Using Global Properties for Qualitative Reasoning: A Qualitative System Theory

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Abstract

All the qualitative simulation algorithms so far proposed depend upon local propagation paradigm, and hence produce spurious states. Recent approaches use topological constraints in phase space diagrams for filtering the spurious states. We propose an alternative method of recognizing global properties in the graphically expressed qualitative model. Qualitative versions of such system-theoretic concepts as stability and observability are used to analyze the global properties of the system. We first present structural conditions for recognizing qualitative stability or instability in the qualitative model. We introduce the new concept of *invariant sign pattern* which is a qualitative version of fixed point. Once the qualitative model becomes the *invariant sign pattern,* it remains in that state. The conditions for the qualitative model to have an *invariant sign pattern* are also characterized. Although our method is restricted to a linear system, the method does not suffer from such restrictions coming from two dimensional phase space diagrams, such as the target system must be a second order system. We also discuss how to extend the method to non-linear systems.

1 Introduction

Qualitative physics [de Kleer and Brown 1984] (or qualitative process theory [Forbus 1984]) has been studied recently. Qualitative reasoning is used for predicting and explaining the behavior of the physical system by using symbolic computation.

De Kleer and Brown [de Kleer and Brown 1984] use logical proof as the explanation for the physical behavior. They pointed out that the logical proof has undesirable features for making causal accouts, and proposed *mythical causality.* While their modeling is rather component oriented, Forbus [Forbus 1984] developed process oriented modeling. Kuipers' theory [Kuipers 1986] , however, starts from abstracting the mathematical model preserving qualitative information in the model. Both Kuipers [Kuipers 1986] , and de Kleer and Bobrow [de Kleer and Bobrow 1984] developed simulation algorithms on their qualitative models. Both of them introduce higher order derivatives to predict change precisely. The problems of these qualitative reasoning methods are:

1. Although these methods provide modeling perspectives, they are not yet ready for the automatic or interactive generation of the qualitative models of large-scale systems such as industrial processing plants.

2. They do not use global properties which do not come from the local propagation of the constraint or state.

Falkenhainer and Forbus [Falkenhainer and Forbus 1988] focus on the problem of the modeling by considering the granularity of the model. We focus on the second problem using structural conditions for global properties. Struss [Struss 1988] and Lee and Kuipers [Lee and Kuipers 1988] also discuss the second problem with reference to the phase space portrait. We will present an alternative method for filtering out spurious behavior using global properties, which can be checked on the graphically expressed qualitative model. The global properties of the system, such that an oscillation will converge on some point or not, can be discriminated to some extent by checking the sign structure of the graph. Our method seems to be more suitable for symbolic computation than with the method using geometric conditions, however ours suffers from the limitation that the target system must be linear.

Section 2 shows the qualitative model of dynamical systems. The qualitative model is a signed directed graph obtained by keeping the qualitative information of sign structures of a linear system. In section 3, global analyses such as stability analysis are made on the qualitative model.

2 Qualitative Model of a Dynamical System

2.1 The qualitative model

Qualitative theory of linear systems, which has been studied extensively in econometrics [Quirk 1965, Quirk 1968] and mathematical ecology [Jeflferies 1974], is used as an analysis tool for the qualitative model. In many systems such as chemical processing plants dynamical behavior is expressed or approximated by a linear differential equation:

(2-1) $dx/dt = Ax, A \in R^{n \times n}$.

We use the qualitative model expressing the signed matrix A_s. [1] In the model, an arc is directed from node i to node j with the sign of $(A_s)_{ij}$. Most of the results of qualitative system theory are obtained for the state-space expression of this linear system. Thus, in order to directly use this qualitative system theory, we transform the model into this state-space expression. All the interactions whose phase lag are $n > 1$ are divided into n sequential interactions of phase lag 1 by introducing n-1 dummy variables (nodes). On the other hand, variables (nodes) connected by the interaction of phase lag 0 are regarded as one variable (node). In the global analysis of section 3, we assume the systems under discussion are already normalized.

Non-linear systems must be first linearized in the following manner. Develop the system around the point of interest, then neglect the higher order non-linear terms. This linear approximation is only valid in the neighborhood of the point. The linearized system must be expressed in state-space expression for later analysis.

Example 2.1

The model for the pressure regulator is shown below.

$$dXs/dt = -a \cdot Po$$
$$dQi/dt = b \cdot (DP - c \cdot Qi^2/Xs)$$
$$dPo/dt = e \cdot (Qo^2 - f \cdot Po)$$
$$DP = Pi - Po$$
$$Qi = Qo$$

Xs : area available for the flow through the valve
Po : pressure at outlet
Pi : pressure at inlet
DP : pressure drop across the valve
Qi : inflow to the valve
Qo : outflow from the valve
a, b, c, e, f: appropriately chosen positive constants.

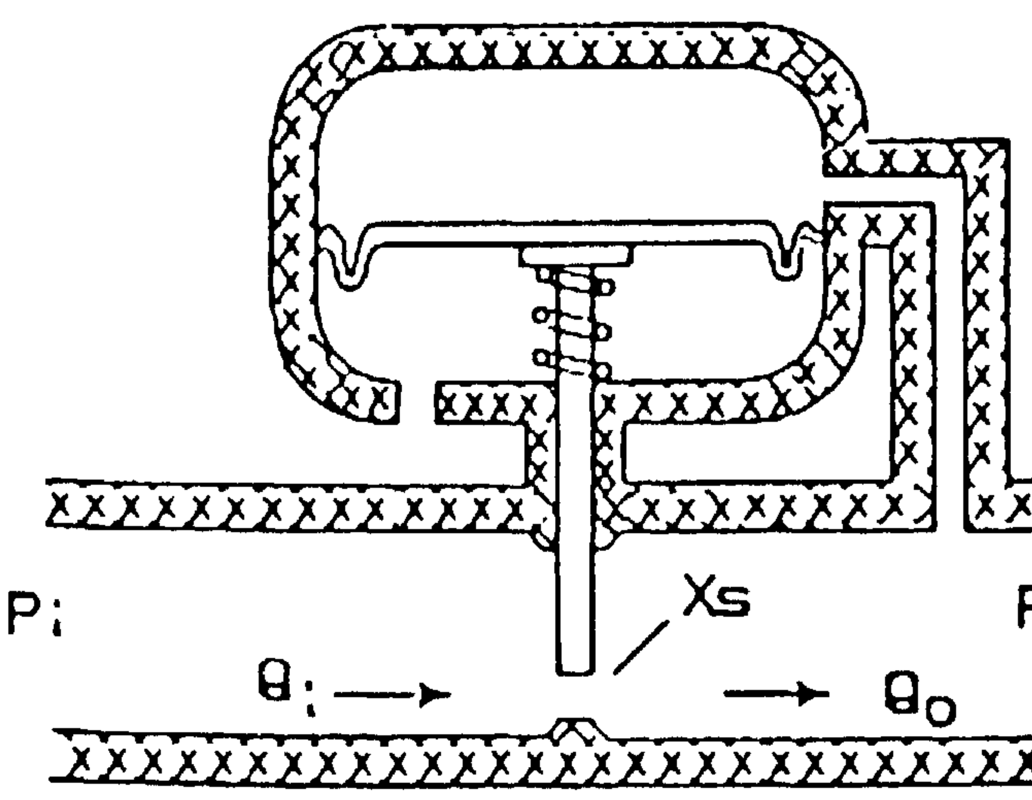

Fig. 1 Diagram of the pressure regulator

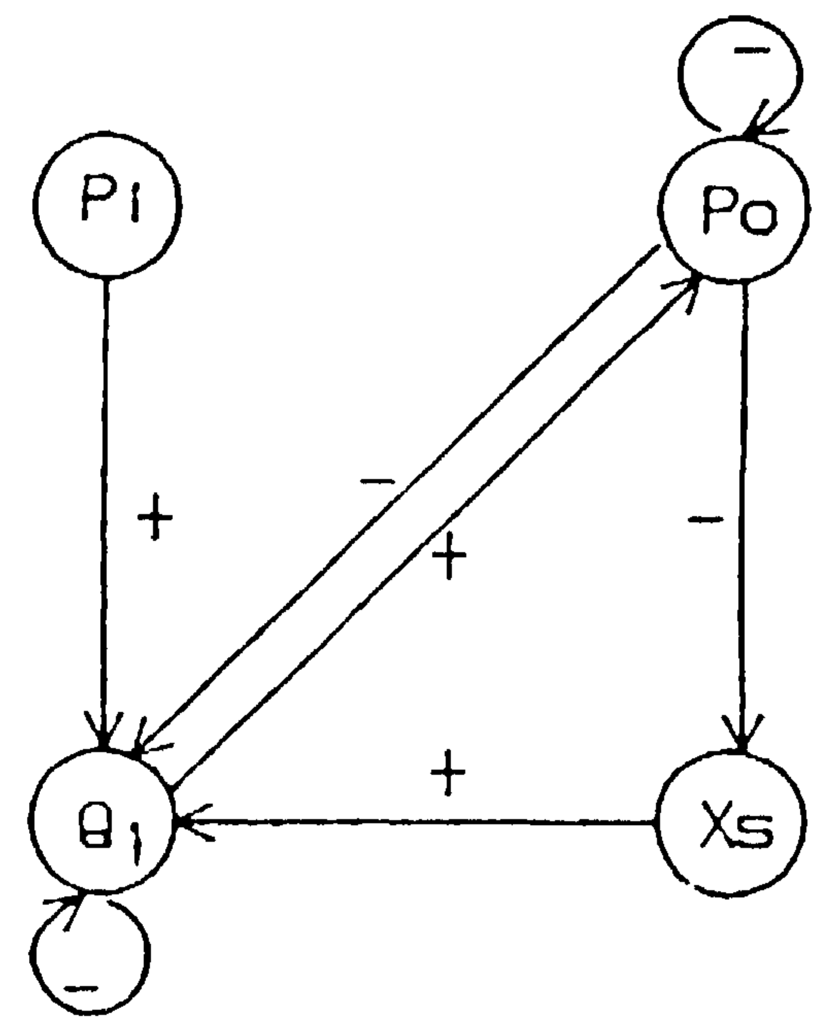

Fig. 2 Qualitative model of the pressure regulator

Fig. 1 shows a diagram of a pressure regulator. Fig. 2 is the signed digraph expressing the qualitative model. Since the phase lag of all the arcs is normalized to 1, only the sign is indicated in the arc.

2.2 Causality and system theoretic concepts

In dynamical system theory, many concepts such as observability and stability have been studied. Since these

[1] Signed matrix A_s of A is a triple value matrix defined as follows:
$(A_s)_{ij} = +, -, 0$ if $(A)_{ij} > 0, < 0, = 0$ respectively.

concepts have intuitive explanatory power, they may be used as aids for causal account. There seems to be an important relation between the concept of observability and causality. In system theory, observability is defined as:

> "A system is said to be observable by an observer if it is possible to determine the initial state by observing the output signal from the observer during a finite time starting from the initial time."

We can use the observability [2] (or its dual concept of controlability) as a tool to check the potential causability. It is not against our intuition to say that the event dX = + (or —) can cause the event dY = + (or —) only when X is observable from Y.

3 Global Analysis

The main advantage of using state-space expression (2-1) of the qualitative model is that it allows many system theoretic analyses, especially the global analysis. This section presents several results which can be used as a tool for a global analysis on the qualitative model.

3.1 Qualitative stability analysis

A property of a system is called qualitative if it is determined only by the sign structure of the qualitative model. In this section, we discuss the qualitative property of the qualitative model. Two kinds of qualitative stabilities, and qualitative observability are defined as follows.

Definition 3.1. (sign stability and potential stability)

A qualitative model A_s is called sign (potential) stable if all (some of) instances of the model are stable [3].

Definition 3.2. (sign observability)

The qualitative model with the observer is said to be sign observable if all instances of the model are observable from the observer.

In the example 2.1, the graph indicates that the model can be decomposed into two strongly connected components corresponding to the subsystem Pi and the subsystem consisting of Po, Q, Xs. Pi is observable from the subsystem of Po, Q, Xs and not in opposite way. Notice, however, that even if the model is decomposed into strongly connected components, the affecting subsystem may not be observable from the affected subsystem in such cases that the affecting subsystem has a constant mode or two effects canceling each other. (Obviously, the affected subsystem is not observable from the affecting subsystem.)

A necessary and sufficient condition for a qualitative model to be sign stable is obtained with the concept of sign observability.

Theorem 3.3.

A qualitative model is sign stable if and only if the qualitative model has the following properties.

(1) There is no positive loop [5] and there exists at least one negative loop,

(2) there is no positive circuit of length two,

(3) there is no circuit of length greater than two, and

(4) by setting the subsystem of negative loops as observer, the rest of the subsystem is sign observable from the observer.

Proof

The conditions (l)-(3) guarantee that the system does not have divergent mode [6]. In order to further guarantee that the system does has neither constant mode nor pure periodical modes we only have to put the condition that that the signals are always observable from the node having a negative loop. In other words, if the system has pure periodical modes then the signal may not be observable by the cancellation of the oscillations of different phase. Likewise, if the system has a constant mode the signal is not observable.

Example 3.4.

To demonstrate the power of the sign stability, let us consider the mass-spring system whose state-space expression is as below:

$$dX/dt = V$$

$dV/dt = —kX — fV$ where k and / are positive constants.

[2] The observability of the linear system can be checked by a matrix. Let $y = Cx, y \in R^{lXm}, C \in R^{nXm}$ be observed output of the linear system (2-1), then the observability from y can be known by testing whether or not the matrix $[C_1 CA_y CA^2{}_{>} \ldots{}_i CA^n{}-{}^1]$ have the full rank.

[3] The solution of the system will asymptotically converge on an equilibrium point.

[4] Strongly connected component is such subgraph that for all the pairs of nodes in the subgraph there exists a path from both sides.

[5] A circuit is a closed path where the path is a graph connecting many nodes by arcs of the same direction sequentially. The sign of a circuit is a multiplication of all the signs of the arcs included in the circuit. The length of the circuit is the number of all the arcs included in the circuit. The circuit of length 1 is called a loop.

[6] For the system (2-1), divergent mode, pure periodical modes, and constant mode are realized when the matrix A has eigenvalues with positive real part, pure imaginaries, and 0 respectively.

Fig. 3 shows the diagram of a mass-spring system with dashpot. The qualitative model of the mass-spring system is shown in Fig. 4a. This system is known to be sign stable, since all the conditions of theorem 3.3 are satisfied. Thus, oscillation will always converge eventually. When / = 0 (when there is no loop at node V), however, the system is always in a pure periodical mode.

The necessary and sufficient conditions for a qualitative model to be potentially stable have not yet been obtained. We present some heuristics which will be used to identify the potentially stable sign structure.

Theorem 3.5.

A qualitative model is potentially stable if the subgraph of the the qualitative model is potentially stable.

Proof

One of the powerful heuristics used in the system theory is that

> "A property of a system is preserved after a change of the system if the property is *locally invariant* to the change."

Since the stability holds even with a small change of parameter, the qualitative model obtained by adding arcs to the sign stable qualitative model has a stable instance supposing the added arcs represent the interactions with small absolute values. This argument is also true when adding arcs to the stable instances *oi* a potentially stable qualitative model. Q.E.D.

We have obtained another sufficient condition for the potential stability.

Theorem 3.6.

A qualitative model of n nodes is potentially stable if the signed digraph has the negative circuit of length exactly k for every integer k = 1, 2, ... ,n.

Proof (see [Ishida *ct al.*, 1981])

As for the necessary condition, we obtained the following theorem.

Theorem 3.7.

If a qualitative model is potentially stable then the signed digraph has a set of negative circuits whose sum of length is equal to k for every integer k — 1 ... n .

Proof (see [Ishida *et al.*, 1981])

Example 3.8.

Consider the qualitative stability of the pressure regulator example. Since the subsystem Pi is always constant, only the subsystem composed of Po, Q, Xs is analyzed. By theorem 3.3, this model is not sign stable because of the circuit of length 3. However, this model is potentially stable, since the graph has negative circuits of length 1, 2, and 3. The model can be made stable by making the effect of negative circuits of length 1 and 2 relatively stronger than that of length 3. Notice, however, these analyses are valid only in the neighborhood of the equilibrium point where the changes around the point are considered. In order to consider the neighborhood of a different point, we must use different models linearized on the other point.

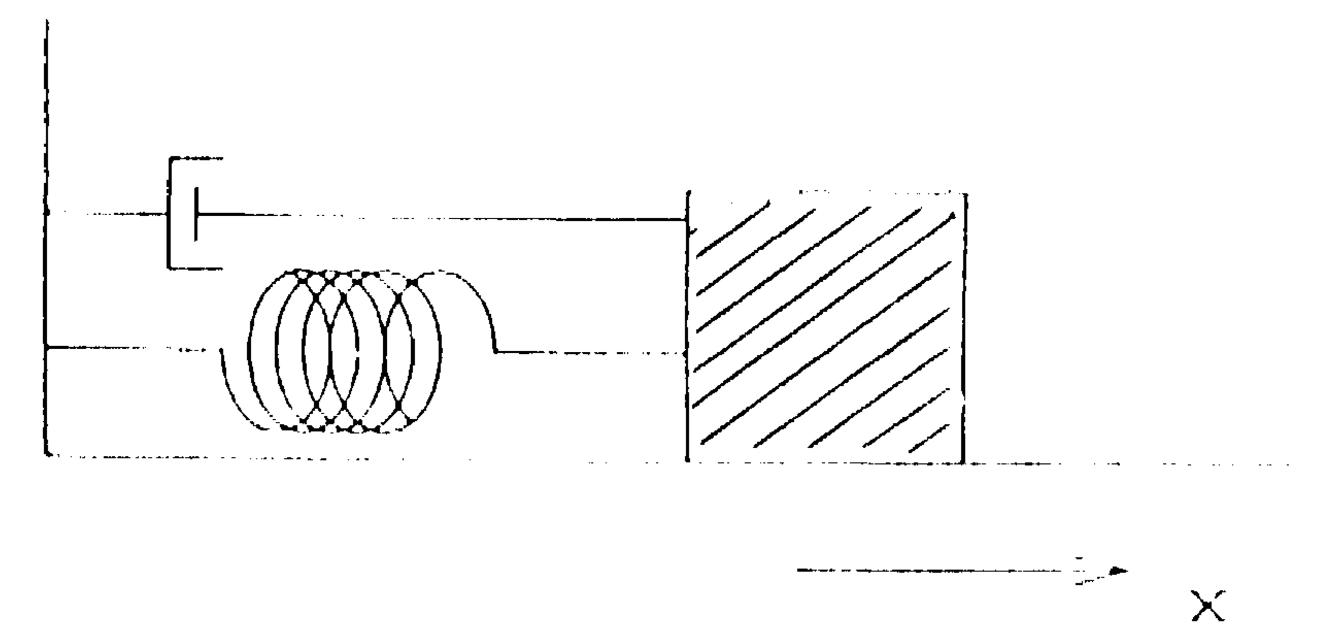

Fig. 3 Mass-spring system with a dashpot.

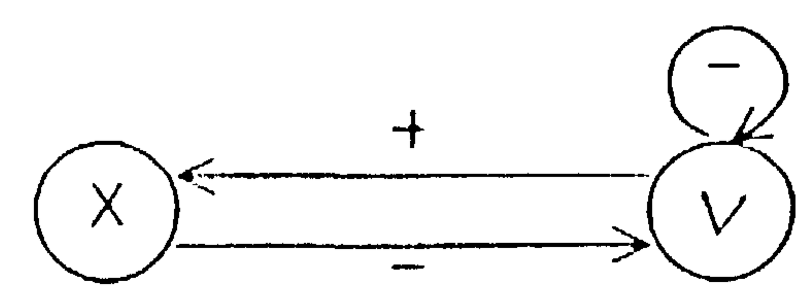

Fig. 4a
Qualitative model of
mass-spring system

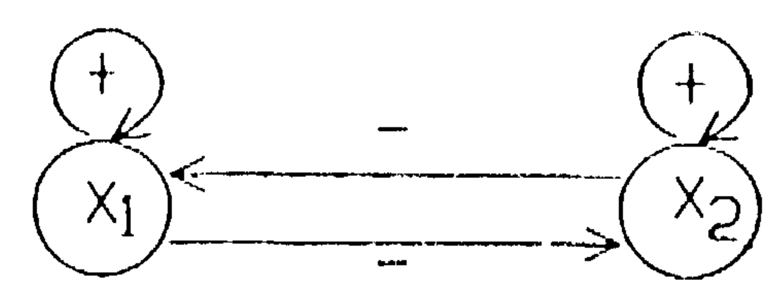

Fig. 4b
Qualitative model
having invariant sign patterns

Other than the conditions for sign stability and potential stability so far proposed, the following condition of sign instability can also be used as a tool to check the qualitative stability, since the potential stable class is the complementary set of the sign unstable class.

Theorem 3.9

The qualitative model obtained by making all the signs of arcs to some nodes opposite in the sign stable model is sign unstable.

Proof (see [Ishid a ct a/., 1981])

3.2 Invariant sign pattern of a system

There is a class of qualitative model in which the initial sign pattern can be specified from the current sign pattern. In this section, we define the new concept of *invariant sign pattern.* Also, we discuss the relation between it and the class of qualitative stability.

Definition 3.10. (invariant sign pattern)

A sign pattern x_s is called *invariant sign pattern* of a qualitative model if the model stays at the sign pattern x_5 all the time, once it attains the state.

It is easily checked whether or not a given qualitative model has *invariant sign patterns.*

Theorem 3.11.

A strongly connected qualitative model has an invariant sign pattern if

(1) All the circuits have positive sign, and

(2) All the reconvergent fanout paths [7] between two nodes have the same sign.

Proof

We first show that the qualitative model has the invariant sign pattern if the sign equation $x_s = A_s x_s$ has a solution *X0* and that this solution *Xo* itself is the invariant sign pattern [8] . The solution must have determined sign mode (4+, — ,or 0) as its elements.

The solution x_s of sign equation satisfies that $\{dx/dt)_i$ and $\{x_s)_i$ are of the same sign. Thus, if x_s is given as the primary sign pattern (sign pattern of the initial value vector of the system (2-1)) it does not change for all the time after. Now we will show that the existence of a solution of the sign equation is equivalent to the conditions for a strongly connected qualitative model.

Necessity: Suppose there exists a negative circuit from node $x_{i,}$ then the change direction imposed through the circuit is opposite to the sign pattern of $(x_s),$. Thus $(x_s)_i$ must be 0 otherwise it becomes undetermined mode. However, if it is equal to 0 then all the patterns of the elements of x_s must be 0, for the model is strongly connected. Therefore, the sign equation has no invariant sign pattern other than a trivial one (0 ... 0).

Suppose next that there are two reconvergent fanout paths whose signs are opposite and both have common initial node $x_{\%}$ and common terminal node *Xj.* Then the sign pattern of $(x_s)j$ is undetermined whether $(x_s)_t$ is -f or —. Again, $(x_s)_t$ must be 0 and hence the sign equation does not have any invariant sign pattern except a trivial one.

Sufficiency: Suppose the sign equation does not have the solution. This is because some sign pattern has become undetermined mode. And the cases where the. undetermined mode cannot be avoided occur when sign of a variable imposes a different sign on other variables. These cases occur only when negative loop or reconvergent fanout paths of the other sign exist. Q.E.D.

The invariant sign pattern itself can be obtained from the sign structure of the qualitative model.

Theorem 3.12.

A sign pattern x_s is an invariant sign pattern of a strongly connected qualitative model if it satisfies

(1) $(x_s)_t$ — -f or — for alh = 1. . . n, and

(2) $(x_f),$ = +(—) if there exists an arc (x^x,) such that *sgn(xk, Xi)(x,)k* = -f(—) wheres^n(xjt, x_t) is the sign associated with the arc (x*,x,).

Proof

Immediate from the sign equation.

Theorem 3.13.

If a strongly connected qualitative model has an invariant sign pattern x, then all the sign subpatterns converge on the invariant sign pattern. Sign subpattern is the sign pattern obtained by replacing some (but not all) of -f or — with 0 in the original sign pattern.

Proof

Since the qualitative model is strongly connected, all the elements of the sign pattern vector will converge on a non-zero pattern except the trivial all zero pattern. Further, they are not undetermined, for the qualitative model has two invariant sign patterns whose sign is opposite to each other. Thus, the primary sign pattern will converge on an invariant sign pattern which has the sign pattern as sign subpattern according to the dynamics of the system. Q.E.D.

In connection with the qualitative stability discussed in the previous section, the next theorem holds.

Theorem 3.14.

If the sign equation $x_8 = A_\$ x_s$ has a solution then the qualitative model of the sign structure A_s is sign unstable.

Proof

If the sign equation has the solution *x,* then the solution of all the instances of the qualitative model with A_9 does not converge on 0. Q.E.D.

Theorem 3.15. [9]

Reconvergent fanout paths are such paths that share the initial and terminal nodes.

[8]This fact can be generalized to the non-linear system, i.e. The invariant sign pattern of the system dx/dt = f(x) is the solution of x = f(x).

This theorem can be generalized to the non-linear system dx/dt = f(x). That is, the subspace where an equilibrium point

If there is a qualitative state assignment for the qualitative model such that the total effect on each node is not definite sign then the qualitative model potentially has an equilibrium point in the subspace specified by the assignment.

Proof

This assignment can be obtained by solving a sign equation of $0 = Ax_s$. This means that the matrix potentially have 0 eigenvalue. Q.E.D.

Example 3.16.

A qualitative model shown in Fig. 4b has an invariant sign pattern x_s — (+—) and hence (— +) (If a qualitative model has a invariant sign pattern x_s, then $—x_s$ also.) For example, $(x_s)1$ — + is preserved for all the time, since feedback circuit from both $x1$ itself and $X2$ keep $x\backslash$ increasing. Similarly, sign patterns of $(x_s)_2$ are also preserved. Thus, the subpattern (+ 0) and (0 —) will fall into the sign pattern (-f —) by theorem 3.13. Further, it is also known by theorem 3.15 that this model has an equilibrium point in subspace (4- -f) or (——) in case the system has a constant mode.

As we have known that the qualitative model of the pressure regulator example is potentially stable, it does not have any invariant sign pattern. The model does not have a non-zero equilibrium point.

4 Conclusion

We have shown that such global properties as stability and observability can be investigated purely from the qualitative information of dynamical interaction.

So far we have discussed a global analysis of a linear system. As often done in system theory, the results of linear system can be used for non-linear systems in the following three manners:

(1) Non-linear systems can be approximated as linear systems in the neighborhood of the equilibrium point as in the example 2.1, and hence the results for linear systems hold there.

(2) The results of linear system $dx/dt - Ax$ holds for the non-linear system $dx/dt = A(t)x$ if the change of $A(t)$ is very slow.

(3) By *locally invariant* heuristics, some properties such as stability of the system $dx/dt = Ax + \varepsilon. F(x, t)$ do not change from that of $dx/dt = Ax$ if ε is sufficiently small.

We can use these approaches to the qualitative analysis for the non-linear system. That is, we divide the non-linear system into a set of linear systems each of which is an approximation of the non-linear system at some point and the neighborhood of the point. Summing up the results of these linear systems, the qualitative aspects of the non-linear systems are analyzed. Implementation of such an inference engine that synthesizes the results of global properties of non-linear systems is left for future work.

can potentially exist is specified by solving the possible sign pattern for $0 = f(x)$.

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