Brief Announcement: Local Approximability of Minimum Dominating Set on Planar Graphs

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ABSTRACT

We show that there is no deterministic local algorithm (constant-time distributed graph algorithm) that finds a $(7-\epsilon)$-approximation of a minimum dominating set on planar graphs, for any positive constant ϵ. In prior work, the best lower bound on the approximation ratio has been $5-\epsilon$; there is also an upper bound of 52.

Categories and Subject Descriptors

C.2.4 [**Computer-Communication Networks**]: Distributed Systems; F.2.2 [**Analysis of Algorithms and Problem Complexity**]: Nonnumerical Algorithms and Problems

Keywords

Approximation algorithm; dominating set problem; local distributed algorithm; lower bound; planar graph

1. INTRODUCTION

This work studies one of the last uncharted corners in the area of deterministic local algorithms: planar graphs.

A *local algorithm* is a distributed graph algorithm that runs in $O(1)$ communication rounds, independently of the size of the network. While the theory of *randomised* local algorithms is still in its infancy, we have nowadays a good understanding of the capabilities of *deterministic* local algorithms.

For many classical graph problems, there are exactly matching upper and lower bounds on the best possible approximation ratio that can be achieved by a deterministic local algorithm [6]. In many cases, we can apply a straightforward two-step procedure to derive tight lower bounds:

1. Prove tight bounds for anonymous networks (without unique identifiers).
2. Apply a simulation argument [2] to show that unique identifiers do not help.

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However, there are some isolated examples of natural questions in which the above two-step procedure fails badly. Perhaps the most intriguing example is *dominating sets on planar graphs*:

1. We do not have tight bounds for this problem in anonymous networks.
2. Planar graphs are not closed under lifts, and therefore the simulation argument [2] cannot be applied.

In this work we are interested in the smallest α such that there is a deterministic local algorithm that finds an α-approximation of a minimum dominating set in any planar graph. The current bounds are very far from being tight:

- $5-\epsilon < \alpha \le 636$ for anonymous networks [1, 7],
- $5-\epsilon < \alpha \le 52$ in the LOCAL model [1, 3, 4, 8].

In this work we give the first improvement on the lower bounds in six years: we prove a lower bound $\alpha > 7-\epsilon$ for both models, for any positive constant ϵ.

2. PROOF OVERVIEW

Let $\mathcal{A}$ be a deterministic distributed algorithm with running time $T = O(1)$ in the LOCAL model. Assume that $\mathcal{A}$ finds a dominating set $D = \mathcal{A}(G)$ in any planar graph G (that is, each node that is not in D is adjacent to at least one node of D).

Pick sufficiently large $m \gg T$ and r. Let $m' = m - 2T$. We will construct a planar graph G with $n = m^2 r$ nodes as shown in Figure 1a. There are r *blocks* with $m \times m$ nodes in each block. The nodes of each block are partitioned to *internal nodes* and *boundary nodes*: there are $m' \times m'$ internal nodes, and they are surrounded by boundary areas of width T. Let B_i be the set of nodes in block i, and let $I_i \subseteq B_i$ be the set of internal nodes in block B_i. We will prove the following lemma.

LEMMA 1. *For any m and any sufficiently large r, we can assign unique identifiers in G so that $I_i \subseteq \mathcal{A}(G)$ for all $1, 2, \ldots, r-\ell$, for some $\ell = o(r)$.*

In other words, all internal nodes of blocks $1, 2, \ldots, r-\ell$ are in the dominating set $D = \mathcal{A}(G)$ produced by algorithm $\mathcal{A}$. Now if we choose large enough m and r, we can make the contributions of the boundary nodes and the contributions of the remaining $o(r)$ blocks arbitrarily small. In particular, for any positive constant ϵ', we can pick m and r such that $|D| \ge (1-\epsilon')n$.

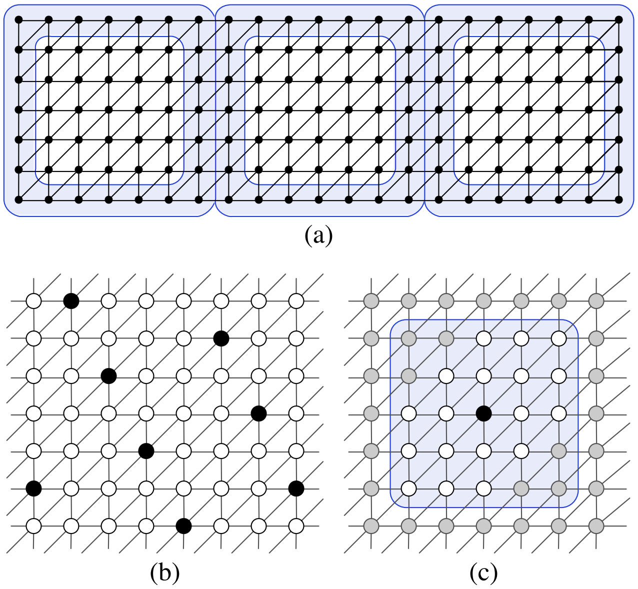

Figure 1: (a) Construction of graph G for $T = 1$, $m = 7$, and $r = 3$. There are 3 blocks. In each block there are 7×7 nodes: 5×5 internal nodes (white area), surrounded by a boundary area of width 1 (shaded). (b) A dominating set D^* of G that contains only a fraction 1/7 of internal nodes—the figure shows only a part of a large triangular grid. (c) The local output of an internal node v (black node) only depends on its radius-T neighbourhood (white nodes, here $T = 2$). In particular, if we know the unique identifiers in the $k \times k$ region R_v around v (shaded area), we know the local output of node v.

On the other hand, there is a dominating set D^* which contains only a fraction 1/7 of the internal nodes; see Figure 1b. Therefore $|D^*| \leq (1/7 + \epsilon')n$, and the claim follows: for any positive constant ϵ we can show that algorithm $\mathcal{A}$ cannot find a factor $7 - \epsilon$ approximation of a minimum dominating set on planar graphs.

3. PROOF OF LEMMA 1

The proof uses the strategy of repeated applications of Ramsey's theorem; cf. Czygrinow et al. [1, Lemma 4]. We will use the notation $\mathcal{A}(G, v) \in \{0, 1\}$ to refer the *local output* of node v when we apply algorithm $\mathcal{A}$ to graph G; we have $\mathcal{A}(G, v) = 1$ if node v is in the dominating set computed by algorithm $\mathcal{A}$. By definition, $\mathcal{A}(G, v)$ only depends on the radius-T neighbourhood of v in G.

Let $k = 2T + 1$, $K = k^2$, and $M = m^2$. Consider any internal node $v \in I_i$ of any block B_i. The structure of graph G in the radius-T neighbourhood does not depend on the choice of v. The local output of node v only depends on the unique identifiers in the local neighbourhood. The local neighbourhood is contained within a rectangular $k \times k$ region $R_v \subseteq B_i$; see Figure 1c.

Let $V = \{1, 2, \dots, n\}$ be the set of unique identifiers. Consider any K-subset of identifiers $X \subseteq V$, $|X| = K$. We will associate a *colour* $c(X) \in \{0, 1\}$ with each such set, as follows:

1. Pick an internal node v.
2. Assign the identifiers from X to region R_v in an increasing order by rows: the smallest k identifiers to the bottom row from left to right, etc. Assign the identifiers from $V \setminus X$ to the remaining nodes arbitrarily.
3. Apply algorithm $\mathcal{A}$, and set $c(X) = \mathcal{A}(G, v)$.

Now we have defined a colouring of all K-subsets of V; by restriction, we also have a colouring of all K-subsets of any $V' \subseteq V$. We say that $Y \subseteq V$ is *monochromatic* if $c(X_1) = c(X_2)$ for any K-subsets X_1 and X_2 of Y. By Ramsey's theorem [5] there exists an integer $N = N(K, M)$ such that the following holds: if V' is any N-subset of V, then there always exists a monochromatic subset $Y \subseteq V'$ of size M.

Now we will pick r and ℓ so that $\ell M > N$ and $\ell = o(r)$. Let $V_1 = V$. For each $i = 1, 2, \dots, r - \ell$, we define the identifiers of block i as follows.

1. As $|V_i| \geq N$, we can find a monochromatic subset $Y_i \subseteq V_i$ of size M.
2. Assign the identifiers from Y_i to block B_i in an increasing order by rows: the smallest m identifiers to the bottom row from left to right, etc.
3. Set $V_{i+1} = V_i \setminus Y_i$.

Finally, assign the remaining ℓM identifiers from $V_{r-\ell+1}$ to blocks $r - \ell + 1, \dots, r$ arbitrarily.

To complete the proof, consider a block i, where $1 \leq i \leq r - \ell$. Let $v \in I_i$ be an internal node of the block. Consider the $k \times k$ region R_v around v, and let X_v be the set of unique identifiers assigned to region R_v. Observe that the identifiers of X_v are assigned in an increasing order by rows. It follows that $\mathcal{A}(G, v) = c(X_v)$, i.e., the local output of the internal node v is simply the colour of subset X_v. Furthermore, $X_v \subseteq Y_i$ and Y_i was monochromatic. Hence all internal nodes of block i produce the same output. The common output cannot be 0; otherwise there would be nodes that are not dominated. Hence $I_i \subseteq \mathcal{A}(G)$.

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