Brief Announcement: Minimum Spanning Trees and Cone-Based Topology Control

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ABSTRACT

Consider a setting where nodes can vary their transmission power thereby changing the network topology, the goal of topology control is to reduce the transmission power while ensuring the communication graph remains connected. Wattenhofer et al. [6] introduced the distributed cone-based topology control algorithm with parameter α ($CBTC(\alpha)$) and proved it correct if $\alpha \leq \frac{2\pi}{3}$. Li et al. [4] proposed performing asymmetric edge removal or increasing α to $\frac{5\pi}{6}$, and proved that when applied separately these optimizations preserve connectivity. Bahramgiri et al. [1] proved that when $\alpha \leq \frac{2\pi}{3}$ it was possible to extend the algorithm to work in three dimensions and described a variation to preserve k-connectivity.

We give a short self-contained proof that when $\alpha \leq \frac{2\pi}{3}$ the minimum spanning tree is contained in the graph produced by $CBTC(\alpha)$. Its interesting to note that by comparison other topology control algorithms are variations of the Gabriel Graph [5], the Relative Neighbor Graph [2] or the Delaunay Triangulation [3]; all of which are structures known to contain the minimum spanning tree. The proof is essentially an application of a lemma proved by Yao [7]. As a consequence of this proof we get as corollaries new short proofs of some of the main technical results of Wattenhofer et al. [6], Li et al. [4] and Bahramgiri et al. [1]. (1) When $\alpha \leq \frac{2\pi}{3}$ the algorithm $CBTC(\alpha)$ preserves connectivity [6]. (2) The asymmetric edge removal operation preserves connectivity [4]. (3) The algorithm can be extended to three dimensions [1], and generally to n-dimensional space.

Categories and Subject Descriptors

C.2.1 [**Computer-Communication Networks**]: Network Architecture and Design—*Network topology, Wireless communication*; G.2.2 [**Discrete Mathematics**]: Graph Theory—*Graph algorithms*

General Terms

Algorithms, Theory

Keywords

Topology Control, Minimum Spanning Trees

1. DEFINITIONS AND ALGORITHM

Consider a set V of n nodes where $\|uv\|$ is the distance from u to v and $p(u,v) : V \times V \rightarrow \mathbb{R}^+$ is the minimum power required to reach node v from node u. The model assumes all nodes can transmit with the same maximum power $p_{\max}$ and $p(u,v) \leq p(u,w)$ iff $\|uv\| \leq \|uw\|$; in other words the power function is symmetric and a non-decreasing function of distance. Furthermore a node has the ability to accurately determine the direction from which another node is transmitting.

Let $G = (V,E)$ be the maximum power communication graph, so $E = \{(u,v) \mid p(u,v) \leq p_{\max}\}$. Running $CBTC$ with parameter α produces some power assignment $\sigma_\alpha : V \rightarrow [p_{\min}, p_{\max}]$. Using σ_α we define the graph $G_\alpha^+ = (V, E_\alpha^+)$ where $E_\alpha^+ = \{(u,v) \mid p(u,v) \leq \sigma_\alpha(u) \vee p(v,u) \leq \sigma_\alpha(v)\}$, and the symmetric version $G_\alpha^- = (V, E_\alpha^-)$ where $E_\alpha^- = \{(u,v) \mid p(u,v) \leq \sigma_\alpha(u) \wedge p(v,u) \leq \sigma_\alpha(v)\}$.

We describe the core of the $CBTC(\alpha)$ algorithm informally; for a detailed description we refer the reader to Wattenhofer et al. [6]. The algorithm proceeds in synchronous rounds, at the beginning of a round every node u broadcasts a HELLO message. Each receiving node replies with an ACK message, and node u collects the replies along with the direction from which they came from. Initially nodes transmit with minimum power $p_{\min}$ and increase (i.e. double) the transmission power when going into the next round. Node u terminates the algorithm when either it reaches maximum power or every cone with apex at u of aperture α contains a neighbor.

2. RESULTS

Assuming the nodes are embedded in the Euclidean plane and $\alpha \leq \frac{2\pi}{3}$ Wattenhofer et al. [6] proved that G_α^+ is a spanning subgraph of G; under the same assumptions Li et al. [4] described the asymmetric edge removal procedure to obtain G_α^- and proved it was a also spanning subgraph of G; Bahramgiri et al. [1] allowed the nodes to be in three-dimensional space and proved G_α^+ preserved connectivity.

We start with a succinct proof of a special case of Yao's lemma using the Euclidean metric; for the general lemma see Yao [7]. An Euclidean minimum spanning tree is a minimum spanning tree were the weight of an edge (u,v) is the Euclidean distance $\|uv\|$.

LEMMA 1 (YAO'S LEMMA). *If an edge (u,v) belongs to the Euclidean minimum spanning tree, then v is u's closest neighbor in every cone with apex at u and aperture $\frac{\pi}{3}$ which contains v.*

PROOF. Suppose not, then there exists an edge (u,v) which belongs to the Euclidean minimum spanning tree T, u has a neighbor w where $\|uw\| < \|uv\|$, and there is a cone with apex at u and aperture $\frac{\pi}{3}$ which contains both v and w.

Removing the edge (u,v) from T creates two disjoint connected components P and Q where $u \in P$ and $v \in Q$. If $w \in Q$ then the joining P and Q with the edge (u,w) creates a tree of smaller weight – a contradiction.

Hence suppose $w \in P$, by the cosine law we have $\|vw\|^2 = \|uv\|^2 + \|uw\|^2 - 2\|uv\|\|uw\|\cos\theta$ where $\theta = \angle v\hat{u}w$. By assumption we have $\|uw\| < \|uv\|$ and since v and w are contained in a cone with apex at u of aperture $\frac{\pi}{3}$ then clearly $\theta \leq \frac{\pi}{3}$. Hence $\cos\theta \in (\frac{1}{2}, 1]$ and thus $\|vw\|^2 < \|uv\|^2$, therefore joining P and Q with the edge (v,w) creates a tree of smaller weight – a contradiction. $\square$

Let $MST(H)$ be the minimum spanning tree of H, now we are ready to prove the main theorem.

THEOREM 2. *If $\alpha \leq \frac{2\pi}{3}$ then $MST(G) \subseteq G_\alpha^-$.*

PROOF. To prove this lemma its sufficient to show that if $\alpha \leq \frac{2\pi}{3}$ then $e \notin G_\alpha^- \Rightarrow e \notin MST(G)$.

Fix some edge $(u,v) \notin G_\alpha^-$; without loss of generality we assume $p(u,v) > \sigma_\alpha(u)$. Consider a cone with apex at u and aperture $\alpha \leq \frac{2\pi}{3}$ with its axis passing through v. By construction of $CBTC(\alpha)$ this cone contains some vertex w and moreover $\|uw\| < \|uv\|$. Since the cone has aperture $\leq \frac{2\pi}{3}$ and v is at its axis, it follows that $\angle v\hat{u}w \leq \frac{\pi}{3}$. Hence there exists a cone with apex at u of aperture $\frac{\pi}{3}$ which contains both v and w, and by Lemma 1 it follows that $(u,v) \notin MST(G)$. $\square$

Now as promised the corollaries trivially follow.

COROLLARY 1. *If $\alpha \leq \frac{2\pi}{3}$ then $CBTC(\alpha)$ enhanced with asymmetric edge removal preserves connectivity of G (Li et al. [4]).*

PROOF. By Theorem 2, $MST(G) \subseteq G_\alpha^-$ and by definition $MST(G)$ is a spanning subgraph of G. $\square$

COROLLARY 2. *If $\alpha \leq \frac{2\pi}{3}$ then $CBTC(\alpha)$ preserves connectivity of G (Wattenhofer et al. [6]).*

PROOF. The asymmetric edge removal procedure only removes edges hence $G_\alpha^- \subseteq G_\alpha^+$, therefore as before we have $MST(G) \subseteq G_\alpha^+$. $\square$

COROLLARY 3. *If $\alpha \leq \frac{2\pi}{3}$ then running $CBTC(\alpha)$ where nodes are in three-dimensional space using three-dimensional cones of aperture α preserves connectivity of G (Bahramgiri et al. [1]).*

PROOF. We prove something stronger; if $\alpha \leq \frac{2\pi}{3}$ then $CBTC(\alpha)$ enhanced with asymmetric edge removal preserves connectivity when nodes are embedded in n-dimensional space using n-dimensional cones of aperture α. This follows since the proof of Theorem 2 only required Yao's Lemma which holds for any number of dimensions (and under different metrics). $\square$

3. REFERENCES

[1] M. Bahramgiri, M. T. Hajiaghayi, and V. S. Mirrokni. Fault-tolerant and 3-dimensional distributed topology control algorithms in wireless multi-hop networks. *Wireless Networks*, 12(2):179–188, 2006.

[2] S.A. Borbash and E.H. Jennings. Distributed topology control algorithm for multihop wireless networks. *Neural Networks, 2002. IJCNN '02. Proceedings of the 2002 International Joint Conference on*, 1:355–360, 2002.

[3] L. Hu. A novel topology control for multihop packet radio networks. *IEEE INFOCOM 1991. Tenth Annual Joint Conference of the IEEE Computer and Communications Societies. Proceedings*, pages 1084–1093 vol.3, Apr 1991.

[4] L. Li, J. Y. Halpern, P. Bahl, Y. M. Wang, and R. Wattenhofer. Analysis of a cone-based distributed topology control algorithm for wireless multi-hop networks. *PODC 2001: Proceedings of the twentieth annual ACM symposium on Principles of distributed computing*, pages 264–273, 2001.

[5] X.Y. Li, P.J. Wan, Y. Wang, and O. Frieder. Sparse power efficient topology for wireless networks. *System Sciences, 2002. HICSS. Proceedings of the 35th Annual Hawaii International Conference on*, pages 3839–3848, Jan. 2002.

[6] R. Wattenhofer, L. Li, P. Bahl, and Y.M. Wang. Distributed topology control for power efficient operation inmultihop wireless ad hoc networks. *IEEE INFOCOM 2001. Twentieth Annual Joint Conference of the IEEE Computer and Communications Societies. Proceedings*, 3, 2001.

[7] A. C. Yao. On constructing minimum spanning trees in k-dimensional spaces and related problems. *SIAM J. Comput.*, 11(4):721–736, 1982.