Prudent Opportunistic Cognitive Radio Access Protocols

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Abstract. In a cognitive radio network, a Primary User (PU) may vacate a channel for intermissions of an unknown length. A substantial amount of research has been devoted to minimizing the disturbance a Secondary User (SU) may cause the PU. We take another step and optimize the throughput of an SU, even when assuming that the disturbance to the PU is indeed avoided using those other methods.

We suggest new optimization parameters the lengths of SU packets. That is, the SU fills up the intermission with consecutive packets. Each packet is associated with some fixed overhead. Hence, using a larger number of smaller packets increases the overhead ratio for each SU packet. On the other hand, it reduces the loss of throughput the SU suffers with the loss of a packet in a collision at the end of the intermission.

As opposed to previous studies, we optimize also the case where the distribution of the channel intermission is unknown. That is, we develop optimal competitive protocols. Those seek to minimize the ratio of the SU's profit compared to a hypothetical optimal algorithm that knows the intermission length in advance. We show how to compute the optimal present packets' sizes for the case that the distribution *is* known (for a *general* distribution). Finally, we show several interesting properties of the optimal solutions for several popular distributions.

1 Introduction

Cognitive Radio Networks (CRN) divide the users into Primary Users (PUs) and Secondary Users (SUs) groups. PUs are the spectrum "license holders" and have the right to use their channel at will. Various techniques have been devised to prevent SUs' transmission from disturbing the PU's transmission, e.g. by having the SU sense the channel and avoid transmission whenever the PU transmits.

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However, collisions between the user groups not only impact the ownership rights of the PUs but also affect the performance of the SUs, reducing the effective channel usage of the SUs even during an *intermission* in the PU's transmissions. To see that, consider the extreme (rather likely) case that the PUs are allowed a much higher transmission power. Hence, an SU packet transmitted at the end of the PU intermission is likely to be lost when the packet level checksum or a CRN integrity test is conducted at the receiver. That is, even the part of the packet that was transmitted during the intermission is lost. Please observe that this SU loss may happen even if the PU's transmission is not disturbed at all (e.g. thanks to the much higher power of the PU's transmissions)!

While our work revisits one of the most basic questions in CRN, we optimize it from a different angle. To highlight that, we stress that our results are meaningful even if a negative impact on the PU is avoided (for example, avoided by using the previous methods). Our main problem is: devise optimal access algorithms that maximize the efficient usage of the channel by the SU, given the possible loss at the end of the intermission. While the above question (and our results) concentrates on the SU's throughput, our model (and some of the results) are more general than that, and can also be used to minimize the negative impact (also on the PU) caused by the collision at the end of the intermission.

Prudent Protocols Our objective function includes a penalty for conflicts between an SU and the PUs. This penalty is traded off against a loss of SU throughput as follows. We allow an SU to break the transmission of its data to smaller packets; each packet is transmitted in a *transmission* interval that is followed by a sensing period of a *a fixed length* and (in case the intermission does not end) by the next interval. The fixed periods between the intervals are viewed as representing the fixed overhead, that may include, beside fixed length sensing, a preamble, headers, checksums, etc., associated with a packet transmission.[4]

The main optimization parameters are the lengths of the transmission intervals. Intuitively, a good sequence is not "too daring" (having too long sequences) on one hand, since we want to avoid the case that the last packet, the one that is lost, is long. On the other hand, a good sequence is not "too hesitant" (having too short messages). This is because a sequence of short packets will suffer from a relatively high overhead per packet (since the overhead per packet is fixed). Hence, we term protocols that achieve a good tradeoff-*Prudent* Optimistic protocols and the problem of finding good sequences– the problem of *Dynamic Interval Cover (DIC)*. The problem is defined formally below. We develop such protocols both for the case that the intermission length is unknown, and for the case that it is taken from a general distribution. We also highlight interesting results for several specific distributions that were not addressed before.

[4] We comment that in earlier work [4], the periods between the intervals were not fixed. This is because they had a different purpose- that of minimizing the probability of a collision at the end of the intermission. Recall that here, we want to emphasize the optimization that is still required for the SU's transmission, even if the collision at the end of the intermission does not harm the PU.

The model The issues considered in this paper are manifested even in a system with a single primary user and a single secondary user that share a single channel. Note that assuming multiple PUs would not change our results at all. (The case of multiple SUs is beyond the scope of this paper; however, we hope in a future work to fit multiple SUs into our model as follows: multiple SUs would coordinate transmissions among themselves using more traditional methods; that way, they would present the face of a single SU to the PU). The PU owns the channel and transmits over it intermittently. The SU cannot start transmitting until the PU stops. It is easy to show that optimizing for a sequence of multiple intermissions can be reduced to optimizing the SU transmission over each intermission separately. Hence, our analysis is performed per intermission.

We assume that the SU always has data. The data is divided by the SU into (variable length) packets. As opposed to some previous studies, we do not assume that the packet sizes are given ahead of time. W.L.O.G., the transmission of x bits takes x time units. When an intermission of the PU starts, the SU starts transmitting its data in packets $p_0, p_1, p_2, ...$ such that p_0 is transmitted using time interval ψ_0, p_1 using time interval ψ_1, etc. The sequences of time intervals $\psi_0, \psi_1, \psi_2, ...$ is thus, the output of the SU access control algorithms addressed in the current paper. Optimizing this sequence is the DIC problem defined below. A *prudent* protocol is an opportunistic access protocol that transmits a sequence of packets whose lengths are the result of this optimization problem.

If the PU resumes transmitting at the time that the SU is transmitting its jth packets, then the SU ceases transmission, and the jth packet is lost (an *unsuccessful transmission*). In the other case (a *successful transmission*), the SU starts and completes a packet's transmission during the intermission. The SU's benefit of such a single PU intermission is a profit for the $j-1$ successful transmissions of the first $j-1$ packets. There is also a penalty for the unsuccessful transmission of the j'th packet. Different benefits and different penalties define different profit models. In this paper, we use the following profit model.

The α-*cost* profit model: Consider a constant $\alpha \geq 0$ (representing the above mentioned fixed overhead per packet). The SU's profit for a successful transmission of a single packet of time length ψ (a packet with ψ bits) is $\psi - \alpha$, i.e., the SU earns ψ and pays a fixed cost of α, for every time length $\psi > 0$. For simplicity, we ignore the cost of the last transmission. That is, for an unsuccessful transmission of a single packet, the SU earns nothing and pays nothing[5].

Definition 1. Dynamic Interval Cover (DIC) *is the optimization problem of generating a sequence of intervals according to the profit model described above.*

Our results A part of the novelty in this paper is the fact that we deal with case that the intermission length may be unknown at all. That is, we develop optimal $\left(1 + \frac{\sqrt{4\alpha - 3\alpha^2} + \alpha}{2(1-\alpha)}\right)$-competitive protocols (these protocols are 2.62-competitive,

[5] Note that alternative penalty assumptions, e.g., a double loss in the case of a collision, to account also for the PU's loss, can be analyzed using this framework but are left for future research.

since $\alpha \leq 1/2$). Protocols that seek to minimize the ratio of the SU's profit compared to an optimal hypothetical algorithm that does know the intermission length in advance. We also show that the competitive ratio for the bounded intermission model is better than the one for the unbounded case, however, it is only slightly better.

For the case where the distribution of the intermission is known, we address the case of a general distribution. Previous studies assumed some specific distribution for the length of the intermission. Most assumed the exponential distribution, see e.g., [4, 10, 11]. Others [1, 5, 12], extend this assumption to distributions derived from specific Markovian system models. For a general distribution, we present (Section 3) an efficient (polynomial) algorithm to compute the optimal length of each transmission interval for the realistic case that the interval length must be discrete, and a fully polynomial-time approximation scheme (FPTAS) for computing a sequence that approximates the optimal solution.

Interestingly, one difference resulting from the general distributions we address, is that (unlike those known studies of memory-less distributions) we show that the length of the optimal intervals in a sequence is not always constant. (Our method can be used also in the case that a constant length is required.) Finally, we also found some interesting properties of an optimal solution under some popular specific distributions.

Some related work A dual problem of DIC is the *buffer management* problem. In that problem, the packets arrive with different sizes. The online algorithm needs to decide which packets to drop while the size of each packet is fixed. The objective is to minimize the total value of lost packets, subject to the buffer space. Lotker and Patt-Shamir [6] studies this problem and present a 1.3-competitive algorithm.

The problem of cognitive access in a network of PUs and SUs was studied intensively. We refer to two surveys [2] and [7]. The problem of designing of sensing and transmission that maximized the throughput of the SU is studied in [4, 5, 7, 8, 10–12] under a model with collision constraints. Recall that our length optimization can be made after, and on top of, the optimizations performed by previous papers, since we optimize different parameters. Hence, a direct comparison of the the performance would not be meaningful.

Preliminaries Consider an output $\Psi : \mathbb{N}^+ \rightarrow \mathbb{R}$ of an SU access control algorithm that defines a sequence of time intervals $\Psi = \langle \psi_0, \psi_1, \psi_2, ... \rangle$. That is, $\psi_0 = \Psi(0), \psi_1 = \Psi(1), \psi_2 = \Psi(2), ...$. Denote by $\Psi_{\text{INF}} = \{\Psi \in \mathbb{R}^{\mathbb{N}} \mid \Psi \text{ is an infinite sequence}\}$ the family of infinite sequences (of time intervals). $\Psi_{\text{FIN}} = \{\Psi \in \mathbb{R}^M \mid M \in \mathbb{N}, \Psi \text{ is a finite sequence}\}$ is the family of finite sequences. Denote by $\Psi_M = \{\Psi \in \Psi_{\text{FIN}} \mid |\Psi| = M\}$ (for every $M \in \mathbb{N}^+$) the family of *M-size* sequences. Let $\mathrm{S}(\Psi, k) = \sum_{i=0}^{k} \psi_i$, and $\mathrm{S}(\Psi) = \mathrm{S}(\Psi, |\Psi|)$, for every $\Psi \in \Psi_{\text{FIN}}$. Let $K_{\langle \Psi, t \rangle}$ be the number of packets that SU transmitted successfully. The profit of a sequence Ψ, with respect to an intermission of duration t, is the sum of the profits of the SU for the time intervals in the sequence Ψ:

$$\textsc{Pfit}_{\langle \alpha \rangle}(\Psi, t) = \sum_{i=0}^{K_{\langle \Psi, t \rangle}} (\psi_i - \alpha). \tag{1}$$

2 Unknown PU intermission length

We begin with a difficult case in which even the distribution of the intermission length $t' \in \mathbb{R}^+$ is unknown to the SU. As is common in analyzing competitive algorithms [3], we measure the quality of the SU protocol by comparing the profit it obtains to the profit obtained by a hypothetical optimal algorithm (the "offline" algorithm) that knows the intermission length in advance. Moreover, we make this comparison in the worst case. Informally, one may envision an "adversary" who knows in advance the sequence Ψ of transmission intervals chosen by the SU and chooses an intermission length for which the profit of the SU from Ψ is minimized relatively to the profit of the optimal offline algorithm. Formal definitions are given below, following standard notations.

It is easy to verify that if the intermission can be shorter than α, then no online protocol can achieve a positive profit. We normalize the lengths and also α such that the minimum intermission length is 1 and $0 < \alpha < 1/2$ (this implies that the first packet is of length 1 in any optimal sequence and an optimal sequence has a positive profit). Let $\Psi_\alpha = \{\Psi \in \Psi_{\text{FIN}} \cup \Psi_{\text{INF}} \mid \psi_0 = 1 \text{ and } \psi_i \geq \alpha \text{ for every } i \geq 1\}$. This family of sequences has a positive profit for any intermission time length. Moreover, it has a nonnegative profit from each interval. It is easy to make the following observation.

Observation 1 *An optimal sequence must belong to* Ψ_α.

For a sequence $\Psi \in \Psi_\alpha$ and an intermission length $t \geq 1$, the *competitive ratio* is

$$\textsc{c-ratio}(\Psi, \alpha, t) = \frac{t - \alpha}{\textsc{Pfit}_{\langle \alpha \rangle}(\Psi, t)}. \tag{2}$$

The numerator in this ratio is the optimal (maximum) profit that could have been made by the SU had it known the intermission time length t (that is, a profit of $t - \alpha$ for the sequence $\langle t \rangle$). The denominator is the actual profit of the SU who selected sequence Ψ. The *competitive ratio* of Ψ is the maximum (over all $t \geq 1$) of the competitive ratio of Ψ with respect to t. That is,

$$\textsc{opt-ratio}(\Psi, \alpha) = \max_{t \geq 1} \textsc{c-ratio}(\Psi, \alpha, t).$$

The goal of the online algorithm is to generate a sequence Ψ that minimizes the competitive ratio. We consider both infinite and finite sequence models as well as the bounded and unbounded intermission time models. Denote the optimal competitive ratio for an infinite sequence model under the unbounded intermission time model (for punishment α) by

$$\textsc{opt-ratio}(\alpha) = \min_{\Psi \in \Psi_\alpha} \textsc{c-ratio}(\Psi, \alpha). \tag{3}$$

We present an optimal competitive ratio sequence and establish the following.

Theorem 1. $\text{OPT-RATIO}(\alpha) = 1 + \frac{\sqrt{4\alpha - 3\alpha^2} + \alpha}{2(1-\alpha)} \leq 2.62.$

For proving the theorem, it is convenient to define also the competitive ratio under a certain strategy of the adversary. Specifically, $f_{\langle \alpha, \Psi \rangle}(k)$ is computed as if the intermission ends just before the $(k+1)$'th interval in Ψ ends. I.e., the intermission length is $t' = \sum_{i=0}^{k+1} \psi_i - \epsilon$, where $\epsilon > 0$ is negligible. Let

$$f_{\langle \alpha, \Psi \rangle}(k) = \lim_{\epsilon \to 0^+} \text{C-RATIO}(\Psi, \alpha, \text{S}(\Psi, k+1) - \epsilon) = \frac{\sum_{i=0}^{k+1} \psi_i - \alpha}{\sum_{i=0}^{k} \psi_i - (k+1)\alpha} \quad (4)$$
$$= 1 + \frac{\psi_{k+1} + k\alpha}{\psi_0 + ... + \psi_k - (k+1)\alpha} ,$$

for every $i \in \mathbb{N}$. The usefulness of $f_{\langle \alpha, \Psi \rangle}$ becomes evident given the following.

Observation 2 *The competitive ratio of $\Psi \in \Psi_\alpha$ is*

$$\text{C-RATIO}(\alpha, \Psi) = \sup\{f_{\langle \alpha, \Psi \rangle}(k) \mid k \in \mathbb{N}\}.$$

By Eq.(3), the observation implies that the optimal competitive ratio is

$$\text{OPT-RATIO}(\alpha) = \min_{\Psi \in \Psi_\alpha} \sup\{f_{\langle \alpha, \Psi \rangle}(k) \mid k \in \mathbb{N}\}.$$

Claim 1 *There exists an optimal sequence Ψ for DIC such that $f_{\langle \alpha, \Psi \rangle}(k)$ is some constant for every $k \in \mathbb{N}$.*

Proof: Consider any optimal sequence Ψ^*. Let $\lambda = \sup\{f_{\langle \alpha, \Psi^* \rangle}(k) \mid k \in \mathbb{N}\}$ and let Ψ' be a sequence obtained from Ψ^* as follows: $\psi'_0 = 1$ and $\psi'_{i+1} = (\lambda - 1)\left(\sum_{j=0}^{i} \psi'_j - (i+1)\alpha\right) - i\alpha$. By induction on i and by Eq. (4) we know that $\psi'_i \geq \psi_i$ and $f_{\langle \alpha, \Psi' \rangle}(k) = \lambda$, for every $k \in \mathbb{N}$. It remains to prove that $\Psi' \in \Psi_\alpha$, so that Observation 2 can be used. For that, we prove by induction, that $\psi'_k \geq \psi^*_k$. By Observation 1, $\psi'_0 = \psi^*_0 = 1$. Assume that $\psi^*_i \leq \psi'_i$, for every $i = 0, ..., k$. On the one hand, we have

$$\lambda \geq f_{\langle \alpha, \psi^* \rangle}(k+1) = 1 + \frac{\psi^*_{k+1} + k\alpha}{\sum_{i=0}^{k} \psi^*_i - (k+1)\alpha} \geq 1 + \frac{\psi^*_{k+1} + k\alpha}{\sum_{i=0}^{k} \psi'_i - (k+1)\alpha},$$

where the left hand equality holds since $\sup\{f_{\langle \alpha, \Psi^* \rangle}(k) \mid k \in \mathbb{N}\} = \lambda$ and right hand inequality holds since $\sum_{i=0}^{k} \psi^*_i \leq \sum_{i=0}^{k} \psi'_i$, by the inductive assumption. On the other hand, we established above $\frac{\psi'_{k+1} + k\alpha}{\sum_{i=0}^{k} \psi'_i - (k+1)\alpha} = f_{\langle \alpha, \psi' \rangle}(k+1) - 1 = \lambda - 1$, By the above inequality, $\lambda - 1 \geq \frac{\psi^*_{k+1} + k\alpha}{\sum_{i=0}^{k} \psi'_i - (k+1)\alpha}$. Thus, $\psi'_{k+1} \geq \psi^*_{k+1}$, as required. Since $\Psi^* \in \Psi_\alpha$, also $\Psi' \in \Psi_\alpha$. The claim follows. ▮

By Eq. (4), Claim 1 implies that, for an optimal solution Ψ,

$$\frac{\psi_k + (k-1)\alpha}{\psi_0 + ... + \psi_{k-1} - k\alpha} = \frac{\psi_{k+1} + k\alpha}{\psi_0 + ... + \psi_k - (k+1)\alpha},$$

hence, for every $k = 1, 2, 3, ...$

$$\psi_{k+1} = \psi_k + \frac{(\psi_k - \alpha)(\psi_k + (k-1)\alpha)}{\psi_0 + ... + \psi_{k-1} - k\alpha} - \alpha. \quad (5)$$

This means that for every Ψ for which $f_{\langle \alpha, \Psi \rangle}(k)$ is some constant, ψ_1 determines Ψ uniquely (since $\psi_0 = 1$). In other words, every such Ψ can be characterized as $\Psi(x) = \langle \psi_0(x), \psi_1(x), \psi_2(x), ... \rangle$ such that $\psi_0(x) = 1$, $\psi_1(x) = x$ and $\psi_{k+1}(x) = \frac{(\psi_k(x)-\alpha)(\psi_k(x)+(k-1)\alpha)}{\psi_0(x)+...+\psi_{k-1}(x)-k\alpha} + \psi_k(x) - \alpha$, for every $k = 1, 2, ...$.

The construction of $\Psi(x)$ implies that, if $\psi_i(x) \geq \alpha$ for every $i \leq k$, then

$$f_{\langle \alpha, \Psi(x) \rangle}(j) = 1 + \frac{x}{1-\alpha}, \text{ for every } j = 0, ..., k. \quad (6)$$

Thus, $f_{\langle \alpha, \Psi(x) \rangle}(k)$ is smaller for smaller values of x. Unfortunately, it might be that $\Psi(x) \notin \Psi_\alpha$. By Observation 1 and by Claim 1, the optimum is,

$$\text{OPT-RATIO}(\alpha) = \min_{x \in [\alpha, \infty)} \{\text{C-RATIO}(\alpha, \Psi(x)) \mid \Psi(x) \in \Psi_\alpha\}.$$

Moreover, the fact that $f_{\langle \alpha, \Psi(x) \rangle}$ is monotonically increasing as a function of $x \in [\alpha, \infty)$, implies that, for $x^* = \min\{x \mid \Psi(x) \in \Psi_\alpha\}$,

$$\text{OPT-RATIO}(\alpha) = f_{\langle \alpha, \Psi(x^*) \rangle}(0). \quad (7)$$

We found that the optimal competitive ratio is achieved for some $x^* \in [\alpha, \infty)$, such that $\Psi(x^*) \in \Psi_\alpha$ and $x^* = \psi_1(x^*) = \psi_2(x^*) = \psi_3(x^*), ...$. Let $x^* = \frac{\alpha + \sqrt{4\alpha - 3\alpha^2}}{2}$. We prove that $\Psi(x^*) \in \Psi_\alpha$ is optimal and $\psi_i(x^*) = x^*$, for every $i = 1, 2, ...$. (It is easy to verify that $x^* \geq \alpha$ for every choice of $0 < \alpha < 1$.) We begin with the following claim.

Claim 2 *$\psi_i(x^*) = x^*$, for every $i = 1, 2, 3, ...$.*

Proof: We prove by induction. For $i = 1$, by definition of $\Psi(x)$, it follows that $\psi_1(x^*) = x^*$, hence, the base of the induction holds. Now, assume that the claim holds for every $i \in \{1, ..., k\}$. By Eq. (5), it suffices to prove that $\frac{(\psi_k(x^*)-\alpha)(\psi_k(x^*)+(k-1)\alpha)}{\psi_0(x^*)+...+\psi_{k-1}(x^*)-k\alpha} - \alpha = 0$. By assigning x^* for $\psi_i(x^*)$ (for every $i = 1, ..., k$, using the induction hypothesis) and $\psi_0(x^*) = 1$, we get $\frac{(\psi_k(x^*)-\alpha)(\psi_k(x^*)+(k-1)\alpha)}{\psi_0(x^*)+...+\psi_{k-1}(x^*)-k\alpha} - \alpha = \frac{(x^*-\alpha)(x^*+(k-1)\alpha)}{1+(k-1)x^*-k\alpha} - \alpha = \frac{(x^*)^2+(k-2)\alpha x^*-(k-1)\alpha^2}{1+(k-1)x^*-k\alpha} - \alpha = 0$, which implies that

$$(x^*)^2 - \alpha x^* + \alpha^2 - \alpha = 0.$$

This implies that $x^* = \frac{\alpha + \sqrt{4\alpha - 3\alpha^2}}{2}$ is the solution to the above quadratic equation under the assumption that $x^* \geq \alpha$. The claim follows. ∎

We now show that the sequence $\Psi(x)$ is monotonically decreasing.

Claim 3 *If $\psi_i(x) \geq \alpha$ for every $i \leq k$, then $\psi_{k+1}(x) < \psi_k(x)$ for every $x \in [\alpha, x^*)$ and every $k = 1, 2, 3, \ldots$.*

(Throughout, due to lack of space, some of the proofs are deferred to the full version of this paper.)

Claim 4 *$\Psi(x) \notin \Psi_\alpha$, for every $x < x^*$.*

Proof: If $x < \alpha$, then $\psi_1(x) = x < \alpha$, and the claim holds. Consider $\alpha \leq x < x^*$, and assume by the way of contradiction that $\Psi(x) \in \Psi_\alpha$. By claim 3, it follows that $\psi_{i+1}(x) < \psi_i(x) < x$, for every $i > 1$. Hence, $\lim_{i\to\infty} \psi_i(x) = x'$, for some $x' \in [\alpha, x)$. Therefore, by Eq. (2), we get that if $x' > \alpha$, then

$$\lim_{t\to\infty} \text{C-RATIO}(\Psi(x), \alpha, t) = \lim_{t\to\infty} \frac{t-\alpha}{\text{PFIT}_{\langle\alpha\rangle}(\Psi(x), t)} = \frac{x'}{x'-\alpha},$$

but $\frac{x'}{x'-\alpha} > \frac{x^*}{x^*-\alpha} = f_{\langle\alpha,\Psi(x^*)\rangle}(0) > f_{\langle\alpha,\Psi(x)\rangle}(0)$, since $\alpha \leq x' < x^*$, (that is the competitive ratio of $\Psi(x)$ is not $f_{\langle\alpha,\Psi(x)\rangle}(0)$), which is contradiction to Observation 2. Hence $\Psi(x) \notin \Psi_\alpha$. If $x' = \alpha$, then $\lim_{t\to\infty} \text{C-RATIO}(\Psi(x), \alpha, t) = \infty$, which leads to a contradiction as well. ∎

We are ready to prove that $\Psi(x^*)$ is an optimal sequence. It is easy to verify that $f_{\langle\alpha,\Psi(x^*)\rangle}(0) > f_{\langle\alpha,\Psi(x)\rangle}(0)$, for every $x > x^*$. Thus, by Observation 2, $\text{C-RATIO}(\alpha, \Psi(x)) > \text{C-RATIO}(\alpha, \Psi(x^*))$, hence $\Psi(x)$ is not optimal. On the other hand, by Claim 4, $\Psi(x) \notin \Psi_\alpha$ for every $x < x^*$. Thus, by Eq. (7), we get that $\Psi(x^*)$ is an optimal sequence. Recall that $\text{C-RATIO}(\alpha, \Psi(x^*)) = 1 + \frac{x^*}{1-\alpha} = 1 + \frac{\sqrt{4\alpha - 3\alpha^2} + \alpha}{2(1-\alpha)}$. This yields Theorem 1.

The bounded intermission time model Above, we have shown that for $\alpha < x < x^*$, the sequence $\Psi(x)$ is monotonically decreasing. (Actually, it can be proven that the sequence is decreasing also for $x \leq \alpha$; recall, $x^* > \alpha$). Informally, the intervals in the infinite suffix of a decreasing sequence that are "short", "pull" the competitive ratio down. Hence, intuitively, if we can stop the decrease, then we can improve the competitive ratio of a sequences $\Psi(x)$ for $\alpha < x < x^*$. In fact, the decrease of $\Psi(x)$ does stop when the intermission length is bounded. In other words, a bound on the length causes the above mentioned suffix of the sequence $\Psi(x)$ to become smaller (at least, it is finite). This allows us to choose x smaller than x^* and still not get a sequence $\Psi(x)$ with a tail of intervals that are "too small". As a result, we show that we can improve the competitive ratio to $1 + \frac{x'}{1-\alpha}$ for the bounded intermission model by choosing a sequence $\Psi(x')$, for some $\alpha < x' < x^*$. (It should be said, though, that the value of x' is, still, close to x^*, since the sequence $\Psi(x)$ decreases very fast when x is much smaller than x^*). Still informally, the more we reduce x, the faster the intervals at the suffix of $\Psi(x)$ drop to a length that is not useful. Hence, the value of x (or "how much can x be smaller than x^*") depends on the bound we are given on the interval. This is illustrated in Figure 1.

Fig. 1. In both parts of the figure, $\alpha = 0.4$. On the left, $T = 15$, and $x \approx x^* - 0.1^{10}$, a negligible improvement. On the right, $T = 5$ and $x \approx x^* - 0.1^3$, allowing a somewhat larger improvement.

Formally, the competitive ratio of Ψ with respect to time bound T is

$$\text{C-RATIO}_{\langle\alpha,T\rangle}(\Psi) = \max_{T \geq t \geq 1} \text{C-RATIO}_{\langle\alpha,T\rangle}(\Psi, t),$$

where $\text{C-RATIO}_{\langle\alpha,T\rangle}(\Psi, t) = \text{C-RATIO}(\Psi, \alpha, t)$. For a given sequence Ψ, such that $\psi_0 = 1$, and a real number x, denote by $pre_x(\Psi)$ the longest prefix of Ψ such that all intervals are of length at least x. That is, $pre_x(\Psi) = \langle\psi_0, \psi_1, ..., \psi_k\rangle$, where $\psi_i \geq x$, for every $0 \leq i \leq k$ and if the $|\Psi| > k + 1$, then $\psi_{k+1} < x$. (Note that, for $x = \alpha$, $pre_\alpha(\Psi) \in \Psi_\alpha$.)

Similarly to Observation 2, the maximum competitive ratio of $pre_\alpha(\Psi)$ is obtained even for a specific strategy of the adversary. In that strategy, the adversary chooses the intermission t to end just before the k'th interval, (for some k), or, alternatively, at $t = T$.

Observation 3 *The competitive ratio of* $pre_\alpha(\Psi) \in \Psi_\alpha$ *is,*
$\max\left(\text{C-RATIO}_{\langle\alpha,T\rangle}(\Psi, t = T), \max\{\lim_{\epsilon\to 0^+} \text{C-RATIO}_{\langle\alpha,T\rangle}(\Psi, \sum_{i=0}^{k} \psi_i - \epsilon) \mid k = 1, ..., |pre_\alpha(\Psi)| - 1\}\right)$.

Now, we prove four claims that help us find an optimal solution.

Claim 5 *Consider any* $x' > x''$ *and an index* $i \geq 1$. *Assume that* $\langle\psi_0(x''), ..., \psi_{i-1}(x'')\rangle \in \Psi_\alpha$ *and* $\psi_i(x'') \geq 0$. *Then,* $\psi_j(x') > \psi_j(x'')$, *for every* $1 \leq j \leq i$.

Claim 6 *There exists a sequence* $x_1, x_2, x_3, ... \in \mathbb{R}$ *, such that*
(P1) $|pre_0(\Psi(x))| = i + 2$, *for every* $x \in [x_i, x_{i+1})$;
(P2) $\psi_i(x_i) = \alpha$, $\psi_{i+1}(x_i) = 0$;
(P3) $x_1 = \alpha < x_2 < x_3 < ...$ *, and* $x_1, x_2, x_3, ... \in [\alpha, \frac{\alpha+\sqrt{4\alpha-3\alpha^2}}{2})$; *and*
(P4) For every $i \leq k+1$, $\psi_i(x)$ *is continuous and strictly increasing in the range* $[x_k, \frac{\alpha+\sqrt{4\alpha-3\alpha^2}}{2}]$.

Proof: We prove properties (P2), (P3) and (P4) by induction on i. The base of the induction holds, since $\psi_1(\alpha) = \alpha$, and by Eq. (5), $\psi_2(x) = x + \frac{(x-\alpha)x)}{1+x-\alpha} - \alpha$. thus, $\psi_2(\alpha) = 0$. Hence, properties (P2) and (P3) hold. In addition, $\psi_2(x)$ is continuous in $[\alpha, \infty]$, since $1 + x - 2\alpha > 0$, for every $x > \alpha$, and by Claim 5, $\psi_2(x)$ is strictly increasing in $[\alpha, \frac{\alpha+\sqrt{4\alpha-3\alpha^2}}{2}]$, hence (P4) holds as well.

Assume that (P2), (P3), and (P4) holds for every $i \leq k$. For every $i = 1,...,k$ and every $x \in [x_k, \frac{\alpha+\sqrt{4\alpha-3\alpha^2}}{2}]$, Properties (P3) and (P4) of the induction assumption imply that $\psi_i(x) \geq \psi_i(x_k) \geq \alpha$. Thus, $\sum_{i=0}^{k} \psi_i(x) - (k+1)\alpha \geq 1-\alpha > 0$. Therefore, by Eq. (5), $\psi_{k+1}(x)$ is continuous, and by Claim 5, $\psi_{k+1}(x)$ is strictly increasing in the range $[x_k, \frac{\alpha+\sqrt{4\alpha-3\alpha^2}}{2}]$. Hence, Property (P4) holds.

In addition, by (P2) of the induction assumption, $\psi_{k+1}(x_k) = 0$, and by Claim 2, $\psi_{k+1}(\frac{\alpha+\sqrt{4\alpha-3\alpha^2}}{2}) = \frac{\alpha+\sqrt{4\alpha-3\alpha^2}}{2} > \alpha$. Hence, there exists a real number $x_{x+1} \in (x_k, \frac{\alpha+\sqrt{4\alpha-3\alpha^2}}{2})$, such that $\psi_{k+1}(x_{k+1}) = \alpha$, and by Eq. (5), $\psi_{k+2}(x_{k+1}) = 0$. Thus, (P2) and (P3) holds.

Finally, consider (P1). On one hand, by (P2), (P3) and (P4), $\psi_i(x) \geq \alpha$, for every $i = 0, 1, ..., k$, and every $x \in [x_k, \frac{\alpha+\sqrt{4\alpha-3\alpha^2}}{2}]$. On the other hand, by (P2), $\psi_{k+1}(x_k) = 0$, $\psi_{k+1}(x_{k+1}) = \alpha$; by (P3) $x_k < x_{k+1}$; and by (P4) $\psi_{k+1}(x)$ is strictly increasing in the range $[x_k, \frac{\alpha+\sqrt{4\alpha-3\alpha^2}}{2}]$. Hence, $0 < \psi_{k+1}(x) < \alpha$, which implies, together with Eq. (5), that $\psi_{k+2}(x) < 0$, for every $x \in [x_k, x_{k+1})$. ∎

Claim 7 $\mathrm{S}(pre_0(\Psi(x)))$ *is continuous and is strictly increasing in the range* $x \in [\alpha, \frac{\alpha+\sqrt{4\alpha-3\alpha^2}}{2})$.

Proof: Let x_i be a real number such that $\psi_i(x_i) = \alpha$. (By property (P2) of Claim 6, there exists such a number.) For every $k \geq 1$, by property (P1) of Claim 6, $\mathrm{S}(pre_0(\Psi(x))) = \sum_{i=0}^{k+1} \psi_i(x)$ in the range $[x_k, x_{k+1})$. By property (P4) of Claim 6, $\sum_{i=0}^{k+1} \psi_i(x)$ is continuous and strictly increasing in the range $[x_k, \frac{\alpha+\sqrt{4\alpha-3\alpha^2}}{2})$. Combining this together with property (P3) of Claim 6, we get that $\mathrm{S}(pre_0(\Psi(x)))$ is continuous and strictly increasing in the ranges $[0, x_1), [x_1, x_2), [x_2, x_3), \ldots$.

It remains to prove that $\lim_{\epsilon \to 0^+} \mathrm{S}(pre_0(\Psi(x_k - \epsilon))) = \mathrm{S}(pre_0(\Psi(x_k)))$. (This also proves that $\mathrm{S}(pre_0(\Psi(x)))$ is strictly increasing.) We have

$$\lim_{\epsilon \to 0^+} \mathrm{S}(pre_0(\Psi(x_k - \epsilon))) = \sum_{i=0}^{k} \psi_i(x_k - \epsilon) = \sum_{i=0}^{k+1} \psi_i(x_k) = \mathrm{S}(pre_0(\Psi(x_k))).$$

where the second equality holds, since $\psi_{k+1}(x_k) = 0$ by property (P2) of Claim 6. The claim follows. ∎

Claim 8 *There exists an optimal sequence Ψ for DIC with a bound T on intermission time, such that $f_{\langle \alpha, \Psi \rangle}(k)$ is some constant for every $k = 0, ..., |\Psi| - 2$. (Similarly to Claim 1.)*

Proof: Consider an optimal solution Ψ^* for DIC with time bound T. Let $\lambda = \max\{f_{\langle \alpha, \Psi^* \rangle}(k) \mid k \in \{0, ..., |\Psi^*| - 1\}\}$ and let Ψ' be a sequence obtained from Ψ^* as follows: $\psi'_0 = 1$ and for every $1 \leq i \leq |\Psi^*| - 2$,

$$\psi'_{i+1} = (\lambda - 1) \left(\sum_{j=0}^{i} \psi'_j - (i+1)\alpha \right) - i\alpha.$$

Note that, $|\Psi^*| = |\Psi'|$, and by induction on i and by Eq. (4) we know that $\psi'_i \geq \psi^*_i$ and $f_{\langle \alpha, \Psi' \rangle}(k) = \lambda$. Thus, $\mathrm{S}(\Psi') \geq \mathrm{S}(\Psi^*)$, with equality if and only if $\Psi^* = \Psi'$. If $\Psi^* = \Psi'$, then the claim follows. Assume by the way of contradiction that $\Psi^* \neq \Psi'$, thus $\mathrm{S}(\Psi') > \mathrm{S}(\Psi^*)$. This implies that, if $\mathrm{S}(\Psi') \leq T$, then the profit of Ψ' is grater than the profit of Ψ^* for time T, hence $\text{C-RATIO}(\Psi', \alpha, T) \leq \text{C-RATIO}(\Psi^*, \alpha, T)$. Otherwise, $\mathrm{S}(\Psi') > T$, and then it follows that $\text{C-RATIO}(\Psi', \alpha, T) \leq \lambda$.

In both cases, $\text{C-RATIO}(\Psi', \alpha, T) \leq \max\{\lambda, \text{C-RATIO}(\Psi^*, \alpha, T)\}$. This implies that

$$\text{C-RATIO}_{\langle \alpha, T \rangle}(\Psi') \leq \max\{\lambda, \text{C-RATIO}_{\langle \alpha, T \rangle}(\Psi^*, t = T)\} = \text{C-RATIO}_{\langle \alpha, T \rangle}(\Psi^*).$$

Therefore, Ψ' is also optimal. In addition $f_{\langle \alpha, \Psi' \rangle}(k) = \lambda$, for every $0 \leq k < |\Psi'| - 1$. ▮

Let $x_{\langle T, \alpha \rangle}$ be real such that $\mathrm{S}(pre_0(\Psi(x_{\langle T, \alpha \rangle}))) = T$. (There exists such real, since by Claim 7 $pre_0(\Psi(x))$ is continuous, $\mathrm{S}(pre_0(\Psi(0))) = 1$ and $\lim_{\epsilon \to 0^+} \mathrm{S}(pre_0(\Psi(\frac{\alpha + \sqrt{4\alpha - 3\alpha^2}}{2} - \epsilon))) = \infty$.)

Theorem 2. *Let $\Psi^{\langle T, \alpha \rangle} = pre_\alpha(\Psi(x_{\langle T, \alpha \rangle}))$. The sequence $\Psi^{\langle T, \alpha \rangle}$ is optimal. The competitive ratio of any access protocol using it is $1 + \frac{x_{\langle T, \alpha \rangle}}{1 - \alpha}$.*

Proof: First, we prove that

$$|\Psi^{\langle T, \alpha \rangle}| = |pre_0(\Psi(x_{\langle T, \alpha \rangle}))| - 1. \tag{8}$$

Let $last = |pre_\alpha(\Psi(x_{\langle T, \alpha \rangle}))| - 1$. By Eq. (5),

$$\begin{aligned} &\text{if } \psi_i(x) \geq \alpha, \text{ then } \psi_{i+1}(x) \geq 0, \text{ and} \\ &\text{if } \psi_i(x) \in [0, \alpha), \text{ then } \psi_{i+1}(x) < 0. \end{aligned}$$

Hence, $\psi_{last}(x_{\langle T, \alpha \rangle}) \in [0, \alpha)$ and $\psi_{last-1}(x_{\langle T, \alpha \rangle}) \geq \alpha$, implying Eq. (8). It is easy to verify that

$$\text{C-RATIO}_T(\Psi^{\langle T, \alpha \rangle}, \alpha) = 1 + \frac{x_{\langle T, \alpha \rangle}}{1 - \alpha}. \tag{9}$$

Let Ψ^* be an optimal solution assuming the conditions of Claim 8. That is, $f_{\langle \alpha, \Psi^* \rangle}(k)$ is a constant. Let $x^* = \psi^*_1$. It follows that $\psi_i(x^*) = \psi^*_i$, for every $i = 0, 1, ..., |\Psi^*| - 1$. If $x^* > x_{\langle T, \alpha \rangle}$, then $\lim_{\epsilon \to 0^+} \text{C-RATIO}_T(\Psi^*, \alpha, 1 + \psi^*_1 - \epsilon) = 1 + \frac{x^*}{1 - \alpha}$. This implies that, $\text{C-RATIO}_T(\Psi^*, \alpha) \geq 1 + \frac{x^*}{1 - \alpha}$, and $1 + \frac{x^*}{1 - \alpha} > 1 + \frac{x_{\langle T, \alpha \rangle}}{1 - \alpha} = \text{C-RATIO}_T(\Psi^{\langle T, \alpha \rangle}, \alpha)$. This, is contradiction to the fact that Ψ^* is optimal. If $x^* < x_{\langle T, \alpha \rangle}$, then $\mathrm{S}(pre_0(\Psi(x^*))) < T$ and $\Psi^* = pre_\alpha(\Psi(x^*))$. Hence, $\text{C-RATIO}_{\langle \alpha, T \rangle}(\Psi(x^*), t = T) > f_{\langle \Psi(x^*), \alpha \rangle}$ and $\mathrm{S}(\Psi^*) < \mathrm{S}(pre_\alpha(\Psi(x_{\langle T, \alpha \rangle})))$. Thus,

$$\text{C-RATIO}_{\langle \alpha, T \rangle}(\Psi^*, t = T) > \text{C-RATIO}_{\langle \alpha, T \rangle}\big(pre_\alpha(\Psi(x_{\langle T, \alpha \rangle})), t = T\big) = 1 + \frac{x_{\langle T, \alpha \rangle}}{1 - \alpha},$$

which is contradiction to the selection of Ψ^* as the optimal. Therefore, $x^* = x_{\langle T, \alpha \rangle}$. The theorem follows. ▮

3 Probabilistic intermission length

We now turn to the case that the intermission time length is taken from a general probability distribution. To be able to deal with any probability distribution P, we assume that P is given as a black box that gets a value x and returns $P(x)$. (It is easy to generate this black box when the distribution follows some known function, e.g., poisson, uniform etc.). We present a polynomial algorithm for DIC for the discrete case, that approximates (as good as we want) the optimal solution for the continuous case. As in the previous case, it is enough to optimize the protocol for each intermission separately (because of the linearity of expectations). Hence, we concentrate on one intermission.

3.1 General probabilistic distribution

Consider a probability distribution $P : \mathbb{N} \to [0,1]$ that represents the intermission time length, i.e., Pr[intermission timelength $\geq t$] $= P(t)$. Sometimes, we assume that a priori upper bound T is known for the intermission time length. Denote by P_T a bounded probability distribution with bound T; That is, $P(t) = 0$, for every $t > T$.

Consider a finite sequence $\Psi = \langle \psi_1, \psi_2, ..., \psi_m \rangle \in \Psi_{\text{FIN}}$ and a probability distribution P_T. By Eq. (1), the expected profit of Ψ with respect to P_T is

$$\text{Epro}_{\langle P_T, \alpha \rangle}(\Psi) = \textstyle\sum_{k=1}^{|\Psi|} (\psi_k - \alpha) \cdot P_T(\text{S}(\Psi, k)).$$

We want to compute the maximal (optimal) expected profit of a sequence with respect to P_T, denoted by- $\text{OPT}(P_T) = \max\{\text{Epro}_{\langle P_T, \alpha \rangle}(\Psi) \mid \Psi \text{ is a sequence}\}$.

Bounded discrete domain First, consider the model where the intermission length is discrete, (consists of t' time slots) and is bounded from above by T. Before addressing the whole intermission, let us consider just its part that starts at $T - \ell$ (for some ℓ) and ends no later than T (note that $T - \ell$ may be empty if $t' \leq T - \ell$). Let $MAX^{TAIL}_{\langle P_T \rangle}(\ell)$ (for every $0 \leq \ell \leq T$) be the expected maximal profit that any sequence may have from this part. That is,

$$MAX^{TAIL}_{\langle P_T \rangle}(\ell) = \max_{\Psi \in \Psi_{\text{FIN}}} \{\sum_{k=1}^{|\Psi|} (\psi_k - \alpha) \cdot P_T\big(T - \ell + \text{S}(\Psi, k)\big)\}.$$

In particular, $\text{OPT}(P_T) = MAX^{TAIL}_{\langle P_T \rangle}(T) = MAX^{TAIL}_{\langle P_T \rangle}(T - 0)$. The recursive presentation of $MAX^{TAIL}_{\langle P_T \rangle}$ is $MAX^{TAIL}_{\langle P_T \rangle}(0) = 0$ and $MAX^{TAIL}_{\langle P_T \rangle}(\ell) = \max_{i \in \{0,1,...,\ell-1\}} \{P_T(T-i) \cdot (\ell - i - \alpha) + MAX^{TAIL}_{\langle P_T \rangle}(i)\}$. Using dynamic programming, we can compute $MAX^{TAIL}_{\langle P_T \rangle}(T) = \text{OPT}(P_T)$ and find an optimal sequence with time complexity $O(T^2)$. Thus, finding an optimal sequence is a polynomial problem in the value of T.[6]

[6] It is reasonable to assume that the time length T is polynomial in the size of the input. Had we assumed that T was say, exponential in the size of the input, this would have meant an intermission whose duration is so long, that in practice, it seems as being infinity, making the whole problem mute.

Bounded continuous domain Let us now consider the case where the probability distribution P_T is continuous. We present a fully polynomial-time approximation scheme (FPTAS) [9] for the case that the optimal solution provides at least some constant profit. We argue that in the other case, where the profit is of a vanishing value, a solution to the problem is useless anyhow. (Still, for completeness, we derived some result for that case: we have shown that if the profit in the optimal case is "too small", then it cannot be approximated at all.)

Consider a real number $\delta > 0$ such that $\mu \equiv T/\delta$ is an integer. Let $\Psi_\delta = \{\Psi \mid \psi_i/\delta$ is an integer, for every $i\}$. Let $MAX^{TAIL}_{\langle\delta,P_T\rangle}(\ell)$ (for every integer $0 \le \ell \le \mu$) be the expected maximum profit over sequences $\Psi \in \Psi_\delta$ from the time period $[T - \ell\cdot\delta, T]$. That is,

$$MAX^{TAIL}_{\langle\delta,P_T\rangle}(\ell) = \max_{\Psi\in\Psi_\delta}\left\{\sum_{k=1}^{|\Psi|}(\psi_k - \alpha)\cdot P_T\left(T - \ell\cdot\delta + \mathrm{S}(\Psi,k)\right)\right\}.$$

We show that the function $MAX^{TAIL}_{\langle\delta,P_T\rangle}$ approximates the value of $\textsc{opt}(P_T)$. In particular, we show that for any optimization parameter $\epsilon > 0$, there exists a $\delta > 0$ such that $MAX^{TAIL}_{\langle\delta,P_T\rangle}(T) \ge (1-\epsilon)\textsc{opt}(P_T)$. Intuitively, we first show that a large fraction of the expected profit of an optimal sequence Ψ^* is made from intervals whose lengths are "sufficient greater" than α. (A long interval can be approximated well by dividing it into smaller intervals, whose lengths are multipliers of δ; dividing a short interval is not profitable because of α.)

Consider an optimal sequence $\Psi^* = \langle\psi^*_1, ..., \psi^*_{|\Psi^*|}\rangle$. Let $\textsc{Pfit}^{TAIL}_\delta(i) = \sum_{j=i}^{|\Psi^*|}\left[(\psi^*_j - \alpha)\cdot P_T(\mathrm{S}(\Psi^*, j))\right]$ be the expected profit gained from the intervals $\psi^*_i, ..., \psi^*_{|\Psi^*|}$ under probability distribution P_T, for every $i \in \{1, ..., |\Psi^*|\}$.

Claim 9 *For any $\lambda > 0$, if $\textsc{Pfit}^{TAIL}_\delta(i) \ge \lambda$, then $\psi^*_i \ge \alpha + \min\{\frac{\lambda}{2T}, \lambda/2\}$.*

We are ready to show that $MAX^{TAIL}_{\langle\delta,P_T\rangle}(T)$ approximates $\textsc{opt}(P_T)$. Assume *first* that we know some constant λ such that $\textsc{opt}(P_T) \ge \lambda$. (We do not need to know how close is λ to $\textsc{opt}(P_T) \ge \lambda$.)

Lemma 1. *Consider $0 < \lambda \le \textsc{opt}(P_T)$ and an optimization parameter $\epsilon > 0$. Let $\mu_\lambda = \max\left\{\left\lceil\frac{16T^2}{\lambda\epsilon^2}\right\rceil, \left\lceil\frac{16}{\lambda\epsilon^2}\right\rceil\right\}$ and $\delta_\lambda = \min\{T/\mu_\lambda, 1/\mu_\lambda\}$. Then, $MAX^{TAIL}_{\langle\delta_\lambda,P_T\rangle}(T) \ge (1-\epsilon)\cdot\textsc{opt}(P_T)$. (For simplicity, we may omit λ from δ_λ and μ_λ.)*

Proof: Let $f_{left}(i)$ be the leftmost point s.t. (1) $f_{left}(i)/\delta$ is an integer, and (2) profit is at least, as large as the sum of all the elements $\psi^*_1, ..., \psi^*_{i-1}$. That is, $f_{left}(1) = 0$ and $f_{left}(i) = \min\{j\cdot\delta \mid j\cdot\delta \ge \sum_{j=1}^{i-1}\psi^*_j\}$. Similarly, $f_{right}(i)$ is the rightmost point such that (1) $f_{right}(i)/\delta$ is an integer, and (2) is at most, the sum of all element $\psi^*_1, ..., \psi^*_i$. It follows that $f_{right}(i) - f_{left}(i) > \psi^*_i - 2\delta$. Thus,

$$\frac{f_{right}(i) - f_{left}(i) - \alpha}{\psi^*_i - \alpha} > \frac{\psi^*_i - 2\delta - \alpha}{\psi^*_i - \alpha} = 1 - \frac{2\delta}{\psi^*_i - \alpha}\ .$$

Let $i^* = \max\{i \mid \textsc{Pfit}_{\delta}^{TAIL}(i) \geq \lambda\epsilon/2\}$. Combining the above inequality together with Claim 9 and the fact that $\delta \leq \min\{\frac{\lambda\epsilon^2}{16T}, \frac{\lambda\epsilon^2}{16}\}$, we get that

$$\frac{f_{right}(i) - f_{left}(i) - \alpha}{\psi_i^* - \alpha} > 1 - \max\left\{\frac{8T\delta}{\lambda\epsilon}, \frac{8\delta}{\lambda\epsilon}\right\} \geq 1 - \epsilon/2,$$

for every $i = 1, ..., i^*$. In addition, it follows that, if $i^* = |\Psi^*|$, then $\sum_{i=1}^{i^*}\big((\psi_i^* - \alpha) \cdot P_T(\mathrm{S}(\Psi^*, i))\big) = \textsc{opt}(P_T)$. Otherwise, $\sum_{i=1}^{i^*}\big((\psi_i^* - \alpha) \cdot P_T(\mathrm{S}(\Psi^*, i))\big) = \textsc{opt}(P_T) - \textsc{Pfit}_{\delta}^{TAIL}(i^* + 1) > \textsc{opt}(P_T) - \lambda\epsilon/2$. In both cases, we get that

$$\sum_{i=1}^{i^*}\big((\psi_i^* - \alpha)P_T(\mathrm{S}(\Psi^*, i))\big) > \textsc{opt}(P_T)(1 - \epsilon/2). \tag{10}$$

Therefore,

$$\begin{aligned}\sum_{i=1}^{i^*}\big(f_{right}(i) - f_{left}(i) - \alpha\big) \cdot P_T(\mathrm{S}(\Psi, i)) &\geq (1 - \epsilon/2) \cdot \left(\sum_{i=1}^{i^*}(\psi_i^* - \alpha) \cdot P_T(\mathrm{S}(\Psi, i))\right) \\ &\geq (1 - \epsilon/2) \cdot \textsc{opt}(P_T)(1 - \epsilon/2) \\ &\geq (1 - \epsilon)\textsc{opt}(P_T).\end{aligned}$$

For the first inequality see Ineq. (10); for the second one see Ineq. (10). Clearly, $MAX_{\langle\delta, P_T\rangle}^{TAIL}(T) \geq \sum_{i=1}^{i^*}[(f_{right}(i) - f_{left}(i) - \alpha) \cdot P_T(\mathrm{S}(\Psi, i))]$. ∎

Let us compute $MAX_{\langle\delta, P_T\rangle}^{TAIL}$ (which, we have shown, approximates the optimal sequence). The recursive presentation of $MAX_{\langle\delta, P_T\rangle}^{TAIL}$ is $MAX_{\langle\delta, P_T\rangle}^{TAIL}(0) = 0$ and $MAX_{\langle\delta, P_T\rangle}^{TAIL}(\ell) = \max_{i \in \{0,1,\ldots,\ell-1\}}\{P_T(T - i\cdot\delta)\cdot((\ell - i)\delta - \alpha) + MAX_{\langle P_T\rangle}^{TAIL}(i)\}$. The $MAX_{\langle\delta, P_T\rangle}^{TAIL}$, as well as the sequence attaining it (observed in the lemma) can now be computed using dynamic programming. The time complexity is $O((T/\delta)^2)$ and $T/\delta = O(\frac{T^2}{\epsilon^2})$; thus, it is polynomial in $1/\epsilon$, $1/\lambda$ and T.

Now, let us get rid of the assumption that λ is known. Let $\lambda_i = 2^{-i}$. We compute $MAX_{\langle\delta_{\lambda_0}, P_T\rangle}^{TAIL}$, then $MAX_{\langle\delta_{\lambda_1}, P_T\rangle}^{TAIL}$,..., as long as $MAX_{\langle\delta_{\lambda_i}, P_T\rangle}^{TAIL} < \lambda_i$. When $MAX_{\langle\delta_{\lambda_k}, P_T\rangle}^{TAIL} \geq \lambda_k$, the algorithm stops and returns a sequence attaining it. (Note that $MAX_{\langle\delta_{\lambda_k}, P_T\rangle}^{TAIL} \geq \lambda_k$ implies that $\textsc{opt}(P_T) \geq \lambda_k$.) The time complexity remains the same as $1/\lambda$ is a constant.

Theorem 3. *There exists a FPTAS for DIC for continuous distribution (for every instance of the problem for which the optimal solution obtains at least some constant positive profit).*

Theorem 4. *If there is no assumption on the minimum value of* $\textsc{opt}(P_T)$*, i.e., it can be negligible, then no approximation algorithm for the DIC problem exists.*

Proof: Consider an algorithm ALG for the DIC problem. An adversary can select P_T after the execution of ALG, such that $\textsc{opt}(P) > 0$ and $A(P_T) = 0$

(the expected profit of the sequence that made by ALG). Recall that algorithm ALG must produce a sequence Ψ_{ALG} of intervals. While producing the sequence, ALG may query the distribution. Let $x_1, x_2, ..., x_z$ be sequence of queries ALG made while producing Ψ_{ALG}, and let $P_T(x_1), P_T(x_2), ..., P_T(x_z)$ be the answers. Let $X_{>\alpha} = \{x_i > \alpha \mid i = 1, ..., z\}$. In the execution of ALG the black box (representing the distribution) return 0, for every $x > \alpha$ and returns 1, for every $x \leq \alpha$. That is, if $x_i > \alpha$, then $P_T(x_i) = 0$, otherwise $P_T(x_i) = 0$, for every $i = 1, ..., z$.

At the end of the execution ALG returns a sequence $\langle \psi_1^{ALG}, ... \rangle$. It is clear that $\psi_1^{ALG} \geq \alpha$ (otherwise, it might have a negative profit). If $\psi_1^{ALG} > \alpha$, then let $x' = \min\{\psi_1^{ALG}, x_i \mid x_i \in X_{>\alpha}\}$. Chose $\epsilon = x' - \alpha$ and $P_T(x) = 1$, for every $x \leq \alpha + \epsilon/2$. Otherwise, $P_T = 0$. We get that $\text{OPT}(P_T) = \epsilon/2 > 0$ and $ALG(P_T) = 0$ as required. If $\psi_1^A = \alpha$, then chose $x' = \min\{1.5\alpha, x_i \mid x_i \in X_{>\alpha}\}$. Set $P_T = 1$, for every $x \leq x'$ and otherwise $P_T = 0$. Thus clearly, $A(P_T) \leq 0$ and $\text{OPT}(P_T) = x' - \alpha > 0$. The Theorem follows. ∎

Finally, we describe some interesting observations on specific distributions (due to lack of space, this section is deferred to the full version).

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