Easy Impossibility Proofs for Distributed Consensus Problems

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ABSTRACT

Easy proofs are given, of the impossibility of solving several consensus problems (Byzantine agreement, weak agreement, Byzantine firing squad, approximate agreement and clock synchronization) in certain communication graphs. It is shown that, in the presence of m faults, no solution to these problems exists for communication graphs with fewer than $3m+1$ nodes or less than $2m+1$ connectivity. While some of these results had previously been proved, the new proofs are much simpler, provide considerably more insight, apply to more general models of computation, and (particularly in the case of clock synchronization) significantly strengthen the results.

Keywords: Consensus, Byzantine agreement, clock synchronization, distributed computing and fault tolerance

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1. Introduction

In this paper, we present easy proofs for the impossibility of solving several consensus problems in particular communication graphs. We prove results for Byzantine agreement, weak agreement, the Byzantine firing squad problem, approximate agreement and clock synchronization. The bounds are all the same: tolerating m faults requires at least 3m + 1 nodes, and requires at least 2m + 1 connectivity in the communication graph. (The connectivity of a graph is the minimum number of nodes whose removal disconnects the graph. Also, we assume throughout that graphs have at least three nodes.) For a given value of m, we call graphs with fewer than 3m+1 nodes or less than 2m + 1 connectivity **inadequate graphs.**

Each of our proofs is an argument by contradiction. We assume that a given problem can be solved in a system with an inadequate communication graph, and construct a set of system behaviors, which cannot all satisfy the correctness conditions for the given problem, although they are required to do so. Versions of many of the results were already known, with proofs of this same general form. Our proofs differ from the earlier proofs in the technique we use to construct the set of behaviors. Our technique is simpler, and applies to more general models of distributed computation.

For Byzantine agreement, both bounds were already known [PSL,D]. The 3m + 1 node lower bound in [PSL] was proved only for a particular synchronous model of computation. Although carefully done, the proof is somewhat complicated and not as intuitive as one might like. In contrast, our proof is simple and transparent, and applies to general models of computation. A proof of the 2m + 1 connectivity lower bound was presented informally in [D]; we prove that bound more formally and for more general models.

For weak Byzantine agreement, the requirement of 3m + 1 nodes was known [L], but was proved using a complicated construction. The new proof is easy and extends to more general models (although not as general as those for Byzantine agreement and approximate agreement). The 2m + 1 connectivity requirement was previously unknown. The result for the Byzantine firing squad problem follows from a reduction to weak agreement in [CDDS]. We provide a direct proof. For approximate agreement, the 3m + 1 bound was noted, but not proved, in [DLPSW], while the 2m + 1 connectivity requirement was previously unknown.

For clock synchronization, the 3m + 1 node bound was proved in [DHS], with a complicated proof. The authors of [DHS] also claimed that they knew how to prove the corresponding 2m + 1 connectivity lower bound, but we presume that such a proof would also be complicated. We prove both the 3m + 1 node and the 2m + 1 connectivity bounds, for a much more general notion of clock synchronization than in [DHS]. These synchronization bounds assume that there is no direct way nodes can measure the passage of time, other than by reading their inaccurate hardware clocks.

Since we obtain the same lower bounds for each problem, one might think that the problems are equivalent in some sense. This is not the case. We see that the bounds for the different problems require different assumptions about the underlying model. For example, the lower bounds for Byzantine and approximate agreement work with virtually any reasonable computational model, while the lower bound for weak agreement requires a special assumption, placing a bound on the rate of propagation of information through the system. The bound for clock synchronization requires a different assumption about how devices can measure time. Many of the results are sensitive to small differences in underlying assumptions (about such factors as communication delay or the behaviors of faulty nodes). This paper helps to clarify these issues.

2. A Model of Distributed Systems

In order to make the impossibility results clear, concise and general, we introduce a simple model of distributed systems.

A **communication graph** is a directed graph G with node set **nodes(G)** and edge set **edges(G)**, such that the directed edges occur in pairs; edge (u,v) $\in$ edges(G) if and only if (v,u) $\in$ edges(G). (We consider a pair of directed edges rather than a single undirected edge in order to model the communication in each direction separately). We call the edge (u,v) an **outedge** of u, and an **inedge** of v. Given U a subset of nodes(G), the **subgraph** G_U induced by U is the graph containing all the nodes in U and all the edges between nodes in U. The **inedge border** of G_U is the set of edges from nodes outside U into U; that is, edges(G) $\cap$ ((nodes(G)\U) $\times$ U).

A system $\mathcal{G}$ is a communication graph G with an assignment of a **device** and an **input** to each node of G. Devices are undefined primitive objects. The specific inputs we consider are encodings of Booleans, real numbers or real-valued functions of time (e.g. local clocks). The particular type of input depends on the agreement problem addressed. If a node is assigned device A in system $\mathcal{G}$, we say that the node **runs** A. A **subsystem** $\mathcal{U}$ of $\mathcal{G}$ is any subgraph G_U of G with the associated devices and inputs.

Every system $\mathcal{G}$ has a **system behavior**, $\mathcal{E}$, which is a tuple containing a **behavior** of every node and edge in G. (We also describe $\mathcal{E}$ as a behavior of the communication graph G. Note that a system has exactly one behavior, while a graph may have several, depending on the devices and inputs assigned to the nodes.) The restriction of a system behavior $\mathcal{E}$ to the behaviors of the nodes and edges of a subgraph G_U of G is the **scenario** $\mathcal{E}_U$ of G_U in $\mathcal{E}$.

For now, we take node and edge behaviors as primitives. In more concrete and familiar models, a node or edge behavior might be a finite or infinite sequence of states, or a mapping from the positive reals to some state set, denoting state as a function of time. (We use the latter interpretation for later results). Less familiar

models might interpret behaviors as mappings from reals to states, or from transfinite ordinals to states. To obtain our first results, the precise interpretation of node and edge behaviors is unimportant. We need only restrict our model so that the following two axioms hold. (We assume these two axioms throughout the paper. Some of the later results require additional assumptions.)

Locality Axiom Let $\mathcal{G}$ and $\mathcal{G}'$ be systems with behaviors $\mathcal{E}$ and $\mathcal{E}'$, respectively, and isomorphic subsystems $\mathcal{U}$ and $\mathcal{U}'$, (with vertex sets U and U'). If the corresponding behaviors of the inedge borders of U and U' in $\mathcal{E}$ and $\mathcal{E}'$ are identical, then scenarios $\mathcal{E}_U$ and $\mathcal{E}_{U'}$ are identical.

At heart, the Locality axiom says that communication only takes place over the edges of the communication graph. In particular, it expresses the following property: The only parameters affecting the behavior of any local portion of a system are the devices and inputs at each local node, together with any information incoming over edges from the remainder of the system. If these parameters are the same in two behaviors, the local behaviors (scenarios) are the same.[1] Clearly, some such locality property must hold, or agreement is trivially achievable by having devices read other device's inputs directly.

Fault Axiom Let A be any device. Let $E_1,...,E_d$ be d edge behaviors, such that each E_i is the behavior of the i'th outedge, in some system behavior $\mathcal{E}^i$, of a node running A. Let u be any node with d outedges $(u,v_1),...,(u,v_d)$. There is a device F such that in any system in which u runs F, the behavior of each outedge (u,v_i) is E_i.

In this case, we write $F_A(E_1,...,E_d)$ for F. This axiom expresses a powerful masquerading capability of failed devices. Any behavior exhibited by a device over different edges in different system behaviors can be exhibited by a failed device in a single system behavior. When this axiom is significantly weakened (say, by adding an unforgeable signature assumption), the following impossibility results do not hold [LSP,PSL].

In order to establish the relevance of our impossibility results to more concrete models of distributed systems, it is sufficient to interpret our definitions in the particular model and then to prove the Locality and Fault axioms.

Our proofs utilize the graph-theoretic notion of a *covering*. For any graph G, let **neighbors** = $\{(u,V) \mid$ u is a node of G and V is the set of all nodes v such that there is an edge from v to u in G$\}$. A graph S *covers* G if there is a mapping φ from the nodes of S to the nodes of G that preserves "neighbors." That is, if node u of S has d neighbors $v_1,...,v_d$, and $\varphi(u) = w$ for a node w of G, then w has d neighbors $x_1,...,x_d$ and $\varphi(v_i) = x_i$ for $1 \leq i \leq d$. Under such a mapping, S looks locally like G.

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[1] For weak agreement and the firing squad problem, we need to extend this locality property to include time, as well.

Graph coverings play an important role in our understanding of the interaction of network topology and distributed computation. A discussion appears in [A], and indeed, some of our proofs are surprisingly similar to Angluin's. Similar techniques also appear in [IR], [B] and elsewhere.

3. Byzantine Agreement

We say that Byzantine agreement is possible in a graph G (with n nodes) if there exist n devices $A_1,...,A_n$ (which we call **agreement devices**), with the following properties.

Each agreement device A_u takes a Boolean input and chooses 1 or 0 as a result. (To model choosing a result, assume there is a function CHOOSE from behaviors of nodes running agreement devices to the set {0,1}.) A node u of G is **correct** in a behavior $\mathcal{E}$ of G if node u runs A_u in $\mathcal{E}$. Any system behavior $\mathcal{E}$ of G in which at least n - m nodes are correct is a **correct system behavior.** Correct system behaviors must satisfy the following conditions.

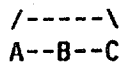
Agreement: Every correct node chooses the same value.

Validity: If all the correct nodes have the same input, that input must be the value chosen.

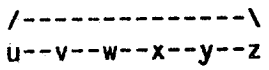
Theorem 1: Byzantine agreement is not possible in inadequate graphs.

3.1. Number of Nodes

We begin with the lower bound of 3m + 1 for the number of nodes required for Byzantine agreement. First consider the case where |G| = n = 3 and m = 1. Assume that the problem can be solved for the communication graph G consisting of three nodes fully connected by communication edges. Let the three nodes of G be a, b and c, and assume that they run agreement devices A, B and C, respectively. We represent each pair of directed edges by a single undirected edge, and label the nodes with the devices they run.

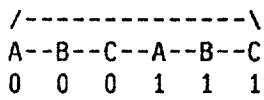


The covering graph S is as follows.



This graph looks locally like G under the mapping φ defined by $\varphi(u) = \varphi(x) = a$, $\varphi(v) = \varphi(y) = b$ and $\varphi(w) = \varphi(z) = c$.

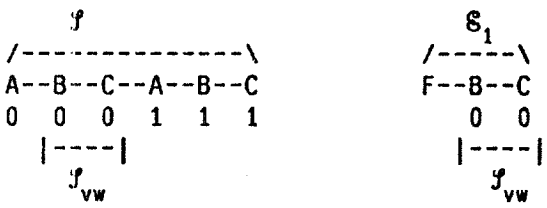
Now specify the system by assigning devices and inputs for the nodes in S as follows.



By this we mean that node u runs device A with input 0, node v runs B with input 0, and so on. Let $\mathcal{S}$ denote the resulting behavior of the system; $\mathcal{S}$ includes a behavior for each of the six nodes and twelve directed edges in S.

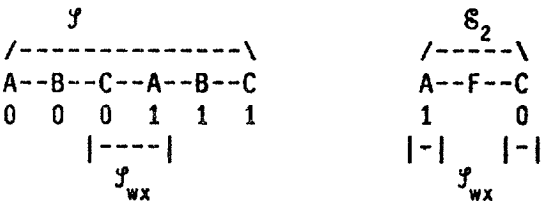
Now consider scenarios $\mathcal{S}_{vw}$, $\mathcal{S}_{wx}$ and $\mathcal{S}_{xy}$ in $\mathcal{S}$, where each consists of the behaviors of the two indicated nodes in S, along with the activity over the two connecting edges. We argue that each of these scenarios is identical to a scenario in a correct behavior of G.

The first scenario $\mathcal{S}_{vw}$ is shown below.



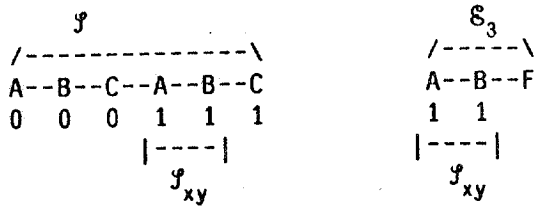
This scenario is the behavior in $\mathcal{S}$ of nodes v and w, together with that of the communication edges between v and w. Now consider the behavior $\mathcal{E}_1$ of G in which node b runs B on input 0, node c runs C on input 0, and node a runs a device that mimics node u in talking to b, and mimics node x in talking to c. Formally, if $E_{(u,v)}$ and $E_{(x,w)}$ are the indicated edge behaviors in $\mathcal{S}$, node a runs device $F_A(E_{(u,v)},E_{(x,w)})$ (we have written just F in the figure). This device exists, by the Fault axiom, and in the resulting behavior, edges from node a to node b and to node c have behaviors $E_{(u,v)}$ and $E_{(x,w)}$, respectively. By the Locality axiom, the scenario containing b and c's behaviors in $\mathcal{E}_1$ is identical to $\mathcal{S}_{vw}$. Validity requirements insure that node b and node c must choose 0 in $\mathcal{E}_1$. Since their behavior is identical in $\mathcal{S}$, v and w choose 0 in $\mathcal{S}$.

Next, consider scenario $\mathcal{S}_{wx}$.



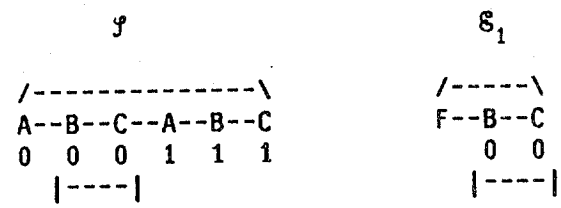
This scenario includes the behavior of nodes w and x in $\mathcal{S}$. It is also the behavior of nodes a and c in a behavior $\mathcal{E}_2$ of G which results when they run their devices A and C on inputs 1 and 0, respectively, and node b is faulty, exhibiting the same behavior to node x that v exhibits to w in $\mathcal{S}$, and the same behavior to node a that y exhibits to x in $\mathcal{S}$. The behavior of node c in $\mathcal{E}_2$ is identical to that of node w in $\mathcal{S}$, so node c chooses 0 in $\mathcal{E}_2$, from the argument above. By agreement, node a decides 0 in $\mathcal{E}_2$. Thus node x decides 0 in $\mathcal{S}$.

Now consider the third scenario, $\mathcal{F}_{xy}$.



This scenario is the behavior of nodes x and y in $\mathcal{F}$. It is also the behavior of nodes a and b in a correct behavior $\mathcal{S}_3$ of G which results when they both run their devices on input 1, and node c is faulty, exhibiting the same behavior to node a that w exhibits to x in $\mathcal{F}$, and the same behavior to node b that z exhibits to y in $\mathcal{F}$. Validity requirements insure that nodes a and b must choose 1. Thus nodes x and y choose 1. But we have already established that node x must choose 0, a contradiction.

Now consider the general case of $|G| = n \leq 3m$. Partition the nodes of G into three sets, a, b and c, so that a, b and c have at least 1 and at most m nodes. This means that any two sets together contain at least n-m nodes. The nodes in each set are running agreement devices, and we denote by A the set of devices running at the nodes in a, and similarly for B and C. Now construct the covering graph S in the obvious way. Briefly, take two copies of G, and label the sets a, b and c in each copy by u, v and w, respectively, in one copy, and x, y and z in the other. Now replace the edges between nodes in u and w and between nodes in x and z by corresponding edges between u and z and between x and w. Assign devices to nodes of S according to their corresponding node in G. We represent the covering graph S and assigned devices exactly as above, so that the edges depicted between two sets of nodes in S, say sets u and v, are now a shorthand representation for all the edges in S between nodes in set u and nodes in set v. The inputs depicted for the sets of devices A, B and C are assigned to all the devices in the respective sets. The arguments proceed exactly as in the preceding pictures. We consider only one in detail.



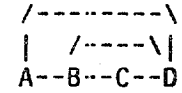
This scenario is now the behavior of the sets of nodes in v and w in the behavior $\mathcal{F}$. It is the same as the behavior of the sets b and c in a behavior $\mathcal{S}_1$ of G in which all nodes in both sets run their devices with input 0 and the nodes in set a exhibit the same behavior to members of b that the corresponding nodes in set u exhibit to the members of v in $\mathcal{F}$, and the same behavior to nodes in c that the corresponding nodes in y exhibit to the members of x in $\mathcal{F}$. Since sets b and c together contain at least n-m correct nodes, $\mathcal{S}_1$ is a correct behavior of G. Thus, all the nodes in b and c must decide 0, by the validity condition, and c contains at least one node, by

construction.

3.2. Connectivity

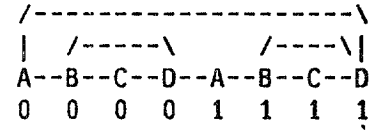
Now we carry out the 2m + 1 connectivity lower bound proof. Let c(G) = connectivity of G. We assume we can achieve Byzantine agreement in a graph G with c(G) $\leq$ 2m, and derive a contradiction.

For now, we consider the case m = 1 and the communication graph G of four nodes a, b, c and d, running devices A, B, C and D, as indicated below.



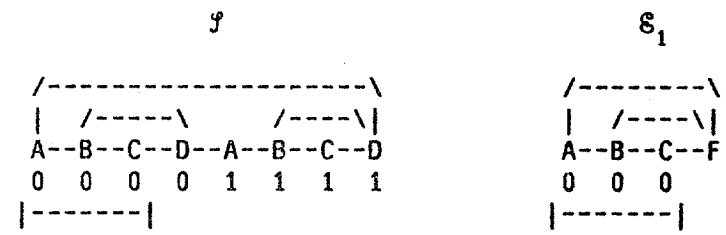
The connectivity of G is two; the two nodes b and d disconnect G into two pieces, the nodes a and c.

We consider the following system, with the eight-node graph S and devices and inputs as indicated.



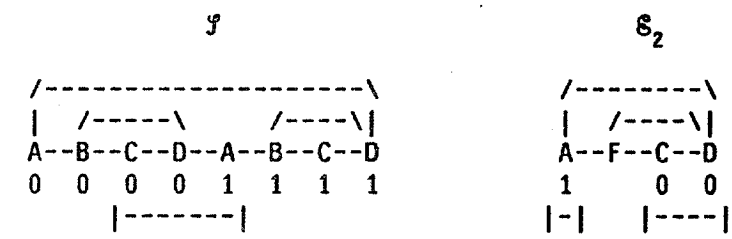
The resulting behavior of the system is $\mathcal{S}$. We consider three scenarios in $\mathcal{S}$: $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$.

The first scenario, $\mathcal{S}_1$, is shown below.



This is also a scenario in a correct behavior $\mathcal{E}_1$ of G. In $\mathcal{E}_1$, nodes a, b and c are correct. Node d is faulty, exhibiting the same behavior to node a as one node running D in the covering graph, and the same behavior to b and c as the other node running D exhibits in the covering graph. Then nodes a, b and c must choose 0 in $\mathcal{E}_1$, and so must the nodes running A, B and C in $\mathcal{S}_1$.

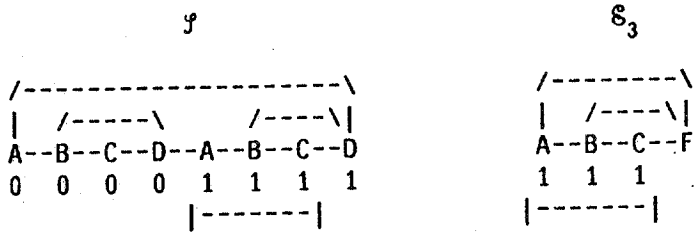
Now consider the second scenario, $\mathcal{S}_2$.



This scenario in $\mathcal{S}$ is also a scenario in a correct behavior $\mathcal{E}_2$ of G in which nodes c, d and a are correct. This

time, node b is faulty, exhibiting the same behavior to nodes c and d as one node running B in the covering, and the same behavior to node a as the other node running B. So nodes a, c and d must agree in $\mathcal{E}_2$, and so do the corresponding nodes in $\mathcal{S}_2$. Since the node running C chooses 0 from the argument above, the nodes running D and A in $\mathcal{S}_2$ choose 0, too.

Finally, consider the last scenario $\mathcal{S}_3$.



This scenario is again the same as a scenario in a behavior $\mathcal{E}_3$ of G in which nodes a, b and c are correct, but have input 1. Node d is faulty, exhibiting the same behavior to node a that one node running D in the covering graph exhibits, and the same behavior to nodes b and c as the other D in the covering exhibits. Then nodes a, b and c choose 1 in $\mathcal{E}_3$, and so must the nodes running A, B and C in $\mathcal{S}_3$, contradicting the argument above that the node running A chooses 0.

The general case for arbitrary $c(G) \leq 2m$ is an easy generalization of the case for $m = 1$. The same pictures are used. Just choose b and d to be sets consisting of at most m nodes each, such that removing the nodes in b and d from G disconnects two nodes u and v of G. Let G' be the graph obtained by removing b and d from G, let the set a contain those nodes connected to u, and the set c contain the remaining nodes of G' (c contains at least one node, v). Construct S as before, by taking two copies of G and rearranging edges between the 'a' sets and their neighbors. The nodes and edges in our figures are now a shorthand for the actual nodes and edges of G and S.

This completes the proof of Theorem 1. □

The succeeding impossibility results for other consensus problems follow the same general form as the two arguments above. We assume a problem can be solved by specific devices in an inadequate graph, G, install the devices in a graph S that covers G, and provide appropriate inputs. Using the Locality and Fault axioms, we argue the existence of a sequence of correct behaviors of G that have node and edge behaviors identical to some of those in the behavior of S. (This sequence was ($\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$), in the arguments above.) By the agreement condition, correct nodes in each of the behaviors of G have to agree. Because each successive pair of system behaviors has a correct node behavior in common, all of the correct nodes in all the behaviors in the sequence have to agree. But by the validity condition, correct nodes in the first behavior in the sequence must choose different values than those in the last behavior, a contradiction.

As we indicated in the introduction, a less general version of Theorem 1 was previously known, and the structure of our proof is very similar to that of earlier proofs [PSL], [D]. Our proof differs in the construction of the system behaviors $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_3$. Earlier results construct these behaviors inductively, in less general models of distributed systems. The detailed assumptions of the models are necessary to carry out the tedious and involved constructions.

Rather than construct the behaviors explicitly, we build them from pieces (node and edge behaviors) extracted from actual runs of the devices in a covering graph. The Locality and Fault axioms imply that scenarios in the covering graph are also found in correct behaviors of the original inadequate graph.

The model used to obtain these results is an extremely general one, but it does assume that systems behave deterministically. (For every set of inputs, a system has a single behavior). By considering a system and inputs as determining a set of behaviors, nondeterminism may be introduced in a straightforward manner. One changes the Locality axiom to express the following: if there exist behaviors of two systems in which the inedge borders of two isomorphic subsystems are identical, there exist such behaviors in which the behaviors of the subsystems are also identical. Using this axiom, the same proofs suffice to show that nondeterministic algorithms cannot guarantee Byzantine agreement.

4. Weak Agreement

Now we give our impossibility results for the weak agreement problem. As in the Byzantine agreement case, nodes have Boolean inputs, and must choose a Boolean output. The agreement condition is the same as for Byzantine agreement--all correct nodes must choose the same output. The validity condition is weaker, however.

Agreement: Every correct node chooses the same value.

Validity: If all nodes are correct and have the same input, that input must be the value chosen.

The weaker validity condition has an interesting impact on the agreement problem. If any correct node observes disagreement or faulty behavior, then all are free to choose a default value, so long as they still agree.

Lamport notes that there are devices for reaching a form of approximate weak consensus, which work when $|G| \leq 3m$. Running these for an infinite time produces exact consensus (at the limit) [L]. In such infinite behaviors, if any correct node observes disagreement or faulty behavior, it has plenty of time to notify the others before they choose a value. Thus, strengthening the choice condition, to prohibit such infinite solutions, is necessary to obtain the lower bound.

We must also bound communication delays away from zero, or a similar type of infinite behavior is possible. In fact, if we assume there is no lower bound on transmission delay, and that devices can control the delay and have synchronized clocks, we have found an algorithm for reaching weak consensus. This algorithm requires at most two broadcasts per node, all with non-zero transmission delay, and works with any number of faults. Again, this is because any correct node which observes disagreement or faulty behavior has plenty of time to notify the others before they choose a value.[2] In more realistic models it is impossible to reach weak consensus in inadequate graphs. To show this, the minimal semantics introduced in the previous sections must be extended to exclude these infinitary solutions. We do this as follows. Previously, behaviors of nodes and edges were elements of some arbitrary set. Henceforth, we consider them to be mappings from $[0,\infty)$, (our definition of time), to arbitrary state sets. Thus, if E is a behavior of node u, then u is in state E(t) at time t.

We add the following condition to the weak agreement problem.

Choice: A correct node must choose 0 or 1 after a finite amount of time.

This means there is a function CHOOSE from behaviors of nodes running weak agreement devices to {0,1}, with the following property: Every such behavior E has a finite prefix E_t (E restricted to the interval [0,t]) such that all behaviors E' extending E_t have CHOOSE(E) = CHOOSE(E').

This choice condition prohibits Lamport's infinite solution. To prohibit the second solution, we bound the rate at which information can traverse the network. To do so, we add the following stronger locality axiom to our model.

Bounded-Delay Locality Axiom

> There exists a positive constant δ such that the following is true. Let $\mathcal{G}$ and $\mathcal{G}'$ be systems with behaviors $\mathcal{E}$ and $\mathcal{E}'$, respectively, and isomorphic subsystems $\mathcal{U}$ and $\mathcal{U}'$, (with vertex sets U and U'). If the corresponding behaviors of the inedge borders of U and U' in $\mathcal{E}$ and $\mathcal{E}'$ are identical through time t, then scenarios $\mathcal{E}_U$ and $\mathcal{E}_{U'}$ are identical through time $t+\delta$.

Thus, news of events k edges away from some subgraph G' takes time at least $k\delta$ to arrive at G'. In a model with explicit messages, this axiom could be proven from an assumption that the transmission delay is at least δ, and the edge behaviors in our model would correspond to state descriptions of the transmitting end of each communications link.

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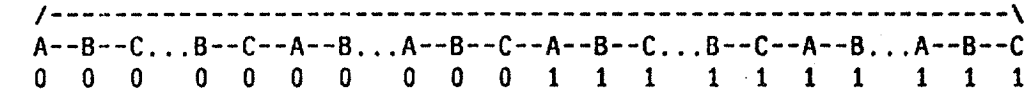
[2] Nodes start at time 0, and decide at time 1. They broadcast their value at time 0, specifying it to arrive at time 1/2. If a node first detects disagreement or failure (at time 1-t), it broadcasts a "failure detected, choose default value" message, specifying it to arrive at time 1-t/2. The obvious decision is made by everyone at time 1.

Theorem 2: Weak agreement is not possible in inadequate graphs for models satisfying the Bounded-Delay Locality axiom.

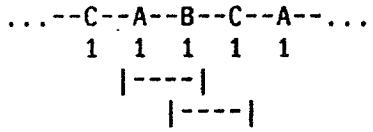
Again, we first sketch the 3m + 1 node bound. In this case, the previously published proof [L] was very difficult. As before, we restrict our attention to the case $|G| = n = 3$, $m = 1$. (The case for general m follows immediately, just as above.)

Assume there are weak agreement devices A, B and C, for the triangle graph G containing nodes a, b and c. Consider the two behaviors of G in which all nodes are correct, and all have input 0 or all have input 1. Let t' be an upper bound on the time it takes all nodes to choose 0 or 1 in both behaviors. Choose $k \geq t'/\delta$ to be a multiple of 3.

The covering graph S consists of 4k nodes, arranged in a ring and assigned devices and inputs as follows:

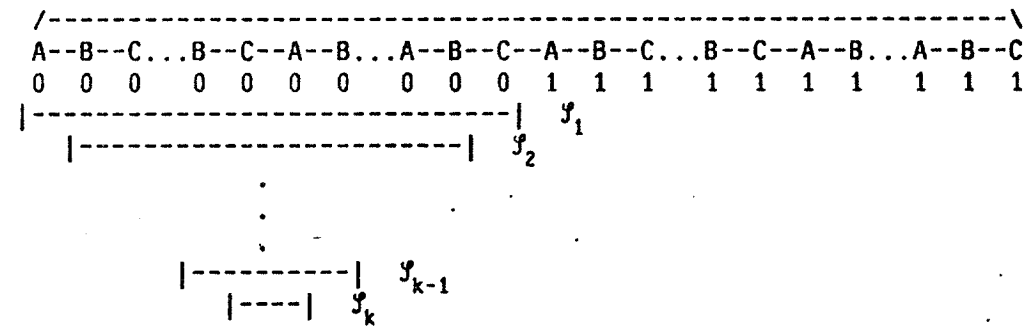


Consider the resulting behavior $\mathcal{S}$, and each pair of successive two-node scenarios, such as the two below.



As before, each scenario is identical to a scenario in a behavior in G of the appropriate two weak consensus devices. Since each pair of successive scenarios overlaps in one node behavior (here, that of the node running B), all the nodes in both scenarios must choose the same value in G and in S. By induction, every node in S must choose the same value. Without loss of generality, assume they choose 1.

Consider the k scenarios indicated below.



Let $\mathcal{E}$ be the behavior of G in which a, b and c are correct and each has input 0, and denote the resulting behaviors of a, b and c by E_a, E_b and E_c, respectively.

Lemma 3: The behavior in scenario $\mathcal{S}_i$ of a node running device A (or B or C) is identical to E_a (or E_b or E_c) through time $i\delta$.

Proof: The proof is an easy induction using the Bounded-Delay Locality axiom. □

By Lemma 3, the nodes running devices C and A in scenario $\mathcal{S}_k$ have behaviors identical to E_c and E_a through time $k\delta$. Since nodes c and a in G have chosen output 0 by this time, so have the corresponding nodes in $\mathcal{S}_k$, a contradiction.

The general case of $|G| \leq 3m$ and the connectivity bound follow as for Byzantine agreement. □

There are strong similarities between this argument and a proof by Angluin, concerning leader elections in rings and arbitrarily long lines of processors [A]. Both results depend crucially on the existence of a lower bound on the rate of information flow. Under this assumption, devices in different communication networks can be shown to see the same local behavior for some fixed time.

5. Byzantine Firing Squad

The Byzantine firing squad problem addresses a form of synchronization in the presence of Byzantine failures. The problem is to synchronize a response to an input stimulus. The response is to enter a designated FIRE state. The problem was studied originally in [BL]. In [CDDS], a reduction of weak agreement to the Byzantine firing squad problem demonstrates that the latter is impossible to solve in inadequate graphs. We provide a direct proof that a simple variant of the original problem is impossible to solve in inadequate graphs. (In the original version, the stimulus can arrive at any time. We require it to arrive at time 0, or not at all. Our validity condition is slightly different.) The proof is very similar to that for weak agreement.

One or more devices may receive a stimulus at time 0. We model the stimulus as an input of 1, and absence of the stimulus as an input of 0. Correct executions must satisfy the following conditions.

Agreement: If a correct node enters the FIRE state at time t, every correct node enters the FIRE state at time t.

Validity: If all nodes are correct and the stimulus occurs at any node, they enter the FIRE state after some finite delay. If the stimulus does not occur and all nodes are correct, no node ever enters the FIRE state.

As in the case of weak agreement, solutions to the Byzantine firing squad problem exist in models in which there is no minimum communication delay. Thus the following result requires the Bounded-Delay Locality axiom, in addition to the Fault axiom.

> **Theorem 4:** The Byzantine firing squad problem cannot be solved in inadequate graphs for models satisfying the Bounded-Delay Locality axiom.

We sketch the 3m + 1 node bound. As before, we examine the case |G| = n = 3, m = 1.

Assume there are Byzantine firing squad devices A, B and C for the triangle graph G containing nodes a, b and c. Consider the two behaviors of G in which all nodes are correct, and all have input 0 or all have input 1. Let t be the time at which the correct devices enter the FIRE state in the case that the stimulus occurred (the input 1 case). Since the correct nodes never enter the FIRE state in the absence of the stimulus, they certainly do not enter the FIRE state at time t. Choose $k \geq t/\delta$ to be a multiple of 3. (Recall that δ is the minimum transmission delay defined in the Bounded-Delay Locality axiom).

The covering graph S consists of 4k nodes, arranged in a ring and assigned devices and inputs as follows:

```
/-------------------------------------------------------------\
A--B--C...B--C--A--B...A--B--C--A--B--C...B--C--A--B...A--B--C
0  0  0   0  0  0  0   0  0  0  1  1  1   1  1  1  1   1  1  1
```

Similarly to the proof for weak agreement, the middle two devices receiving the stimulus enter the FIRE state at time t, as their behavior through time t is the same as that of the correct nodes in G which have received the stimulus and fire at time t. Because of the communication delay, there is not enough time for "news" from the distant nodes to reach these devices. By repeated use of the agreement property, all the devices in S must fire at time t. But through time t, the middle two devices not receiving the stimulus behave exactly as correct nodes in G which do not receive the stimulus (the input 0 case). Thus they do not fire at time t, a contradiction. □

6. Approximate Agreement

Next, we turn to two versions of the approximate agreement problem [DLPSW,MS]. We call them *simple approximate agreement* and *$(\varepsilon,\delta,\gamma)$-agreement.* In these problems, nodes have real values as inputs and choose real numbers as a result. The goal is to have the results close to each other and to the inputs. In order to obtain the strongest possible impossibility result, we formulate very weak versions of the problems.

For the following two theorems we use only the Locality and Fault axioms. We do not need the Bounded-Delay Locality axiom used for the weak agreement and firing squad results.

6.1. Simple Approximate Agreement

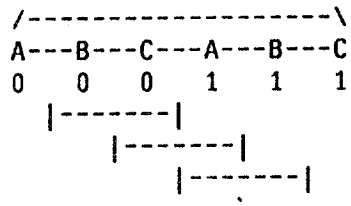
First, we turn to the simple approximate agreement problem [DLPSW]. The version we examine is based on that in [DLPSW]. Each correct node has a real value from the interval [0,1] as input, runs its device and chooses a real value. Correct behaviors (those in which at least n - m nodes are correct) must satisfy the following conditions.

Agreement: The maximum difference between values chosen by correct nodes must be strictly smaller than the maximum difference between the inputs, or be equal to the latter difference if it is zero.

Validity: Each correct node chooses a value within the range of the inputs of the nodes.

Theorem 5: Simple approximate agreement is not possible in inadequate graphs.

The proof is almost exactly that for Byzantine agreement. Here, we consider devices which take as inputs numbers from the interval [0,1], and choose a value from [0,1] to output. (Outputs are modeled by a function CHOOSE from behaviors of nodes running the devices to the interval [0,1].) As before, assume simple approximate agreement can be reached in the triangle graph G. Consider the following three scenarios from the indicated behavior in the covering graph S.



Again, each scenario is also a scenario in a correct behavior of G. In the first scenario, the only value C can choose is 0. In the third, the only value A can choose is 1. This means the values chosen by A and C in the the second scenario are 0 and 1, so that the outputs are no closer than the inputs, violating the agreement condition.

The general case of $|G| \leq 3m$ and the connectivity bounds follow as for Byzantine agreement.

6.2. (, ,)-Agreement

This version of approximate agreement is based on that in [MS]. Let ϵ, δ and γ be positive real numbers. The correct nodes receive real numbers as inputs, with r_{min} and r_{max} the smallest and largest such inputs, respectively. These inputs are all at most δ apart (i.e. the interval of inputs $[r_{min}, r_{max}]$ has length at most δ). They must choose a real number as output, such that correct behaviors (those in which at least n - m nodes are correct) satisfy the following conditions.

Agreement: The values chosen by correct nodes are all at most ϵ apart.

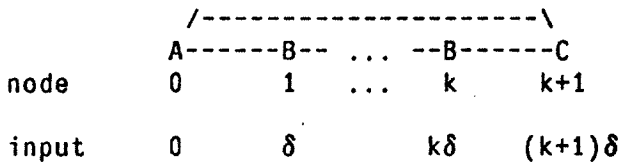
Validity: Each correct node chooses a value in the interval $[r_{min}-\gamma, r_{max}+\gamma]$.

Note that if $\epsilon \geq \delta$, (ϵ,δ,γ)-agreement can be achieved trivially by choosing the input value as output.

Theorem 6: If $\varepsilon < \delta$, $(\varepsilon,\delta,\gamma)$-agreement is not possible in inadequate graphs.

Proof: Let ε, δ and γ be positive real numbers with $\varepsilon < \delta$. We prove only the $3m+1$ bound on the number of nodes. Assume that devices A, B and C exist which solve the $(\varepsilon,\delta,\gamma)$-approximate agreement problem in the complete graph G on three nodes, for particular values of ε, δ and γ, where $\varepsilon < \delta$.

Choose k sufficiently large that $\delta > 2\gamma/(k-1) + \varepsilon$, and $k+2$ is divisible by three. The covering graph S contains $k+2$ nodes arranged in a ring, with devices and inputs assigned to create the following system.



Let $\mathcal{S}_i$, for $0 \leq i \leq k$, denote the two-node scenario in $\mathcal{S}$ containing the behaviors of nodes i and $i+1$. By the Fault Axiom, each scenario $\mathcal{S}_i$ is a scenario of a correct behavior of G, in which the largest input value to a correct node is $(i+1)\delta$.

Lemma 7: For $0 \leq i \leq k$, the value chosen by the device at node $i+1$ is at most $\delta + \gamma + i\varepsilon$.

Proof: The proof is a simple induction. The device at node 1 chooses at most $\delta + \gamma$, by validity applied to scenario $\mathcal{S}_0$. Assume inductively that the device at node i chooses at most $\delta + \gamma + (i-1)\varepsilon$, for $0 < i < k+1$. By agreement applied to scenario $\mathcal{S}_i$, the device at node $i+1$ chooses at most $\delta + \gamma + i\varepsilon$. □

In particular, Lemma 7 implies the device at node k chooses at most $\delta + \gamma + (k-1)\varepsilon$. But validity applied to scenario $\mathcal{S}_k$ implies the device at node k chooses at least $k\delta - \gamma$. So $k\delta - \gamma \leq \delta + \gamma + (k-1)\varepsilon$. This implies $\delta \leq 2\gamma/(k-1) + \varepsilon$, a contradiction.

The general case of $|G| \leq 3m$ and the connectivity bounds follow as in previous proofs. □

7. Clock Synchronization

Each node has a hardware clock and maintains a logical clock. The hardware clocks are real-valued, invertible and increasing functions of time. In general, different hardware clocks run at different rates, and the nodes wish to synchronize their logical clocks more closely than their hardware clocks. We also want the logical clocks to be reasonably close to real time--setting them to be constantly zero should probably be forbidden. Thus, we require the logical clocks to stay within some **envelope** of the hardware clocks.

This problem was studied in [DHS] for the case of linear clock and envelope functions, where it was shown that it is impossible to synchronize to within a constant in inadequate graphs. Some questions concerning more general synchronization problems were raised. It was pointed out, for example, that diverging linear

clocks can easily be synchronized to within a constant if nodes can run their logical clocks as the logarithm of their hardware clocks. For a large class of clock and envelope functions (increasing and invertible clocks, non-decreasing envelopes), we are able to characterize the best synchronization possible in inadequate graphs. This synchronization requires no communication whatsoever.

We model node i's hardware clock, D_i, as an input to the device at node i that has value $D_i(t)$ at time t. The value of the hardware clock at time t is assumed to be part of the state of the node at time t. The time on node i's logical clock at real time t is given by a function of the entire state of node i. Thus, if E_i is a behavior of node i (such that node i is in state $E_i(t)$ at time t), then we express i's logical clock value at time t as $C_i(E_i(t))$.

We assume that any aspect of the system which is dependent upon time (such as transmission delay, minimum step time, maximum rate of message transmission) is a function of the states of the hardware clocks. Having made this assumption, it is clear that speeding up or slowing down the hardware clocks uniformly in different behaviors cannot be observable to the nodes, so the only impact on the behaviors should be that they speed up or slow down in the same way as the hardware clocks.

To formalize this assumption, we need to talk about **scaling** clocks and behaviors. Let h be any invertible function of time. If E is a behavior (of a edge or node), then Eh, the behavior E **scaled by h**, is such that $Eh(t) = E(h(t))$, for all times t. Similarly, Dh is the hardware clock D scaled by h: $Dh(t) = D(h(t))$. If $\mathcal{E}$ is a system behavior or scenario, $\mathcal{E}h$ is the system behavior or scenario obtained by scaling every node and edge behavior in $\mathcal{E}$ by h. Similarly, if $\mathcal{S}$ is a system, then $\mathcal{S}h$ is the system obtained by scaling every clock in $\mathcal{S}$ by h. Intuitively, a scaled clock or behavior is in the state at time t that the corresponding unscaled clock or behavior is in at time h(t).

Scaling Axiom If $\mathcal{E}$ is the behavior of system $\mathcal{S}$, then $\mathcal{E}h$ is the behavior of system $\mathcal{S}h$. □

If this axiom is significantly weakened, as by bounding the transmission delay, clock synchronization may be possible in inadequate graphs.

In the following we use the Locality, Fault and Scaling axioms. We do not need the Bounded-Delay Locality axiom used for the weak agreement and firing squad results.

The synchronization problem can be stated as follows. Let correct hardware clocks run either at f(t) or g(t), where f and g are increasing, invertible functions, with $f(t) \leq g(t)$, for all t. Let the envelope functions l and u be non-decreasing functions such that $l(t) \leq u(t)$, for all t.

Consider what happens if everyone runs their logical clocks at the lower envelope, $C(E(t)) = l(D(t))$. Then

the logical clocks are synchronized to within $l(g(t))-l(f(t))$. The goal then, is to improve this trivial synchronization. We show that logical clocks cannot be synchronized to within $l(g(t))-l(f(t))-\alpha$, for any positive α.

That is, **nontrivial synchronization** is achieved by synchronization devices in G if there exist positive constant α and time t' such that every correct system behavior $\mathcal{E}$ satisfies the following conditions.

Agreement: For any two correct nodes i and j in $\mathcal{E}$,
$|C_i(E_i(t)) - C_j(E_j(t))| \leq l(g(t)) - l(f(t)) - \alpha$, for all times $t \geq t'$.

Validity: For any correct node i in $\mathcal{E}$, with hardware clock D_i and resulting behavior E_i, $l(f(t)) \leq C_i(E_i(t)) \leq u(g(t))$.

> **Theorem 8:** Nontrivial synchronization is not possible in inadequate graphs for models satisfying the Scaling axiom.

We show that for every integer $k>2$, there is a behavior $\mathcal{E}$ of G in which node i is correct, has hardware clock $D_i = f$ (that is, $D_i(t) = f(t)$), and in which $C_i(E_i(t')) \geq l(f(t')) + k\alpha$. For k big enough, this violates the upper envelope condition, $C_i(E_i(t')) \leq u(g(t'))$.

Define $h = f^{-1}g$. (That is, $h(t) = f^{-1}(g(t))$.) Then $h^{-1} = g^{-1}f$. Note that $h(t) \geq t$ for all t, since $f(t) \leq g(t)$.

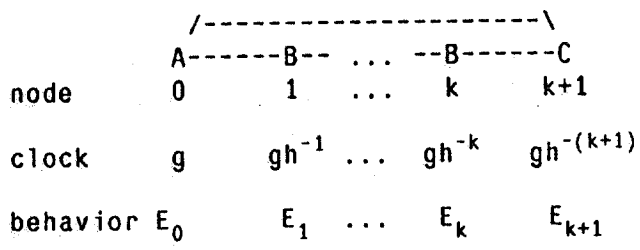
We begin with the three node, one fault case. The argument is very similar to the proof of Theorem 6.

Assume the existence of devices A, B and C, time t' and positive constant α such that logical clocks of correct nodes obey the agreement and validity conditions:

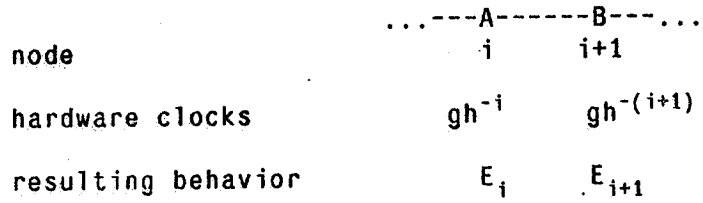
$|C_i(E_i(t)) - C_j(E_j(t))| \leq l(g(t)) - l(f(t)) - \alpha$, for all times $t \geq t'$.

$l(f(t)) \leq C(E_i(t)) \leq u(g(t))$, for all times t.

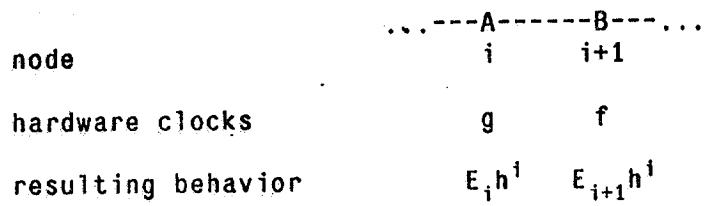
Choose an integer $k > 2$, such that $k+2$ is a multiple of three, and such that $l(f(t')) + k\alpha > u(g(t'))$. The covering graph S contains $k+2$ nodes arranged in a ring, with devices and clock inputs assigned to create the following system.



Let $\mathcal{S}$ be the behavior of this system. An initially troubling concern is that the hardware clocks in $\mathcal{S}$ are much slower in most of the devices in the $\mathcal{S}$ than they would be in a correct behavior in G. But consider $\mathcal{S}_i$, the two-node scenario containing the behaviors of nodes i and i+1, where $0 \leq i \leq k$.



Now consider $\mathcal{S}_i h^i$, the scenario $\mathcal{S}_i$ scaled by h^i.



In this scenario, the hardware clocks have values within the constraints for correct behaviors of G. Thus we have the following.

Lemma 9: Scenario $\mathcal{S}_i h^i$, for $0 \leq i \leq k$, is a scenario containing the behaviors of two correct nodes in a correct behavior of G.

Lemma 10: For all i, $0 \leq i \leq k$, and all $t \geq h^i(t')$, $|C_{i+1}(E_{i+1}(t)) - C_i(E_i(t))| \leq l(g(h^{-i}(t))) - l(f(h^{-i}(t))) - \alpha$.

Proof: Fix $t \geq h^i(t')$. Then $h^{-i}(t) \geq t'$. By Lemma 9, i and i+1 are correct in $\mathcal{S}_i h^i$, so by the agreement assumption $|C_{i+1}(E_{i+1} h^i(h^{-i}(t))) - C_i(E_i h^i(h^{-i}(t)))| \leq l(g(h^{-i}(t))) - l(f(h^{-i}(t))) - \alpha$. The result is immediate. □

Let time $t'' = h^k(t')$. Note that $t'' \geq h^i(t')$, for $i \leq k$.

Lemma 11: For all i, $1 \leq i \leq k+1$, $C_i(E_i(t'')) \geq l(gh^{-(i)}(t'')) + (i-1)\alpha$

Proof: The proof is by induction on i. By Lemma 9, scenario $\mathcal{S}_0$ is a scenario in G of correct nodes a and b,

with hardware clocks g and f, respectively. From the validity condition, for all t, $C_1(E_1(t)) \geq l(f(t))$. Setting t = t'', and substituting gh^{-1} for f, we have the basis step: $C_1(E_1(t'')) \geq l(gh^{-1}(t''))$.

Now make the inductive assumption $C_i(E_i(t'')) \geq l(gh^{-i}(t'')) + (i-1)\alpha$, for $1 \leq i \leq k$.

Since $t'' \geq h^i(t')$, from Lemma 10, we know $|C_{i+1}(E_{i+1}(t'')) - C_i(E_i(t''))| \leq l(gh^{-i}(t'')) - l(fh^{-i}(t'')) - \alpha$.

This implies $C_{i+1}(E_{i+1}(t'')) \geq C_i(E_i(t'')) - l(gh^{-i}(t'')) + l(fh^{-i}(t'')) + \alpha$.

Substituting for $C_i(E_i(t''))$ using the inductive assumption gives us $C_{i+1}(E_{i+1}(t'')) \geq l(gh^{-i}(t'')) - l(gh^{-i}(t'')) + l(fh^{-i}(t'')) + i\alpha = l(fh^{-i}(t'')) + i\alpha$. Noting that $f = gh^{-1}$, we have the result, $C_{i+1}(E_{i+1}(t'')) \geq l(gh^{-(i+1)}(t'')) + i\alpha$. □

Proof of Theorem 8:

Lemma 11 implies $C_{k+1}(E_{k+1}(t'')) \geq l(gh^{-(k+1)}(t'')) + k\alpha$. Since $t'' = h^k(t')$, we have $C_{k+1}(E_{k+1}(t'')) = C_{k+1}(E_{k+1}(h^k(t'))) = C_{k+1}(E_{k+1}h^k(t')) \geq l(gh^{-(k+1)}h^k(t')) + k\alpha = l(f(t')) + k\alpha$.

But the upper envelope constraint for the scaled scenario $\mathcal{S}_k h^k$ (in which k+1 is correct and has hardware clock f(t)) implies that $C_{k+1}(E_{k+1}h^k(t')) \leq u(g(t'))$. Thus, $l(f(t')) + k\alpha \leq u(g(t'))$. This violates the assumed bound on k, $l(f(t')) + k\alpha > u(g(t'))$.

Once again, the general case of $|G| \leq 3m$ is a simple extension of this argument. The connectivity bound also follows easily, as with the earlier results. □

7.1. Linear Envelope Synchronization and other Corollaries

Linear envelope synchronization, as defined in [DHS], examines the synchronization problem when the clocks and envelope functions are linear functions ($g(t)=rt$, $f(t)=t$, $l(t)=at+b$ and $u(t)=ct+d$). It requires correct logical clocks to remain within a constant of each other, so that the agreement condition is $|C_i(E_i(t)) - C_j(E_j(t))| \leq \alpha$, for all times t, instead of our weaker condition $|C_i(E_i(t)) - C_j(E_j(t))| \leq art - at - \alpha$, for all times $t \geq t'$. Our validity condition is slightly weaker, as well. Thus, the proof of [DHS] shows that logical clocks cannot be synchronized to within a constant; we show that that the synchronization of logical clocks cannot be improved by a constant over the synchronization (art - at) that can be achieved trivially. Thus the following corollary follows immediately from Theorem 8. (Each of the four corollaries below holds for models satisfying the Scaling axiom.)

Corollary 12: Linear envelope synchronization is not possible in inadequate graphs.

We also get the following results immediately from Theorem 8, by choosing specific values for the clock and lower envelope functions. Note that the particular choice of the upper envelope function does not affect the minimal synchronization possible in inadequate graphs, although the existence of *some* upper envelope function is necessary to obtain our impossibility proofs.

Corollary 13: If $f(t)=t$, $g(t)=rt$, and $l(t)=at+b$, no devices can synchronize a constant closer than $a(r-1)t$ in inadequate graphs.

Corollary 14: If $f(t)=t$, $g(t)=t+c$ and $l(t)=at+b$, no devices can synchronize a constant closer than ac in inadequate graphs.

Corollary 15: If $f(t)=t$, $g(t)=rt$ and $l(t)=\log_2(t)$, no devices can synchronize a constant closer than $\log_2(r)$ in inadequate graphs.

In general, the best possible synchronization in inadequate graphs can be achieved without any communication at all. The best nodes can do is run their logical clocks as slowly as they are permitted, $C(E(t)) = l(D(t))$.

8. Conclusion

Most of the results we have presented were previously known. Our proofs are simpler than earlier proofs, and hold in more general models, but this is not their main contribution. While simplicity and generality are important goals, in this instance they are the welcome byproduct of our attempt to identify the fundamental issues and assumptions behind a collection of similar results.

One important contribution is to elucidate the relationship between the unrestricted, or Byzantine failure assumption, and inadequate graphs. As is clear from our proofs, this fault assumption permits faulty nodes to mimic executions of disparate network topologies. If the network is inadequate, a covering graph can be constructed so that correct devices cannot distinguish the execution in the original graph from one in the covering graph.

A second contribution is related to the generality of our results. Nowhere do we restrict state sets or transitions to be finite, or even to reflect the outcome of effective computations. The inability to solve consensus problems in inadequate graphs has nothing to do with computation per se, but rather with distribution. It is the distinction between local and global state, and the uncertainty introduced by the presence of Byzantine faults, which result in this limitation.

Finally, we have identified a small, natural set of assumptions upon which the impossibility results depend.

For example, in the case of weak agreement and the firing squad problem, the correctness conditions are sensitive to the actions of faulty nodes. Instantaneous notification of the detection of fault events would allow one to solve these problems. An assumption that there are minimum delays in discovering and relaying information about faults is sufficient to make these problems unsolvable.

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