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Accessibility of Values as a Determinant of Relative Complexity in Algebras*

NANCY A. LYNCH

School of Information and Computer Science, Georgia Institute of Technology, Atlanta, Georgia 30332

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I. INTRODUCTION

The present paper is a companion to [2], in which a general definition of straight-line program length is proposed as a size parameter for complexity analysis in an arbitrary finitely generated algebra. In [2], the general size parameter is defined and its basic properties derived. Relative time complexity of algebras is defined in terms of the size parameter, using alternative definitions based on two different abstract programming languages (flowcharts and the very rudimentary "expression assignment" language). Relative time complexity defined using expression assignments is seen to be a lower bound on relative time complexity defined using flowcharts (or practically any other programming language), because the former simply measures the relative accessibility of elements in the respective algebras. Composition theorems are proved for relative complexity according to these definitions. Finally, an extended example is given to show how these measures can be used as the basis for a complexity theory for finitely generated groups.

In this paper, we classify several numeric and bit-string algebras by the relative expression assignment complexity defined in [2]. These results are useful primarily as coding-independent lower bounds on computation time in ordinary programming languages. Results are seen to be fairly tight. Also in this paper, an apparent tradeoff between expression assignment complexity and number of representations is examined.

This paper is not intended to be self-contained. Reference [2] is used for many necessary definitions and theorems. The reader is also referred to [2] for examples which should clarify the formalism.

Reference [3] is a preliminary paper including earlier versions of the present results as well as the results of [2].

II. UPPER BOUNDS

We examine seven natural algebras arising frequently in mathematics and computer science, and obtain upper bounds on the relative expression assignment

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complexity of these algebras, as defined in [2]. We assume for the theorems of this paper that $\mathrm{Fun}_{\mathscr{A}}$ is a finite set.

The seven algebras are as follows: (Here, $\mathscr{N}$ represents the set of nonnegative integers, while $\mathscr{Z}$ represents the set of integers.)

$$\mathscr{N} = \langle N; 0, \mathrm{suc}\rangle, \qquad \text{where } suc(x) = x + 1,$$
$$\mathscr{Z} = \langle Z; 0, \mathrm{suc}, \mathrm{pred}\rangle, \qquad \text{where } pred(x) = x - 1,$$
$$\mathscr{G}^+ = \langle N \times N; 0, \mathrm{suc}, \mathrm{up}\rangle, \qquad \text{where } suc(x, y) = (x + 1, y) \text{ and } up\ (x, y) = (x, y + 1),$$
$$\mathscr{G} = \langle Z \times Z;, 0, \mathrm{suc}, \mathrm{pred}, \mathrm{up}, \mathrm{down}\rangle, \qquad \text{where } pred\ (x, y) = (x - 1, y) \text{ and } down\ (x, y) = (x, y - 1),$$
$$\mathscr{E} = \langle \{0, 1\}^*; \lambda, 0\mathrm{suc}, 1\mathrm{suc}\rangle, \qquad \text{where } \lambda \text{ is the empty string, } 0suc(x) = x0,\ 1suc(x) = x1,$$
$$\mathscr{E}' = \langle \{0, 1\}^*; \lambda, 0\mathrm{suc}, 1\mathrm{suc}, \mathrm{pred}\rangle, \qquad \text{where } pred(\lambda) = \lambda,\ pred(x0) = pred(x1) = x,$$
$$\mathscr{N}' = \langle N; 0, 1, +\rangle.$$

Thus, $\mathscr{G}^+$ and $\mathscr{G}$ are algebraic abstractions of matrices. $\mathscr{E}$ and $\mathscr{E}'$ are algebras based on binary trees.

We use the definition [2] for "$\mathscr{A} \leqslant_\tau^{\exp} \mathscr{A}'$ with complexity t" in order to classify these algebras. We write $\mathscr{A} \leqslant^{\exp} \mathscr{A}'$ *with complexity* t to represent $(\exists \tau)$ [$\mathscr{A} \leqslant_\tau^{\exp} \mathscr{A}'$ with complexity t]. For each pair $\mathscr{A}$, $\mathscr{A}'$ chosen from the above algebras, it is not difficult to show that $\mathscr{A} \leqslant^{\exp} \mathscr{A}'$ with complexity t for t with order of magnitude as given in the following table:

$\mathscr{A}$ ↓ \ $\mathscr{A}'$ →	$\mathscr{N}$	$\mathscr{Z}$	$\mathscr{G}^+$	$\mathscr{G}$	$\mathscr{E}$	$\mathscr{E}'$	$\mathscr{N}'$
$\mathscr{N}$	1	1	1	1	1	1	1
$\mathscr{Z}$	n	1	1	1	1	1	1
$\mathscr{G}^+$	n	n	1	1	1	1	1
$\mathscr{G}$	n^2	n	n	1	1	1	1
$\mathscr{E}$	2^n	2^n	$2^{n/2}$	$2^{n/2}$	1	1	1
$\mathscr{E}'$	2^n	2^n	$2^{n/2}$	$2^{n/2}$	1	1	1
$\mathscr{N}'$	2^n	2^n	$2^{n/2}$	$2^{n/2}$	n	n	1

All of these bounds can be calculated directly, or else some can be calculated directly and then others inferred from the main composition theorem of [2, Theorem 3.5]:

THEOREM 2.1 [2]. *Assume* $\mathscr{A} \leqslant_\tau^{\exp} \mathscr{A}'$ *with complexity* t, *where* t *is nondecreasing and not identically equal to* 0. *Assume* $\mathscr{A}' \leqslant_{\tau'}^{\exp} \mathscr{A}''$ *with complexity* t', *where* t' *is nondecreasing. Then* $\mathscr{A} \leqslant_{\tau \circ \tau'}^{\exp} \mathscr{A}''$ *with complexity* t'', *where* $t''(n) = \sum_{i=0}^{\lfloor t(n) \rfloor - 1} t'(\lfloor \sum_{j=0}^{n-1} t(j) \rfloor + i)$.

An outline of those upper bound proofs which are not entirely trivial is as follows.

$\mathscr{Z} \leqslant_\tau^{\exp} \mathscr{S}^+$: Use $\tau((x, y)) = x - y$. Then 0 is a τ-simulator of 0; suc is a τ-simulator of suc, and up is a τ-simulator of pred.

$\mathscr{S} \leqslant_\tau^{\exp} \mathscr{E}$: Define τ so that a τ-simulator of 0 is λ, and τ-simulators of the four unary operations are $\lambda x[x00]$, $\lambda x[x01]$, $\lambda x[x10]$ and $\lambda x[x11]$, respectively.

$\mathscr{E}' \leqslant_\tau^{\exp} \mathscr{N}'$: Define τ so that a τ-simulator of λ is 1 and τ-simulators of the three unary operations are $\lambda x[3x]$, $\lambda x[3x+1]$ and $\lambda x[3x+2]$, respectively.

All of the other codings above the major diagonal either are similar to or else follow from these using Theorem 2.1.

$\mathscr{S}^+ \leqslant_\tau^{\exp} \mathscr{N}$: Define $\tau((x+y)^2 + 3x + y/2) = (x, y)$; τ is Cantor's projection function.

Now, the bounds for $\mathscr{Z} \leqslant^{\exp} \mathscr{N}$ and $\mathscr{S}^+ \leqslant^{\exp} \mathscr{Z}$ follow from the previous bound and Theorem 2.1.

$\mathscr{S} \leqslant_\tau^{\exp} \mathscr{N}$: Define τ to be a bijection between elements of N and pairs in $Z \times Z$, by regarding the $Z \times Z$ elements as points in the Euclidean plane as usual, and enumerating these points in a "spiral" working outward from $(0, 0)$. The reader is referred to the definition of "$\mathscr{A} \leqslant_\tau^{\exp} \mathscr{A}'$ with weak complexity t" [2, Sect. 2] and Theorem 2.3 [2] in order to verify this bound. Note that this bound is quadratic (rather than linear) because of the necessity of simulating the four unary operations by generating their representing values from 0 (rather than from representations of their arguments).

$\mathscr{S} \leqslant_\tau^{\exp} \mathscr{Z}$: This is similar to the previous case, with the same spiral numbering used to code elements of $\mathscr{S}$ as *nonnegative* integers. This time, however, the bound is linear, because the representations of pred(x) and *up* (x) can be generated quickly within $\mathscr{Z}$ from the representation of the argument x.

The bound for $\mathscr{S} \leqslant^{\exp} \mathscr{S}^+$ now follows by Theorem 2.1.

$\mathscr{E} \leqslant_\tau^{\exp} \mathscr{S}^+$: Simply correspond the set of all strings in order according to length, with elements of $N \times N$ in order of increasing sum of coordinates. Generating values from 0 suffices.

The bound for $\mathscr{E} \leqslant^{\exp} \mathscr{S}$, and also those for $\mathscr{E} \leqslant^{\exp} \mathscr{N}$ and $\mathscr{E} \leqslant^{\exp} \mathscr{Z}$ follow by Theorem 2.1. A bit of calculation is required for the second of these bounds. Namely, Theorem 2.1 immediately yields that for a constant c, $\mathscr{E} \leqslant^{\exp} \mathscr{N}$ with complexity

$$\sum_{i=0}^{\lfloor c \cdot 2n/2 \rfloor - 1} c\left(\left\lfloor \sum_{j=0}^{n-1} c \cdot 2^{j/2} \right\rfloor + i\right)$$

$$\leqslant c\left[\lfloor c \cdot 2^{n/2} \rfloor \cdot c \cdot \sum_{j=0}^{n-1} 2^{j/2} + \sum_{i=0}^{\lfloor c \cdot 2^{n/2} \rfloor - 1} i\right]$$

$$= c\left[\lfloor c \cdot 2^{n/2} \rfloor \cdot c \cdot \sum_{j=0}^{n-1} 2^{j/2} + \frac{(\lfloor c \cdot 2^{n/2} \rfloor - 1)(\lfloor c \cdot 2^{n/2} \rfloor)}{2}\right],$$

which is $0(2^n)$. The results for $\mathscr{E}' \leqslant^{\exp} \mathscr{S}^+$, $\mathscr{E}' \leqslant^{\exp} \mathscr{S}$, $\mathscr{E}' \leqslant^{\exp} \mathscr{N}$ and $\mathscr{E}' \leqslant^{\exp} \mathscr{Z}$ are analogous, and the result for $\mathscr{E}' \leqslant^{\exp} \mathscr{E}$ is easy.

$\mathscr{N}' \leqslant_\tau^{\exp} \mathscr{S}^+$: Once again, Cantor's pairing function is used, but this time to represent elements of N by elements of $N \times N$. We show this bound in some detail, since $\mathscr{N}'$ is the only algebra in this paper having a binary operation. If $\text{size}_{\mathscr{F}\!ree(\mathscr{N}')}(\{e, e'\}) = n$, then $\text{size}_{\mathscr{F}\!ree(\mathscr{N}')}(\{e + e'\}) \leqslant n + 1$, and so $\text{val}(e + e') \leqslant 2^n$. Then $\text{size}_{\mathscr{S}^+}(\rho_{\mathscr{E}}(e + e')) \leqslant c \cdot 2^{n/2}$ for some c.

Now, we can infer the bounds $\mathscr{N}' \leqslant^{\exp} \mathscr{S}$, $\mathscr{N}' \leqslant^{\exp} \mathscr{N}$ and $\mathscr{N}' \leqslant^{\exp} \mathscr{Z}$, as before.

$\mathscr{N}' \leqslant_\tau^{\exp} \mathscr{E}$: The argument is similar to the preceding case, using a correspondence between elements of N in increasing order and strings in the natural order given in the case for $\mathscr{E} \leqslant_\tau^{\exp} \mathscr{S}^+$.

The bound for $\mathscr{N}' \leqslant_\tau^{\exp} \mathscr{E}'$ then follows.

The extent to which accessibility upper bounds such as those of this section can actually be realized, as upper bounds on computation time in programming languages based on the given algebras, depends both on the control structures of the programming languages and on the relations allowed.

III. Lower Bounds

In contrast to the significance of accessibility for upper bounds, accessibility lower bounds are a priori lower bounds for flowchart and other program running time. We now prove lower bounds for the pairs of given algebras, which are tight up to order of magnitude in all cases but one. The exception is $\mathscr{S} \leqslant^{\exp} \mathscr{S}^+$, where the best lower bound we have is of the form $\Omega(n^{1/2})$ rather than $\Omega(n)$.

We require some definitions for lower bounds. An i.o. (infinitely often) style lower bound seems most appropriate, so we use the following. We say $t: N \to R^+$ *is a lower bound for* $\mathscr{A} \leqslant^{\exp} \mathscr{A}'$ if t is total, nondecreasing and unbounded, and for any $t' = t$ a.e. (that is, on all but finitely many arguments), it is false that $\mathscr{A} \leqslant^{\exp} \mathscr{A}'$ with complexity t.

An equivalent definition is provided by the following easily proved theorem.

THEOREM 3.1. *Let $t: N \to R^+$ be total, nondecreasing and unbounded. Then t is a lower bound for $\mathscr{A} \leqslant^{\exp} \mathscr{A}'$ iff the following condition holds.*

For any τ, $\mathscr{E}$ having $\mathscr{A} \leqslant_{\tau,\mathscr{E}}^{\exp} \mathscr{A}'$, there is some $f \in \text{Fun}_{\mathscr{A}}$ *and there are infinitely many $n \in N$ for which $(\exists e_1,..., e_m \in \text{Dom}_{\mathscr{F}\!ree(\mathscr{A})})[\text{size}_{\mathscr{F}\!ree(\mathscr{A})}(\{e_1,..., e_n\}) \leqslant n$ and $f(e_1,..., e_m)$ is defined, and $\text{size}_{\mathscr{A}'}(\rho_{\mathscr{E}}(f(e_1,..., e_m)): \rho_{\mathscr{E}}(\{e_1,..., e_m\})) > t(n)]$.*

Proof. Straightforward. (Note that the finiteness of $\text{Fun}_{\mathscr{A}}$ is used here.) ∎

The given definition is most suitable for application of Theorem 2.1, while the equivalent version provided by Theorem 3.1 is usually more convenient for direct lower bound proofs.

With the exception of the single entry for $\mathscr{S} \leqslant^{\exp} \mathscr{S}^+$, we can show that each nonconstant entry in the table of Section II is a lower bound for the given pair of algebras, in the following sense. If $t(n) \not\equiv 1$ appears as the entry in the row for $\mathscr{A}$ and column for $\mathscr{A}'$, then for some $c > 0$, the function ct is a lower bound for $\mathscr{A} \leqslant^{\exp} \mathscr{A}'$. Also, for some $c > 0$, the function $cn^{1/2}$ is a lower bound for $\mathscr{G} \leqslant^{\exp} \mathscr{G}^+$.

The techniques are generally size-of-neighborhood arguments somewhat like those in [5], with some complications introduced by the fact that an algebra's constants are always available in one step, by the presence of binary operations, and by "directionality" considerations. Once again, some results are proved directly and others using Theorem 2.1. An outline of the lower bound proofs follows.

$\mathscr{Z} \leqslant^{\exp} \mathscr{N}$: We state this bound as a theorem, since its proof is fairly complicated and since its proof ideas are also used in the proofs of other lower bounds. It is a typical example of a coding-independent tradeoff lower bound for embedding a two-directional system in a corresponding one-directional system.

THEOREM 3.2. *There exists $c > 0$ for which $\lambda n[cn]$ is a lower bound for $\mathscr{Z} \leqslant^{\exp} \mathscr{N}$.*

Proof. Choose c sufficiently small. ($c = (5 - \sqrt{21})/8$ suffices.) We show that, if $\lambda n[cn]$ is not a lower bound, then there are many pairs of expressions e, $e' \in \mathrm{Dom}_{\mathscr{F}\!ree\,(\mathscr{Z})}$, having $\mathrm{val}(e) = \mathrm{val}(e')$, but $\rho_{\mathscr{g}}(e) \neq \rho_{\mathscr{g}}(e')$. Then neighborhood size limitations are used to give the result. Although these basic ideas are not difficult, some care must be taken because of the one-step accessibility of 0 in $\mathscr{N}$.

Assume $\lambda n[cn]$ is not a lower bound for $\mathscr{Z} \leqslant^{\exp} \mathscr{N}$. That is, assume $\mathscr{Z} \leqslant^{\exp}_{\tau,\mathscr{g}} \mathscr{N}$, $a \in N$, $a \geqslant 1/c$, and for all $e \in \mathrm{Dom}_{\mathscr{F}\!ree\,(\mathscr{Z})}$, $f \in \{\mathrm{suc}, \mathrm{pred}\}$ and $n \geqslant a$, it is the case that $\mathrm{size}_{\mathscr{F}\!ree\,(\mathscr{Z})}(e) \leqslant n$ implies $\mathrm{size}_{\mathscr{N}}(\rho_{\mathscr{g}}(f(e)) : \rho_{\mathscr{g}}(e)) \leqslant cn$.

Write $\mathrm{suc}^{(z)}(0)$ for the expression

$$\underbrace{\mathrm{suc}(\ldots\mathrm{suc}}_{z}(0)\ldots).$$

Let $b \in N$ be greater than a, and also greater than $\mathrm{size}_{\mathscr{N}}(\rho_{\mathscr{g}}(\mathrm{suc}^{(a)}(0)))$.

Let $A = \{e : \mathrm{size}_{\mathscr{F}\!ree\,(\mathscr{Z})}(e) \leqslant 2b$ and $b \leqslant \mathrm{val}(e)$ and $\mathrm{suc}^{(b)}(0)$ is a subexpression of $e\}$. Note that $\mathrm{val}(A) = \{b, \ldots, 2b-1\}$. Then (1) and (2) follow.

$$\text{For all } e \in A, f \in \{\mathrm{suc}, \mathrm{pred}\}, \text{ it is the case that}$$
$$\mathrm{size}_{\mathscr{N}}(\rho_{\mathscr{g}}(f(e)) : \rho_{\mathscr{g}}(e)) \leqslant 2bc. \tag{1}$$

$$\text{For all } k < b, \text{ it is the case that}$$
$$\mathrm{size}_{\mathscr{N}}(\rho_{\mathscr{g}}(\mathrm{suc}^{(k+1)}(0)) : \rho_{\mathscr{g}}(\mathrm{suc}^{(k)}(0))) \leqslant bc. \tag{2}$$

By the Triangle Inequality for the size measure (Theorem 2.1(c) of [2]), applied $b - a$ times to (2), and by choice of b, it is easy to see that

$$\mathrm{size}_{\mathscr{N}}(\rho_{\mathscr{g}}(\mathrm{suc}^{(b)}(0))) < b + (b-a)(bc) \leqslant b^2 c. \tag{3}$$

There are two cases to consider.

Case 1. For some $e \in A, f \in \{\text{suc}, \text{pred}\}$, it is the case that $\text{size}_{\mathcal{N}}(\rho_{\mathscr{g}}(f(e))) > 2bc$ and $\rho_{\mathscr{g}}(f(e)) < \rho_{\mathscr{g}}(e)$.

But in this case, $\text{size}_{\mathcal{N}}(\rho_{\mathscr{g}}(f(e)): \rho_{\mathscr{g}}(e)) = \text{size}_{\mathcal{N}}(\rho_{\mathscr{g}}(f(e))) > 2bc$, contradicting (1).

Therefore, the only remaining possibility is the second case.

Case 2. For all $e \in A$, $f \in \{\text{suc}, \text{pred}\}$, if $\text{size}_{\mathcal{N}}(\rho_{\mathscr{g}}(f(e))) > 2bc$, then $\rho_{\mathscr{g}}(f(e)) > \rho_{\mathscr{g}}(e)$.

(Note that equality cannot hold.)

Now, $|\{x: \text{size}_{\mathcal{N}}(x) \leqslant 2bc\}| \leqslant 2bc$; therefore, also $|\tau(\{x: \text{size}_{\mathcal{N}}(x) \leqslant 2bc\})| \leqslant 2bc$. But $|\text{val}(A)| = b$. Thus, if $B = \text{val}(A) - \tau(\{x: \text{size}_{\mathcal{N}}(x) \leqslant 2bc\}) \subseteq Z$, it follows that $|B| \geqslant b(1-2c)$.

B has the following property.

$$f \in \{\text{suc}, \text{pred}\}, e \in A \text{ and } f(\text{val}(e)) \in B \text{ together imply that } \rho_{\mathscr{g}}(f(e)) > \rho_{\mathscr{g}}(e). \tag{4}$$

To see (4), let $f \in \{\text{suc}, \text{pred}\}$, assume $e \in A$ and $f(\text{val}(e)) \in B$. Then $\tau(\rho_{\mathscr{g}}(f(e))) = \text{val}(f(e)) \notin \tau(\{x: \text{size}_{\mathcal{N}}(x) \leqslant 2bc\})$. Thus, $\text{size}_{\mathcal{N}}(\rho_{\mathscr{g}}(f(e))) > 2bc$. Then the definition for Case 2 yields (4).

Let $C = \{x \in B: x+1 \text{ or } x-1 \in B\}$. Then $|C| \geqslant b(1-2c) - 2bc = b - 4bc$ because at most $2bc$ members x of B have the property that $x+1 \notin B$ and $x-1 \notin B$. Note that $x \in C$ implies $x+1$ or $x-1 \in C$.

Next, we make the key claim which gives the large number of representations which much be squeezed into a small neighborhood.

$$|\{\rho_{\mathscr{g}}(e): e \in A \text{ and } \text{val}(e) \in C\}| \geqslant b^2(\tfrac{1}{2} - 2c)^2. \tag{5}$$

We show (5). Consider any $e \in A$ with $\text{size}_{\mathscr{Free}(\mathscr{Z})}(e) < 2b$ and $\text{val}(e) \in C$. By definition of C, either $\text{val}(e)+1$ or $\text{val}(e)-1$ is in C. Assume $\text{val}(e)+1 \in C$. Property (4) implies that $\rho_{\mathscr{g}}(\text{suc}(e)) > \rho_{\mathscr{g}}(e)$. Now, $\text{suc}(e) \in A$ (since $\text{size}_{\mathscr{Free}(\mathscr{Z})}(e) < 2b$). Also, $\text{val}(\text{suc}(e)) - 1 = \text{val}(e) \in B$, so (4) implies that $\rho_{\mathscr{g}}(\text{pred}(\text{suc}(e))) > \rho_{\mathscr{g}}(\text{suc}(e))$. Thus, $\rho_{\mathscr{g}}(\text{pred}(\text{suc}(e))) > \rho_{\mathscr{g}}(e)$; that is, e and $\text{pred}(\text{suc}(e))$ have two distinct $\rho_{\mathscr{g}}$-images. Analogously, if $\text{val}(e)-1 \in C$, then $\rho_{\mathscr{g}}(\text{suc}(\text{pred}(e))) > \rho_{\mathscr{g}}(e)$.

This argument can be repeated to show that if $\text{val}(e)+1 \in C$, then e, $\text{pred}(\text{suc}(e))$, $\text{pred}(\text{suc}(\text{pred}(\text{suc}(e))))$,... (as long as the sizes of these expressions are all at most $2b+1$) all have distinct $\rho_{\mathscr{g}}$-images. Analogously, if $\text{val}(e)-1 \in C$, then e, $\text{suc}(\text{pred}(e))$, $\text{suc}(\text{pred}(\text{suc}(\text{pred}(e))))$,... have distinct $\rho_{\mathscr{g}}$-images. So consider any $x \in C$. Let $e = \text{suc}^{(x)}(0)$, so that $e \in A$ and $\text{size}_{\mathscr{Free}(\mathscr{Z})}(e) = x+1$. Repeated application of the argument in the preceding paragraph, beginning with e, shows that

$$|\{\rho_{\mathscr{g}}(e'): e' \in A \text{ and } \text{val}(e') = x\}| \geqslant b - \left\lceil \frac{x-1}{2} \right\rceil.$$

Now consider all x in C. It follows from the number of distinct elements in C and the preceding paragraph that

$$|\{\rho_{\mathscr{g}}(e)\colon e \in A \text{ and } \mathrm{val}(e) \in C\}| \geqslant \underbrace{1+1+2+2+3+\cdots}_{\lceil(1-4c)b\rceil \text{terms}} \geqslant b^2(\tfrac{1}{2}-2c)^2.$$

(This sum represents the worst case—where all the elements of C are greater than all the elements of $B-C$.) Thus, we have shown (5).

The rest of the argument is a consequence of the bound on neighborhood size in $\mathscr{N}$. Namely, by (5), there must exist $x \in \{\rho_{\mathscr{g}}(e)\colon e \in A$ and $\mathrm{val}(e) \in C\}$ with $\mathrm{size}_{\mathscr{N}}(x) \geqslant b^2(\frac{1}{2}-2c)^2$. Fix $e \in A$ with $\mathrm{val}(e) \in C$ and $\mathrm{size}_{\mathscr{N}}(\rho_{\mathscr{g}}(e)) \geqslant b^2(\frac{1}{2}-2c)^2$. Now, $\mathrm{size}_{\mathscr{N}}(\rho_{\mathscr{g}}(e)) \leqslant \mathrm{size}_{\mathscr{N}}(\rho_{\mathscr{g}}(e)\colon \rho_{\mathscr{g}}(\mathrm{suc}^{(b)}(0))) + \mathrm{size}_{\mathscr{N}}(\rho_{\mathscr{g}}(\mathrm{suc}^{(b)}(0)))$. Thus $b^2(\frac{1}{2}-2c)^2 < \mathrm{size}_{\mathscr{N}}(\rho_{\mathscr{g}}(e)\colon \rho_{\mathscr{g}}(\mathrm{suc}^{(b)}(0))) + b^2 c$ by (3). That is, $b^2(\frac{1}{4}-3c+4c^2) < \mathrm{size}_{\mathscr{N}}(\rho_{\mathscr{g}}(e)\colon \rho_{\mathscr{g}}(\mathrm{suc}^{(b)}(0)))$. By choice of c, the left-hand side of this inequality is at least $b^2(2c)$.

The Triangle Inequality is now used to expand the right-hand side completely, using successive substrings of e, all of which are in A. There are at most $b-1$ terms in this expansion, so (1) implies that $\mathrm{size}_{\mathscr{N}}(\rho_{\mathscr{g}}(e)\colon \rho_{\mathscr{g}}(\mathrm{suc}^{(b)}(0))) \leqslant (b-1)\,2bc$. This contradicts the lower bound obtained in the preceding paragraph.

Thus, the initial assumption, that $\lambda n[cn]$ is not a lower bound, is false. ∎

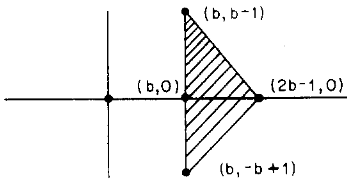
$\mathscr{G}^+ \leqslant^{\exp} \mathscr{Z}$: We show that $t(n) = n/8$ is a lower bound for $\mathscr{G}^+ \leqslant^{\exp} \mathscr{Z}$. For if not, then let $\mathscr{G}^+ \leqslant^{\exp}_{\tau,\mathscr{g}} \mathscr{Z}$, $a \geqslant 1$ be such that for all $e \in \mathrm{Dom}_{\mathscr{F}ree\,(\mathscr{G}^+)}$, $f \in \{\mathrm{suc}, \mathrm{up}\}$ and $n \geqslant a$ it is the case that $\mathrm{size}_{\mathscr{F}ree\,(\mathscr{G}^+)}(e) \leqslant n$ implies $\mathrm{size}_{\mathscr{Z}}(\rho_{\mathscr{g}}(f(e))\colon \rho_{\mathscr{g}}(e)) \leqslant n/8$. Then if $\mathrm{size}_{\mathscr{F}ree\,(\mathscr{G}^+)}(e) \leqslant a+1$, it follows by the Triangle Inequality applied a times that $\mathrm{size}_{\mathscr{Z}}(\rho_{\mathscr{g}}(e)\colon \rho_{\mathscr{g}}(0)) \leqslant a^2/8$. But there are $(a+1)(a+2)/2$ distinct elements x in $N \times N$ with $\mathrm{size}_{\mathscr{G}^+}(x) \leqslant a+1$. For each x, there is some $e \in \mathrm{Dom}_{\mathscr{F}ree\,(\mathscr{G}^+)}$ with $\mathrm{val}(e) = x$ and $\mathrm{size}_{\mathscr{F}ree\,(\mathscr{G}^+)}(e) \leqslant a+1$, and since $\mathrm{val}(e) = \tau(\rho_{\mathscr{g}}(e))$, it follows that all $\rho_{\mathscr{g}}(e)$ for these x are distinct. Thus, there are at least $(a+1)(a+2)/2$ distinct values y in Z with $\mathrm{size}_{\mathscr{Z}}(y\colon \rho_{\mathscr{g}}(0)) \leqslant a^2/8$. But this is impossible because of the structure of $\mathscr{Z}$.

We can now use the first version of the lower bound definition, Theorem 2.1 and the upper bounds of Section II to derive the lower bounds for $\mathscr{G}^+ \leqslant^{\exp} \mathscr{N}$ and $\mathscr{G} \leqslant^{\exp} \mathscr{Z}$. For instance, consider $\mathscr{G}^+ \leqslant^{\exp} \mathscr{N}$. We know that $\mathscr{N} \leqslant^{\exp} \mathscr{Z}$ with complexity c for some constant $c > 0$, and that there exists $c' > 0$ with no t having both $t(n) = c'n$ a.e. and $\mathscr{G}^+ \leqslant^{\exp} \mathscr{Z}$ with complexity t. Let $c'' = c'/c > 0$, and show that no t having $t(n) = c''n$ a.e. can have $\mathscr{G}^+ \leqslant^{\exp} \mathscr{N}$ with complexity t. For if so, then Theorem 2.1 implies that $\mathscr{G}^+ \leqslant^{\exp} \mathscr{Z}$ with complexity t', where $t'(n) = c\lfloor t(n) \rfloor$; then $t'(n) \leqslant c'n$ a.e. This is a contradiction. A similar argument is used for the other result.

$\mathscr{G} \leqslant^{\exp} \mathscr{G}^+$: This is the case for which our upper and lower bounds differ. The argument is fairly complicated, so we present it as a separate theorem. The proof follows the style of the proof of Theorem 3.2.

THEOREM 3.3. *There exists $c < 0$ for which $\lambda n[c\sqrt{n}]$ is a lower bound for $\mathscr{G} \leqslant^{\exp} \mathscr{G}^+$.*

Proof. Write $\pi_1(a, b) = a$ and $\pi_2(a, b) = b$ for $(a, b) \in N \times N$ or $Z \times Z$. Let $c = 1/10$. Assume $\lambda n[c\sqrt{n}]$ is not a lower bound for $\mathscr{G} \leqslant^{\exp} \mathscr{G}^+$. That is, assume $\mathscr{G} \leqslant^{\exp}_{\tau,g} \mathscr{G}^+$, $a \in N$, $a \geqslant 1/c$ and for all $e \in \mathrm{Dom}_{\mathscr{T}ree(\mathscr{G})}$, $f \in \mathrm{Fun}_{\mathscr{G}} - \{0\}$, and $n \geqslant a$ it is the case that $\mathrm{size}_{\mathscr{T}ree(\mathscr{G})}(e) \leqslant n$ implies $\mathrm{size}_{\mathscr{G}^+}(\rho_g(f(e))\colon \rho_g(e)) \leqslant c\sqrt{n}$. Choose $b \in N$ with $b > a^2$, and $b > \mathrm{size}_{\mathscr{G}^+}(\rho_g(\mathrm{suc}^{(a)}(0)))$. Let $A = \{e\colon \mathrm{size}_{\mathscr{T}ree(\mathscr{G})}(e) \leqslant 2b$ and $\pi_1(\mathrm{val}(e)) \geqslant b$ and $\mathrm{suc}^{(b)}(0)$ is a subexpression of $e\}$. Thus, $\mathrm{val}(A)$ can be represented by the lattice points in the shaded area depicted below:



Note that $|\mathrm{val}(A)| = b^2$. Then (1) and (2) hold.

(1) For all $e \in A, f \in \mathrm{Fun}_{\mathscr{G}} - \{0\}$, it is the case that

$$\mathrm{size}_{\mathscr{G}^+}(\rho_g(f(e))\colon \rho_g(e)) \leqslant c\sqrt{2b}.$$

(2) For all $k < b$, it is the case that

$$\mathrm{size}_{\mathscr{G}^+}(\rho_g(\mathrm{suc}^{(k+1)}(0))\colon \rho_g(\mathrm{suc}^{(k)}(0))) \leqslant c\sqrt{b}.$$

Then $\mathrm{size}_{\mathscr{G}^+}(\rho_g(\mathrm{suc}^{(b)}(0))) < b + (b-a)(c\sqrt{b}) \leqslant 2bc\sqrt{b}$ by the Triangle Inequality and assumptions on the magnitudes of a and b.

We consider two cases.

Case 1. *For some e in A,* $f \in \mathrm{Fun}_{\mathscr{G}} - \{0\}$, it is the case that $\mathrm{size}_{\mathscr{G}^+}(\rho_g(f(e))) > c\sqrt{2b}$ and either $\pi_1(\rho_g(f(e))) < \pi_1(\rho_g(e))$ or $\pi_2(\rho_g(f(e))) < \pi_2(\rho_g(e))$.

But a contradiction to (1) can be reached in this case, so this is impossible. Thus, Case 2 holds.

Case 2. For all e in A, f in $\mathrm{Fun}_{\mathscr{G}} - \{0\}$, if $\mathrm{size}_{\mathscr{G}^+}(\rho_g(f(e))) > c\sqrt{2b}$, then $\pi_1(\rho_g(f(e))) \geqslant \pi_1(\rho_g(e))$ and $\pi_2(\rho_g(f(e))) \geqslant \pi_2(\rho_g(e))$, and at least one of these two last inequalities is strict.

Now, $|\{x\colon \mathrm{size}_{\mathscr{G}^+}(x) \leqslant c\sqrt{2b}\}| \leqslant 2c^2b$, so $|\tau(\{x\colon \mathrm{size}_{\mathscr{G}^+}(x) \leqslant c\sqrt{2b}\})| \leqslant 2c^2b$. But $|\mathrm{val}(A)| = b^2$. So if $B = \mathrm{val}(A) - \tau(\{x\colon \mathrm{size}_{\mathscr{G}^+}(x) \leqslant c\sqrt{2b}\})$, then $|B| \geqslant b^2 - 2c^2b$. B has the following property.

(3) $f \in \mathrm{Fun}_{\mathscr{G}} - \{0\}$, $e \in A$ and $f(\mathrm{val}(e)) \in B$ together imply that $\rho_g(f(e))$ has both its components at least as great as the corresponding components of $\rho_g(e)$, and at least one is strictly greater.

Let $C = \{x \in B\colon f(x) \in B$ for some $f \in \mathrm{Fun}_{\mathscr{G}} - \{0\}\}$. Then $|C| \geqslant b^2 - 4c^2b$ (since at most $2c^2b$ members x of B have the property that $f(x) \notin B$ for all $f \in \mathrm{Fun}_{\mathscr{G}} - \{0\}$) and $[x \in C$ implies $f(x) \in C$ for some $f \in \mathrm{Fun}_{\mathscr{G}} - \{0\}]$.

The key claim is now the following.

(4) $|\{\rho_{\mathscr{g}}(e): e \in A \text{ and } \mathrm{val}(e) \in C\}| \geqslant (\frac{1}{6} - 2c^2)\, b^3$.

The bound is calculated as in the proof of Theorem 3.2, where now the sum to be bounded is:

$$\underbrace{1 + \cdots + 1}_{4(b-1)\text{ terms}} + \underbrace{2 + \cdots + 2}_{4(b-3)\text{ terms}} + \underbrace{3 + \cdots + 3}_{4(b-5)\text{ terms}} + \cdots,$$

where the total number of terms is $\lceil b^2 - 4c^2 b \rceil$. But this sum can be seen (using an estimate by integration) to be at least $\frac{1}{6}b^3 - 4c^2 b(\lceil b/2 \rceil)$, which is at least $(\frac{1}{6} - 2c^2)\, b^3$.

Thus, there must exist $x \in \{\rho_{\mathscr{g}}(e): e \in A$ and $\mathrm{val}(e) \in C\}$ with $\mathrm{size}_{\mathscr{G}^+}(x) \geqslant \sqrt{2(\frac{1}{6} - 2c^2)\, b^3} - 1$. Fix $e \in A$ with $\mathrm{val}(e) \in C$ and $\mathrm{size}_{\mathscr{G}^+}(\rho_{\mathscr{g}}(e)) \geqslant \sqrt{2(\frac{1}{6} - 2c^2)\, b^3} - 1$. Now, $\mathrm{size}_{\mathscr{G}^+}(\rho_{\mathscr{g}}(e)) \leqslant \mathrm{size}_{\mathscr{G}^+}(\rho_{\mathscr{g}}(e): \rho_{\mathscr{g}}(\mathrm{suc}^{(b)}(0))) + \mathrm{size}_{\mathscr{G}^+}(\rho_{\mathscr{g}}(\mathrm{suc}^{(b)}(0)))$. Thus, $\sqrt{2(\frac{1}{6} - 2c^2)\, b^3} - 1 < \mathrm{size}_{\mathscr{G}^+}(\rho_{\mathscr{g}}(e): \rho_{\mathscr{g}}(\mathrm{suc}^{(b)}(0))) + 2bc\sqrt{b}$. That is, $(\sqrt{\frac{1}{3} - 4c^2} - 2c)\, b^{3/2} - 1 < \mathrm{size}_{\mathscr{G}^+}(\rho_{\mathscr{g}}(e): \rho_{\mathscr{g}}(\mathrm{suc}^{(b)}(0)))$. By choice of c and the bounds on b, the left side of this inequality is at least $c\sqrt{2}\, b^{3/2}$.

The Triangle Inequality is now used to expand the right-hand side completely by successive substrings of e, all of which are in A. There are at most $b - 1$ terms in this expansion, so (1) implies that $\mathrm{size}_{\mathscr{G}^+}(\rho_{\mathscr{g}}(e): \rho_{\mathscr{g}}(\mathrm{suc}^{(b)}(0))) \leqslant (b-1)\, c\sqrt{2b}$. But this contradicts the lower bound obtained in the preceding paragraph. ∎

$\mathscr{G} \leqslant^{\exp} \mathscr{N}$: This lower bound is shown by a proof similar to those for Theorems 3.2 and 3.3.

$\mathscr{E} \leqslant^{\exp} \mathscr{Z}$: For this argument, we use the first lower bound definition. Let $t(n) = 2^n/8$. We show that $t(n)$ is a lower bound for $\mathscr{E} \leqslant^{\exp} \mathscr{Z}$. If not, then assume $t' = t$ a.e. and $\mathscr{E} \leqslant^{\exp}_{\tau,\mathscr{g}} \mathscr{Z}$ with complexity t'. For any n, there are $2^n - 1$ elements $x \in \{0, 1\}^*$ with $\mathrm{size}_{\mathscr{E}}(x) \leqslant n$. Therefore, for $n \geqslant 2$, there is some e with $\mathrm{size}_{\mathscr{F}_{\tau s e}(\mathscr{E})}(e) \leqslant n$, for which $\mathrm{size}_{\mathscr{Z}}(\rho_{\mathscr{g}}(e): \rho_{\mathscr{g}}(\lambda)) \geqslant 2^{n-2}$. But using the given upper bound and the Triangle Inequality, we see that $\mathrm{size}_{\mathscr{Z}}(\rho_{\mathscr{g}}(e): \rho_{e}(\lambda)) \leqslant \sum_{i=1}^{n-1} t'(i)$. For sufficiently large n, this sum will be strictly less than 2^{n-2}, a contradiction.

The bounds for $\mathscr{E}' \leqslant^{\exp} \mathscr{Z}$, $\mathscr{E} \leqslant^{\exp} \mathscr{N}$ and $\mathscr{E}' \leqslant^{\exp} \mathscr{N}$ now follow easily.

To conclude several of the other bounds, we use a lemma.

LEMMA 3.1. *There is a constant d with the following property. If $t(n) = c \cdot 2^{n/2}$* a.e. *and $t'(n) = c'n$, $\mathscr{A} \leqslant^{\exp} \mathscr{A}'$ with complexity t and $\mathscr{A}' \leqslant^{\exp} \mathscr{A}''$ with complexity t', then $\mathscr{A} \leqslant^{\exp} \mathscr{A}''$ with complexity t'', where $t'' = dc^2c'2^n$* a.e.

Proof. By a careful analysis of the bound yielded by Theorem 3.5 of [2]. Namely, we see immediately that $\mathscr{A} \leqslant^{\exp} \mathscr{A}''$ with complexity t'', where $t''(n) = c' \sum_{i=0}^{\lfloor t(n) \rfloor - 1} (\lfloor \sum_{j=0}^{n-1} t(j) \rfloor + i)$ for all n, $\leqslant c' \sum_{i=0}^{\lfloor c \cdot 2^{n/2} \rfloor - 1} (\lfloor \sum_{j=0}^{n-1} c \cdot 2^{j/2} \rfloor + i + k)$ for some constant k and all except finitely many n. This expression is at most

$$c'\left[\sum_{i=0}^{\lfloor c\cdot 2^{n/2}\rfloor-1}\left[\sum_{j=0}^{n-1} c\cdot 2^{j/2}\right]+\sum_{i=0}^{\lfloor c\cdot 2^{n/2}\rfloor-1} i+\sum_{i=0}^{\lfloor c\cdot 2^{n/2}\rfloor-1} k\right]$$

$$=c'\lfloor c\cdot 2^{n/2}\rfloor\left[c\sum_{j=0}^{n-1} 2^{j/2}\right]+\frac{c'(\lfloor c\cdot 2^{n/2}\rfloor-1)(\lfloor c\cdot 2^{n/2}\rfloor)}{2}+c'k\lfloor c\cdot 2^{n/2}\rfloor$$

$$\leqslant dc^2c'\cdot 2^n \text{ a.e., for some } d \text{ independent of } c,\ c' \text{ and } k. \quad \blacksquare$$

Using Lemma 3.1 and lower bounds already proved, we can now infer the bounds for $\mathscr{E}\leqslant^{\exp}\mathscr{G}^+$, $\mathscr{E}\leqslant^{\exp}\mathscr{G}$, $\mathscr{E}'\leqslant^{\exp}\mathscr{G}^+$ and $\mathscr{E}'\leqslant^{\exp}\mathscr{G}$.

$\mathscr{N}'\leqslant^{\exp}\mathscr{Z}$: Similar to the proof for $\mathscr{E}\leqslant^{\exp}\mathscr{Z}$.

The bounds for $\mathscr{N}'\leqslant^{\exp}\mathscr{N}$, $\mathscr{N}'\leqslant^{\exp}\mathscr{G}^+$ and $\mathscr{N}'\leqslant^{\exp}\mathscr{G}$ follow.

$\mathscr{N}'\leqslant^{\exp}\mathscr{E}$: We show that $t(n)=n/8$ is a lower bound for $\mathscr{N}'\leqslant^{\exp}\mathscr{E}$. For if not, then let $\mathscr{N}'\leqslant^{\exp}_{\tau,\mathscr{E}}\mathscr{E}$, $a\geqslant 32$ be such that for all e, $e'\in \mathrm{Dom}_{\mathscr{F}ree(\mathscr{N}')}$, and all $n\geqslant a$ it is the case that $\mathrm{size}_{\mathscr{F}ree(\mathscr{N}')}(\{e,e'\})\leqslant n$ implies $\mathrm{size}_{\mathscr{E}}(\rho_{\mathscr{E}}(e+e')\colon \rho_{\mathscr{E}}(\{e,e'\}))\leqslant n/8$.

Let $A'=\{x\in N\colon \mathrm{size}_{\mathscr{N}'}(x)\leqslant a/2\}$. Then $|A'|\geqslant 2^{a/4}$, so choose an arbitrary $A\subseteq A'$, with $|A|=2^{a/4}$. For each $x\in A$, fix $e_x\in\mathrm{Dom}_{\mathscr{F}ree(\mathscr{N}')}$ with $\mathrm{size}_{\mathscr{F}ree(\mathscr{N}')}(e_x)\leqslant a/2$ and $\mathrm{val}(e_x)=x$. Note that $\mathrm{size}_{\mathscr{F}ree(\mathscr{N}')}(\{e_x,e_y\})\leqslant a$ for all x, $y\in A$.

For each $x\in A$, let $B_x=\{y\in A\colon \mathrm{size}_{\mathscr{E}}(\rho_{\mathscr{E}}(e_x+e_y)\colon\rho_{\mathscr{E}}(e_x))\leqslant a/8\}$. Then $|B_x|\leqslant 1(2^{a/8+1}-1)+(2^{a/8}-1)=3.2^{a/8}-2$. (The first term counts the values accessible from $\rho_{\mathscr{E}}(e_x)$ and the second counts those accessible from λ; note that λ itself requires one step.)

Note that all x and y in A have $\mathrm{size}_{\mathscr{E}}(\rho_{\mathscr{E}}(e_x+e_y)\colon\rho_{\mathscr{E}}(\{e_x,e_y\}))\leqslant a/8$, by assumption. Thus, we can define a function α with domain $\{(x,y)\colon x,y\in A$ and $x\neq y\}$ as follows. α assigns to (x,y) a pair (z,b) with $z\in\{x,y\}$, such that $\mathrm{size}_{\mathscr{E}}(\rho_{\mathscr{E}}(e_x+e_y)\colon\rho_{\mathscr{E}}(e_z))\leqslant a/8$. (This is possible since $\mathscr{E}$ has only unary operations.) $b\in\{\lambda,0,1\}^*$ codes a sequence of $\mathrm{Fun}_{\mathscr{E}}$ operations which can be applied to $\rho_{\mathscr{E}}(e_z)$ to yield $\rho_{\mathscr{E}}(e_x+e_y)$, with the length of the sequence at most $a/8$.

For each x, any y for which $\alpha(x,y)=(x,b)$ for some b, must be in B_x. Thus, at least $|A|-|B_x|-1\geqslant 2^{a/4}-(3.2^{a/8}-2)-1=2^{a/4}-3.2^{a/8}+1$ of the values $y\in A$, $y\neq x$ must have $\alpha(x,y)=(y,b)$ for some b.

Now, we show that α is "almost one-to-one," more precisely, that α is injective on $C=\{(x,y)\colon x\in A, y\in A-B_x\}$. If $\alpha(x,y)=\alpha(x',y')$, $y\notin B_x$ and $y'\notin B_{x'}$, then $\alpha(x,y)=(y,b)=(y',b)=\alpha(x',y')$. Thus, $y=y'$ and the same sequence of operations can be applied to $\rho_{\mathscr{E}}(e_y)$ to obtain either $\rho_{\mathscr{E}}(e_x+e_y)$ or $\rho_{\mathscr{E}}(e_{x'}+e_{y'})$. But then $\rho_{\mathscr{E}}(e_x+e_y)=\rho_{\mathscr{E}}(e_{x'}+e_{y'})$, so that $\tau(\rho_{\mathscr{E}}(e_x+e_y))=\tau(\rho_{\mathscr{E}}(e_{x'}+e_{y'}))$, or $\mathrm{val}(e_x+e_y)=\mathrm{val}(e_{x'}+e_{y'})$, or $x+x'=y+y'$. But then $x=x'$ and $y=y'$.

Now, $|C|\geqslant 2^{a/4}(2^{a/4}-3.2^{a/8}+1)$. Thus, $|\mathrm{range}(\alpha)|\geqslant 2^{a/4}(2^{a/4}-3.2^{a/8}+1)$. But $|\mathrm{range}(\alpha)|\leqslant 2^{a/4}(3.2^{a/8}-2)$, by definition of α. Thus, $2^{a/4}(2^{a/4}-3.2^{a/8}+1)\leqslant 2^{a/4}(3.2^{a/8}-2)$, so that $2^{a/4}+3\leqslant 6.2^{a/8}$, a contradiction.

Finally, the bound for $\mathscr{N}'\leqslant^{\exp}\mathscr{E}'$ follows from Theorem 2.1 and the bounds on $\mathscr{N}'\leqslant\mathscr{E}$ and $\mathscr{E}'\leqslant\mathscr{E}$.

IV. Multiplicity of Representation

In Section III, relative complexity was classified purely on the basis of accessibility. Here we consider the multiplicity of representations often used in Section III to achieve fast accessibility. There appear to be inherent tradeoffs between accessibility and number of representations in three cases: the codings of $\mathscr{Z}$ in $\mathscr{S}^+$, $\mathscr{E}'$ in $\mathscr{E}$ and $\mathscr{S}$ in $\mathscr{E}'$. We express these tradeoffs in this section in terms of our size parameter. Similar tradeoffs (for the third case) were demonstrated in [4, 1], in a framework of finite graphs and uniform bounds rather than algebras and size parameter bounds.

We begin with definitions expressing simultaneous bounds on accessibility and number of representations. The style is chosen to parallel the bound definitions of Sections II and III. Let s, t: $N \to R^+$ be nondecreasing and unbounded, τ, $\mathscr{E}$ as usual. We say $\mathscr{A} \leqslant^{\exp}_{\tau,\mathscr{E}} \mathscr{A}'$ *with time-value complexity* (t, s) provided $\mathscr{A} \leqslant^{\exp}_{\tau,\mathscr{E}} \mathscr{A}'$ with complexity t and if for all n, x, it is the case that $|\{\rho_{\mathscr{E}}(e)\colon \text{size}_{\mathscr{T}ree(\mathscr{A})}(e) \leqslant n$ and $\text{val}(e) = x\}| \leqslant s(n)$. We say $\mathscr{A} \leqslant^{\exp}_{\tau} \mathscr{A}'$ *with time-value complexity* (t, s) provided $(\exists\mathscr{E})[\mathscr{A} \leqslant^{\exp}_{\tau,\mathscr{E}} \mathscr{A}'$ with time-value complexity $(t, s)]$, and $\mathscr{A} \leqslant^{\exp} \mathscr{A}'$ *with time-value complexity* (t, s) provided $(\exists\tau)[\mathscr{A} \leqslant^{\exp}_{\tau} \mathscr{A}'$ with time-value complexity $(t, s)]$. We say (t, s) *is a time-value lower bound for* $\mathscr{A} \leqslant^{\exp} \mathscr{A}'$ if t, $s\colon N \to R^+$ are total, nondecreasing and unbounded, and for any $t' = t$ a.e., $s' = s$ a.e., it is false that $\mathscr{A} \leqslant^{\exp} \mathscr{A}'$ with time-value complexity (t', s'). As before, an equivalent formulation is the following. For any τ, $\mathscr{E}$ having $\mathscr{A} \leqslant^{\exp}_{\tau,\mathscr{E}} \mathscr{A}'$, there are infinitely many $n \in N$ for which at least one of (a), (b) holds.

(a) There is some $f \in \text{Fun}_{\mathscr{A}}$ for which $(\exists e_1, \ldots, e_m) \in \text{Dom}_{\mathscr{T}ree(\mathscr{A})}$ $[\text{size}_{\mathscr{T}ree(\mathscr{A})}(\{e_1, \ldots, e_m\}) \leqslant n$ and $f(e_1, \ldots, e_m)$ is defined and $\text{size}_{\mathscr{A}'}(\rho_{\mathscr{E}}(f(e_1, \ldots, e_m))\colon \rho_{\mathscr{E}}(\{e_1, \ldots, e_m\})) > t(n)]$.

(b) There is some $x \in \text{Dom}_{\mathscr{A}}$ with $|\{\rho_{\mathscr{E}}(e)\colon \text{size}_{\mathscr{T}ree(\mathscr{A})}(e) \leqslant n$ and $\text{val}(e) = x\}| > s(n)$.

THEOREM 4.1. *There exists a nonzero constant c such that $(\lambda n[c\sqrt{n}], \lambda n[cn])$ is a time-value lower bound for $\mathscr{Z} \leqslant^{\exp} \mathscr{S}^+$.*

Proof. We follow the method of Theorems 3.2 and 3.3. Choose c to be sufficiently small. Assume the contrary, so $\mathscr{Z} \leqslant^{\exp}_{\tau,\mathscr{E}} \mathscr{S}^+$, a is sufficiently large (with respect to c), such that for all $e \in \text{Dom}_{\mathscr{T}ree(\mathscr{Z})}$, $f \in \{\text{suc}, \text{pred}\}$ and $n \geqslant a$ it is the case that $\text{size}_{\mathscr{T}ree(\mathscr{Z})}(e) \leqslant n$ implies $\text{size}_{\mathscr{S}^+}(\rho_{\mathscr{E}}(f(e))\colon \rho_{\mathscr{E}}(e)) \leqslant c\sqrt{n}$, and also such that for all $x \in Z$ and $n \geqslant a$ it is the case that $|\{\rho_{\mathscr{E}}(e)\colon \text{size}_{\mathscr{T}ree(\mathscr{Z})}(e) \leqslant n$ and $\text{val}(e) = x\}| \leqslant cn$. Choose b sufficiently large (with respect to a). Let $A = \{e\colon \text{size}_{\mathscr{T}ree(\mathscr{Z})}(e) \leqslant 2b$ and $b \leqslant \text{val}(e)$ and $\text{suc}^{(b)}(0)$ is a subexpression of $e\}$. Then (1) and (2) follow.

(1) For all $e \in A$, $f \in \{\text{suc}, \text{pred}\}$, it is the case that $\text{size}_{\mathscr{S}^+}(\rho_{\mathscr{E}}(f(e))\colon \rho_{\mathscr{E}}(e)) \leqslant c\sqrt{2b}$.

(2) For all $x \in Z$, $|\{\rho_{\mathscr{E}}(e)\colon \text{size}_{\mathscr{T}ree(\mathscr{Z})}(e) \leqslant 2b$ and $\text{val}(e) = x\}| \leqslant 2bc$.

We consider two cases, identical to those of Theorem 3.3 except that $\{\text{suc}, \text{pred}\}$ is used in place of $\mathrm{Fun}_{\mathscr{G}}$. As before, Case 1 leads to a contradiction, so Case 2 holds. Construct B from A as in Theorem 3.3, and let $C = \{x \in B: f(x) \in B$ for some $f \in \{\mathrm{suc}, \mathrm{pred}\}\}$. Then $|C| \geqslant (1 - 4c^2)\, b$. Thus, since $C \subseteq \{b, \ldots, 2b - 1\}$, there is some $x \leqslant (1 + 4c^2)\, b$ in C. Fix such an x.

But then the argument for the justification of (5) in Theorem 3.2 shows that $|\{\rho_{\mathscr{E}}(e): e \in A$ and $\mathrm{val}(e) = x\}| \geqslant b - \lceil (x-1)/2 \rceil \geqslant b - \lceil (1+4c^2)(b-1)/2 \rceil > 2cb$ (since c is sufficiently small). Thus, $|\{\rho_{\mathscr{E}}(e): \mathrm{size}_{\mathscr{Tree}(\mathscr{X})}(e) \leqslant 2b$ and $\mathrm{val}(e) = x\}| > 2bc$, which contradicts (2). ∎

THEOREM 4.2. *There exists a nonzero constant c such that* $(\lambda n[cn], \lambda n[2^{cn}])$ *is a time-value lower bound for* $\mathscr{E}' \leqslant^{\exp} \mathscr{E}$.

Proof. This is again similar (but somewhat more difficult). Choose c sufficiently small. Assume $\mathscr{E}' \leqslant^{\exp}_{\tau, \mathscr{E}} \mathscr{E}$, a is sufficiently large, such that for all $e \in \mathrm{Dom}_{\mathscr{Tree}(\mathscr{E}')}$, $f \in \{0\mathrm{suc}, 1\mathrm{suc}, \mathrm{pred}\}$ and $n \geqslant a$ it is the case that $\mathrm{size}_{\mathscr{Tree}(\mathscr{E}')}(e) \leqslant n$ implies $\mathrm{size}_{\mathscr{E}}(\rho_{\mathscr{E}}(f(e)): \rho_{\mathscr{E}}(e)) \leqslant cn$, and also such that for all $x \in \{0, 1\}^*$ and $n \geqslant a$ it is the case that $|\{\rho_{\mathscr{E}}(e): \mathrm{size}_{\mathscr{Tree}(\mathscr{E}')}(e) \leqslant n$ and $\mathrm{val}(e) = x\}| \leqslant 2^{cn}$. Choose b sufficiently large. Let $A = \{e: \mathrm{size}_{\mathscr{Tree}(\mathscr{E}')}(e) \leqslant 2b$ and $\mathrm{val}(e)$ has 0^b as a prefix and $0\mathrm{suc}^{(b)}(\lambda)$ is a subexpression of $e\}$. Then $|\mathrm{val}(A)| = 2^b - 1$. Now (1) and (2) follow.

(1) For all $e \in A$, $f \in \{0\mathrm{suc}, 1\mathrm{suc}, \mathrm{pred}\}$, it is the case that $\mathrm{size}_{\mathscr{E}}(\rho_{\mathscr{E}}(f(e)): \rho_{\mathscr{E}}(e)) \leqslant 2bc$.

(2) For all $x \in \{0, 1\}^*$, $|\{\rho_{\mathscr{E}}(e): \mathrm{size}_{\mathscr{Tree}(\mathscr{E}')}(e) \leqslant 2b$ and $\mathrm{val}(e) = x\}| \leqslant 2^{2bc}$.

We consider two cases, the first of which leads to a contradiction of (1).

Case 1. For some $e \in A$, $f \in \{0\mathrm{suc}, 1\mathrm{suc}, \mathrm{pred}\}$, it is the case that $\mathrm{size}_{\mathscr{E}}(\rho_{\mathscr{E}}(f(e))) > 2bc$ and $\rho_{\mathscr{E}}(e)$ is not a prefix of $\rho_{\mathscr{E}}(f(e))$.

Case 2. For all $e \in A, f \in \{0\mathrm{suc}, 1\mathrm{suc}, \mathrm{pred}\}$, if $\mathrm{size}_{\mathscr{E}}(\rho_{\mathscr{E}}(f(e))) > 2bc$, then $\rho_{\mathscr{E}}(e)$ is a proper prefix of $\rho_{\mathscr{E}}(f(e))$.

Now, $|\{x: \mathrm{size}_{\mathscr{E}}(x) \leqslant 2bc\}| \leqslant 2^{2bc} - 1$, so if $B = \mathrm{val}(A) - \tau(\{x: \mathrm{size}_{\mathscr{E}}(x) \leqslant 2bc\})$, then $|B| \geqslant 2b - 2^{2bc}$. Then there is some $x \in \{0, 1\}^*$ such that $|x| = \lfloor (1 + 2c)\, b + 1 \rfloor$ and such that all strings xy with $|xy| \leqslant 2b - 1$ are in B. (That is, some element very near the root of the tree of values represented by A must be in B, along with all of its descendants in that tree of values.) Fix such an x. Also fix $e \in A$ with $\mathrm{size}_{\mathscr{Tree}(\mathscr{E}')}(e) = \lfloor (1 + 2c)\, b + 2 \rfloor$ and $\mathrm{val}(e) = x$.

Let $D = \{e' \in A$: $\mathrm{size}_{\mathscr{Tree}(\mathscr{E}')}(e') = \lfloor \frac{3}{2} b \rfloor$ and e' consists solely of applications of $0\mathrm{suc}$ and $1\mathrm{suc}$ to $e\}$. Then $|\rho_{\mathscr{E}}(D)| = |D| = 2^{\lfloor (3/2) b \rfloor - \lfloor (1+2c)b+2 \rfloor}$. The Triangle Inequality and (1) imply that all $e' \in D$ have $\mathrm{size}_{\mathscr{E}}(\rho_{\mathscr{E}}(e'): \rho_{\mathscr{E}}(e)) \leqslant b^2 c$. By definition of B and the condition for Case 2, we see that $\rho_{\mathscr{E}}(e)$ is a prefix of $\rho_{\mathscr{E}}(e')$ for all $e' \in D$. The three preceding statements together imply that there exists $E \subseteq D$, $|E| \geqslant 2^{\lfloor (3/2) b \rfloor - \lfloor (1+2c)b+2 \rfloor}/b^2 c$ such that no element of $\rho_{\mathscr{E}}(E)$ is a prefix of any other. But since c is small, we see that $|E| > 2^{2bc}$.

Now let $F = \{e' \in \mathrm{Dom}_{\mathscr{Tree}(\mathscr{E}')}$: $\mathrm{val}(e') = x$ and e' consists solely of applications of pred to elements of $E\}$. By definition of B and the condition for Case 2, it follows that each $\rho_{\mathscr{E}}(e')$, for $e' \in F$, has some $\rho_{\mathscr{E}}(e'')$ as a prefix, where $e'' \in E$; moreover, no two values of e' have the same value of e''. Thus, by incomparability of elements of E, it follows that $|\rho_{\mathscr{E}}(F)| > 2^{2bc}$. But this contradicts (2). ∎

Unlike the previous two bounds, the final bound we consider, that for $\mathscr{G} \leqslant^{\exp} \mathscr{E}'$, is probably not as sharp as it ought to be. The result that seemingly ought to hold is the following.

Conjecture. There exists a nonzero constant c such that $(\lambda n[c \log n], \lambda n[2^{cn}])$ is a time-value lower bound for $\mathscr{G} \leqslant^{\exp} \mathscr{E}'$.

We have attempted to redo the tradeoff arguments from [4.1] in terms of the size parameter. However, this effort has so far only succeeded for the degenerate case which proves a lower bound for time when the number of representation is limited to one. (The conference version of this paper, [3], contains a Theorem 9 which claims a more general tradeoff. The proof of this theorem has since been found to be incorrect.) The best result we have for this case is the following.

THEOREM 4.3. *There exists a nonzero constant c such that* $(\lambda n[c \log n], \lambda n[1])$ *is a time-value lower bound for* $\mathscr{G} \leqslant^{\exp} \mathscr{E}'$. *(Note: We take the liberty of temporarily regarding* $\log 0$ *as* 0.)

Proof. A version of an argument in [2] can be redone in terms of the size parameter. ∎

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