Q

1. SVP: Compute $\operatorname{Min} Q$ and describing inequalities of the polyhedral cone

$$\mathcal{P}(Q) \;=\; \{\; Q' \in \mathcal{S}^n \;:\; Q'[x] \geq 1 \text{ for all } x \in \operatorname{Min} Q \;\}$$

2. PolyRepConv: Enumerate extreme rays $R_1, \ldots, R_k$ of $\mathcal{P}(Q)$
3. SVPs: Determine contiguous perfect forms $Q_i = Q + \alpha R_i$, $i = 1, \ldots, k$

Voronoi's algorithm



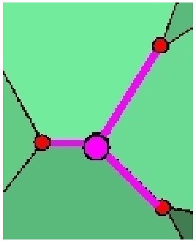
Start with a perfect form Q

1. SVP: Compute $\mathrm{Min}\,Q$ and describing inequalities of the polyhedral cone

$$\mathcal{P}(Q) \;=\; \{\; Q' \in \mathcal{S}^n \;:\; Q'[x] \geq 1 \text{ for all } x \in \mathrm{Min}\,Q \;\}$$

2. PolyRepConv: Enumerate extreme rays $R_1, \ldots, R_k$ of $\mathcal{P}(Q)$
3. SVPs: Determine contiguous perfect forms $Q_i = Q + \alpha R_i$, $i = 1, \ldots, k$
4. ISOMs: Test if Q_i is arithmetically equivalent to a known form

Voronoi's algorithm



Start with a perfect form Q

1. SVP: Compute $\operatorname{Min} Q$ and describing inequalities of the polyhedral cone

$$\mathcal{P}(Q) \;=\; \{\; Q' \in \mathcal{S}^n \;:\; Q'[x] \geq 1 \text{ for all } x \in \operatorname{Min} Q \;\}$$

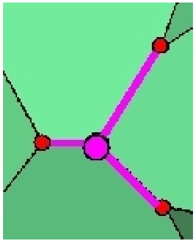
2. PolyRepConv: Enumerate extreme rays $R_1, \ldots, R_k$ of $\mathcal{P}(Q)$
3. SVPs: Determine contiguous perfect forms $Q_i = Q + \alpha R_i$, $i = 1, \ldots, k$
4. ISOMs: Test if Q_i is arithmetically equivalent to a known form
5. Repeat steps 1.–4. for new perfect forms

Enumeration of perfect forms

- **BOTTLENECK**: Computing rays of polyhedra!

  EX: Rays of a 36-dim. polyhedral cone given by

  120 linear inequalities yield „neighbors“ of E_8

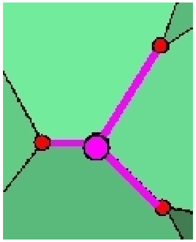


Enumeration of perfect forms

- **BOTTLENECK**: Computing rays of polyhedra!

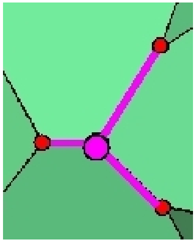
  EX: Rays of a 36-dim. polyhedral cone given by

  120 linear inequalities yield „neighbors" of E_8



n	# perfect forms	# extreme forms	author(s)
2	1	1	Lagrange, 1773
3	1	1	Gauß, 1840
4	2	2	Korkine & Zolotareff, 1877
5	3	3	Korkine & Zolotareff, 1877
6	7	6	Barnes, 1957
7	33	30	Jaquet-Chiffelle, 1991

Enumeration of perfect forms



- **BOTTLENECK**: Computing rays of polyhedra!

  EX: Rays of a 36-dim. polyhedral cone given by

  120 linear inequalities yield „neighbors“ of E_8

n	# perfect forms	# extreme forms	author(s)
2	1	1	Lagrange, 1773
3	1	1	Gauß, 1840
4	2	2	Korkine & Zolotareff, 1877
5	3	3	Korkine & Zolotareff, 1877
6	7	6	Barnes, 1957
7	33	30	Jaquet-Chiffelle, 1991
8	10916	2408	Dutour Sikirić, Sch. & Vallentin, 2005; Riener, 2005
9	> 500000		

Computer assisted proof with *Recursive Adj. Decomp. Method*
for ray enumeration under symmetries

(showing that the “E_8-cone” has 25075566937584 rays in 83092 orbits)

Equivariant theory

For a finite group $G \subset \mathsf{GL}_n(\mathbb{Z})$ the space of invariant forms

$$T_G \quad := \quad \{\, Q \in \mathcal{S}^n \ : \ G \subset \operatorname{Aut} Q \,\}$$

is a linear subspace of $\mathcal{S}^n$; $T_G \cap \mathcal{S}^n_{>0}$ is called Bravais space

Equivariant theory

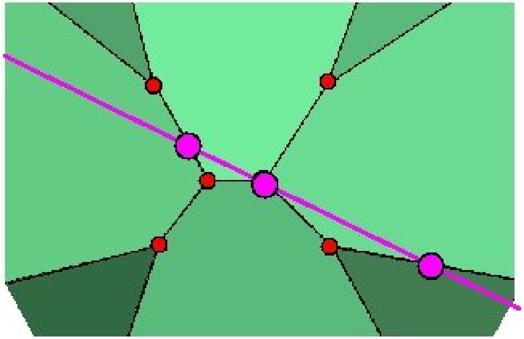
For a finite group $G \subset \mathsf{GL}_n(\mathbb{Z})$ the space of invariant forms

$$T_G \quad := \quad \{\, Q \in \mathcal{S}^n \ : \ G \subset \operatorname{Aut} Q \,\}$$

is a linear subspace of $\mathcal{S}^n$; $\qquad T_G \cap \mathcal{S}^n_{>0}$ is called Bravais space

IDEA (Bergé, Martinet, Sigrist, 1992):

Intersect Ryshkov polyhedron $\mathcal{R}$ with a linear subspace $T \subset \mathcal{S}^n$



$\mathbb{T}$-perfect and $\mathbb{T}$-extreme forms

DEF: $Q \in T \cap \mathcal{S}^n_{>0}$

- is T-extreme if it attains a loc. max. of δ within T

$\mathbb{T}$-perfect and $\mathbb{T}$-extreme forms

DEF: $Q \in T \cap \mathcal{S}^n_{>0}$

- is T-extreme if it attains a loc. max. of δ within T
- is T-perfect if it is a vertex of $\mathcal{R} \cap T$

$\mathbb{T}$-perfect and $\mathbb{T}$-extreme forms

DEF: $Q \in T \cap \mathcal{S}^n_{>0}$

- is T-extreme if it attains a loc. max. of δ within T
- is T-perfect if it is a vertex of $\mathcal{R} \cap T$
- is T-eutactic if $Q^{-1} \mid T \in \operatorname{relint}(\mathcal{V}(Q) \mid T)$

$\mathbb{T}$-perfect and $\mathbb{T}$-extreme forms

DEF: $Q \in T \cap \mathcal{S}^n_{>0}$

- is T-extreme if it attains a loc. max. of δ within T
- is T-perfect if it is a vertex of $\mathcal{R} \cap T$
- is T-eutactic if $Q^{-1} \mid T \in \operatorname{relint}(\mathcal{V}(Q) \mid T)$

THM (BMS, 1992): Q T-extreme $\Leftrightarrow$ Q T-perfect and T-eutactic

$\mathbb{T}$-perfect and $\mathbb{T}$-extreme forms

DEF: $Q \in T \cap \mathcal{S}^n_{>0}$

- is T-extreme if it attains a loc. max. of δ within T
- is T-perfect if it is a vertex of $\mathcal{R} \cap T$
- is T-eutactic if $Q^{-1} \mid T \in \operatorname{relint}(\mathcal{V}(Q) \mid T)$

THM (BMS, 1992): Q T-extreme $\Leftrightarrow$ Q T-perfect and T-eutactic

- $Q, Q' \in T \cap \mathcal{S}^n_{>0}$ are called T-equivalent, if $\exists U \in \mathrm{GL}_n(\mathbb{Z})$ with

$$Q' = U^t Q U \quad \text{and} \quad T = U^t T U$$

$\mathbb{T}$-perfect and $\mathbb{T}$-extreme forms

DEF: $Q \in T \cap \mathcal{S}^n_{>0}$

- is T-extreme if it attains a loc. max. of δ within T
- is T-perfect if it is a vertex of $\mathcal{R} \cap T$
- is T-eutactic if $Q^{-1} \mid T \in \operatorname{relint}(\mathcal{V}(Q) \mid T)$

THM (BMS, 1992): Q T-extreme $\Leftrightarrow$ Q T-perfect and T-eutactic

- $Q, Q' \in T \cap \mathcal{S}^n_{>0}$ are called T-equivalent, if $\exists U \in \mathrm{GL}_n(\mathbb{Z})$ with

$$Q' = U^t Q U \quad \text{and} \quad T = U^t T U$$

THM (Jaquet-Chiffelle, 1995): $\{ \, T_G\text{-perfect } Q \; : \; \lambda(Q) = 1 \, \} \; / \sim_{T_G}$ finite

$\mathbb{T}$-perfect and $\mathbb{T}$-extreme forms

DEF: $Q \in T \cap \mathcal{S}^n_{>0}$

- is T-extreme if it attains a loc. max. of δ within T
- is T-perfect if it is a vertex of $\mathcal{R} \cap T$
- is T-eutactic if $Q^{-1} \mid T \in \mathrm{relint}(\mathcal{V}(Q) \mid T)$

THM (BMS, 1992): Q T-extreme $\Leftrightarrow$ Q T-perfect and T-eutactic

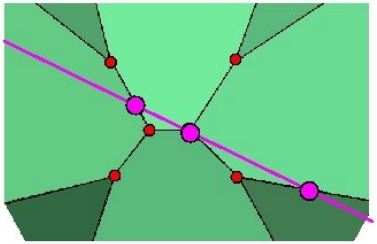
- $Q, Q' \in T \cap \mathcal{S}^n_{>0}$ are called T-equivalent, if $\exists U \in \mathrm{GL}_n(\mathbb{Z})$ with

$$Q' = U^t Q U \quad \text{and} \quad T = U^t T U$$

THM (Jaquet-Chiffelle, 1995): $\{ T_G\text{-perfect } Q \ : \ \lambda(Q) = 1 \} \ / \sim_{T_G}$ finite

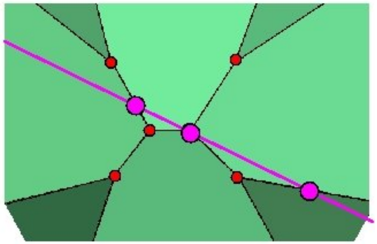
$\Rightarrow$ Voronoi's algorithm can be applied to $\mathcal{R} \cap T_G$

$\mathbb{T}$-Algorithm



SVPs: Obtain a T-perfect form Q

$\mathbb{T}$-Algorithm

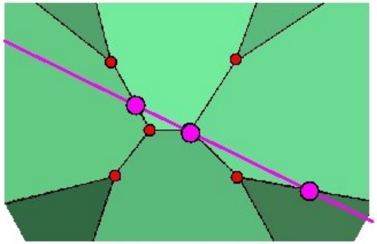


SVPs: Obtain a T-perfect form Q

1. SVP: Compute $\operatorname{Min} Q$ and describing inequalities of the polyhedral cone

$$\mathcal{P}(Q) \;=\; \{\; Q' \in T \;:\; Q'[x] \geq 1 \text{ for all } x \in \operatorname{Min} Q \;\}$$

$\mathbf{T}$-Algorithm



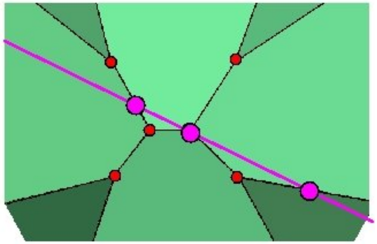
SVPs: Obtain a T-perfect form Q

1. SVP: Compute $\operatorname{Min} Q$ and describing inequalities of the polyhedral cone

$$\mathcal{P}(Q) \;=\; \{\; Q' \in T \;:\; Q'[x] \geq 1 \text{ for all } x \in \operatorname{Min} Q \;\}$$

2. PolyRepConv: Enumerate extreme rays $R_1, \ldots, R_k$ of $\mathcal{P}(Q)$

T-Algorithm



SVPs: Obtain a T-perfect form Q

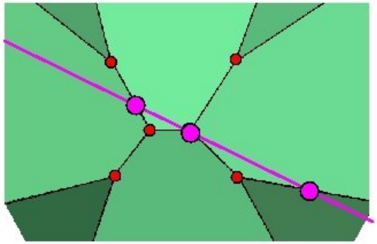
1. SVP: Compute $\operatorname{Min} Q$ and describing inequalities of the polyhedral cone

$$\mathcal{P}(Q) \;=\; \{\; Q' \in T \;:\; Q'[x] \geq 1 \text{ for all } x \in \operatorname{Min} Q \;\}$$

2. PolyRepConv: Enumerate extreme rays $R_1, \ldots, R_k$ of $\mathcal{P}(Q)$
3. For the indefinite R_i, $i = 1, \ldots, k$

   SVPs: Determine contiguous perfect forms $Q_i = Q + \alpha R_i$

$\mathbb{T}$-Algorithm



SVPs: Obtain a T-perfect form Q

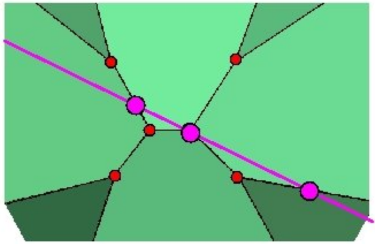
1. SVP: Compute $\operatorname{Min} Q$ and describing inequalities of the polyhedral cone

$$\mathcal{P}(Q) \;=\; \{\, Q' \in T \;:\; Q'[x] \geq 1 \text{ for all } x \in \operatorname{Min} Q \,\}$$

2. PolyRepConv: Enumerate extreme rays $R_1, \ldots, R_k$ of $\mathcal{P}(Q)$
3. For the indefinite R_i, $i = 1, \ldots, k$

   SVPs: Determine contiguous perfect forms $Q_i = Q + \alpha R_i$
4. T-ISOMs: Test if Q_i is T-equivalent to a known form

T-Algorithm



SVPs: Obtain a T-perfect form Q

1. SVP: Compute $\operatorname{Min} Q$ and describing inequalities of the polyhedral cone

$$\mathcal{P}(Q) \;=\; \{\; Q' \in T \;:\; Q'[x] \geq 1 \text{ for all } x \in \operatorname{Min} Q \;\}$$

2. PolyRepConv: Enumerate extreme rays $R_1, \ldots, R_k$ of $\mathcal{P}(Q)$
3. For the indefinite R_i, $i = 1, \ldots, k$

   SVPs: Determine contiguous perfect forms $Q_i = Q + \alpha R_i$
4. T-ISOMs: Test if Q_i is T-equivalent to a known form
5. Repeat steps 1.–4. for new perfect forms

Examples/Applications

n	2	4	6	8	10	12
# $\mathcal{E}$-perfect	1	1	2	5	1628	?
maximum δ	$0.9069\ldots$	$0.6168\ldots$	$0.3729\ldots$	$0.2536\ldots$	$0.0360\ldots$	

Perfect Eisenstein forms

Examples/Applications

n	2	4	6	8	10	12
# $\mathcal{E}$-perfect	1	1	2	5	1628	?
maximum δ	$0.9069\ldots$	$0.6168\ldots$	$0.3729\ldots$	$0.2536\ldots$	$0.0360\ldots$	

Perfect Eisenstein forms

n	2	4	6	8	10	12
# $\mathcal{G}$-perfect	1	1	1	2	≥ 8192	?
maximum δ	$0.7853\ldots$	$0.6168\ldots$	$0.3229\ldots$	$0.2536\ldots$		

Perfect Gaussian forms

Examples/Applications

n	2	4	6	8	10	12
# $\mathcal{E}$-perfect	1	1	2	5	1628	?
maximum δ	$0.9069\ldots$	$0.6168\ldots$	$0.3729\ldots$	$0.2536\ldots$	$0.0360\ldots$	

Perfect Eisenstein forms

n	2	4	6	8	10	12
# $\mathcal{G}$-perfect	1	1	1	2	≥ 8192	?
maximum δ	$0.7853\ldots$	$0.6168\ldots$	$0.3229\ldots$	$0.2536\ldots$		

Perfect Gaussian forms

n	4	8	12	16
# $\mathcal{Q}$-perfect	1	1	8	?
maximum δ	$0.6168\ldots$	$0.2536\ldots$	$0.03125\ldots$	

Perfect Quaternion forms

Extension from Lattices to Periodic Sets

Extension from Lattices to Periodic Sets

$$\Lambda = A\left(\bigcup_{i=1}^{m} t_i + \mathbb{Z}^n\right)$$ with $A \in \mathrm{GL}_n(\mathbb{R})$, $t_i \in \mathbb{R}^n$ and $t_m = 0$

is identified (up to orthogonal transformations) with

$$\left(A^t A, t_1, \ldots, t_{m-1}\right) \in \mathcal{S}^{n,m}_{>0} := \mathcal{S}^n_{>0} \times \mathbb{R}^{n\times(m-1)}$$

Extension from Lattices to Periodic Sets

$$\Lambda = A\left(\bigcup_{i=1}^{m} t_i + \mathbb{Z}^n\right) \text{ with } A \in \mathrm{GL}_n(\mathbb{R}),\ t_i \in \mathbb{R}^n \text{ and } t_m = 0$$

is identified (up to orthogonal transformations) with

$$\left(A^tA, t_1, \dots, t_{m-1}\right) \in \mathcal{S}^{n,m}_{>0} := \mathcal{S}^{n}_{>0} \times \mathbb{R}^{n\times(m-1)}$$

- For fixed m and $t = (t_1, \dots, t_{m-1})$, the set of periodic sets with points at min. dist. $\geq \lambda > 0$ is identified with a locally finite polyhedron $\mathcal{R}$ in $\mathcal{S}^n_{>0}$

Extension from Lattices to Periodic Sets

$$\Lambda = A\left(\bigcup_{i=1}^{m} t_i + \mathbb{Z}^n\right)$$ with $A \in \mathrm{GL}_n(\mathbb{R})$, $t_i \in \mathbb{R}^n$ and $t_m = 0$

is identified (up to orthogonal transformations) with

$$\left(A^t A, t_1, \ldots, t_{m-1}\right) \in \mathcal{S}^{n,m}_{>0} := \mathcal{S}^n_{>0} \times \mathbb{R}^{n\times(m-1)}$$

- For fixed m and $t = (t_1, \ldots, t_{m-1})$, the set of periodic sets with points at min. dist. $\geq \lambda > 0$ is identified with a locally finite polyhedron $\mathcal{R}$ in $\mathcal{S}^n_{>0}$

THM: For rational and fixed t,

there exist only finitely many *inequivalent* vertices of $\mathcal{R}$

Periodic extreme sets

Periodic extreme sets

DEF: $X = (Q, t) \in \mathcal{S}^{n,m}_{>0}$ (and a corresponding periodic pointset)

is called periodic extreme,

if it is m'-extreme for all possible representations $X' \in \mathcal{S}^{n,m'}_{>0}$

(attains a local maximum of δ on $\mathcal{S}^{n,m'}_{>0}$)

Periodic extreme sets

DEF: $X = (Q, t) \in \mathcal{S}^{n,m}_{>0}$ (and a corresponding periodic pointset)

is called periodic extreme,

if it is m'-extreme for all possible representations $X' \in \mathcal{S}^{n,m'}_{>0}$

(attains a local maximum of δ on $\mathcal{S}^{n,m'}_{>0}$)

DEF: $Q \in \mathcal{S}^n_{>0}$ (and a corresponding lattice) is called strongly eutactic, if

$$Q^{-1} = \underbrace{\alpha}_{>0} \sum_{v \in \operatorname{Min} Q} v v^t$$

Periodic extreme sets

DEF: $X = (Q, t) \in \mathcal{S}^{n,m}_{>0}$ (and a corresponding periodic pointset)

is called periodic extreme,

if it is m'-extreme for all possible representations $X' \in \mathcal{S}^{n,m'}_{>0}$

(attains a local maximum of δ on $\mathcal{S}^{n,m'}_{>0}$)

DEF: $Q \in \mathcal{S}^{n}_{>0}$ (and a corresponding lattice) is called strongly eutactic, if

$$Q^{-1} = \underbrace{\alpha}_{>0} \sum_{v \in \operatorname{Min} Q} vv^t$$

THM: (Sch. 2007) Perfect and strongly eutactic forms are periodic extreme

Periodic extreme sets

DEF: $X = (Q, t) \in \mathcal{S}^{n,m}_{>0}$ (and a corresponding periodic pointset)

is called periodic extreme,

if it is m'-extreme for all possible representations $X' \in \mathcal{S}^{n,m'}_{>0}$

(attains a local maximum of δ on $\mathcal{S}^{n,m'}_{>0}$)

DEF: $Q \in \mathcal{S}^{n}_{>0}$ (and a corresponding lattice) is called strongly eutactic, if

$$Q^{-1} = \underbrace{\alpha}_{>0} \sum_{v \in \operatorname{Min} Q} vv^t$$

THM: (Sch. 2007) Perfect and strongly eutactic forms are periodic extreme

COR: A$_n$, D$_n$, E$_n$ and Λ_{24} are periodic extreme

ToDo

- Systematic searches for interesting perfect and extreme forms / lattices (in suitable subspaces)

ToDo

- Systematic searches for interesting perfect and extreme forms / lattices (in suitable subspaces)

- Systematic searches for dense periodic (non-lattice) sets

ToDo

- Systematic searches for interesting perfect and extreme forms / lattices (in suitable subspaces)
- Systematic searches for dense periodic (non-lattice) sets

Challenges

- Prove for some non-lattice sphere packing that it is denser than any lattice packing in its dimension

ToDo

- Systematic searches for interesting perfect and extreme forms / lattices (in suitable subspaces)
- Systematic searches for dense periodic (non-lattice) sets

Challenges

- Prove for some non-lattice sphere packing that it is denser than any lattice packing in its dimension
- Determine Hermite's constant for some $n \geq 9$ ($n \neq 24$)

Muchas Gracias!

http://www.math.uni-magdeburg.de/lattice_geometry/