Gradient Clock Synchronization in Dynamic Networks

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Abstract

Over the last years, large-scale decentralized computer networks such as peer-to-peer and mobile ad hoc networks have become increasingly prevalent. The topologies of many of these networks are often highly dynamic. This is especially true for ad hoc networks formed by mobile wireless devices.

In this paper, we study the fundamental problem of clock synchronization in dynamic networks. We show that there is an inherent trade-off between the skew $\mathcal{S}$ guaranteed along sufficiently old links and the time needed to guarantee a small skew along new links. For any sufficiently large initial skew on a new link, there are executions in which the time required to reduce the skew on the link to $O(\mathcal{S})$ is at least $\Omega(n/\mathcal{S})$.

We show that this bound is tight for moderately small values of $\mathcal{S}$. Assuming a fixed set of n nodes and an arbitrary pattern of edge insertions and removals, a weak dynamic connectivity requirement suffices to prove the following results. We present an algorithm that always maintains a skew of $O(n)$ between any two nodes in the network. For a parameter $\mathcal{S} = \Omega(\sqrt{\rho n})$, where ρ is the maximum hardware clock drift, it is further guaranteed that if a communication link between two nodes u, v persists in the network for $\Theta(n/\mathcal{S})$ time, the clock skew between u and v is reduced to no more than $O(\mathcal{S})$.

1 Introduction

Establishing coordination between participants is at the core of many algorithmic challenges in distributed computing. A fundamental coordination task, and a basic prerequisite for many applications, is achieving a common notion of time. Typically every node in the network has access to a local hardware clock, but the hardware clocks of different nodes run at slightly different rates, and the rates can change over time. In addition, although a bound on the message delays in the network may be known, specific message delays are unpredictable. As a consequence it is generally not possible for any node in the network to get an accurate estimate of the clock values of neighboring nodes.

Operating under these uncertainties, a distributed clock synchronization algorithm computes logical clocks at every node, with the goal of synchronizing these clocks as tightly as possible. Traditionally, distributed clock synchronization algorithms tried to minimize the maximal difference between any two logical clocks in the network. We call this quantity the *global skew* of a clock synchronization algorithm. It is well-known that no algorithm can guarantee a global skew better than $\Omega(D)$, where D is the diameter of the network [3].

In many cases it is more important to tightly synchronize the logical clocks of nearby nodes in the network than it is to minimize the global skew. For example, to run a time division multiple access (TDMA) protocol for coordinating access to the shared communication medium in a wireless network, one only needs to synchronize the clocks of nodes that interfere with each other when transmitting. The problem of achieving synchronization that depends on the distance between the two nodes is called *gradient clock synchronization*. It was introduced in a seminal paper by Fan and Lynch, where it is also shown that surprisingly, a clock skew of $\Omega(\log D/\log\log D)$ cannot be prevented even between immediate neighbors in the network [8]. The maximal difference between the two logical clocks of adjacent nodes in the network is called the *local skew* of a clock synchronization algorithm; for static networks, Lenzen et. al. have recently proven an asymptotically tight bound of $\Theta(\log D)$ for the best possible local skew an algorithm can achieve [12, 13]. For other related work on clock synchronization, see Section 2.

Most existing work on clock synchronization considers static networks. However, many modern networks are inherently dynamic. Typically formed by autonomous agents without central control, nodes can join and leave the network in an arbitrary pattern. In ad hoc networks where often the devices are even assumed to be mobile, the resulting network topology can be highly dynamic even if the set of participating nodes remains stable. Coordination in dynamic networks is challenging, and due to the increasing significance of such networks, it is also particularly important.

In this paper we study the gradient clock synchronization problem in dynamic networks. Because the distance between nodes in the network can change over time, the problem becomes significantly harder in a dynamic setting. Consequently, unlike the static case, the requirements we make on the skew between the logical clocks of different nodes can also change over time. Every new edge that is formed induces a new and stronger constraint on the skew between its endpoints; the algorithm must adapt by reducing the skew on the edge until the new constraint is satisfied.

Hence, we distinguish between two requirements: a *stable local skew* bound applies, conceptually, to edges that exist for a long time. This is analogous to the local skew guaranteed by gradient clock synchronization algorithms for static networks. In practice, we impose a weaker *dynamic local skew* bound on all the edges, including new ones. The dynamic local skew bound is a function of how long the edge has existed: the bound starts out weak and grows stronger with time, until in the limit it converges to the stable local skew bound.

The following intuitive example shows that in general, the clock skew on a new edge cannot be reduced too quickly without violating the stable local skew bound on edges that were formed a long time before. Let u and v be two nodes at distance k from each other. As no algorithm can prevent a skew of $\Omega(k)$ between nodes at distance k, a newly formed edge between nodes u and v can carry $\Omega(k)$ local skew. To reduce the skew on the new edge, whichever node is behind must increase its logical clock by a large amount. However, a sudden increase in u or v's clocks will create a large skew along the edges of the old path that connects them. Specifically, if the algorithm guarantees a stable local skew of $\mathcal{S}$, neither u nor v can instantaneously increase their logical clocks to more than $\mathcal{S}$ ahead of their next neighbor along the old path. In turn, when this neighbor realizes it must increase its clock, it cannot increase it to more than $\mathcal{S}$ ahead of *its* next neighbor, and so on. It takes $\Omega(k/\mathcal{S})$ time until the skew can be reduced, as information about the new edge can take time to propagate through the path.

Somewhat surprisingly, the example above is not the worst one possible: adjusting the local skew on a newly formed edge can require even more than $\Omega(k/\mathcal{S})$ time, where k is the previous distance between the endpoints of the new edge. We show that (almost) independent of the initial skew on a new edge, the time required to reduce the initial skew to $\mathcal{S}$ is at least $\Omega(n/\mathcal{S})$ where n is the number of nodes in the system. This is shown in Section 4.

In Section 5 we show that this lower bound is asymptotically tight for moderately small values of $\mathcal{S}$ by extending a simple gradient clock synchronization algorithm described in [14] to the dynamic case. In a static setting, the algorithm of [14] guarantees a local skew of $O(\sqrt{\rho D})$ where ρ is the maximum hardware clock drift. When modeling a dynamic network, we assume that the set of nodes remains fixed, but edges can appear and disappear in a completely arbitrary pattern. If a weak connectivity requirement is satisfied, the algorithm guarantees a global skew of $O(n)$ at all times. Further, for a parameter $\mathcal{S} \geq \sqrt{\rho n}$ and a sufficiently large constant λ, the algorithm guarantees a local skew of at most $\mathcal{S}$ on all edges that are present for at least $\lambda \cdot n/\mathcal{S}$ time. It will be interesting to see whether techniques used in the recent strong static gradient clock synchronization algorithms in [12, 13] can be adapted to the dynamic setting, in order to obtain similar results for smaller values of $\mathcal{S}$.

2 Related Work

Being a fundamental problem, it is not surprising that there is a rich literature on clock synchronization algorithms and lower bounds. Until recently, the work on clock synchronization focused on global synchronization, i.e., on minimizing the maximal clock difference between any two nodes in the system. Essentially all lower bounds on distributed clock synchronization use the *shifting* technique introduced in [15], which exploits uncertainty resulting from unknown message delays, the *scaling* technique from [5], which uses uncertainty that arises as a consequence of different clock rates, or a combination of the two techniques. Using the shifting technique, it is shown in [3] that even if clocks experience no drift, a clock skew of $D/2$ can not be avoided in a network of diameter D. In light of this result, the algorithm described in [20] which guarantees a global skew of $O(D)$ is asymptotically optimal.

A number of related algorithms and lower bounds for varying models and with different properties have been described (see e.g. [1, 2, 7, 18, 19]). The algorithms described in these papers do not guarantee a skew between neighboring nodes that is better than $O(D)$. The gradient clock synchronization problem was introduced in [8], where it is shown that on a path of length D, no clock synchronization algorithm can avoid having a skew of $\Omega(\log D/\log\log D)$ between adjacent nodes. This lower bound has recently been improved to $\Omega(\log D)$ in [13]. The first algorithm to guarantee a non-trivial local skew was described by Locher and Wattenhofer in [14]. The algorithm in [14] guarantees a local skew of $O(\sqrt{\rho D})$ between any two neighbors in a network of diameter D, where ρ denotes the maximal hardware clock drift. The algorithm of [14] forms the basis for the dynamic gradient clock synchronization algorithm described in this paper. For static networks, the upper bound was recently improved to an asymptotically optimal bound of $O(\log D)$ by Lenzen et. al. [12, 13].

Most closely related to the dynamic clock synchronization problem considered in this work are algorithms that cope with faulty nodes (e.g. [4, 5, 11, 17]). While this line of work goes far beyond studying crash failures and describes algorithms that even cope with Byzantine faults, a topic that is out of the scope of the present paper, none of these papers consider a truly dynamic setting. In particular, the results rely on the fact that a considerable part of the network remains non-faulty and stable. Moreover, all the described algorithms and lower bounds focus solely on global synchronization. To the best of our knowledge, the present paper is the first to look at gradient clock synchronization in dynamic networks.

3 Preliminaries

3.1 Notation

Given an undirected static graph $G = (V, E)$, we denote by $\mathcal{P}$ the set of all (undirected) paths in G. For convenience in notation we regard each path $P \in \mathcal{P}$ as a set of edges $P \subseteq E$. We use $\mathcal{P}(u, v)$ to denote all paths between two nodes $u, v \in V$. The distance between two nodes u and v is defined by

$$\mathrm{dist}(u, v) := \min_{P \in \mathcal{P}(u,v)} |P|.$$

The definitions above are used only in the context of a static graph. (We use static graphs in the proof of the lower bound in Section 4). In this work we are often concerned with dynamic graphs, which do not have a static set of edges. We use $V^{(2)} := \{\{u,v\} \mid u,v \in V\}$ to denote the set of all *potential* edges over a static set V of nodes.

3.2 Network Model

We model a dynamic network over a static set V of nodes using Timed I/O Automata (TIOA) [9]. Each node in the network is modelled as a TIOA, and the environment is also modelled as a TIOA. The dynamic behavior of the network is modelled using events of the form $\mathsf{add}(\{u,v\})$ and $\mathsf{remove}(\{u,v\})$ for $u,v \in V$, which correspond to the formation and failure (respectively) of a link between u and v. It is assumed that no edge is both added and removed at the same time.

The history of link formations and failures in a particular execution α, together with an initial set of edges E_0^α, induces a *dynamic graph* $G = (V, E^\alpha)$, where $E^\alpha : \mathbb{R}^+ \to V^{(2)}$ is a function that maps a time $t \geq 0$ to the set of edges (links) that exist in α at time t. We define $E^\alpha(t)$ to be the set of edges that are added no later than time t, and not removed between the last time they are added and time t (inclusive). This includes edges that appear in E_0^α and are not removed by time t. We say that an edge e *exists throughout the interval* $[t_1, t_2]$ in α if $e \in E^\alpha(t_1)$ and e is not removed at any time during the interval $[t_1, t_2]$.

A *static execution* is one in which no edges are added or removed. Formally, α is a static execution if for all $t_1, t_2 \in \mathbb{R}^+$ we have $E^\alpha(t_1) = E^\alpha(t_2)$.

We consider a very general model, in which edges can be inserted or removed arbitrarily, subject only to the following connectivity constraint.

Definition 3.1 (T-interval connectivity). *We say that a dynamic graph* $G = (V, E^\alpha)$ *is* T-interval connected *if for all* $t \geq 0$, *the static subgraph* $G_{[t,t+T]} = (V, E^\alpha|_{[t,t+T]})$ *is connected, where* $E^\alpha|_{[t,t+T]}$ *is the set of all edges that exist throughout the interval* $[t, t+T]$.

In the sequel we omit the superscript α when it is clear from the context.

We assume that nodes do not necessarily find out immediately about edge insertions and removals[1]. Instead, we assume that there is a parameter $\mathcal{D}$, such that if an edge appears or disappears at time t in an execution, and the change is not reversed by time $t + \mathcal{D}$, the endpoints of the edge find out no later than time $t + \mathcal{D}$. Transient link formations or failures, which do not persist for $\mathcal{D}$ time, may or may not be detected by the nodes affected. We model the discovery by node u of a link formation or failure $X \in \{\mathsf{add}(\{u,v\}), \mathsf{remove}(\{u,v\}) \mid v \in V\}$ by an event $\mathsf{discover}(X)$ that occurs at node u. (A $\mathsf{discover}(X)$ event is always preceded by event X itself.)

We also assume reliable FIFO message delivery[2], with message delays bounded by $\mathcal{T}$. This is modelled using events of the form $\mathsf{send}(u,v,m)$ and $\mathsf{receive}(u,v,m)$ that occur at node u. If node

[1]Otherwise reference-broadcast-style synchronization would be possible using these events [6]. In general, whenever some event is guaranteed to occur at two nodes at the same (or roughly the same) time, the nodes can use this event to synchronize their clocks by exchanging the clock values they each had at the time the event occurs. This type of synchronization circumvents shifting and scaling lower bounds of the type we use in Section 4.

[2]We assume FIFO message delivery to simplify the presentation, but this is not necessary. In the algorithm of Section 5 each message carries a timestamp. If FIFO is not assumed, nodes can remember the latest timestamp they have seen from each neighbor, and discard messages that carry older timestamps.

u sends a message to node v at time t, the environment guarantees the following. If edge $\{u, v\}$ exists throughout the interval $[t, t + \mathcal{T}]$, then node v is guaranteed to receive the message no later than time $t+\mathcal{T}$. If edge $\{u, v\}$ exists at time t but is removed at some point in the interval $[t, t+\mathcal{T}]$, there are two possible outcomes: either the message is delivered before the edge is removed, or the message is not delivered and node u discovers the edge removal no later than time $t + \mathcal{D}$. Finally, if edge $\{u, v\}$ does not exist at time t, the message is not delivered, and node u discovers that the edge does not exist no later than time $t + \mathcal{D}$. These definitions correspond to an abstract version of MAC layer acknowledgements.

In the sequel we assume that $\mathcal{D} > \mathcal{T}$, that is, nodes do not necessarily find out about changes to the network within $\mathcal{T}$ time units. This is a reasonable assumption because even if nodes transmit very frequently, as much as $\mathcal{T}$ time may pass without any message being received on a link, leaving the link formation or failure undiscovered.

3.3 The Clock Synchronization Problem

In the clock synchronization problem, each node $u \in V$ has access to a continuous *hardware clock* $H_u(t)$, which may progress at a different rate than real time. The hardware clocks suffer from *bounded drift* ρ: although they progress at a variable rate, their rate is always between $1 - \rho$ and $1 + \rho$ the rate of real time, so that for any node u and times $t_1 < t_2$ we have

$$(1 - \rho)(t_2 - t_1) \leq H_u(t_2) - H_u(t_1) \leq (1 + \rho)(t_2 - t_1).$$

For simplicity we assume that at the beginning of any execution the hardware clock values are all 0. We also assume for the analysis that the hardware clocks are differentiable.

The goal of a dynamic clock synchronization algorithm (DCSA) is to output a *logical clock* $L_u(t)$ such that the logical clocks of different nodes are close to each other. In particular we consider two requirements. A *global skew constraint* bounds the difference between the logical clocks of any two nodes in the network at all times in the execution. A *dynamic local skew constraint* requires that if an edge exists for sufficiently long, the skew between the two endpoints of the edge should not be too large. These requirements are formally defined as follows.

Definition 3.2 (Global skew). *A DCSA guarantees a* global skew of $\bar{\mathcal{G}}(n)$ *if in any execution of the algorithm in a network of n nodes, for any two nodes u, v and time $t \geq 0$ we have*

$$L_u(t) - L_v(t) \leq \bar{\mathcal{G}}(n).$$

To represent the local skew guaranteed by the algorithm after an edge has existed for some time, we use a function $s(n, I, t)$, where n is the number of nodes, I is the initial skew on the edge when it appeared, and t is the time that has passed since the edge appeared. The skew function must satisfy the following technical requirements.

Definition 3.3 (Skew function). *A function $s : \mathbb{N} \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ (where $\mathbb{R}^+$ are the positive reals) is said to be a* skew function *if the following conditions hold.*

1. *The function $s(n, I, t)$ is non-decreasing in I and non-increasing in t; and*
2. *For all $n \in \mathbb{N}$, $I \in \mathbb{R}^+$, the limit $\lim_{t\to\infty} s(n, I, t)$ is defined and finite; and*
3. *For all $I_1, I_2 \in \mathbb{R}^+$ we have*

$$\lim_{t\to\infty} s(n, I_1, t) = \lim_{t\to\infty} s(n, I_2, t).$$

We note that the third requirement above essentially means that the skew on any edge converges to the same stable local skew, regardless of the initial skew on that edge. Now let us define what it means for an algorithm to have a local skew of s.

Definition 3.4 (Dynamic local skew)**.** *A DCSA guarantees a* dynamic local skew *of $s : \mathbb{N} \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, where s is a skew function, if in every execution of the algorithm in a network over n nodes, for any edge $e = \{u, v\}$ and times $t_1 \leq t_2$ such that e exists throughout the interval $[t_1, t_2]$ in the execution, we have*

$$|L_u(t_2) - L_v(t_2)| \leq s(n, |L_u(t_1) - L_v(t_1)|, t_2 - t_1).$$

Definition 3.5 (Stabilizing DCSA)**.** *A DCSA $\mathcal{A}$ is said to be* stabilizing *if there is a skew function s such that $\mathcal{A}$ guarantees a dynamic local skew of s. In this case we say that $\mathcal{A}$ guarantees a* stable local skew *of $\bar{s}(n) = \lim_{t\to\infty} s(n, I, t)$ for some $I \in \mathbb{R}^+$ (recall that this limit is the same for all $I \in \mathbb{R}^+$).*

Finally, logical clocks have to be strictly increasing and are thus not allowed to temporarily stop. In particular, we require the rate of each logical clock to be at least half the rate of real time; that is, for any node u and times $t_1 \leq t_2$ we require

$$L_u(t_2) - L_u(t_1) \geq \frac{1}{2}(t_2 - t_1).$$

4 Lower Bound

We begin our analysis of dynamic clock synchronization algorithms with a lower bound on the time needed to adjust the local skew on a newly formed edge. Specifically, we show that for every sufficiently large initial skew I (a small constant times the stable local skew $\bar{s}(n)$ suffices), the time needed to reduce the skew by a factor of $\Theta(n/\bar{\mathcal{G}}(n))$ is $\Omega(n/\bar{s}(n))$. Thus, there is an inherent tradeoff between the stable skew guaranteed by the algorithm and the time the algorithm requires to reduce the skew on new edges.

Theorem 4.1. *Let $\mathcal{A}$ be a stabilizing DCSA that guarantees a global skew of $\bar{\mathcal{G}}(n)$ and a dynamic local skew of s with a stable local skew of $\bar{s}(n) = o(n)$. Then there exist constants $\lambda, \zeta \geq 0$ such that for all sufficiently large n and for all $I \geq 3\bar{s}(n)$ we have*

$$s(n, I, \lambda \cdot \frac{n}{\bar{s}(n)}) \geq \zeta \frac{n}{\bar{\mathcal{G}}(n)} \cdot I.$$

Most static clock synchronization algorithms in the literature guarantee a global skew of $O(D)$ in networks of diameter D. Moreover, all gradient clock synchronization algorithms of which we are aware *rely* on having a global skew of $O(D)$ in order to prove their gradient property [14, 12, 13].

In dynamic graphs the diameter is undefined, and the natural extention is to require a global skew of $\bar{\mathcal{G}}(n) = O(n)$. This is achieved by the algorithm presented in Section 5, and here, too, this fact is used to prove the local skew guarantee. It therefore seems most interesting to consider algorithms with this global skew guarantee. For such algorithms, Theorem 4.1 shows that it takes $\Omega(n/\bar{s}(n))$ time to reduce the initial skew on a new edge by a *constant* factor.

Corollary 4.2. *Let $\mathcal{A}$ be a stabilizing DCSA that guarantees a global skew of $\bar{\mathcal{G}}(n) = O(n)$ and a dynamic local skew of s with a stable local skew of $\bar{s}(n) = o(n)$. Then there exist constants $\lambda, \zeta \geq 0$ such that for all sufficiently large n and for all $I \geq 3\bar{s}(n)$ we have*

$$s(n, I, \lambda \cdot \frac{n}{\bar{s}(n)}) \geq \zeta \cdot I.$$

Note that the lower bound asserts the *existence* of a time $\lambda \cdot n/\bar{s}(n)$ after which the skew is reduced by no more than a constant factor (ζ). It is not necessarily the case that *for all* times $t = \Theta(n/\bar{s}(n))$ we still have a large local skew. Indeed, the algorithm we give in Section 5 makes a sharp transition after an edge exists for $\Theta(n/\bar{s}(n))$ time: before the transition the algorithm provides no non-trivial local skew guarantee on the edge (beyond what the global skew already guarantees), and afterwards the algorithm guarantees the stable skew, $\bar{s}(n)$. The trade-off shows that this transition can only be made when the edge exists for $\Theta(n/\bar{s}(n))$ time; essentially, it asserts that the algorithm must wait $\Theta(n/\bar{s}(n))$ time before it acts to drastically reduce the skew on a new edge.

4.1 Proof overview

The main idea in the proof of Theorem 4.1 is to show that because of the local skew guarantee, even nodes that are distant from a new edge may prevent the skew on it from being reduced. These distant nodes require time to "find out" about the new edge, and thus they limit the speed with which the algorithm can react.

As an informal overview, consider the network shown in Fig. 1(a), which consists of two parallel chains, A and B, joined at both ends w_0 and w_n. The two chains exist throughout the execution; new edges are eventually added along the B-chain, but no edges are ever removed. We wait until the algorithm has stabilized to some degree. For the purpose of this overview, let us suppose that we reach some time T_s such that for all $t \geq T_s$, the local skew guarantee is $s(n, 0, t) \leq \bar{s}(n)$. (This is an over-simplification, since the local skew guarantee only converges to $\bar{s}(n)$ in the limit; however, we can get arbitrarily close to $\bar{s}(n)$ by waiting sufficiently long.)

Next we select two sufficiently large times $T_1, T_2 \geq T_s$ where $T_2 - T_1 = \lambda(n/\bar{s}(n))$ (for a constant λ). Our goal is to add new edges at time T_1, each with a skew of at most I (see Fig. 1(b)), and cause at least one new edge to still have a skew of $\Omega(I \cdot n/\bar{\mathcal{G}}(n))$ at time T_2. This is achieved by

1. Adding $O(\bar{\mathcal{G}}(n)/I)$ new edges at time T_1, and
2. Creating a skew of $\Omega(n)$ between w_0 and w_n at time T_2.

The *average* skew on the new edges at time T_2 must then be $\Omega(I \cdot n/\bar{\mathcal{G}}(n))$, which implies that at least one new edge has a skew of $\Omega(I \cdot n/\bar{\mathcal{G}}(n))$ at time T_2.

First we show how to create $\Omega(n)$ skew between w_0 and w_n at time T_2. Note that because of the new edges, the distance between w_0 and w_n at time T_2 is reduced to $O(\bar{\mathcal{G}}(n)/I)$. Standard shifting arguments create a skew proportional to the distance, and this is not enough in our case. Hence we must use a more roundabout way. Essentially, we want to show that w_0 and w_n cannot react quickly enough to the new edges, or they would violate the local skew guarantee with respect to some distant nodes u, v on the A-chain that have not yet discovered the new edges.

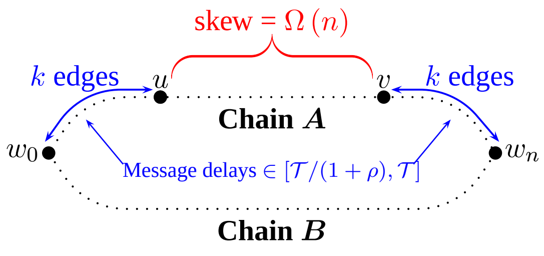
To this end, we choose two nodes u, v on the A-chain such that $\mathrm{dist}(w_0, u) = \mathrm{dist}(w_n, v) = k$, where $k = \Theta\left(n/\bar{s}(n)\right)$, and where $\mathrm{dist}(u, v) = \Omega(n)$. Nodes u and v are "shielded" from events on the B-chain by large message delays (see Fig. 1(a)). We first consider an execution α in which the network is static and no new edges are added at time T_1. Using a modified shifting argument (Lemma 4.3 below), we create a skew of $\Omega(n)$ between u and v at time T_2 in α, while keeping delays of at least $\mathcal{T}/(1+\rho)$ on all links between w_0 and u and between w_n and v.

Nodes u, v act as a barrier between w_0 and w_n: the local skew guarantee implies that the clocks of w_0 and w_n cannot be more than $k \cdot \bar{s}(n) = \Theta(n)$ removed from the clocks of u and v respectively. Hence, whenever the skew between u and v is $\Omega(n)$, the skew between w_0 and w_n is also $\Omega\left(n\right)$ (see Fig. 1(d)).

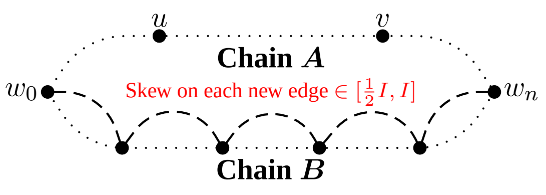
Finally, we create a new execution β, which is identical to α until time T_1. At time T_1 we add new edges as shown in Fig. 1(b). Recall that the skew between w_0 and w_n is bounded by $\bar{\mathcal{G}}(n)$ at all times, and at time T_1 the skew on each edge of the B-chain is at most $\bar{s}(n)$. Thus, it is possible to find a set of edges as shown in Fig. 1(b), such that each edge carries a skew in the range $[I - \bar{s}(n), I]$, and the skews (in absolute value) sum to at most $\bar{\mathcal{G}}(n)$. When $I \geq 2\bar{s}(n)$, the number of edges required is at most $2\bar{\mathcal{G}}(n)/I$.

By time T_2 in β, the skew on each new edge must be reduced to at most $s(n, I, T_2 - T_1) = s(n, I, \lambda(n/\bar{s}(n)))$, and consequently the total skew between nodes w_0 and w_n cannot exceed $(2\bar{\mathcal{G}}(n)/I) \cdot s(n, I, \lambda(n/\bar{s}(n)))$ (see Fig. 1(c)).

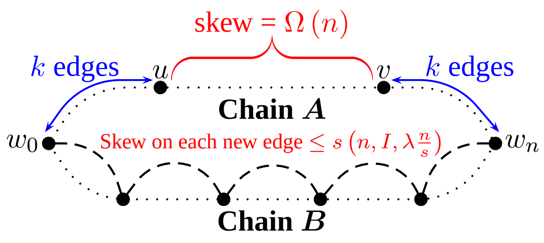
However, in addition to this upper bound on the skew, we can also show that the skew between w_0 and w_n at time T_2 in β is at least $\Omega(n)$: nodes u and v cannot distinguish between α and β until time T_2, since they are shielded from the B-chain by $k = \Theta(n/\bar{s}(n))$ edges with large message delays. At time T_2 in β, nodes u, v have the same skew of $\Omega(n)$ that they have in α, and as argued above, this implies that w_0 and w_n also have $\Omega(n)$ skew between them. Combining the upper and lower bound on the skew between w_0 and w_n we see that $s(n, I, \lambda(n/\bar{s}(n)))$ cannot be less than $\Omega(I \cdot (n/\bar{\mathcal{G}}(n)))$. This concludes the proof.



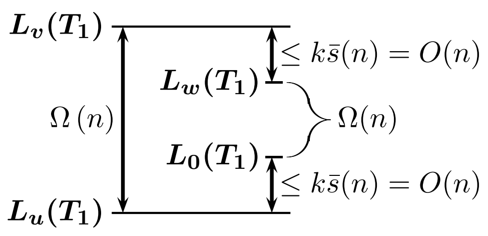
(a) Execution α at time T_2.



(b) Execution β at time T_1 (new edges shown as dashed lines)



(c) Execution β at time T_2



(d) The logical clocks of w_0, u, v, w_n at time T_2 in executions α and β (assuming $L_u(T_1) \le L_v(T_1)$)

Figure 1: Illustrations for the proof of Theorem 4.1

4.2 Formal proof of the tradeoff

As explained above, as part of the proof we create a large skew between certain nodes, while also maintaining large message delays in parts of the network. The skew is created using shifting (see, e.g., [16]). A standard shifting argument shows that two nodes cannot avoid having a large skew between them, by adjusting message delays so that the nodes cannot tell the difference between an execution in which the skew is large and an execution in which it is not. In the resulting execution the message delays on some links are zero, and in the standard construction it is not possible to control which links these will be.

In our proof we require large message delays along certain specific links. A straightforward modification of the argument from [1] and [8] allows us to create large skews while maintaining a predefined pattern of message delays. The following definitions capture this notion more formally.

Definition 4.1 (Delay pattern)**.** *Given a network over a set V of nodes, a* delay pattern *for N is a pair $M = (E^{\mathrm{C}}, P)$, where $E^{\mathrm{C}} \subseteq V^{(2)}$ is a set of* constrained links *and $P : E^{\mathrm{C}} \rightarrow [0, \mathcal{T}]$ is a* delay pattern *assigning a message delay to every constrained link.*

Definition 4.2 (Constrained executions)**.** *An execution is said to be M-constrained until time t, for a delay pattern $M = (E^{\mathrm{C}}, P)$, if the delay of messages sent on a link $e \in E^{\mathrm{C}}$ and received by time*

*t is in the range $[\frac{1}{1+\rho}P(e), P(e)]$. We say that an execution is M-*constrained *if for all times $t \geq 0$ the execution is M-constrained until time t.*

Definition 4.3 (Flexible distance)**.** *Given a delay pattern $M = (E^{\mathrm{C}}, P)$, the M-*flexible distance *between two nodes $u, v \in V$, denoted $\mathrm{dist}_M(u, v)$, is defined as the minimum number of unconstrained edges on any path between u and v.*

Lemma 4.3 (Masking Lemma)**.** *Let $N = (V, E)$ be a static network, and let $M = (E^{\mathrm{C}}, P)$ be a delay pattern for N. For any $u, v \in V$ and for any time $T_0 > \mathcal{T} \cdot \mathrm{dist}_M(u, v)(1 + 1/\rho)$, there is a time $t \geq T_0$ and an M-constrained static execution in which*

$$|L_u(t) - L_v(t)| \geq \frac{1}{4}\mathcal{T}\,\mathrm{dist}_M(u, v).$$

Proof. The proof is a fairly straightforward application of the scaling and shifting proof techniques (see, e.g., [5] and [15]). It is similar to the proof from [1], where it is shown that the worst-case skew between any two nodes in the network is proportional to the shortest-path distance between them. However, in the current proof we use only unconstrained links to build up the skew, and as a result we can build up a skew between two nodes that is proportional to their *flexible* distance.

Definitions and setup. Let $d = \mathrm{dist}_M(u, v)$ and let $D = \max_{w \in V} \mathrm{dist}_M(u, w)$. We partition the graph into layers $L_0, \ldots, L_D$, where each layer is given by $L_i = \{w \in V \mid \mathrm{dist}_M(u, w) = i\}$. In particular, $L_0 = \{u\}$ and $v \in L_d$. We define a total order $\prec$ on nodes by $x \prec y$ iff $\mathrm{dist}_M(u, x) < \mathrm{dist}_M(u, y)$. We write $x \equiv y$ if $\mathrm{dist}_M(u, x) = \mathrm{dist}_M(u, y)$, and $x \preceq y$ if $x \prec y$ or $x \equiv y$.

Note the following properties of the relations defined above: for any edge $\{x, y\} \in E$,

1. If $\{x, y\} \in E^{\mathrm{C}}$ then $x \equiv y$: if $\{x, y\} \in E^{\mathrm{C}}$, then any path from u to x can be extended to a path from u to y that has the same number of unconstrained edges, and vice-versa. It follows that $\mathrm{dist}_M(u, x) = \mathrm{dist}_M(u, y)$.
2. If $x \prec y$ then $\mathrm{dist}_M(u, x) = \mathrm{dist}_M(u, y) - 1$.

We define two executions, α and β. In α, all hardware clocks progress at the rate of real time, and message delays on each edge e are defined as follows:

- If $e \in E^{\mathrm{C}}$ then messages on e are delayed by $P(e)$;
- If $e = \{x, y\} \in E \setminus E^{\mathrm{C}}$ and $x \prec y$, then messages from x to y are delayed by $\mathcal{T}$ and messages from y to x are delayed by 0.
- If $e = \{x, y\} \in E \setminus E^{\mathrm{C}}$ and $x \equiv y$, then messages from x to y and vice-versa are delayed by 0.

Execution α is M-constrained by definition.

In execution β, we slowly increase the skew of the hardware clocks of nodes at different layers, while keeping the difference small enough that it can be disguised by altering message delays. We begin by keeping u's hardware clock rate at 1 and letting nodes in layers $L_1, \ldots, L_D$ run at a rate of $1 + \rho$, until a skew of $\mathcal{T}$ is built up between the hardware clock of u and any node in L_1. Then we let u and all L_1-nodes run at a rate of 1 while nodes in layers $L_2, \ldots, L_D$ run at a rate of $1 + \rho$, until a skew of $\mathcal{T}$ is built up between nodes in L_1 and nodes in L_2. At this point the hardware clock skew between u and any node in L_2 is $2\mathcal{T}$. We continue in this manner until we have built up a skew of $d \cdot \mathcal{T}$ between u and any node in layer L_d, including v.

More formally, β is constructed as a sequence of segments $\beta_0\beta_1 \ldots \beta_{d-1}\beta_*$, where

- $\beta_* := \left[\frac{d}{\rho}\mathcal{T}, \infty\right]$ is an infinite suffix, and
- For all $0 \leq i \leq d-1$, $\beta_i := \left[\frac{i}{\rho}\mathcal{T}, \frac{i+1}{\rho}\mathcal{T}\right)$ is a finite segment of duration $\mathcal{T}/\rho$. (This is the time required to build a skew of $\mathcal{T}$ between the hardware clocks of nodes in adjacent layers when one node runs at a rate of 1 and the other at $1 + \rho$.)

In β_0 and β_* all hardware clocks run at a rate of 1 and all messages are delivered with no delay. In each middle segment β_i, the hardware clock rate of a node $x \in L_j$ is given by

$$\frac{d}{dx}H_x^\beta = \begin{cases} 1 & \text{if } i \leq j, \\ 1 + \rho & \text{otherwise.} \end{cases}$$

Message delays throughout β are adjusted so that β is indistinguishable from α to all nodes. In particular, if $t_s^\alpha, t_s^\beta, t_r^\alpha$ and t_r^β are times such that

1. At time t_s^α in α node x sends a message that node y receives at time t_r^α, and
2. $H_x^\alpha(t_s^\alpha) = H_x^\beta(t_s^\beta)$ and $H_y^\alpha(t_r^\alpha) = H_y^\beta(t_r^\beta)$,

then in β, node x will send the same message at time t_s^β and node y will receive it at time t_r^β.

From the definition of β, at any time t we have

$$H_x^\beta(t) = \begin{cases} (1+\rho)t & \text{if } t \in \beta_i \text{ where } \operatorname{dist}_M(u,x) > i, \\ t + \mathcal{T} \cdot \operatorname{dist}_M(u,x) & \text{otherwise.} \end{cases}$$

That is,

$$H_x^\beta(t) = t + \min\left\{\rho t, \mathcal{T} \cdot \operatorname{dist}_M(u,x)\right\}. \tag{4.1}$$

In α, where all hardware clocks run at a rate of 1, $H_x^\alpha(t) = t$ for all $x \in V$.

β **is an** M**-constrained execution.** Next we claim that β is a legal M-constrained execution, that is, all message delays are in the range $[0, \mathcal{T}]$, and for all $e \in E^{\mathrm{C}}$, message delays on e are in the range $\left[\frac{1}{1+\rho}P(e), P(e)\right]$. Consider a message sent by node x at time t_s^β and received by node y at time t_r^β. Let t_s^α, t_r^α be the send and receive times (respectively) of the same message in execution α; that is,

$$H_x^\beta(t_s^\beta) = H_x^\alpha(t_s^\alpha) = t_s^\alpha \qquad \text{and} \qquad H_y^\beta(t_r^\beta) = H_y^\alpha(t_r^\alpha) = t_r^\alpha.$$

Using (4.1) we obtain

$$\begin{aligned} t_r^\alpha - t_s^\alpha &= H_y^\beta(t_r^\beta) - H_x^\beta(t_s^\beta) = \\ &= t_r^\beta + \min\left\{\rho t_r^\beta, \mathcal{T} \cdot \mathrm{dist}_M(u,y)\right\} - t_s^\beta - \min\left\{\rho t_s^\beta, \mathcal{T} \cdot \mathrm{dist}_M(u,x)\right\}. \end{aligned} \tag{4.2}$$

We divide into cases.

- $\rho t_s^\beta \le \mathcal{T} \cdot \mathrm{dist}_M(u,x)$ and $\rho t_r^\beta \le \mathcal{T} \cdot \mathrm{dist}_M(u,y)$. In this case (4.2) implies

  $$t_r^\alpha - t_s^\alpha = (1+\rho)(t_r^\beta - t_s^\beta).$$

  By the definition of α we have $t_r^\alpha - t_s^\alpha \in [0, \mathcal{T}]$, and hence $t_r^\beta - t_s^\beta \in [0, \mathcal{T}]$ as well. In addition, if $\{x,y\} \in E^{\mathrm{C}}$ then $t_r^\alpha - t_s^\alpha = P(e)$ (again by definition of α); in this case we have $t_r^\beta - t_s^\beta = P(e)/(1+\rho) \in [P(e)/(1+\rho), P(e)]$, as required.

- $\rho t_s^\beta > \mathcal{T} \cdot \mathrm{dist}_M(u,x)$ and $\rho t_r^\beta > \mathcal{T} \cdot \mathrm{dist}_M(u,y)$. In this case (4.2) implies

  $$t_r^\alpha - t_s^\alpha = t_r^\beta - t_s^\beta + \mathcal{T}(\mathrm{dist}_M(u,y) - \mathrm{dist}_M(u,x)).$$

  If $\{x,y\} \in E^{\mathrm{C}}$ or $x \equiv y$, then $\mathrm{dist}_M(u,x) = \mathrm{dist}_M(u,y)$, and hence $t_r^\beta - t_s^\beta = t_r^\alpha - t_s^\alpha$. Thus, the message delay in β is the same as in α. The delay in α is legal and respects the delay pattern, and the same holds for the delay in β.

  Otherwise, either $x \prec y$ and $\mathrm{dist}_M(u,y) - \mathrm{dist}_M(u,x) = 1$, or $y \prec x$ and $\mathrm{dist}_M(u,y) - \mathrm{dist}_M(u,x) = -1$. In the first case we have $t_r^\beta - t_s^\beta = t_r^\alpha - t_s^\alpha - \mathcal{T} = \mathcal{T} - \mathcal{T} = 0$ (by definition of α), and in the second case, $t_r^\beta - t_s^\beta = t_r^\alpha - t_s^\alpha + \mathcal{T} = 0 + \mathcal{T} = \mathcal{T}$. In both cases the delays are legal.

- $\rho t_s^\beta > \mathcal{T} \cdot \mathrm{dist}_M(u,x)$ and $\rho t_r^\beta \le \mathcal{T} \cdot \mathrm{dist}_M(u,y)$. In this case (4.2) implies

  $$t_r^\alpha - t_s^\alpha = t_r^\beta - t_s^\beta + \rho t_r^\beta - \mathcal{T} \cdot \mathrm{dist}_M(u,x).$$

  Since $\rho t_r^\beta \le \mathcal{T} \cdot \mathrm{dist}_M(u,y)$ and $\mathcal{T} \cdot \mathrm{dist}_M(u,x) < \rho t_s^\beta$, we can write

  $$(1+\rho)(t_r^\beta - t_s^\beta) < t_r^\alpha - t_s^\alpha \le t_r^\beta - t_s^\beta + \mathcal{T}(\mathrm{dist}_M(u,y) - \mathrm{dist}_M(u,x)).$$

If $\{x,y\} \in E^{\mathrm{C}}$ or $x \equiv y$, then $\mathrm{dist}_M(u,x) = \mathrm{dist}_M(u,y)$, and we obtain

$$(1+\rho)(t_r^\beta - t_s^\beta) < t_r^\alpha - t_s^\alpha \le t_r^\beta - t_s^\beta,$$

which is impossible, because $t_r^\alpha - t_s^\alpha \ge 0$ and $\rho \ge 0$.

Otherwise, if $x \prec y$, then $\mathrm{dist}_M(u,y) = \mathrm{dist}_M(u,x) + 1$, and we have $t_r^\alpha - t_s^\alpha = \mathcal{T}$ and

$$(1+\rho)(t_r^\beta - t_s^\beta) < \mathcal{T} \le t_r^\beta - t_s^\beta + \mathcal{T}.$$

It follows that $t_r^\beta - t_s^\beta \in [0, \mathcal{T}/(1+\rho)) \subseteq [0, \mathcal{T}]$.

Finally, if $y \prec x$, then $\mathrm{dist}_M(u,x) = \mathrm{dist}_M(u,y) + 1$, and we have $t_r^\alpha - t_s^\alpha = 0$ and

$$(1+\rho)(t_r^\beta - t_s^\beta) < 0 \le t_r^\beta - t_s^\beta - \mathcal{T}.$$

But this is impossible, because it implies both $t_r^\beta - t_s^\beta < 0$ and $t_r^\beta - t_s^\beta \ge \mathcal{T}$.

- $\rho t_s^\beta \le \mathcal{T} \cdot \mathrm{dist}_M(u,x)$ and $\rho t_r^\beta > \mathcal{T} \cdot \mathrm{dist}_M(u,y)$. In this case (4.2) implies

$$t_r^\alpha - t_s^\alpha = t_r^\beta - t_s^\beta + \mathcal{T} \cdot \mathrm{dist}_M(u,y) - \rho t_s^\beta.$$

This time, we can re-write this to obtain

$$t_r^\beta - t_s^\beta + \mathcal{T}(\mathrm{dist}_M(u,y) - \mathrm{dist}_M(u,x)) \le t_r^\alpha - t_s^\alpha < (1+\rho)(t_r^\beta - t_s^\beta).$$

If $\{x,y\} \in E^{\mathrm{C}}$ or $x \equiv y$, then again $\mathrm{dist}_M(u,x) = \mathrm{dist}_M(u,y)$, and we have

$$t_r^\beta - t_s^\beta \le t_r^\alpha - t_s^\alpha < (1+\rho)(t_r^\beta - t_s^\beta).$$

If $\{x,y\} \in E^{\mathrm{C}}$ then this implies that $t_r^\beta - t_s^\beta \in [P(e)/(1+\rho), P(e)] \subseteq [0, \mathcal{T}]$, as required. Otherwise, if $x \equiv y$ but $\{x,y\} \notin E^{\mathrm{C}}$, then we have both $t_r^\beta - t_s^\beta \le 0$ and $t_r^\beta - t_s^\beta > 0$, which is impossible. (Recall that for this case we defined $t_r^\alpha - t_s^\alpha = 0$.)

If $x \prec y$ and $\mathrm{dist}_M(u,y) = \mathrm{dist}_M(u,x) + 1$, then we have

$$t_r^\beta - t_s^\beta + \mathcal{T} \le \mathcal{T} < (1+\rho)(t_r^\beta - t_s^\beta),$$

which is a contradiction.

And finally, if $y \prec x$ and $\mathrm{dist}_M(u,x) = \mathrm{dist}_M(u,y) + 1$, then

$$t_r^\beta - t_s^\beta - \mathcal{T} \le 0 < (1+\rho)(t_r^\beta - t_s^\beta)$$

and it follows that $t_r^\beta - t_s^\beta \in (0, \mathcal{T}]$.

The skew between u and v. It remains to show that in either α or β, the skew between u and v at some time $t \geq T_0$ is large.

Let $T_1 := T_0 + \mathcal{T} \cdot \mathrm{dist}_M(u,v)$. Since $T_0 \geq (1/\rho)\,\mathrm{dist}_M(u,v) \cdot \mathcal{T}$, at time T_0 we have

$$\begin{aligned} H_v^\beta(T_0) &= T_0 + \min\{\rho T_0, \mathcal{T} \cdot \mathrm{dist}_M(u,v)\} = T_0 + \mathcal{T} \cdot \mathrm{dist}_M(u,v) = \\ &= H_v^\alpha(T_0 + \mathcal{T} \cdot \mathrm{dist}_M(u,v)) = H_v^\alpha(T_1), \end{aligned}$$

while

$$H_u^\beta(T_0) = T_0 + \min\{\rho T_0, \mathcal{T} \cdot \mathrm{dist}_M(u,u)\} = T_0 = H_u^\alpha(T_0).$$

No node in the network can distinguish between α and β, and consequently, for all nodes $w \in V$ and times t_1, t_2 we have $L_w^\alpha(t_1) = L_w^\beta(t_2)$ iff $H_w^\alpha(t_1) = H_w^\beta(t_2)$. In particular,

$$L_u^\alpha(T_0) = L_u^\beta(T_0) \tag{4.3}$$

and

$$L_v^\alpha(T_1) = L_v^\beta(T_0). \tag{4.4}$$

Since u increases its logical clock at a rate of at least $1/2$,

$$L_u^\alpha(T_1) \geq L_u^\alpha(T_0) + \frac{1}{2}(T_1 - T_0) \overset{(4.3)}{=} L_u^\beta(T_0) + \frac{1}{2}\mathcal{T} \cdot \mathrm{dist}_M(u,v), \tag{4.5}$$

and subtracting (4.4) from (4.5) yields

$$L_u^\alpha(T_1) - L_v^\alpha(T_1) \geq L_u^\beta(T_0) - L_v^\beta(T_0) + \frac{1}{2}\mathcal{T} \cdot \mathrm{dist}_M(u,v). \tag{4.6}$$

This implies that either $|L_u^\alpha(T_1) - L_v^\alpha(T_1)| \geq \frac{1}{4}\mathcal{T} \cdot \mathrm{dist}_M(u,v)$, or $\left|L_u^\beta(T_0) - L_v^\beta(T_0)\right| \geq \frac{1}{4}\mathcal{T} \cdot \mathrm{dist}_M(u,v)$. Since $T_1 \geq T_0$ and both executions are M-constrained, this proves the claim.

□

The following technical lemma is used in the proof of Theorem 4.1 to select the new edges that appear along the B-chain.

Lemma 4.4. *Let $X = x_1, \ldots, x_n$ be a sequence of numbers where $x_1 \leq x_n$ and for all $1 \leq i < n$, $|x_i - x_{i+1}| \leq d$ for some $d > 0$. Then for any $c > d$, there is a subsequence $X' = x_{i_1}, \ldots, x_{i_m} \subseteq X$ such that*

1. *$m \leq \frac{x_n - x_1}{c-d} + 1$, and*
2. *for all $1 \leq j \leq m-1$ we have $\left|x_{i_{j+1}} - x_{i_j}\right| \in [c-d, c]$.*

Proof. We construct a sequence $i_1, i_2, \ldots$ inductively, starting with $i_1 := 1$. Given i_j, we define

$$i_{j+1} := \min \left(\{n\} \cup \left\{ \ell \mid i_j < \ell < n \text{ and } x_\ell - x_{i_j} \geq c - d \text{ and } x_\ell \leq x_n \right\} \right) \tag{4.7}$$

The sequence $i_1, i_2, \ldots$ is strictly increasing, and eventually it reaches n and remains there. Let $m = \max\{j \mid i_j < n\}$. The sequence we return is $X' = x_{i_1}, \ldots, x_{i_m}$.

By construction, $x_1 = x_{i_1} \leq x_{i_2} \leq \ldots \leq x_{i_m} \leq x_n$, and for all $1 \leq i \leq m-1$ we have $x_{i_{j+1}} - x_{i_j} \geq c - d > 0$. It remains to prove the following.

1. $m \leq \frac{x_n - x_1}{c-d} + 1$: because

$$x_n - x_1 \geq x_{i_m} - x_{i_1} = \sum_{1 \leq j \leq m-1} \left(x_{i_{j+1}} - x_{i_j} \right) \geq (m-1) \cdot (c-d).$$

2. For all $1 \leq j \leq m-1$ we have $\left| x_{i_{j+1}} - x_{i_j} \right| \in [c-d, c]$: since $x_{i_{j+1}} - x_{i_j} \geq c - d > 0$, we need only to show that $x_{i_{j+1}} - x_{i_j} \leq c$. We consider two cases.

   I. $i_{j+1} = i_j + 1$: in this case we already know that $|x_{i_{j+1}} - x_{i_j}| \leq d$. Since $c > d$ the claim follows.

   II. $i_{j+1} > i_j + 1$: let $\ell > i_j$ be the minimal index such that $x_\ell - x_{i_j} \geq c - d$. By construction, $i_{j+1} > i_j$ is the minimal index that satisfies both $x_{i_{j+1}} - x_{i_j} \geq c - d$ and $x_{i_{j+1}} \leq x_n$; hence, $i_{j+1} \geq \ell$, and if $i_{j+1} > \ell$ then $x_\ell > x_n$. It follows that $x_{i_{j+1}} \leq x_\ell$. Since ℓ is the minimal index for which $x_\ell - x_{i_j} \geq c - d$, for index $\ell - 1$ we have $x_{\ell-1} - x_{i_j} < c - d$. In addition, $x_\ell - x_{\ell-1} \leq d$. Together we have $x_{i_{j+1}} - x_{i_j} \leq x_\ell - x_{i_j} = x_\ell - x_{\ell-1} + x_{\ell-1} - x_{i_j} \leq d + c - d = c$, as required.

□

Now we are ready to prove the tradeoff theorem.

Proof of Theorem 4.1. Let $\delta = \frac{\mathcal{T}}{128}$ and $\xi = 1 + \frac{\mathcal{T}}{3\bar{s}(n)}$, and define $k = \delta \frac{n}{\bar{s}(n)}$. We assume that n is large enough that the following requirements are satisfied.

- $k \geq 1$: since $\bar{s}(n) = o(n)$, we can choose n large enough so that $\bar{s}(n) \leq \delta n$ and $k \geq 1$.
- $\bar{s}(n) \geq \mathcal{T}$: since $\bar{s}(n) = \Omega(\mathcal{T} \cdot \log n)$ in a network with diameter $\Omega(n)$ [13], for sufficiently large n we have $\bar{s}(n) \geq \mathcal{T}$.
- $\xi \in (1, \frac{4}{3}]$: this follows from the previous requirement.
- $n/2 - 2(k+1) > 0$: it is sufficient to require $\bar{s}(n) > 4\delta$, which is implied by $\bar{s}(n) \geq \mathcal{T}$.

Setup. Consider the network N shown in Fig. 1(a), over nodes $V = \{w_0, w_n\} \cup (I_A \times \{A\}) \cup (I_B \times \{B\})$ (here A and B are merely symbols used to distinguish the nodes of the two chains), where

$$I_A = \{1, \ldots, \lfloor n/2 \rfloor - 1\}\,, \text{ and}$$
$$I_B = \{1, \ldots, \lceil n/2 \rceil - 1\}\,.$$

For the sake of convenience we also use $\langle 0, A\rangle$ and $\langle 0, B\rangle$ to refer to node w_0, and we use $\langle \lfloor n/2 \rfloor, A\rangle$ and $\langle \lceil n/2 \rceil, B\rangle$ to refer to node w_n.

Using this notation, the initial set of edges is given by

$$E = \{(\langle i, A\rangle, \langle i+1, A\rangle) \mid i \in I_A \text{ or } i+1 \in I_A\} \cup$$
$$\cup\, \{(\langle i, B\rangle, \langle i+1, B\rangle) \mid i \in I_B \text{ or } i+1 \in I_B\}\,.$$

Let $u = \langle \lceil k \rceil, A\rangle$ and $v = \langle \lfloor n/2 - k \rfloor, A\rangle$. The distance between u and v is at least $n/2 - 2(k+1)$, and the distance between nodes w_0 and u and between nodes v and w_n is at least k.

We use E^{block} to denote the set of edges on the shortest path between nodes w_0 and u and between nodes v and w_n (these edges are shown covered by double-sided arrows in Fig. 1(a)). Formally,

$$E^{\text{block}} = E \cap \{(\langle i, A\rangle, \langle j, A\rangle) \mid j \in \{i-1, i+1\}\,, \text{ and either } 0 \le i \le \lceil k \rceil \text{ or } \lfloor n/2 - k \rfloor \le i \le n\}\,.$$

Construction of execution α. Let $S = \xi \cdot \bar{s}(n)$. By definition, $\bar{s}(n) = \lim_{t\to\infty} s(n, 0, t)$. In particular, there is some time T_s such that for all $t \ge T_s$ we have $s(n, 0, t) \le S$. In the proof we focus on the suffix of the execution starting from T_s.

Consider a delay mask $M = (E^{\text{block}}, P)$ where $P(e) = \mathcal{T}$ for all $e \in E^{\text{block}}$. By Lemma 4.3, there is an M-constrained execution α and a time $T_2 \ge T_s$ in which

$$|L_u(T_2) - L_v(T_2)| \ge \frac{1}{4}\mathcal{T} \cdot \text{dist}_M(u, v) \ge \frac{1}{4}\mathcal{T}\left(\frac{n}{2} - 2(k+1)\right). \tag{4.8}$$

Define $T_1 := T_2 - k \cdot \mathcal{T}/(1+\rho)$. We will eventually add new edges to the network at time T_1, and show that the algorithm cannot reduce the skew on them much by time T_2. The new edges must be added "in the past" ($T_1 < T_2$), as we require a large skew between w_0 and w_n "in the present" (time T_2) to show that at least one new edge still has a large skew.

The skew between nodes w_0 and w_n. We argue that the large skew between u and v at time T_2 in α implies a large skew between nodes w_0 and w_n at the same point in time (see Fig. 1(d) for an illustration[3]). Let $S_2 = |L_{w_0}(T_2) - L_{w_n}(T_2)|$. We proceed to bound S_2 from below.

[3]Note that the figure actually depicts the best-case scenario for the algorithm; it could be, for example, that the skew between w_0 and w_n is even larger than the skew between u and v.

Since $T_2 \geq T_s$ we have $s(n, 0, T_2) \leq S$. Because s is non-decreasing in the initial skew, this implies that the skew on each of the edges between nodes w_0 and u and between nodes v and w_n is at most S. There are at most $k+1$ edges between each pair, and hence

$$|L_{w_0}(T_2) - L_u(T_2)| \leq S \cdot (k+1) = \xi \bar{s}(n)(k+1) \tag{4.9}$$

and

$$|L_v(T_2) - L_{w_n}(T_2)| \leq S \cdot (k+1) = \xi \bar{s}(n)(k+1). \tag{4.10}$$

Using (4.8), (4.9) and (4.10) we obtain

$$\begin{aligned} S_2 = |L_{w_0}(T_2) - L_{w_n}(T_2)| &\geq |L_u(T_2) - L_v(T_2)| - |L_{w_0}(T_2) - L_u(T_2)| - |L_v(T_2) - L_{w_n}(T_2)| \\ &\geq \frac{1}{8} n\mathcal{T} - \frac{1}{2}\mathcal{T}(k+1) - 2\xi\bar{s}(n)(k+1) && (\xi\bar{s}(n) \geq \mathcal{T}) \\ &\geq \frac{1}{8} n\mathcal{T} - 3\xi\bar{s}(n)(k+1) && (k \geq 1, \xi \leq \tfrac{4}{3}) \\ &\geq \frac{1}{8} n\mathcal{T} - 8k\bar{s}(n) && (k = \delta \tfrac{n}{\bar{s}(n)}) \\ &= \left(\frac{1}{8}\mathcal{T} - 8\delta\right) \cdot n && (\delta = \tfrac{\mathcal{T}}{128}) \\ &= \frac{1}{16} n\mathcal{T}. \end{aligned}$$

Construction of execution β. We now construct another execution β, in which new edges E^{new} appear at time $T_1 = T_2 - k \cdot \frac{\mathcal{T}}{1+\rho}$ (see Fig. 1(b)). Formally, the network in execution β is defined by

$$E^{\beta}(t) = \begin{cases} E^{\alpha}(t) & \text{if } t < T_1, \\ E^{\alpha}(t) \cup E^{\text{new}} & \text{if } t \geq T_1. \end{cases}$$

In β, a $\mathsf{discover}(\{u, v\})$ event occurs at time $T_1 + \mathcal{D}$ at every node u such that $\{u, v\} \in E^{\text{new}}$ for some $v \in V$. All message delays on edges in E and all hardware clock rates are the same in α and in β. Message delays on edges in E^{new} in β are chosen arbitrarily. Note that since α is M-constrained, β is M-constrained as well.

The new edges E^{new} are chosen between nodes on the B-chain using Lemma 4.4. For any adjacent nodes x, y on the B-chain we have $|L_x(T_1) - L_y(T_1)| \leq S$. Therefore, by Lemma 4.4, there is a sequence $X' = x_1, \ldots, x_m$ of B-chain nodes such that

1. For all $1 \leq i \leq m-1$ we have $\left|L_{x_i}(T_1) - L_{x_{i+1}}(T_1)\right| \in [I - S, I]$, and
2. $m \leq \frac{|L_0(T_1) - L_n(T_1)|}{I - S} + 1$.

Set $E^{\text{new}} = \{\{x_i, x_{i+1}\} \mid 1 \leq i \leq m-1\}$. Then

$$|E^{\text{new}}| = m - 1 \leq \frac{|L_0(T_1) - L_n(T_1)|}{I - S} \leq \frac{\bar{\mathcal{G}}(n)}{I - S},$$

where in the last step we used the fact that the global skew is bounded by $\bar{\mathcal{G}}(n)$.

Indistinguishability of α and β. We show by induction on i that for all $0 \leq i \leq k$, executions α and β are indistinguishable up to time $t_i := T_1 + i \cdot \frac{\mathcal{T}}{1+\rho} + \mathcal{D}$, exclusive, in the eyes of all nodes in the set

$$Y_i := \{\langle j, A\rangle \mid i \leq j \leq \lfloor n/2 \rfloor - i\}\,.$$

- (Base.) For $i = 0$ the claim follows from the fact that α and β are identical up to time T_1 (exclusive), and no node finds out about the new edges until time $T_1 + \mathcal{D}$.

- (Step.) Suppose that up to time t_i, exclusive, executions α and β are indistinguishable in the eyes of all nodes in the set $Y_i = \{\langle j, A\rangle \mid i \leq j \leq \lfloor n/2 \rfloor - i\}$. Let $u \in Y_{i+1}$. From the definition of Y_i and Y_{i+1}, node u and its neighbors are in Y_i. Thus, at any time $t < t_i$, neither u nor its neighbors can distinguish between α and β.

  Since message delays and the hardware clocks of all nodes are the same in α and in β, and no nodes in Y_i experience link formations or failures, the only way a node in Y_i could distinguish between executions α and β is by receiving a message from a node that previously could distinguish between α and β. We show that no node in Y_{i+1} can receive a message from a node that distinguishes α from β until time t_{i+1} (exclusive).

  Consider first messages sent by a node $v \in Y_i \setminus Y_{i+1}$ and received by $u \in Y_{i+1}$ at some time $t_r < t_{i+1}$. Let t_s be the time at which v sent the message. Because $i + 1 \leq k$, the edge $\{u, v\}$ must be in E^{block}, and since β is M-constrained this means that $t_s \leq t_r - \frac{\mathcal{T}}{1+\rho} < t_{i+1} - \frac{\mathcal{T}}{1+\rho} = t_i$. Thus, the message was sent prior to time t_i, and node v could not distinguish between α and β when it sent the message.

  As for messages sent between nodes in Y_{i+1}, it is easy to show by induction on the number of such messages received that neither sender nor recipient can distinguish between α and β.

Since $u, v \in Y_k$ and $T_2 = T_1 + k\frac{\mathcal{T}}{1+\rho} < T_1 + k\frac{\mathcal{T}}{1+\rho} + \mathcal{D}$, nodes u and v cannot distinguish between α and β at any time $t \leq T_2$. It follows that u and v will have the same logical clocks at time T_2 in β as they do in α, and the skew between them will be S_2.

The skew on the new edges at time T_2. At time T_2, every edge in E^{new} carries a skew of no more than $s(n, I, T_2 - T_1)$, since the initial skew on every edge was no more than I and s is non-decreasing in the initial skew. Consequently, the total skew between the endpoints at time T_2 satisfies $S_2 \leq |E^{\text{new}}| \cdot s(n, I, T_2 - T_1)$. However, we have shown that $S_2 \geq \frac{1}{16} n\mathcal{T}$, and hence

$$\frac{1}{16} n\mathcal{T} \leq S_2 \leq |E^{\text{new}}| \cdot s(n, I, T_2 - T_1) \leq \frac{\bar{\mathcal{G}}(n)}{I - S} \cdot s(n, I, k \cdot \frac{\mathcal{T}}{1+\rho}).$$

Rearranging the terms and substituting $k = \delta\frac{n}{\bar{s}(n)}$, $\delta = \frac{\mathcal{T}}{128}$ and $I \geq 3\bar{s}(n) \geq 2S$ yields

$$s(n, I, \frac{\mathcal{T}}{128(1+\rho)} \cdot \frac{n}{\bar{s}(n)}\mathcal{T}) \geq \frac{n}{16\bar{\mathcal{G}}(n)}\mathcal{T}(I - S) \geq \frac{n}{32\bar{\mathcal{G}}(n)}\mathcal{T} \cdot I.$$

This concludes the proof. We note that while the bound applies to any initial skew $I \geq 3\bar{s}(n)$, it is perhaps more meaningful when $\frac{n}{32\bar{\mathcal{G}}(n)}\mathcal{T} \cdot I > \bar{s}(n)$, that is, $I > \frac{32\bar{\mathcal{G}}(n)\bar{s}(n)}{\mathcal{T}n}$. $\square$

5 A Dynamic Clock Synchronization Algorithm

In this section we give a simple DCSA that achieves the tradeoff demonstrated in the previous section. Alg. 1 gives the algorithm in pseudocode, and a detailed description follows.

5.1 Overview

The algorithm is based on the $O(\sqrt{\rho D})$-gradient clock synchronization algorithm from [14]. In the original algorithm, each node attempts to catch up with the maximum clock among its neighbors, under the following constraint: if v is a neighbor of u, then u's clock is not allowed to exceed u's estimate for v's clock by more than B, where $B = \Theta(\sqrt{\rho D})$ is a parameter. Intuitively, the value of B governs how much each node has to wait for its slowest neighbor.

Our dynamic algorithm uses the same general idea; however, instead of treating all edges equally, we use a dynamic weight $B_u^v(\Delta t)$ to determine the amount by which node u's clock is allowed to exceed node v's clock when the link $\{u, v\}$ has existed for Δt time. The tradeoff from Section 4 shows that nodes must not wait for new neighbors as much as they would wait for old neighbors; if they tried to do so they might violate the local skew guarantee along old links. Accordingly we set an initial value of $B_u^v(0) = \infty$, meaning that nodes are allowed to get arbitrarily far ahead of new neighbors. After edge $\{u, v\}$ exists for a "long enough" period of time, the value of B_u^v drops down instantaneously to its final value of $B_0 = \Theta(\sqrt{\rho n})$, which roughly corresponds to the stable local skew of the algorithm (see Theorem 6.12 below). The amount of time before B_u^v drops from ∞ is $\Theta(n/B_0)$, matching the $\Omega(n/\bar{s}(n))$ lower bound from Section 4[4].

5.2 Events and timing

Throughout the algorithm, nodes send each other periodic updates containing their own logical clock value and their estimate for the maximal logical clock in the network. Updates are sent to all neighbors every ΔH subjective time units; that is, if node u sends an update to all its neighbors at real time t, the next time it will send an update is real time t' such that $H_u(t') = H_u(t) + \Delta H$.

During the execution nodes keep track of their dynamic set of neighbors, and remember how much time has elapsed since they last received a message from each neighbor. If a long time passes and a message is not received along an edge, the node concludes that the edge must have failed, and its endpoint is removed from the set of neighbors. Since all hardware clocks progress at a rate of at least $1 - \rho$, each node sends updates to all its neighbors at least once every $\Delta H/(1 - \rho)$ real time

[4]In the conference version of this paper the weights $B_u^v(\Delta t)$ were continuous, starting from a large initial value and decreasing linearly with Δt until reaching the final value of B_0. Here we use a simpler function which nevertheless yields the same local skew guarantee.

units. Therefore, the longest period of real time that can pass between the receipt of two messages along an edge that does not fail is given by

$$\Delta\mathcal{T} := \mathcal{T} + \frac{\Delta H}{1-\rho}.$$

Since nodes do not have access to real time, they use their hardware clocks to conservatively estimate when $\Delta\mathcal{T}$ time has passed. The amount of subjective time they wait is

$$\Delta\mathcal{T}' := (1+\rho)\,\Delta\mathcal{T}.$$

Nodes interact with the network using the following primitives and events.

- $\mathsf{receive}(u, v, m)$: node u receives message m from node v.
- $\mathsf{send}(u, v, m)$: node u sends message m to node v.
- $\mathsf{discover}(X)$, where $X \in \{\mathsf{add}(\{u, v\}), \mathsf{remove}(\{u, v\}) \mid v \in V\}$: node u discovers a change in the status of edge $\{u, v\}$. (See Section 3.2 for a detailed description of the network model.)
- Timers and alarms: nodes can set a timer to trigger a delayed event using the primitive $\mathsf{set\_timer}(\Delta t, \text{timer-ID})$. If node u calls $\mathsf{set\_timer}(\Delta t, \text{timer-ID})$ at real time t, then at real time t' such that $H_u(t') = H_u(t) + \Delta t$, an $\mathsf{alarm}(\text{timer-ID})$ event is triggered at node u. A timer can be cancelled by calling $\mathsf{cancel}(\text{timer-ID})$.

The algorithm uses two types of timers:

- The tick timer is set to go off every subjective ΔH time. When it goes off, the node sends updates to all its neighbors.
- For every neighbor v of u, the $\mathsf{lost}(v)$ timer is set to go off $\Delta\mathcal{T}'$ subjective time units after a message from v is received. If the $\mathsf{lost}(v)$ timer goes off and a new message from v has not been received, node u concludes that the edge $\{u, v\}$ has failed.

5.3 Local variables

Throughout the run of the algorithm each node u maintains two sets Γ_u, Υ_u such that $\Gamma_u \subseteq \Upsilon_u$.

The set Υ_u contains all the nodes v such that a $\mathsf{discover}(\mathsf{add}(\{u, v\}))$ event occurred at u and was not yet followed by a $\mathsf{discover}(\mathsf{remove}(\{u, v\}))$ event. The criterion for membership in Γ_u is more restrictive: the nodes in Γ_u are those nodes of Υ_u that u has heard from at most $\Delta\mathcal{T}'$ subjective time units ago. If $\Delta\mathcal{T}'$ subjective time units pass and u does not receive a message from v, then v is removed from Γ_u (but not from Υ_u). The nodes in Γ_u are the only ones used to determine u's logical clock value, since they are the ones for which u has an accurate estimate. However, u sends (or tries to send) periodic updates to all nodes in Υ_u.

In addition to Γ_u and Υ_u, node u maintains the following local variables.

L_u	Node u's logical clock.
$L_u^{\max}$	Node u's estimate for the maximal logical clock in the network.
C_u^v for $v \in \Gamma_u$	The value of node u's hardware clock when v was last added to Γ_u.
L_u^v for $v \in \Gamma_u$	Node u's estimate for node v's current logical clock.

The local variables are modified upon processing the various events as shown in Algorithm 1. Between events, the variables L_u, $L_u^{\max}$ and L_u^v for all $v \in \Gamma_u$ are increased at the rate of u's hardware clock.

5.4 Updating the logical clock

Node u uses its local estimate L_u^v to estimate the skew on every edge $\{u, v\}$ for $v \in \Gamma_u$. A function $B : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{\infty\}$ governs how much perceived skew node u is willing to tolerate on any edge. The argument to the function B is $(H_u - C_u^v)$, the subjective amount of time that has passed since u discovered edge $\{u, v\}$. Given parameters B_0, W and $W' := (1 + \rho)W$, the function is defined by

$$B(\Delta t) := \begin{cases} \infty & \text{if } \Delta t < W', \\ B_0 & \text{otherwise.} \end{cases}$$

The parameter B_0 corresponds roughly to the stable skew of the algorithm: for sufficiently old edges, nodes try to maintain a perceived skew of at most B_0. However, the real skew may be larger than B_0, in part because the node's estimates for its neighbors' clocks are not perfectly accurate. We bound the real skew in Section 6.

An edge is considered to be "sufficiently old" if the node discovered it at least W *real* time units ago, where

$$W := \left(4\frac{\mathcal{G}(n)}{B_0} + 1\right)\tau \qquad \text{and} \qquad \tau := \frac{1+\rho}{1-\rho}\Delta\mathcal{T} + \mathcal{T} + \mathcal{D}, \tag{5.1}$$

and where $\mathcal{G}(n) = \Theta(n)$ is the bound on the global skew derived in Theorem 6.9 in Section 6.3. The waiting time W corresponds to the $\Omega(n/\bar{s}(n))$ lower bound shown in the previous section. As with $\Delta\mathcal{T}$ and $\Delta\mathcal{T}'$, nodes use $W' = (1 + \rho)W$ to conservatively estimate the subjective time they must wait to ensure that W real time units have passed.

We defer the choice of a value for B_0 until Section 6, and note only that for correctness we require

$$B_0 > 2(1 + \rho)\tau. \tag{5.2}$$

The logical clock of each node is adjusted after every event. In each adjustment, node u increases L_u to the largest value that it can, subject to the following constraints:

(1) L_u is never decreased,

(2) L_u cannot exceed $L_u^{\max}$, and

(3) The perceived skew on every edge $\{u, v\}$ such that $v \in \Gamma_u$ cannot exceed the value of B for that edge. That is, for all $v \in \Gamma_u$ we require $L_u - L_u^v \leq B\,(H_u - C_u^v)$.

If the constraints cannot be met (e.g., if u has a neighbor that is very far behind), node u cannot make a discrete increase to its logical clock. However, the logical clock continues to increase at the rate of u's hardware clock. The update rule is given by

Procedure `AdjustClock`

```
1 $L_u \leftarrow \max\{L_u, \min\{L_u^{\max}, \min_{v\in\Gamma_u}\{L_u^v + B(H_u - C_u^v)\}\}\}$
```

Algorithm 1: Responses to events that occur at node u

```
when discover(add($\{u, v\}$)) occurs at $u$
    send($u, v, \langle L_u, L_u^{\max} \rangle$)
    $\Upsilon_u \leftarrow \Upsilon_u \cup \{v\}$
    AdjustClock()
end
when discover(remove($\{u, v\}$)) occurs at $u$
    $\Gamma_u \leftarrow \Gamma_u \setminus \{v\}$
    $\Upsilon_u \leftarrow \Upsilon_u \setminus \{v\}$
    AdjustClock()
end
when alarm(lost($v$)) occurs at $u$
    $\Gamma_u \leftarrow \Gamma_u \setminus \{v\}$
    AdjustClock()
end
when receive($u, v, \langle L_v, L_v^{\max} \rangle$) occurs at $u$
    cancel(lost($v$))
    if $v \notin \Gamma_u$ then
        $\Gamma_u \leftarrow \Gamma_u \cup \{v\}$
        $C_u^v \leftarrow H_u$
    end
    $L_u^v \leftarrow L_v$
    $L_u^{\max} \leftarrow \max\{L_u^{\max}, L_v^{\max}\}$
    AdjustClock()
    set_timer($\Delta\mathcal{T}'$, lost($v$))
end
when alarm(tick) occurs at $u$
    forall $v \in \Upsilon_u$ do
        send($u, v, \langle L_u, L_u^{\max} \rangle$)
    end
    AdjustClock()
    set_timer($\Delta H$, tick)
end
```

We assume that all nodes know (upper bounds on) the maximum hardware clock drift ρ, the propagation delay $\mathcal{T}$, as well as the bound $\mathcal{D}$ on the time between topology changes and the nodes discovering these changes. Depending on how edge insertions and deletions are discovered, $\mathcal{D}$ typically is a function of ρ, $\mathcal{T}$, as well as the parameter ΔH. Throughout the remainder of the paper, we assume that $\mathcal{D} > \max\{\mathcal{T}, \Delta H/(1-\rho)\}$. We also assume that all nodes know n, the number of nodes participating in the system. With these assumptions, each node u knows enough to compute the value of B_u^v for every $v \in \Gamma_u$. In particular, all nodes can compute the bound $\mathcal{G}(n)$ on the global skew. Note that the same asymptotic results can be achieved if all nodes know n up to a constant factor. This would allow to generalize the setting and also adapt to nodes joining and leaving the system as long as n only changes at a constant rate.

6 Analysis of the Algorithm

In this section we show that when the parameter B_0 is set appropriately, the DCSA from Section 5 achieves $O(n)$ global skew and $O(\sqrt{\rho n})$ stable local skew.

6.1 Basic properties

We begin by establishing several simple properties of the algorithm, most of which concern the quality of information that nodes have about their neighbors.

Lemma 6.1. *If edge $\{u, v\}$ exists throughout the interval $[t_1, t_2]$ where $t_2 \geq t_1 + \Delta\mathcal{T} + \mathcal{D}$, then for any time t such that $t_1 + \Delta\mathcal{T} + \mathcal{D} \leq t \leq t_2$,*

1. *$u \in \Gamma_v(t)$ and $v \in \Gamma_u(t)$,*
2. *Node u receives at least one message from v in the interval $[t - \Delta\mathcal{T}, t]$, and*
3. *Node v receives at least one message from u in the interval $[t - \Delta\mathcal{T}, t]$.*

Proof. Since the edge $\{u, v\}$ exists throughout the interval $[t_1, t_2]$ where $t_2 \geq t_1 + \mathcal{D}$, it is discovered by u and v at times t_u^0, t_v^0 respectively such that $t_u^0, t_v^0 \leq t_1 + \mathcal{D}$.

Upon discovering the edge nodes u and v add each other to Υ_u and Υ_v respectively. No $\mathsf{discover}(\mathsf{remove}(u, v))$ event can occur at u or at v between times t_u^0, t_v^0 (respectively) and time t_2, because the edge exists throughout the interval $[t_1, t_2]$. Therefore, for all $t \in [t_1 + \mathcal{D}, t_2]$ we have $v \in \Upsilon_u(t)$ and $u \in \Upsilon_v(t)$. It follows that nodes u and v send each other updates every subjective ΔH time units at most throughout the interval $[t_1 + \mathcal{D}, t_2]$. This in turn implies that u and v send each other updates every objective $\frac{\Delta H}{1-\rho}$ time units at most throughout this interval.

Let t be a time such that $[t - \Delta\mathcal{T}, t] \subseteq [t_1 + \mathcal{D}, t_2]$. Since v sends u a message at least once every $\frac{\Delta H}{1-\rho}$ time units throughout the interval $[t_1 + \mathcal{D}, t_2]$, there is some $t_s \in [t - \Delta\mathcal{T}, t - \Delta\mathcal{T} + \frac{\Delta H}{1-\rho}]$ such that v sends u a message at time t_s. The message is received by u at time t_r such that

$$t - \Delta\mathcal{T} \leq t_s \leq t_r \leq t_s + \mathcal{T} \leq t - \Delta\mathcal{T} + \frac{\Delta H}{1-\rho} + \mathcal{T} = t.$$

Therefore, condition 2 of the lemma is satisfied. Condition 3 is similar.

Condition 1 of the lemma follows from Conditions 2 and 3: from lines 6–23 of the algorithm, if u received a message from v at time t_r such that $H_u(t) - H_u(t_r) \leq (1 + \rho)\Delta\mathcal{T}$

and no discover(remove($\{u, v\}$)) event occurs during the interval $[t_r, t]$, then $v \in \Gamma_u(t)$, as desired. Let t be a time such that $[t - \Delta\mathcal{T}, t] \subseteq [t_1 + \mathcal{D}, t_2]$. Condition 2 of the lemma shows that node u receives a message from node v at some time $t_r \in [t - \Delta\mathcal{T}, t]$. In particular, $H_u(t) - H_u(t_r) \leq (1 + \rho)(t - t_r) \leq (1 + \rho)(t - (t - \Delta\mathcal{T})) = (1 + \rho)\Delta\mathcal{T}$. Finally, we know that no discover(remove($\{u, v\}$)) event occurs during the interval $[t_r, t]$, because the edge $\{u, v\}$ exists throughout the interval $[t_1, t_2]$ and $[t_r, t] \subseteq [t - \Delta\mathcal{T}, t] \subseteq [t_1 + \mathcal{D}, t_2]$. This shows that $v \in \Gamma_u(t)$, and $u \in \Gamma_v(t)$ is proven in a similar manner. □

Lemma 6.2. *If $v \in \Gamma_u(t)$, then by time t node u has received at least one message that node v sent at time $t_s \geq t - \tau$.*

Proof. If $v \in \Gamma_u(t)$ then u has received a message from v at some time t_r such that $H_u(t) - H_u(t_r) \leq (1 + \rho)\Delta\mathcal{T}$, otherwise u would have removed v from Γ_u prior to time t. Since the hardware clock rate of u is at least $1 - \rho$,

$$H_u(t) - H_u(t_r) \geq (1 - \rho)(t - t_r).$$

Thus, $t - t_r \leq \frac{1+\rho}{1-\rho}\Delta\mathcal{T}$. The message was sent at some time $t_s \geq t_r - \mathcal{T} \geq t - \frac{1+\rho}{1-\rho}\Delta\mathcal{T} - \mathcal{T} \geq t - \tau$, so the lemma holds. □

Lemma 6.3 (Max estimates). *For all $u \in V$ and times $t \geq 0$,*

$$L_u^{\max}(t) \geq L_u(t).$$

Proof. The variables $L_u^{\max}$ and L_u are modified in three circumstances: in line 22 of the algorithm, which is executed when u receives a message; in procedure `AdjustClock()`, which is called after every event; and in between discrete events. It is sufficient to show that all of these preserve the invariant $L_u^{\max} \geq L_u$.

Between processing discrete events, $L_u^{\max}$ and L_u are both increased at the rate of u's hardware clock, and the invariant is preserved. Suppose then that $L_u^{\max} \geq L_u$ prior to the execution of line 22 or of procedure `AdjustClock()`. In line 22 the value of $L_u^{\max}$ can only be increased, so the invariant is preserved. In `AdjustClock()`, node u sets

$$L_u \leftarrow \max\left\{L_u, \min\left\{L_u^{\max}, \ldots\right\}\right\}.$$

Since we assume that $L_u^{\max} \geq L_u$ prior to the execution of `AdjustClock()`, both terms in the $\max$ are no greater than $L_u^{\max}$. Following the assignment we still have $L_u \leq L_u^{\max}$.

□

Lemma 6.4 (Estimate quality). *If $v \in \Gamma_u(t)$ then $L_v(t - \tau) \leq L_u^v(t) \leq L_v(t) + 2\rho\tau$.*

Proof. Let t_s be the latest time that node v sends a message $\langle L_v(t_s), L_v^{\max}(t_s)\rangle$ which is received by u at some time $t_r \leq t$. From Lemma 6.2 we have $t_s \geq t - \tau$. Upon receiving the message, node u sets $L_u^v \leftarrow L_v(t_s)$ (line 21).

Since messages are delivered in FIFO fashion, node u does not receive another message from v during the interval $(t_r, t]$; during this interval L_u^v is increased at the rate of u's hardware clock, and in particular, L_u^v is not decreased. Thus,

$$L_u^v(t) \geq L_u^v(t_r) = L_v(t_s) \geq L_v(t - \tau),$$

giving us the first side of the inequality. In addition, since the rate of u's hardware clock is always at most $(1 + \rho)$,

$$L_u^v(t) \leq L_u^v(t_r) + (t - t_r)(1 + \rho) \leq L_v(t_s) + (t - t_s)(1 + \rho). \tag{6.1}$$

During the interval $[t_s, t]$, node v also increases its logical clock: even if no discrete changes are made, the logical clock increases at least at the rate of v's hardware clock, which is no less than $(1 - \rho)$. Thus,

$$L_v(t_s) \leq L_v(t) - (t - t_s)(1 - \rho). \tag{6.2}$$

Combining (6.1) and (6.2) yields

$$L_u^v(t) \leq L_v(t) + 2\rho(t - t_s) \leq L_v(t) + 2\rho\tau. \tag{6.3}$$

□

6.2 Discrete updates and blocked nodes

To analyze the algorithm it is important to understand what conditions prevent nodes from making discrete changes to their logical clocks. These conditions are captured by the following definitions and properties.

Let

$$B_u^v(t) := B(H_u(t) - C_u^v(t))$$

be the amount of perceived skew node u is willing to tolerate on edge $\{u, v\}$ at real time t.

Definition 6.1 (Blocked nodes)**.** *We say that a node u is* blocked by node v at time t *if*

1. $L_u^{\max}(t) > L_u(t)$, *and*
2. $v \in \Gamma_u(t)$, *and*
3. $L_u(t) - L_u^v(t) > B_u^v(t)$.

In this case we also say that node v blocks node u at time t and that node u is blocked at time t.

It is easy to see that being blocked prevents nodes from increasing their logical clock value in Procedure `AdjustClock()`. The next lemma shows that being blocked is in fact the *only* reason that can prevent a node from increasing its logical clock to its max estimate.

Lemma 6.5. *If $L_u^{\max}(t) > L_u(t)$, then node u is blocked at time t.*

Proof. Let $t' \leq t$ be the most recent time a discrete event occurs at node u up to (and including) time t.

Between time t' and time t node u increases $L_u^{\max}$ and L_u at the rate of its hardware clock, and therefore $L_u^{\max}(t') - L_u(t') = L_u^{\max}(t) - L_u(t)$. Since we assume that $L_u^{\max}(t) > L_u(t)$ it follows that $L_u^{\max}(t') > L_u(t')$.

Node u must be blocked following the last event that occurs at time t', otherwise it would have set $L_u(t') \leftarrow L_u^{\max}(t')$ in Procedure `AdjustClock()`after processing the last event. Thus, there is some neighbor $v \in \Gamma_u(t')$ such that $L_u(t') - L_u^v(t') > B_u^v(t')$.

Between time t' and time t node v was not removed from Γ_u, because nodes are only removed from Γ_u following discrete events, and no discrete event occurs at node u between the last event that occurs at time t' and time t. Thus, $v \in \Gamma_u(t)$. Also, between times t' and t, the values L_u and L_u^v were both increased at the rate of u's hardware clock, and hence $L_u(t) - L_u^v(t) = L_u(t') - L_u^v(t') > B_u^v(t') \geq B_u^v(t)$. This shows that node v blocks node u at time t. □

Each node u decides whether or not to increase its clock based on its estimates of its neighbors' clocks, aiming to keep the skew on edge $\{u, v\}$ no greater than B_u^v. Since the estimate may be larger than the real value of the neighbor's clock, node u may overshoot the mark, but the following lemma shows that it does not overshoot it by much.

Lemma 6.6. *If u's logical clock made a discrete jump at time t, then immediately following the jump, for all $v \in \Gamma_u(t)$ we have $L_u(t) - L_v(t) \leq B_u^v(t) + 2\rho \cdot \tau$.*

Proof. If u's logical clock made a discrete jump at time t, then following the jump in Procedure `AdjustClock()`we have

$$L_u(t) \leq \min_{v \in \Gamma_u} \left(L_u^v(t) + B_u^v(t) \right) \leq L_u^v(t) + B_u^v(t).$$

Applying Lemma 6.4 we obtain

$$L_u(t) \leq L_u^v(t) + B_u^v(t) \leq L_v(t) + B_u^v(t) + 2\rho\tau.$$

□

6.3 Global Skew

The basic strategy to bound the global skew of our dynamic clock synchronization algorithm is the same as the one used in a static network (see [14]). We first show that for any two nodes u and v, the estimates $L_u^{\max}(t)$ and $L_v^{\max}(t)$ of the maximum clock value in the system are not too far apart. Second, we show that if the global skew exceeds a certain value at time t, the node v with the smallest logical clock value $L_v(t)$ cannot be blocked at time t. By Lemma 6.5, we then have $L_v(t) = L_v^{\max}(t)$ and thus the bound on the maximal difference between two estimates $L_u^{\max}(t)$ and $L_v^{\max}(t)$ also yields a bound on the global skew.

For any $t \geq 0$, define

$$L^{\max}(t) := \max_{u \in V} L_u^{\max}(t). \tag{6.4}$$

Lemma 6.7 (Rate of $L^{\max}$)**.** *The value of $L^{\max}$ increases at a rate at most $1+\rho$. That is, for all $t_2 \geq t_1 \geq 0$ we have*

$$L^{\max}(t_2) - L^{\max}(t_1) \leq (1+\rho)(t_2 - t_1).$$

Proof. Informally, we wish to argue that any node that has the largest max estimate only increases it at the rate of its hardware clock, because it never hears larger clock values from its neighbors. Thus, the overall maximum of the max estimates in the network increases at an average rate of at most $(1+\rho)$, the rate of the fastest hardware clock. However, this argument is complicated by the fact that $L^{\max}$ is not differentiable everywhere. We require the following easy lemma.

($\star$) Let $F = \{f_1, \ldots, f_n\} \subseteq \mathbb{R}^{\mathbb{R}}$ be a set of functions and let $b \in \mathbb{R}$ be a bound such that for all $f \in F$ and for all $x \in \mathbb{R}$, if $f(x) = \max_{g \in F} g(x)$, then f is differentiable at x and $f'(x) \leq b$. Then for all $x_1, x_2 \in \mathbb{R}$ we have

$$\max_{f \in F} f(x_2) \leq \max_{f \in F} f(x_1) + b(x_2 - x_1).$$

The proof is technical and is not included here. To apply the lemma we must show that for all nodes u and times t, if $L_u^{\max}(t) = L^{\max}(t)$ then $L_u^{\max}$ is differentiable at t and $dL_u^{\max}/dt \leq 1+\rho$.

Let u be a node such that $L_u^{\max}(t) = L^{\max}(t)$. Between discrete updates node u increases $L_u^{\max}$ at the rate of its hardware clock, which is differentiable and has a rate of at most $1+\rho$; thus, it is sufficient to show that u does not make a discrete update to $L_u^{\max}$ at time t. The only place where u might make a discrete update to $L_u^{\max}$ is line 22 of the algorithm, which is executed upon receiving a message from a neighbor. Thus, suppose that at time t node u receives a message $\langle L_v(t_s), L_v^{\max}(t_s) \rangle$ that node v sent at time $t_s \leq t$. Since $L_v^{\max}$ is non-decreasing, we have $L_v^{\max}(t_s) \leq L_v^{\max}(t) \leq L_u^{\max}(t)$; hence the value of $L_u^{\max}$ does not change upon execution of line 22, and a discrete update does not occur. $\square$

The accuracy of the estimates $L_u^{\max}(t)$ can be bounded by applying the interval connectivity property of the dynamic network graph. Informally, suppose we "freeze" the value of $L^{\max}$ at some time t, and let us track the propagation of this value throughout the network. Let $M = L^{\max}(t)$. Consider the cut $(S(t), V \setminus S(t))$, where $S(t) := \{u \in V \mid L_u^{\max}(t) \geq M\}$ is the set of nodes that have heard of M or a greater clock value. The $(\mathcal{T} + \mathcal{D})$-interval connectivity of the graph guarantees that there is some edge in the cut that persists long enough for its endpoints to get at least one message across. Thus, at every "step", at least one node in $V \setminus S(t)$ learns a value that is at least M, and increases its max estimate accordingly. This node will then be added to $S(t+\mathcal{T}+\mathcal{D})$.

After at most $O(n)$ such steps we will reach a time t' $(= t+O(n))$ such that $S(t') = V$, that is, for all $u \in V$ we have $L_u^{\max}(t') \geq M$. In other words, after $O(n)$ time, all nodes catch up to the *old* value $M = L^{\max}(t)$. But Lemma 6.7 shows that $L^{\max}$ does not "run away" during this interval: $L^{\max}(t') \leq L^{\max}(t) + (1+\rho)(t-t') = M + O(n)$. Therefore the difference between $L^{\max}(t')$ and the max estimate $L_u^{\max}(t')$ of any node u is at most $O(n)$. This argument is formalized in the following lemma.

Lemma 6.8 (Max Propagation Lemma)**.** *If the dynamic graph $G(t)$ is $(\mathcal{T}+\mathcal{D})$-interval connected, then for all $t \geq 0$ and all $u \in V$ it holds that*

$$L^{\max}(t) - L_u^{\max}(t) \leq ((1+\rho)\cdot\mathcal{T} + 2\rho\cdot\mathcal{D})\cdot(n-1).$$

Proof. All hardware clocks and max-estimates are initialized to 0 at time 0, and hence $L^{\max}(0) - L_u^{\max}(0) = 0$. The max clock $L^{\max}$ increases at a rate of no more than $1+\rho$, and the max-estimate $L_u^{\max}(t)$ of any node u increases at a rate of at least $1-\rho$. Consequently, the difference $L^{\max}(t) - L_u^{\max}(t)$ grows at a rate of no more than $(1+\rho)-(1-\rho) = 2\rho$, and because $\rho < 1$, the claim holds at least until time

$$t = \frac{(1+\rho)\mathcal{T} + 2\rho\cdot\mathcal{D}}{2\rho}\cdot(n-1) > (\mathcal{T}+\mathcal{D})\cdot(n-1).$$

Thus, it is sufficient to consider times t such that $t > (\mathcal{T}+\mathcal{D})\cdot(n-1)$.

For $i \in \{1,\ldots,n\}$, define

$$t_i := t - (n-i)(\mathcal{T}+\mathcal{D})$$

and

$$V_i := \left\{v \in V \mid L_v^{\max}(t_i) \geq L^{\max}(t_1) + (i-1)(1-\rho)\mathcal{D}\right\}.$$

We prove by induction on i that for all $i \in \{1,\ldots,n\}$ we have $|V_i| \geq i$.

- (Base) By definition, $V_1 = \{v \in V \mid L_v^{\max}(t_1) \geq L^{\max}(t_1)\}$. There exists some node v such that $L_v^{\max}(t_1) = L^{\max}(t_1)$, and consequently $|V_1| \geq 1$.

- (Step) Suppose that $|V_{i-1}| \geq i-1$. By definition, for all $v \in V_{i-1}$ we have

$$L_v^{\max}(t_{i-1}) \geq L^{\max}(t_1) + (i-2)(1-\rho)\mathcal{D}. \tag{6.5}$$

  The max estimate of each node increases at least at the rate of its hardware clock. Consequently, for all $v \in V_{i-1}$,

$$\begin{aligned} L_v^{\max}(t_i) &\geq L_v^{\max}(t_{i-1}) + (t_i - t_{i-1})(1-\rho) \\ &\overset{(6.5)}{\geq} L^{\max}(t_1) + (i-2)(1-\rho)\mathcal{D} + (t_i - t_{i-1})(1-\rho) \\ &\geq L^{\max}(t_1) + (i-1)(1-\rho)\mathcal{D}, \end{aligned}$$

and hence $V_{i-1} \subseteq V_i$.

If $V \setminus V_{i-1} = \emptyset$, then $|V_i| \geq |V_{i-1}| = n$ and we are done. Otherwise by $(\mathcal{T} + \mathcal{D})$-interval connectivity of $G(t)$ there exists an edge $e = \{v, w\}$, where $v \in V_{i-1}$ and $w \in V \setminus V_{i-1}$, such that e exists throughout the interval $[t_{i-1}, t_i]$. By Lemma 6.1, there are times $t_{\text{snd}} \geq t_{i-1}$ and $t_{\text{rcv}} \leq t_i$ such that node v sends node w a message containing $L_v^{\max}(t_{\text{snd}})$ at time t_{snd}, and node w receives the message at time t_{rcv} and updates its max estimate. Thus we have

$$\begin{aligned}
L_w^{\max}(t_i) &\geq L_w^{\max}(t_{\text{rcv}}) + (1-\rho)(t_i - t_{\text{rcv}}) \\
&\geq L_v^{\max}(t_{\text{snd}}) + (1-\rho)(t_i - t_{\text{rcv}}) \\
&\geq L_v^{\max}(t_{i-1}) + (1-\rho)(t_i - t_{\text{rcv}}) + (1-\rho)(t_{\text{snd}} - t_{i-1}) \\
&\geq L_v^{\max}(t_{i-1}) + (1-\rho)(t_i - t_{i-1} - \mathcal{T}) \\
&= L_v^{\max}(t_{i-1}) + (1-\rho)\mathcal{D} \\
&\overset{(6.5)}{\geq} L^{\max}(t_1) + (i-1)(1-\rho)\mathcal{D}.
\end{aligned}$$

It follows that $w \in V_i$. Since $w \notin V_{i-1}$ and $V_{i-1} \cup \{w\} \subseteq V_i$ we have $|V_i| \geq |V_{i-1}| + 1 \geq i$.

The claim we proved implies that $V_n = V$; that is, for all $v \in V$, at time $t_n = t$ we have

$$L_v^{\max}(t) \geq L^{\max}(t_1) + (n-1)(1-\rho)\mathcal{D}. \tag{6.6}$$

From Lemma 6.7,

$$L^{\max}(t) \leq L^{\max}(t_1) + (1+\rho)(t - t_1) = L^{\max}(t_1) + (1+\rho)(n-1)(\mathcal{T} + \mathcal{D}), \tag{6.7}$$

and combining (6.6) and (6.7) yields

$$L^{\max}(t) - L_v^{\max}(t) \leq (n-1)\left((1+\rho)\mathcal{T} + 2\rho \cdot \mathcal{D}\right).$$

□

Using the approach sketched above, Lemma 6.8 allows us to prove the following theorem, which bounds the global skew of our algorithm.

Theorem 6.9 (Global skew)**.** *The algorithm guarantees a global skew of*

$$\mathcal{G}(n) := ((1+\rho) \cdot \mathcal{T} + 2\rho \cdot \mathcal{D}) \cdot (n-1).$$

Proof. We show the stronger statement that at all times t,

$$\forall v \in V \quad : \quad L^{\max}(t) - L_v(t) \leq \mathcal{G}(n)$$

and the claim then follows from Lemma 6.3 and the definition of $L^{\max}$.

For the sake of contradiction, assume that this is not the case. Then there is some time t, node $v \in V$ and $\varepsilon > 0$ such that

$$L^{\max}(t) - L_v(t) \geq \mathcal{G}(n) + \varepsilon \tag{6.8}$$

Let $\bar{t}$ be the infimum of times when (6.8) holds for some node v. By Lemma 6.8, we have $L^{\max}(\bar{t}) - L_v^{\max}(\bar{t}) \leq \mathcal{G}(n)$ and thus $L_v(\bar{t}) < L_v^{\max}(\bar{t})$. Hence, as a consequence of Lemma 6.5, v is blocked at time $\bar{t}$. Therefore by Definition 6.1, there is a node $u \in \Gamma_v(\bar{t})$ such that $L_v(\bar{t}) - L_v^u(\bar{t}) > B_v^u(\bar{t}) \geq B_0$. By Lemma 6.4, it therefore holds that $L_u(\bar{t} - \tau) < L_v(\bar{t}) - B_0$. By Lemma 6.7, we have $L^{\max}(\bar{t} - \tau) \geq L^{\max}(\bar{t}) - (1 + \rho)\tau$. We therefore obtain

$$L^{\max}(\bar{t} - \tau) - L_u(\bar{t} - \tau) > L^{\max}(\bar{t}) - L_v(\bar{t}) - (1 + \rho)\tau + B_0.$$

Because we assume that $B_0 \geq (1 + \rho)\tau$, this is a contradiction to the assumption that $\bar{t}$ is the infimum of times when (6.8) is satisfied for the first time for some node v. $\square$

6.4 Local Skew

The local skew guarantee of the algorithm hinges on the fact that for a long time after an edge appears, the skew on it is unconstrained and its endpoints do not need to wait for each other. Specifically, we can show that at least W real time units must pass before two newly adjacent nodes can block one another.

Lemma 6.10. *If node v blocks node u at time t, then $v \in \Gamma_u(t')$ for all $t' \in [t - W, t]$.*

Proof. Let $t_0 \leq t$ be the last time in which node u added node v to Γ_u. (Such a time must exist, because at time t node v blocks node u, and in particular $v \in \Gamma_u(t)$.) From the algorithm, $C_u^v(t) = H_u(t_0)$, and by choice of t_0 we have $v \in \Gamma_u(t')$ for all $t' \in [t_0, t]$. Our goal now is to show that $t_0 \leq t - W$.

From the definition of blocked nodes, if node v blocks node u at time t then $B_u^v(t) < \infty$. This implies that $H_u(t) - C_u^v(t) \geq W'$, that is, $H_u(t) - H_u(t_0) \geq W'$. Since the hardware clock progresses at a rate of $(1 + \rho)$ at most, we can write

$$(1 + \rho)W = W' \leq H_u(t) - H_u(t_0) \leq (1 + \rho)(t - t_0).$$

Thus, $t_0 \leq t - W$, as desired. $\square$

We use the lemma above to show that by the time two nodes *can* block each other, they have been in communication for a long time, and have up-to-date information about each other. Specifically, the node that lags behind has a max estimate that reflects the clock value of the faster node. We will later argue that if there is a large skew between the nodes, then the slower node must itself be blocked, otherwise it would have increased its clock to match its max estimate (Lemma 6.5).

Lemma 6.11 (Edge reversal)**.** *If node v blocks node u at time t then for all $t' \in [t - W + \Delta\mathcal{T}, t - \mathcal{D}]$ we have $L_v^{\max}(t') \geq L_u^{\max}(t' - \tau)$.*

Proof. From Lemma 6.10, if v blocks u at time t, then for all $t' \in [t-W,t]$ we have $v \in \Gamma_u(t')$. Since $\Gamma_u(t') \subseteq \Upsilon_u(t')$, this implies that $v \in \Upsilon_u$ throughout the interval. Hence, throughout the interval $[t-W,t]$, node u sends node v an update every $\frac{\Delta H}{1-\rho}$ real time units at most.

The model guarantees that if a message sent by u to v at time t' is not delivered, node u experiences a $\mathsf{discover}(\mathsf{remove}(\{u,v\}))$ event no later than time $t' + \mathcal{D}$, which would lead u to remove v from Γ_u (line 7). Since $v \in \Gamma_u$ throughout the interval $[t-W,t]$, all messages sent from u to v during the interval $[t-W, t-\mathcal{D}]$ are delivered. It follows that during the interval $[t-W+\Delta\mathcal{T}, t-\mathcal{D}]$, node v receives a message from u at least once every $\Delta\mathcal{T}$ time units, and hence throughout the interval we have $u \in \Gamma_v$. Lemma 6.2 implies that $L_v^{\max}(t') \geq L_u^{\max}(t'-\tau)$ for all $t' \in [t-W+\Delta\mathcal{T}, t-\mathcal{D}]$. □

The local skew guarantee of the algorithm is as follows.

Theorem 6.12. *For any two nodes u,v and time t such that $v \in \Gamma_u(t)$,*

$$L_u(t) - L_v(t) \leq B_u^v(t-W) + 2\rho W = B_u^v(t-W) + 2\rho\tau\left(4\frac{\mathcal{G}(n)}{B_0}+1\right)$$

Proof. Suppose by way of contradiction that at time t there are two nodes $u,v \in V$ such that $v \in \Gamma_u(t)$, but

$$L_u(t) - L_v(t) > B_u^v(t-W) + 2\rho W.$$

We will show that this implies a contradiction to the global skew guarantee (Theorem 6.9) at some earlier time in the execution.

There are two parts to the proof. First, we show that since the skew between u and v is very large, u has been blocked for a long time, and its logical clock has not increased by much. More formally, since B_u^v is non-increasing, for all $t' \in [t-W,t]$ we have

$$B_u^v(t') \leq B_u^v(t-W). \tag{6.9}$$

From Lemma 6.6 and Lemma 6.10, at any time $t' \in [t-W,t]$ node u's logical clock cannot jump to a value that exceeds $L_v(t') + B_u^v(t') + 2\rho\tau \leq L_v(t') + B_u^v(t-W) + 2\rho\tau$. Thus, the excess skew of $2\rho W - 2\rho\tau$ was built up by increasing u's logical clock at the rate of u's hardware clock, which is at most $1+\rho$, while v's clock increased at a rate of at least $1-\rho$. In other words, as long as the skew is greater than $B_u^v(t-W) + 2\rho\tau$ it increases at a rate of at most 2ρ, which implies that u's clock cannot make a discrete jump throughout the interval $[t-W+\tau, t]$. Thus, for all $t' \in [t-W+\tau, t]$ we have

$$L_u(t') \geq L_u(t) - (1+\rho)(t-t'). \tag{6.10}$$

In the second part of the proof we argue that node v would not have fallen so far behind node u unless it was itself blocked until very recently by some other node u_2, which lags far behind v.

And since u_2 lags far behind v, it must *also* have been blocked recently, and so on. In this way we construct a chain $u_0, u_1, \ldots, u_{\ell+1}$ of nodes, where $u_0 = u$, $u_1 = v$, and each node in the chain is blocked by the next node.

Formally, we define a sequence of decreasing times $t_0, t_1, \ldots, t_{\ell+1}$, where

$$t_i := t - i \cdot \tau \qquad \text{and} \qquad \ell := \lfloor \frac{W - \tau}{2\tau} \rfloor. \tag{6.11}$$

We construct the chain $u_0, \ldots, u_{\ell+1}$ so that each node u_i satisfies the following properties.

(1) If $i \neq 0$ then $L_u(t) - L_{u_i}(t_i) > i \cdot B_0$.

(2) For all $t' \in [t_{2\ell-i+1}, t_i]$ we have $L_{u_i}^{\max}(t') \geq L_u(t' - i\tau)$.

(3) If $i \leq \ell$ then node u_i is blocked at time t_i.

The first property is the one we are truly interested in: we will use it to obtain the contradiction to the global skew guarantee. The other two properties are necessary for the inductive construction of the chain. We use property (2) to show that u_i is "aware" of a large clock value in the network, by bounding its max estimate from below. Then we combine properties (1) and (2) to show that u_i's clock value is smaller than its max estimate, which means it must be blocked (Lemma 6.5). The node that blocks u_i will be chosen as u_{i+1}.

Before showing the construction of the chain in detail, let us show how we use the chain to derive a contradiction. Suppose we already have a chain $u_0, \ldots, u_{\ell+1}$ that satisfies properties (1)–(3). In particular,

$$L_u(t) - L_{u_{\ell+1}}(t_{\ell+1}) > (\ell+1)B_0 \overset{(5.2)}{>} (\ell+1) \cdot 2(1+\rho)\tau. \tag{6.12}$$

In order to obtain a contradiction to the global skew, we must relate the clock values of u and $u_{\ell+1}$ *at the same time*. Now the first part of the proof comes into play: it shows that u's clock value at time $t_{\ell+1}$ is not much less than it is at time t, specifically,

$$L_u(t_{\ell+1}) \geq L_u(t) - (1+\rho)(t - t_{\ell+1}) = L_u(t) - (1+\rho)(\ell+1)\tau. \tag{6.13}$$

(We use the fact that $t_{\ell+1} = t - (\ell+1) \cdot \tau > t - W + \tau$, which allows us to apply (6.10).) Combining (6.12) with (6.13), we obtain

$$\begin{aligned} L_u(t_{\ell+1}) - L_{u_{\ell+1}}(t_{\ell+1}) &> 2(1+\rho)(\ell+1)\tau - (1+\rho)(\ell+1)\tau \\ &\overset{(6.11)}{=} (1+\rho)\tau \cdot \left(\lfloor \frac{W-\tau}{2\tau} \rfloor + 1 \right) \\ &\geq (1+\rho)\tau \cdot \frac{W-\tau}{2\tau} \\ &\overset{(5.1)}{=} 2(1+\rho)\tau \cdot \frac{\mathcal{G}(n)}{B_0} > \mathcal{G}(n). \end{aligned}$$

This is the contradiction we sought.

It remains to show how the chain $u_0, \dots, u_{\ell+1}$ is constructed. The base case, $u_0 = u$, is immediate. Suppose that we have constructed the chain up to node u_i, where $i \leq \ell$ and u_i satisfies properties (1)–(3). If $i = 0$, we choose $u_1 = v$. (We know that v blocks u at time $t_0 = t$.) Otherwise, from property (3) we know that u_i is blocked at time t_i. Thus, there is some node $u_{i+1} \in \Gamma_{u_i}(t_i)$ such that

$$L_{u_i}(t_i) - L_{u_i}^{u_{i+1}}(t_i) > B_{u_i}^{u_{i+1}}(t_i) \geq B_0.$$

We show that u_{i+1} satisfies properties (1)–(3).

Property (1). Since $u_{i+1} \in \Gamma_{u_i}(t_i)$, Lemma 6.4 shows that $L_{u_i}^{u_{i+1}}(t_i) \geq L_{u_{i+1}}(t_{i+1})$ (recall that $t_{i+1} = t_i - \tau$ by definition). Thus we have

$$L_{u_i}(t_i) - L_{u_{i+1}}(t_{i+1}) > B_0. \tag{6.14}$$

Now there are two cases. If $i > 0$, then property (1) gives us $L_u(t) - L_{u_i}(t_i) > i \cdot B_0$, which together with (6.14) yields

$$L_u(t) - L_{u_{i+1}}(t_{i+1}) > (i+1)B_0. \tag{6.15}$$

If $i = 0$, the property we must show is $L_u(t) - L_v(t - \tau) > B_0$. We assumed that u and v violate the local skew guarantee at time t, and in particular, $L_u(t) - L_v(t) > B_0$. But L_v is non-decreasing, and therefore $L_u(t) - L_v(t-\tau) \geq L_u(t) - L_v(t) > B_0$, as desired.

Property (2). Let $t' \in [t_{2\ell-i}, t_{i+1}]$. Since $t_{2\ell-i} = t_{2\ell-i+1} + \tau$ and $t_{i+1} = t_i - \tau$, we have $t' - \tau \in [t_{2\ell-i+1}, t_i - 2\tau]$, and property (2) applied to u_i at time $t' - \tau$ gives us

$$L_{u_i}^{\max}(t' - \tau) \geq L_u(t' - (i+1)\tau). \tag{6.16}$$

In addition we also have $t' \in [t_i - W + \Delta\mathcal{T}, t_i - \tau]$, and since u_{i+1} blocks u_i at time t_i, we can apply Lemma 6.11 to obtain

$$L_{u_{i+1}}^{\max}(t') \geq L_{u_i}^{\max}(t' - \tau) \overset{(6.16)}{\geq} L_u(t' - (i+1)\tau), \tag{6.17}$$

as required.

Property (3). It remains to verify that node u_{i+1} is blocked at time t_{i+1}.

Recall that by definition $t_i = t - i \cdot \tau$. In particular,

$$t_{2\ell} = t - 2\ell\tau \overset{(6.11)}{\geq} t - 2\tau \cdot \frac{W - \tau}{2\tau} = t - W + \tau.$$

Thus, we can apply (6.13) to any time $t' \in [t_{2\ell}, t]$, obtaining

$$L_u(t) - L_u(t') \leq (1+\rho)(t - t'). \tag{6.18}$$

Applying (6.17) with $t' = t_{i+1}$ yields

$$\begin{aligned} L^{\max}_{u_{i+1}}(t_{i+1}) &\geq L_u(t_{i+1} - (i+1)\tau) = L_u(t_{2(i+1)}) \\ &\overset{(6.18)}{\geq} L_u(t) - (1+\rho)(t - t_{2(i+1)}) = L_u(t) - (1+\rho)(2i+1)\tau \\ &\overset{(6.15)}{>} L_{u_{i+1}}(t_{i+1}) + (i+1)B_0 - (1+\rho)(2i+1)\tau \\ &\overset{(5.2)}{>} L_{u_{i+1}}(t_{i+1}). \end{aligned}$$

From Lemma 6.5, node u_{i+1} is blocked at time t_{i+1}.

□

Theorem 6.12 describes the local skew guarantee from a point of view that is subjective to node u: the statement of the theorem assumes that $v \in \Gamma_u$, and the value of B_u^v depends on the local variables C_u^v and H_u. The following corollary states the "objective" local skew guarantee of the algorithm.

Corollary 6.13. *The algorithm guarantees a dynamic local skew of*

$$s(n, I, \Delta t) := s(n, \Delta t) := B\left(\max\left\{(1-\rho)(\Delta t - \Delta\mathcal{T} - \mathcal{D} - W), 0\right\}\right) + 2\rho W,$$

regardless of the initial skew I on the edge.

Proof. Let $e = \{u, v\}$ be an edge that exists throughout an interval $[t, t+\Delta t]$. If $\Delta t - \Delta\mathcal{T} - \mathcal{D} - W \leq 0$, then $s(n, \Delta t) = B(0) + 2\rho W = \infty$, and all edges carry less than $s(n, \Delta t)$ skew. Suppose then that $\Delta t - \Delta\mathcal{T} - \mathcal{D} - W > 0$, that is, $t + \Delta t - W > t + \Delta\mathcal{T} + \mathcal{D}$.

Since the edge exists throughout the interval $[t, t+\Delta t]$, from Lemma 6.1, at any time $t' \in [t + \Delta\mathcal{T} + \mathcal{D}, t + \Delta t]$ we have $v \in \Gamma_u(t')$. Thus, the last time v was added to Γ_u prior to time $t + \Delta t$ is some time $t_1 \leq t + \Delta\mathcal{T} + \mathcal{D} < t + \Delta t - W$, and from the algorithm, $C_u^v(t + \Delta t - W) = H_u(t_1) \leq H_u(t + \Delta\mathcal{T} + \mathcal{D})$. Since B is non-increasing,

$$\begin{aligned} B_u^v(t + \Delta t - W) &= B\left(H_u(t+\Delta t - W) - C_u^v(t + \Delta t - W)\right) \leq \\ &\leq B\left(H_u(t+\Delta t - W) - H_u(t + \Delta\mathcal{T} + \mathcal{D})\right) \leq \\ &\leq B\left((1-\rho)(t + \Delta t - t - \Delta\mathcal{T} - \mathcal{D} - W)\right) = \\ &= B\left((1-\rho)(\Delta t - \Delta\mathcal{T} - \mathcal{D} - W)\right). \end{aligned}$$

Now we can use Theorem 6.12 to obtain

$$L_u(t+\Delta t) - L_v(t + \Delta t) \leq B_u^v(t + \Delta t - W) + 2\rho W \leq s(n, \Delta t),$$

and similarly we can show that $L_v(t+\Delta t) - L_u(t+\Delta t) \leq s(n, \Delta t)$ as well. Together we have $|L_u(t+\Delta t) - L_v(t + \Delta)| \leq s(n, \Delta t)$, as required. □

Corollary 6.14. *If the parameter B_0 is chosen as $B_0 \geq \lambda\sqrt{\rho n}$ for a constant $\lambda > 0$, the stable local skew of the algorithm is $O(B_0)$. Further, the time to reach this stable skew on a new edge is $O(n/B_0)$. Hence, for this choice of B_0, the trade-off achieved by the algorithm asymptotically matches the trade-off established by the lower bound in Theorem 4.1.*

7 Conclusion

We have established fundamental trade-offs for gradient clock synchronization algorithms in dynamic networks. First, the time to adjust the skew on a newly formed edge is inversely proportional to the skew one is willing to tolerate on well-established edges. Hence, having a stronger skew requirement in stable conditions impairs the ability to adapt to dynamic changes. Second, contrary to what one might initially think, reducing the skew on edges with a small initial skew turns out to be as hard as reducing the skew on edges with a large initial skew. The time needed in both cases is linear in the global skew bound of the algorithm and is thus at least linear in n.

The algorithm we gave in Section 5 achieves a stable local skew of $O(\sqrt{\rho n})$ and has optimal stabilization time, $O(\sqrt{n/\rho})$. In subsequent work [10], we showed that it is possible for a DCSA to achieve a stable skew of $O(\log_{1/\rho} n)$, matching the best possible local skew of a static algorithm (in light of the lower bound from [13]). The improved stable skew necessarily comes at the cost of adaptability; the stabilization time of the algorithm in [10] is $O(n)$. Note that the tight lower bound we give in the current paper would show that no algorithm with a stable skew of $O(\log_{1/\rho} n)$ can have a stabilization time better than $\Omega(n/\log_{1/\rho} n)$, seemingly indicating that the algorithm of [10] has sub-optimal stabilization time. However, in [10] we refine the lower bound and show that for "true gradient" algorithms — algorithms that guarantee a skew smaller than the global skew between *any* two nodes at distance less than the diameter of the graph — the stabilization time cannot be better than $\Omega(n)$. Thus the algorithm in [10] is optimal in both the stable skew and the time until that stable skew is reached. (We note that the algorithm from Section 5 is not subject to the refined lower bound from [10], because for nodes at distance $\Omega(\sqrt{n/\rho})$ from each other, the only skew guarantee it provides is $\Omega(\sqrt{n/\rho} \cdot \sqrt{\rho n}) = \Omega(n)$, no better than the global skew guarantee. In this sense this algorithm is not a "true gradient" algorithm. This property allows it, however, to achieve a stabilization time of $O(\sqrt{n/\rho})$ instead of $\Omega(n)$.)

An interesting generalization of these results would be to incorporate node insertions and deletions in the dynamic graph model. As long as nodes join and leave at a constant rate, it might be possible to adapt all the parameters used sufficiently quickly in order to still guarantee the same basic results. The details of such a protocol as well as possible limitations on how fast one can adapt to changes of the network size remain open questions.

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