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# The Euler-Gompertz Constant

## and its relations to Wallis' Hypergeometric series

Master Thesis Mathematics


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November 21, 2016

# Abstract

Basic rules and definitions for summing divergent series, regularity, linearity and stability of a summation method. Examples of common summation methods: averaging methods, analytic continuation of a power series, Borel's summation methods.
Introducing a formal totally divergent power series $F(x) = 0! - 1!x + 2!x^2 - 3!x^3 + \dots$; the main interest is the value at $x = 1$ called Wallis' hypergeometric series (WHS). Examine the four summation methods used by Euler to assign a finite value $\delta \approx 0.59$ (Euler-Gompertz constant) to this series: (1) Solving an ordinary differential equation that has a formal power series solution $F(x)$; (2) Repeated application of Euler transform - a regular summation method useful to accelerate oscillating divergent series; (4) Extrapolating a polynomial $P(n)$ which formally gives WHS at $n = 0$; (3) Expanding $F(x)$ as a continued fraction and inspecting its convergence.
Multiple connections among the four methods are established, mainly by notions of asymptotic series and Borel summability. The value of $\delta$ is approximated by 3 methods, at most to a precision of several thousand decimal places.

Keywords: Euler-Gompertz constant, Wallis' hypergeometric series, divergent series, averaging summation methods, Borel summation, Euler transform, asymptotic series, continued fractions

# Acknowledgements

First of all I would like to thank my supervisor Dr. Alef Sterk for his invaluable advice and expertise. He allowed this paper to be my own work while providing thorough feedback and pointing me in the right direction whenever I needed it.

I would also like to thank my second supervisor Prof. Dr. Jaap Top, who helped me with the choice of the topic and offered his advice on multiple occasions.

To Karin Rozeboom, Bas Nieraeth and Jelmer van der Schaaf goes my heartfelt thanks, as they spent hours of their own free time searching for typos and giving me suggestions on how to improve the readability of the paper.

Finally, I must express my profound gratitude to my parents and to my boyfriend Robert Beerta for their unceasing support and continuous encouragement throughout my years of study. Without them this accomplishment would not have been possible.

# Contents

# Introduction

*"Divergent series are the invention of the devil, and it is shameful to base on them any demonstration whatsoever."* – N. H. Abel

This quote from Abel's letter to his friend Holmboe is a fitting description of how rigorists, who began to dominate mathematical research towards the end of 19th century, felt about divergent series. Despite having been investigated before by many, including Euler, Poisson or Fourier, and by that time having lots of successful arguments in applied physics and astronomy, they spiked controversy and were generally frowned upon. Part of the problem of assigning a value to a series that did not converge might have been the fact that after Cauchy formally defined what a sum of convergent series is, nobody yet made a proper generalisation for divergent series.

This distaste towards divergent series was not as prominent in France. In Paris around 1886, Poincaré and Stieltjes created the theory of asymptotic series. Earlier, Frobenius and Hölder began developing a summation method that was later completed by Cesàro. It summed a large class of divergent series. The sums defined this way turned out to make sense both in applications and in theoretical work[†].

Nowadays, the theory of summing divergent series is fairly well-developed, one of the greatest contributions undoubtedly being the book "Divergent Series" (1949) by G. H. Hardy. If a summation method is well defined, consistent with convergent series and adhering to certain reasonable rules, it may furnish a natural generalisation of the sum to divergent series that can be manipulated in many ways typical to convergent series. Even the notion of approximating a function can be extended to divergent series by means of asymptotic expansions.

In this thesis we will not pick apart the general theory of summing divergent series, but rather have a look at a particular one: Wallis' hypergeometric series. We will also consider its many connections to a constant usually referred to as the Euler-Gompertz constant and denoted $\delta$.

Define a hypergeometric power series in a complex variable $z$

$$F(z) = \sum_{n=0}^{\infty}(-1)^n n!\,z^n = 0! - 1!\,z + 2!\,z^2 - 3!\,z^3 + 4!\,z^4 - 5!\,z^5 + \dots \tag{1}$$

We will write $F(z)$ for complex variable $z$ and $F(x)$ if we only consider $x \geq 0$. Obviously $F(z)$ only converges for $z = 0$ and for $z < 0$ it is a series with positive unbounded terms, hence diverging to infinity. By several summation methods this series is assigned a finite value $f(z)$ of the following form:

$$f(z) = \int_0^{\infty} \frac{e^{-t}}{1+zt}\mathrm{d}t. \tag{2}$$

[†]More on the history of divergent series can be found in Jahnke (2003)

At $z=1$ the series is referred to by Euler as Wallis' hypergeometric series (WHS); its formal sum (later defined in several ways) will be denoted by $\delta$:

$$\delta = \int_0^\infty \frac{e^{-t}}{1+t}\mathrm{d}t \quad \leftrightarrow \quad F(1) = \sum_{n=0}^{\infty}(-1)^n n! = 0!-1!+2!-3!+4!-5!+\dots$$

This series caught the interest of Leonhard Euler who then wrote a paper "On Divergent Series" (1760) entirely dedicated to its summation. It is worth noting that at that time dealing with divergent series was quite controversial, which compelled Euler to devote the first 13 paragraphs (out of total 27) to carefully convincing the reader that what he is doing is not a complete heresy. In spite of being hardly rigorous, his work is almost entirely correct, proving once again his marvelous mathematical intuition.

Euler summed the series using 4 different methods; our goal will be to address and examine each of them separately and find connections among them. We will consult more recent literature to find out more about these and other useful summation methods.

In the first chapter (Preliminaries) we acquaint ourselves with basic rules and definitions for summing divergent series and a few well known regular summation methods. Section 1.3 introduces a powerful method developed by Borel, which is the first method capable of summing the hypergeometric series (1) and will play an important role in following chapters. The last section of the chapter defines a class of summation methods using weighted averages to transform a given series. One simple example is inspected more closely in subsection 1.4.1 (Midpoint method), with many examples of divergent series summed by this method.

The remaining four chapters deal with the four different approaches by Euler, listed in a different order from his original paper for our convenience:

1. The third method: solving an ordinary differential equation that is formally satisfied by the series (1). The first approach is by G. H. Hardy as laid out in his book Divergent Series, after that we solve the equation in a more rigorous manner and explain the connection between the two solutions by means of asymptotic series.

2. The first method: Euler method $(E,1)$ or Euler transform. Its repeated application to WHS accelerates the series and gives an approximation of $\delta$. The generalised method $(E,q)$ for $q>0$ and its relation to repeated application of $(E,1)$ will be defined and it will be shown that Borel method is consistent with each of these methods and still stronger, being the limiting case of $(E,q)$ as $q\to\infty$.

3. The second method: define a polynomial $P(n) = 1+(n-1)+(n-1)(n-2)+(n-1)(n-2)(n-3)+\dots$, which has finitely many terms for each $n\in\mathbb{N}$. Then $P(0)$ gives WHS. Euler used tried to extrapolate $P(n)$ at 0 to approximate $\delta$ using Newton's extrapolation method. We will show that this does not work and introduce instead an extrapolating function obtained from the Borel sum of the series. This function again assigns the value $\delta$ to $P(0)$.

4. The fourth method: formal continued fraction expansion of a class of series including (1) will be shown to converge; in case of (1) to the function $f(z)$. By means of a simple transformation we will define a proper summation method by continued fractions, attributed to Stieltjes, and obtain another continued fraction representing WHS and $\delta$. This continued fraction will be used to compute 8683 decimal places of the constant.

In the conclusion there will be a short summary of all found connections between the four methods and also all expressions representing $\delta$.

# Chapter 1

# Preliminaries

## 1.1 Notation and conventions

Throughout the work we will use the same notation for the following things whenever possible:

- $n, m, i, j$ for **indices** starting from 0 unless specified otherwise, i.e. $n, m, i, j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$;
- $a_0, a_1, a_2, \ldots, a_n, \ldots$ for the **terms** of a series;
- bold letters $\mathbf{v} = \{v_0, v_1, v_2, \ldots, v_n \ldots\}$ denote (usually infinite) **vectors**;
- $\mathbf{s} = \{s_0, s_1, s_2, \ldots, s_n, \ldots\}$ is the sequence of the **partial sums** of a series; $\mathbf{s}$ can be also treated as an infinite vector. The series itself can be referred to as series $\mathbf{s}$;
- **series transformations** will use capital calligraphic letters $\mathcal{M}, \mathcal{T}, \ldots$. If a transformation has a matrix representation, these will be denoted and considered the same.
  If $\mathcal{T}$ is a series transformation, we denote $\mathcal{T}^k\mathbf{s}$ the series resulting from $\mathcal{T}$ applied on $\mathbf{s}$ $k$-times. The partial sums will be then denoted $\mathcal{T}^k\mathbf{s}(n)$ or, in case there is no confusion as to which transformation is used, $s_n^{(k)}$. Similarly the $n$-th term of the $k$-times transformed series will be denoted as $a_n^{(k)}$. In agreement with the original notation for $\mathbf{s}$, $s_n^{(0)} = s_n$ and $a_n^{(0)} = a_n$ for all $n \in \mathbb{N}_0$;
- unless specified otherwise, $z$ will stand for a complex variable and $x$ for a real variable;

Standard definitions and their notations throughout the work:

- Difference operator:
  For a sequence $\{a_n\}_{n \in \mathbb{N}_0}$ define the differences $\Delta a_n = a_{n+1} - a_n$.
- Small o notation:
  Let $f(x)$, $g(x)$ be real functions and $x_0 \in \mathbb{R}$. We say that $f(x)$ is *asymptotically smaller* than $g(x)$ and write $f(x) = o(g(x))$ as $x \to x_0$ provided that for any $\varepsilon > 0$ there is $\delta > 0$ such that
  $$|f(x)| \leq \varepsilon |g(x)|$$
  whenever $|x - x_0| < \delta$. Equivalently, if $g(x)$ is non-zero in some neighbourhood of $x_0 \in \mathbb{R} \cup \{-\infty, \infty\}$ (except possibly at $x_0$),
  $$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0.$$

- Big O notation:
  We say $f(x)$ is *asymptotically bounded* by $g(x)$ and write $f(x) = O(g(x))$ as $x \to x_0$ if there is a constant $M \in \mathbb{R}^+$ such that
  $$|f(x)| \leq M|g(x)|$$
  in some neighbourhood of $x_0 \in \mathbb{R}$ (or, in case $x_0 = \pm\infty$, for sufficiently large $x$).

- We say $f(x)$ is *asymptotically equivalent* to $g(x)$ and write $f(x) \sim g(x)$ as $x \to x_0$ provided that
  $$f(x) = g(x) + o(g(x))$$
  as $x \to x_0$, or equivalently, provided that $g(x) \neq 0$ in some neighbourhood of $x_0$ (resp. for sufficiently large $x$ in case $x_0 \pm \infty$), if
  $$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1.$$

Unless mentioned otherwise, the series considered in the thesis are always complex. By "regular convergence" we mean the convergence of partial sums in $\mathbb{C}$ with respect to the Euclidian topology.

## 1.2 Basic rules and definitions for summing divergent series†

Defining a "sum" of a divergent series sounds vague and counter-intuitive, but we can treat it simply as an extension of the theory of convergent series. Thus intuitively we should want it to obey some natural rules to be consistent with that theory. Most of the definitions of a sum of a divergent series should therefore adhere to at least one of the following rules:

(I) Multiplication by a constant:
if $\sum\limits_{n=0}^{\infty} a_n = s$ and $c \in \mathbb{C}$ is a constant, then $\sum\limits_{n=0}^{\infty} ca_n = cs$.

(II) Term by term addition:
if $\sum\limits_{n=0}^{\infty} a_n = s$ and $\sum\limits_{n=0}^{\infty} b_n = t$ then $\sum\limits_{n=0}^{\infty} (a_n + b_n) = s + t$.

(III) Subtraction of a constant:
if $\sum\limits_{n=0}^{\infty} a_n = s$ then $\sum\limits_{n=1}^{\infty} a_n = s - a_0$ and vice versa.

The first two rules define linearity of a method, while the third can be described as stability. Using only these rules we can compute the "natural sum" for many divergent series. As an example consider the series $\sum_{n=0}^{\infty}(-1)^n = 1 - 1 + 1 - 1 + 1 - \dots$, that has its partial sums oscillating between 0 and 1, therefore it is divergent. If $s$ is the sum of this series, then by rules 3 and 1 we have:

$$s = 1 - 1 + 1 - 1 + \dots = 1 + (-1 + 1 - 1 + 1 - \dots) = 1 - (1 - 1 + 1 - 1 + \dots) = 1 - s$$

†This theory in this section follows Hardy, Sections 1.3 and 1.4

and so $s = \frac{1}{2}$.

We will naturally never write $\sum_{n=0}^{\infty} a_n = s$ for a divergent series, as it does not have a sum in the conventional sense, but employ the following notation instead: if $A$ is a notation for a summation method assigning a number $s$ to a series $\sum_{n=0}^{\infty} a_n$, we say the series is $A$-*summable* or *summable* $(A)$, call $s$ the $A$-sum of $\sum_{n=0}^{\infty} a_n$ and write $\sum_{n=0}^{\infty} a_n = s \ (A)$.

The following definitions explain regularity of a method.

**Definition 1. (Regular method):** A summation method is said to be regular if it sums every convergent series to its ordinary sum.

**Definition 2. (Totally regular method):** A method is said to be totally regular if in addition to being regular it gives $s = \infty$ for a series $\sum_{n=0}^{\infty} a_n$ where $a_n \in \mathbb{R}$ and $s_n \to \infty$.

A regular method has the ability to transform a divergent series into a function that has a finite limit at infinity, while not disrupting the finite limit of a sequence that is already convergent, thus we can think of it as a "taming" transformation (Enyeart (RDSTT)).

Notice that a (totally) regular summation method must oblige rules (I)-(III) for convergent series, but it is not granted that the same holds for divergent series summable by the given method. As a simple example consider a method that assigns to convergent series their regular value and a fixed constant to all other series. Similarly, a method consistent with rules (I)-(III) might not be regular; method $\mathfrak{E}$ defined below is one such case. The best methods are naturally those both regular and adhering to rules (I)-(III), as they can preserve useful properties known to convergent series.

Now we can introduce some basic summation methods which are (totally) regular and, as can be easily verified in most cases, obey rules (I)-(III).

**Definition 3. (Cesàro summation):** If $s_n = a_0 + a_1 + a_2 + \cdots + a_n$ for $n \in \mathbb{N}_0$ and

$$\lim_{n\to\infty} \frac{s_0 + s_1 + \cdots + s_n}{n+1} = s\,,$$

then we call $s$ the $(\mathcal{C},1)$-sum of $\sum_{n=0}^{\infty} a_n$ and the $(\mathcal{C},1)$-limit of $s_n$.

The method of Cesàro is an example from a class of summation methods all using some averaging process. They are addressed closely in Section 1.4.

Abel summation is consistent with but more powerful than Cesàro:

**Definition 4. (Abel summation):** If $\sum_{n=0}^{\infty} a_n x^n$ is convergent for $0 \le x < 1$ (and thus for all $|z| < 1$ complex) with $g(x)$ its sum and

$$\lim_{x\to 1^-} g(x) = s\,,$$

then we call $s$ the $\boldsymbol{\mathcal{A}}$-sum of $\sum_{n=0}^{\infty} a_n$.

Some explanation is needed before we define Euler's summation method $(E,1)$.

Suppose $\sum_{n=0}^{\infty} a_n x^n$ converges to $g(x)$ for small $x$ and let $y = \frac{x}{1+x}$, so $y = \frac{1}{2}$ corresponds to $x = 1$. Then for small $x$ and $y$ we have

$$\begin{aligned} xg(x) &= \sum_{n=0}^{\infty} a_n x^{n+1} = a_0 \frac{y}{1-y} + a_1 \frac{y^2}{(1-y)^2} + a_2 \frac{y^3}{(1-y)^3} + \dots \\ &= \sum_{n=0}^{\infty} a_n \sum_{k=0}^{\infty} \binom{n+k}{k} y^{n+k+1} = \sum_{n=0}^{\infty} a_n \sum_{m=n}^{\infty} \binom{m}{m-n} y^{m+1}, \end{aligned}$$

where the second line is derived from the Taylor expansion of $\frac{1}{(1-y)^{n+1}} = \sum_{k=0}^{\infty} \binom{n+k}{k} y^k$. Changing the order of summation we find that

$$xg(x) = \sum_{m=0}^{\infty} y^{m+1} \sum_{n=0}^{m} \binom{m}{m-n} a_n = \sum_{m=0}^{\infty} y^{m+1} \sum_{n=0}^{m} \binom{m}{n} a_n = \sum_{m=0}^{\infty} b_m y^{m+1}$$

where $b_0 = a_0$, $b_m = a_0 + \binom{m}{1} a_1 + \binom{m}{2} + \cdots + a_m$.

**Definition 5. (Euler's summation):** Define the power series in $x$ and $y$ as above. If the $y$-series is convergent for $y = \frac{1}{2}$, that is, if $\sum_{m=0}^{\infty} 2^{-m-1} b_m = s$, then we call $s$ the $(E,1)$-sum of $\sum_{n=0}^{\infty} a_n$.

Euler's summation is an accelerating method, as it "tames" the growth of the series. More interestingly, even if the resulting series does not converge for $y = \frac{1}{2}$, it can be applied again.

This definition relies on convergence for small $x$ and $y$ and hence is not applicable in case of a series like $\sum_{n=0}^{\infty} (-1)^n n! x^n$, which does not converge for values other than 0. However, a weaker version called Euler transform (in essence the same transformation formally defined and omitting the requirement of convergence for small values) can be applied to any divergent series. It was used by Euler to approximate $\delta$ and will be closely addressed in Chapter 3, together with the generalised Euler's summation $(E,q)$ for $q > 0$.

**Definition 6. (Analytic continuation of power series):** If $\sum_{n=0}^{\infty} a_n z^n$ is convergent for small $z$ and converges to a function $g(z)$ of the complex variable $z$, one-valued and regular in an open and connected region containing the origin and the point $z = 1$, and $g(1) = s$, then we call $s$ the $\mathfrak{C}$-sum of $\sum_{n=0}^{\infty} a_n$. The value of $s$ may depend on the region chosen.

Similarly this can be defined with paths instead of regions. This last method is consistent with rules (I)-(III) but it is not totally regular (not even regular as $s$ might depends on the chosen region), and, as an interesting fact, assigns a rather confusing sum $s = -1$ to the series $1 + 2 + 4 + 8 + \ldots$.

The following section introduces a powerful summation method attributed to Borel, which will be an important tool throughout this work as it connects different approaches to summing (1) and WHS in particular.

## 1.3 Borel's summation methods

We define three different gradually stronger methods, in the sense that they can be applied on more series while being consistent with the previous ones. We prove they are regular, linear and partially stable.

Denote $A(z)$ a formal complex series $A(z) = \sum_{n=0}^{\infty} a_n(z)$. Define its partial sums as $s_n(z) = \sum_{i=0}^{n} a_i(z)$.

**Definition 7. (Weak Borel summability):** Define the weak Borel sum for a series $A(z)$ as

$$\lim_{x \to \infty} e^{-x} \sum_{n=0}^{\infty} \frac{s_n(z) x^n}{n!}.$$

If this converges at $z \in \mathbb{C}$ to some $h(z) \in \mathbb{C}$, we say that the weak Borel sum of $A(z)$ converges at $z$ and write $\sum_{n=0}^{\infty} a_n(z) = h(z)$ $(wB)$.

Notice the necessary condition for weak Borel sum to converge at $z$ is that the series $\sum_{n=0}^{\infty} \frac{s_n(z)t^n}{n!}$ converges at $z$ for sufficiently large $t$.

**Definition 8. (Integral Borel summability):** For a series $A(z)$ define its Borel transform as

$$\mathcal{B}A(z)(t) = \sum_{n=0}^{\infty} \frac{a_n(z)t^n}{n!}.$$

If $\mathcal{B}A(z)(t)$ converges for $t \geq 0$ and the integral

$$\int\limits_0^{\infty} e^{-t}\mathcal{B}A(z)(t)\,\mathrm{d}t$$

is well defined and converges at $z \in \mathbb{C}$ to some $h(z)$, we say that the Borel sum of $A(z)$ converges at $z$ and write $\sum_{n=0}^{\infty} a_n(z) = h(z)\ \ (B)$.

**Definition 9. (Integral Borel transform with analytic continuation):** Let the Borel transform $\mathcal{B}A(z)(t)$ converge for $t$ in some neighbourhood of the origin to an analytic function that can be analytically continued to all $t > 0$ and denote this analytic continuation $\mathcal{B}_cA(z)(t)$. Then if the integral

$$\int_0^{\infty} e^{-t}\mathcal{B}_cA(z)(t)\,\mathrm{d}t$$

converges at $z \in \mathbb{C}$ to some $h(z)$, we say that the $B^*$ sum of $A(z)$ converges at $z$ and write $\sum_{n=0}^{\infty} a_n(z) = h(z)\ \ (B^*)$.

*Remark* 1. In case $A(z) = \sum_{n=0}^{\infty} a_n z^n$ is a power series with a positive radius of convergence, each method (if convergent) furnishes an analytic continuation of $A(z)$.

The following lemma will be needed to prove regularity of the Borel methods and will also be utilized multiple times throughout the thesis.

**Lemma 1.1.** *Let* $I_n = \int_0^{\infty} e^{-w}w^n\,\mathrm{d}w$. *Then* $I_n = n!$ *for all* $n \in \mathbb{N}_0$.

*Proof.* $I_0 = 1$ and simple integration by parts shows that $I_{n+1} = (n+1)I_n$. By induction, $I_n = n!$. □

*Remark* 2. This is a special case of the generalized factorial function called Gamma function, defined as

$$\Gamma(a) = \int_0^{\infty} e^{-w}w^{a-1}\,\mathrm{d}w.$$

For $a > 0$ by the same approach as above we can derive the formula $\Gamma(a+1) = a\Gamma(a)$.

**Theorem 1.2.** *Methods* $B$ *and* $B^*$ *are regular.*

*Proof.* Assume the series $A(z) = \sum_{n=0}^{\infty} a_n(z)$ converges at $z$. Then using Lemma 1.1 to express $n!$ as an integral we write

$$A(z) = \sum_{n=0}^{\infty} a_n(z) = \sum_{n=0}^{\infty} \frac{a_n(z)}{n!} \int\limits_0^{\infty} e^{-t}t^n\,\mathrm{d}t = \int\limits_0^{\infty} e^{-t} \sum_{n=0}^{\infty} \frac{a_n(z)t^n}{n!}\,\mathrm{d}t = \int\limits_0^{\infty} e^{-t}\mathcal{B}A(z)(t)\,\mathrm{d}t,$$

where reversing the order of integration and summation is justified by convergence of $A(z)$. □

As can be seen in Example 1.8, the methods are not totally regular. The weak Borel's summation method is regular as well, but we will not need to prove this, as it is a simple consequence of Theorem 1.5 below. Before that we will need a few prerequisites.

**Lemma 1.3.** *Let $\phi(x) \in C^1\{(M,\infty)\}$ for some $M \in \mathbb{R}\cup\{-\infty\}$. If $\lim_{x\to\infty}(\phi(x)+\phi'(x)) = A$, then $\lim_{x\to\infty}\phi(x) = A$ and $\lim_{x\to\infty}\phi'(x) = 0$.*

*Proof.* Without loss of generality we can assume $A = 0$ (otherwise let $\psi(x) = \phi(x) - A$ and continue the proof with $\psi(x)$). There are two possible cases:

- if the derivative $\phi'(x)$ keeps the same sign for large enough $x$, then $\phi(x)$ is eventually monotone so it either converges to a finite value $l$ or it is unbounded. For a finite limit $l$ $\phi'(x)$ must converge to 0 and at the same time to $-l$, therefore $l = 0$. For $\phi(x)$ unbounded the derivative diverges with the same sign, but then the condition of the theorem is not satisfied so this case is impossible.
- if $\phi'(x)$ changes signs an infinite number of times, there is a sequence of arbitrarily large $x_n$ such that $\phi'(x_n) = 0$ and these are local extremes. This implies that $\lim_{n\to\infty}\phi(x_n) = 0$ and so $\phi(x)$ converges to 0 bounded by its extremes.

The assertion is proven. □

**Lemma 1.4.** *For a sequence of complex numbers $\{a_n\}_{n\in\mathbb{N}_0}$ and their corresponding partial sums $\{s_n\}_{n\in\mathbb{N}_0}$ define formally two series*

$$a(x) = \sum_{n=0}^{\infty}\frac{a_n x^n}{n!}, \qquad s(x) = \sum_{n=0}^{\infty}\frac{s_n x^n}{n!}.$$

*If one series converges for all $x > 0$, so does the other.*

*Remark* 3. Note that this means they converge for all $z \in \mathbb{C}$. If the radius of convergence of $s(x)$ is finite, then $a(x)$ has the same finite radius of convergence, which is clear from the proof below.

*Proof.* Assume first that the series $s(x)$ is convergent, then differentiating term-by-term yields again a convergent series $s'(x) = \sum_{n=0}^{\infty}\frac{s_{n+1}x^n}{n!}$ and the difference $s'(x) - s(x) = \sum_{n=0}^{\infty}\frac{a_{n+1}x^n}{n!}$ converges for all $x$ as well. Integrating term-by-term and adding $a_0$ results in $a(x)$, which is therefore convergent.

The other direction is little bit more complicated. Let $a(x) = \sum_{n=0}^{\infty}\frac{a_n x^n}{n!}$ converge for all $x > 0$ so that $a(x)$ is analytic and consider the linear differential equation

$$y'(x) - y(x) = a'(x) \tag{1.1}$$
$$y(0) = a_0. \tag{1.2}$$

The general solution to (1.1)-(1.2) is

$$y(x) = a_0 e^x + e^x \int_0^x e^{-t}a'(t)\,\mathrm{d}t,$$

which is again an analytic function with its series centred at 0 converging to $y(x)$ for all $x > 0$, since both $a'(x)$ and $e^x$ have that property and products, sums and integrals of analytic functions are analytic again with radius of convergence at least the minimum of all radii of convergence involved. Now that we know the solution $y(x)$ is analytic, we can compute its Taylor series coefficients from (1.1)-(1.2). First, notice that

$$a^{(k)}(x) = \sum_{n=0}^{\infty} \frac{a_{n+k}\, x^n}{n!} \qquad \text{hence} \qquad a^{(k)}(0) = a_k \quad \forall k \in \mathbb{N}_0.$$

From the initial condition we have

$$y(0) = a_0 = s_0$$

and from (1.1)

$$y'(0) = y(0) + a'(0) = s_0 + a_1 = s_1.$$

Differentiating (1.1) gives the second derivative $y''(x)$ and so

$$y''(0) = y'(0) + a''(0) = s_1 + a_2 = s_2.$$

In general, $y^{(n+1)}(x) = y^{(n)}(x) + a^{(n+1)}(x)$, hence if we assume that $y^{(n)}(0) = s_n$, then

$$y^{(n+1)}(0) = y^{(n)}(0) + a^{(n+1)}(0) = s_n + a_{n+1} = s_{n+1},$$

proving by induction that $y^{(n)}(0) = s_n$ for all $n \in \mathbb{N}_0$ and so the Taylor series of $y(x)$ at $x = 0$ is give as

$$y(x) = \sum_{n=0}^{\infty} \frac{s_n x^n}{n!} = s(x),$$

converging for all $x > 0$. This concludes the proof. □

Now we can prove that methods $wB$ and $B$ are equivalent under a certain condition.

**Theorem 1.5.** *Let $A(z) = \sum\limits_{n=0}^{\infty} a_n(z)$ be a formal series and fix $z \in \mathbb{C}$, then:*

*(i) if $\sum\limits_{n=0}^{\infty} a_n(z) = A\ (wB)$, then $\sum\limits_{n=0}^{\infty} a_n(z) = A\ (B)$;*

*(ii) if $\sum\limits_{n=0}^{\infty} a_n(z) = A\ (B)$ and $\lim\limits_{x\to\infty} e^{-x} \sum\limits_{n=0}^{\infty} \frac{a_n(z)x^n}{n!} = \lim\limits_{x\to\infty} e^{-x}\mathcal{B}A(z)(x) = 0$, then $\sum\limits_{n=0}^{\infty} a_n(z) = A\ (wB)$.*

*Proof.* For simplicity we will fix $z \in \mathbb{C}$ and drop it from the notation. We define series $a(x)$ and $s(x)$ as in Lemma 1.4, then the weak Borel sum converges if the limit

$$\lim_{x\to\infty} e^{-x} s(x)$$

exists and the integral Borel sum converges if the limit

$$\lim_{x\to\infty} \int_0^x e^{-t} a(t)\,\mathrm{d}t$$

exists, therefore to begin with at least one of the series $a(x)$, $s(x)$ must converge for all $x > 0$. Lemma 1.4 then asserts that both series converge for all $x > 0$ and we can freely differentiate and integrate them term by term. In particular,

$$s'(x) \;=\; \sum_{n=0}^{\infty} \frac{s_{n+1}x^n}{n!} \qquad \text{and} \qquad a'(x) \;=\; \sum_{n=0}^{\infty} \frac{a_{n+1}x^n}{n!}. \tag{1.3}$$

Integrating the following expression by parts implies

$$\int_0^x e^{-t}a'(t)\,\mathrm{d}t \;=\; e^{-x}a(x) - a(0) + \int_0^x e^{-t}a(t)\,\mathrm{d}t \;=\; e^{-x}a(x) - a_0 + \int_0^x e^{-t}a(t)\,\mathrm{d}t \tag{1.4}$$

and utilising (1.3) yields another equivalent expression

$$\begin{aligned}\int_0^x e^{-t}a'(t)\,\mathrm{d}t \;&=\; \int_0^x e^{-t}\sum_{n=0}^{\infty} a_{n+1}\frac{t^n}{n!}\,\mathrm{d}t \;=\; \int_0^x e^{-t}\sum_{n=0}^{\infty}(s_{n+1}-s_n)\frac{t^n}{n!}\,\mathrm{d}t \;=\; \int_0^x e^{-t}\Big(s'(t)-s(t)\Big)\,\mathrm{d}t \\ &=\; \int_0^x \frac{\mathrm{d}}{\mathrm{d}t}\Big(e^{-t}s(t)\Big)\,\mathrm{d}t \;=\; e^{-x}s(x) - s(0) \;=\; e^{-x}s(x) - a_0. \end{aligned} \tag{1.5}$$

Hence, combining (1.4) and (1.5) we have for all $x > 0$

$$e^{-x}s(x) \;=\; e^{-x}a(x) + \int_0^x e^{-t}a(t)\,\mathrm{d}t,$$

showing that if $\lim_{x\to\infty} e^{-x}a(x) = 0$ and $A(z)$ is $B$-summable, then it is also $wB$-summable with the same value, hence (ii) is proved. Furthermore from the above equation we can deduce that if

$$\int_0^x e^{-t}a(t)\,\mathrm{d}t \;=\; \phi(x),$$

then $\phi(x) \in C^1\{(0,\infty)\}$ and by the Fundamental Theorem of Calculus $e^{-x}s(x) = \phi'(x) + \phi(x)$. If the series is $wB$-summable to a sum $A$, then

$$\lim_{x\to\infty}(\phi'(x) + \phi(x)) = A.$$

By Lemma 1.3 $\phi(x)$ converges to $A$, thus the series is $B$-summable to the same value, concluding the proof of (i). □

**Corollary 1.6.** *Method $wB$ is regular.*

For an example of a series that is $B$-summable but not $wB$-summable see Hardy (1949), p.183.

Apart from being regular, Borel's methods maintain their good behaviour for divergent series too, as indicated by the following corollary.

**Corollary 1.7.** *All three Borel methods are consistent with rules (I) and (II) and partially with rule (III), in the sense that if $a_1 + a_2 + a_3 + \ldots = A - a_0$ $(B)$ then $a_0 + a_1 + a_2 + \ldots = A$ $(B)$ but the converse is not true. The assertion is analogous for $wB$ and $B^*$.*

*Proof.* Conditions (I) and (II), i.e. linearity, are straightforward from the definition of each method, since integrals, sums and limits are linear. Thanks to the uniqueness of analytic continuation on a connected domain, the same argument works even for method $B^*$. For (III), observe from (1.5) that the following assertions are equivalent:

$$a_0+a_1+a_2+\ldots = A\ (wB) \qquad \Longleftrightarrow \qquad a_1+a_2+a_3+\ldots = A-a_0\ (B). \tag{1.6}$$

Using this equivalence and Theorem 1.5(i) we deduce the following:

$$\begin{aligned} a_1+a_2+\ldots = A-a_0\ (B) &\overset{(1.6)}{\Longrightarrow} a_0+a_1+a_2+\ldots = A\ (wB) \\ &\overset{1.5(i)}{\Longrightarrow} a_0+a_1+a_2+\ldots = A\ (B), \end{aligned} \tag{1.7}$$

and similarly for $wB$

$$\begin{aligned} a_1+a_2+\ldots = A-a_0\ (wB) &\overset{1.5(i)}{\Longrightarrow} a_1+a_2+\ldots = A-a_0\ (B) \\ &\overset{(1.6)}{\Longrightarrow} a_0+a_1+a_2+\ldots = A\ (wB). \end{aligned}$$

To see that the converse is not always true, assume a series $\sum_{n=0}^{\infty} a_n$ is $B$-summable but not $wB$-summable. If $a_0+a_1+a_2+\ldots = A\ (B)$ would imply $a_1+a_2+\ldots = A-a_0\ (B)$, then by (1.6) $a_0+a_1+a_2+\ldots = A\ (wB)$, contradicting the assumption.

Similarly, let a series $\sum_{n=1}^{\infty} a_n$ be $B$-summable but not $wB$-summable. By (1.6) then $a_0+a_1+a_2+\ldots = A\ (wB)$, but if this would imply that $a_1+a_2+\ldots = A-a_0\ (wB)$, then by Theorem 1.5(i) also $a_1+a_2+\ldots = A-a_0\ (B)$, which contradicts our assumption.

To prove (III) for method $B^*$, notice that in the case that $s(x)$ (and so $a(x)$ as well) has only finite positive radius of convergence, the equations (1.4)-(1.5) are still true for their analytic continuations to $x>0$, since it is a connected domain. Therefore all the steps leading to the proof of (III) for method $B$ can be used for method $B^*$ as well. □

**Example 1.8.** Consider the geometric series $A(z)=\sum_{n=0}^{\infty} z^n$, convergent only for $|z|<1$ to the analytic function $\frac{1}{1-z}$. The Borel transform of the series is

$$\mathcal{B}A(z)(t) = \sum_{n=0}^{\infty} \frac{(zt)^n}{n!} = e^{zt}$$

for any $z\in\mathbb{C}$ and $t\geq 0$, so the Borel sum is defined as

$$\int_0^{\infty} e^{-t}e^{zt}\mathrm{d}t = \lim_{x\to\infty} \frac{e^{(z-1)x}}{1-z} - \frac{1}{z-1},$$

convergent for $Re(z)<1$ to function $h(z)=\frac{1}{1-z}$.

Furthermore, since the limit $\lim_{x\to\infty} e^{-x}e^{zx}=0$ for $Re(z)<1$, the weak Borel sum should converge on the same domain. Indeed,

$$\lim_{x\to\infty} e^{-x}\sum_{n=0}^{\infty}\frac{s_n(z)x^n}{n!} = \lim_{x\to\infty} e^{-x}\sum_{n=0}^{\infty}\frac{1-z^{n+1}}{1-z}\frac{x^n}{n!} = \lim_{x\to\infty}\frac{e^{-x}}{1-z}(e^x - ze^{zx}) = \lim_{x\to\infty}\frac{1-ze^{x(z-1)}}{1-z},$$

which converges to $h(z)$ for $Re(z)<1$. ▲

**Example 1.9.** It should not come as a surprise that Borel's method is powerful enough to sum the series $F(z) = \sum_{n=0}^{\infty}(-1)^n n!\, z^n$. Its Borel transform is

$$\mathcal{B}F(z)(t) \;=\; \sum_{n=0}^{\infty}(-zt)^n,$$

which converges at any $z$ complex and $|t| < 1/|z|$ to the analytic function $\frac{1}{1+zt}$. This can be analytically continued to $t > 0$ and so the $B^*$-sum of the series is the function (2), i.e.

$$f(z) \;=\; \int\limits_0^{\infty} \frac{e^{-t}}{1+zt}\,\mathrm{d}t \;\; (B^*),$$

convergent for all $z$ not real and negative. In particular, the Borel sum at $z = 1$ converges to $\int_0^{\infty} \frac{e^{-t}}{1+t}\,\mathrm{d}t$. This integral is connected with WHS in many ways and will appear several times throughout this work, always denoted as $f(z)$ (or $f(x)$). ▲

## 1.4 Averaging methods

The definitions and theorems in this section can be found in Enyeart (RDSTT).

As mentioned earlier, Cesàro summation is an example of a particular class of summation methods. They are all characterized by taking a (weighted) average of the partial sums in some manner, which is closely explained in the following definition.

**Definition 10.** For every $m \in \mathbb{N}_0$ consider the sequence of weights $\mathbf{w}(m) = \{w_0(m), w_1(m), w_2(m), w_3(m), \dots\}$ satisfying

$$w_n(m) \geq 0,\ \forall m,n \in \mathbb{N}_0 \quad \text{and} \quad \sum_{n=0}^{\infty} w_n(m) = 1,\ \forall m \in \mathbb{N}_0.$$

Given any sequence $\mathbf{s} = \{s_0, s_1, s_2, s_3, \dots\}$ we define a sequence of transformations $\mathcal{T}s$ as

$$\mathcal{T}\mathbf{s}(m) = w_0(m)s_0 + w_1(m)s_1 + w_2(m)s_3 + \dots \;=\; \sum_{n=0}^{\infty} w_n(m)s_n.$$

If $\lim_{m\to\infty} \mathcal{T}\mathbf{s}(m) = c$ is a finite constant, we say that the sequence $\mathbf{s}$ is $\mathcal{T}$-convergent and thus the series $\sum_{n=0}^{\infty} a_n$ is $\mathcal{T}$-summable with $\mathcal{T}$-sum $c$.

This transformation can be expressed by an infinite matrix of weights. It is defined as follows:

**Definition 11. (Averaging matrix):** Let $\mathcal{M} = (w_n(m))$ be an infinite matrix with rows numbered $m \in \mathbb{N}_0$ and columns $n \in \mathbb{N}_0$. We call it an averaging matrix if the terms are non-negative and the sum of each row is 1.

The corresponding transformation is then obtained by multiplying an infinite vector $\mathbf{s}$ by an averaging matrix $\mathcal{M}$, i.e. $\mathcal{M}$ is the matrix representation of $\mathcal{T}$:

$$\mathcal{T}\mathbf{s} = \mathcal{M}\mathbf{s} = \begin{bmatrix} w_1(1) & w_2(1) & w_3(1) & \cdots \\ w_1(2) & w_2(2) & w_3(2) & \cdots \\ w_1(3) & w_2(3) & w_3(3) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ \vdots \end{bmatrix}$$

From now on, we will refer to $\mathcal{M}$ as both the transformation and its matrix representation.

**Example 1.10. Identity summation method**
The weights are simply $w_n(m) = \delta_{nm}$ and are represented by the (infinite) identity matrix $I$. This method is the usual summation and its domain is therefore the set of convergent series. ▲

**Example 1.11. Cesàro method**
With the weights

$$w_n(m) = \begin{cases} \frac{1}{m+1} & \text{if } n \leq m \\ 0 & \text{otherwise} \end{cases}$$

the averaging matrix will become

$$\mathcal{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \\ \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

One can verify easily that multiplying vector $\mathbf{s}$ by this matrix gives the Cesàro averages. ▲

More examples can be found in Enyeart (RDSTT), as well as the details and proofs of the following theorems.

**Theorem 1.12.** *If $\mathcal{A}$ and $\mathcal{B}$ are both lower triangular averaging matrices, then $\mathcal{AB}$ will also be a lower triangular averaging matrix.*

To clarify the importance of this statement, notice that it shows that higher order Hölder summations $(H,k)$, which are in essence $k$-times repeated Cesàro summations $(\mathcal{C},1)$, are again averaging summations represented by matrices $\mathcal{C}^k$.

The following two theorems give conditions on the regularity of an averaging summation method:

**Theorem 1.13.** *Suppose $\mathcal{T}$ is a summation method given by averaging matrix $\mathcal{M} = (w_n(m))$. Then this method is regular if and only if*

$$\lim_{m\to\infty} w_n(m) = 0, \quad \forall n \in \mathbb{N}_0.$$

**Theorem 1.14.** *If $\mathcal{A},\mathcal{B}$ are both regular averaging matrices, then $\mathcal{AB}$ will be a regular matrix as well.*

Hence we can see all the above mentioned methods and their iterations are regular. Another simple example is covered in the following subsection.

### 1.4.1 Midpoint method

**Definition 12.** Define matrix $\mathcal{P}$ as follows:

$$\mathcal{P} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

The summation method represented by $\mathcal{P}$ will be called the midpoint method.

This summation method and all its iterations $(\mathcal{P},k)$ represented by $\mathcal{P}^k$ for any $k \in \mathbb{N}$ are regular as an immediate consequence of the previous theorems.

*Remark* 4. This method represents how I see the "sum" of a divergent series intuitively: a limit of the line connecting the points that are equally distanced from two subsequent partial sums. If this limit does not exist, the process will be repeated again with the newly created points (possibly an infinite number of times). Figure 1.1 illustrates this approach.

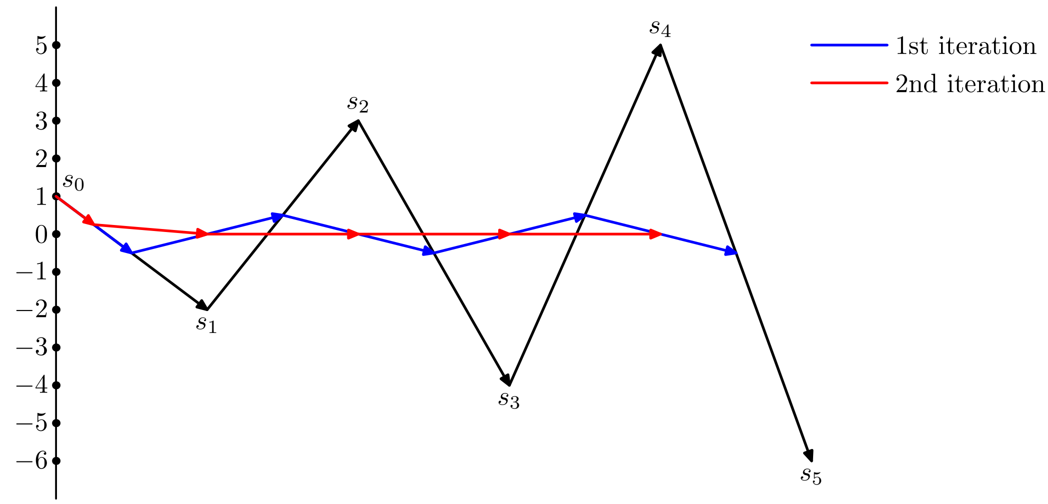


**Figure 1.1:** Midpoint method applied twice to series $\sum_{n=0}^{\infty}(-1)^n(2n+1)$

It works for (oscillating) divergent series with up to a certain magnitude of oscillation growth (i.e. the growth of the terms $a_n$ in an alternating series $\sum_{n=0}^{\infty}(-1)^n a_n$), as will be shown later.

Let us first list a few examples of divergent series with their sum computed by this method applied a finite number of times. Computations were done in Maxima (the source code can be found in Appendix B, Example B.1).

**Example 1.15.** $\sum_{n=0}^{\infty}(-1)^n = 1-1+1-1+1-1+\ldots$

The partial sums are $\mathbf{s} = \{1,0,1,0,1,0,\ldots\}$ with their midpoints $\mathcal{P}\mathbf{s} = \{\frac{1}{2},\frac{1}{2},\frac{1}{2},\ldots\}$. The limit is then $\frac{1}{2}$, which is therefore the $(M,1)$-sum of the series. ▲

**Example 1.16.** $\sum_{n=0}^{\infty}(-1)^n(2n+1) = 1-3+5-7+9-\ldots$

The sequence of partial sums is $\mathbf{s} = \{1,-2,3,-4,5,-6,\ldots\}$. After the first iteration we get $\mathcal{P}\mathbf{s} = \{1,-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\ldots\}$ which still does not have a limit, but looks very similar to the first example. Indeed, applying the method second time we get $\mathcal{P}^2\mathbf{s} = \{1,\frac{1}{4},0,0,0,\ldots\}$ with the $(M,2)$-limit 0. ▲

**Example 1.17.** $\sum_{n=0}^{\infty}(-1)^n n = 1-2+3-4+5-6+\ldots$

with its partial sums $\mathbf{s} = \{1,-1,2,-2,3,-3,\ldots\}$. Again, applying the method twice, we first get $\mathcal{P}\mathbf{s} = \{1,0,\frac{1}{2},0,\frac{1}{2},0,\ldots\}$ and then $\mathcal{P}^2\mathbf{s} = \{1,\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{1}{4},\ldots\}$ with the $(M,2)$-limit $\frac{1}{4}$. ▲

**Example 1.18.** $\sum_{n=0}^{\infty}(-1)^n(2n+1)^7 = 1-3^7+5^7-7^7+9^7-\ldots = 0 \quad (\mathcal{P},8)$

More generally, $\sum_{n=0}^{\infty}(-1)^n(2n+1)^p$ needs $p+1$ iterations to give a finite result ($s_n^{(p+1)}$ being all equal for large enough $n$) and for $p$ odd the $(\mathcal{P},p+1)$-sum is always 0 (computed for $p$ up to 20). This leads to an interesting identity (1.9) addressed at the end of this chapter. ▲

These results are consistent with the results in Hardy (1949) computed by several methods (in Chapter 1).

It is easy to see that each iteration reduces the oscillation of a series by the factor $\frac{1}{2}$, so $(\mathcal{P},k)$ tames the growth at least by the factor $\frac{1}{2^k}$, which will be shown properly in the following proposition.

**Proposition 1.19.** *For a given divergent alternating series* $\sum_{n=0}^{\infty}(-1)^n a_n$ *with* $a_{n+1} \geq a_n \geq 0$, *the oscillation after k-th iteration is reduced by factor at least* $\frac{1}{2^k}$, *that is, for* $k \geq 1$

$$\left| s_{n+1}^{(k)} - s_n^{(k)} \right| < \frac{a_{n+1}}{2^k}.$$

*Proof.* First we establish that after each iteration $k$ the resulting series will still be alternating with non-decreasing terms, i.e. $\mathcal{P}^k\mathbf{s} = \sum_{n=0}^{\infty}(-1)^n a_n^{(k)}$ with $a_{n+1}^{(k)} \geq a_n^{(k)} \geq 0$ for all $n \in \mathbb{N}_0$. For $k=0$ this is true by assumption. Now assume this holds for some $k$, then for $k+1$ and any $n \in \mathbb{N}_0$ the difference between two consecutive partial sums is

$$\begin{aligned} &= s_{n+1}^{(k+1)} - s_n^{(k+1)} = \frac{s_{n+1}^{(k)} + s_n^{(k)}}{2} - \frac{s_n^{(k)} + s_{n-1}^{(k)}}{2} = \frac{s_{n+1}^{(k)} - s_{n-1}^{(k)}}{2} \\ &= \frac{(-1)^{n+1}a_{n+1}^{(k)} + (-1)^n a_n^{(k)}}{2} = (-1)^{n+1}\frac{a_{n+1}^{(k)} - a_n^{(k)}}{2} =: (-1)^{n+1}a_{n+1}^{(k+1)}, \end{aligned}$$

which means that $\mathcal{P}^{k+1}\mathbf{s}(n) = \sum_{i=0}^{n}(-1)^n a_n^{(k+1)}$ with the terms $a_{n+1}^{(k+1)} \geq 0$ and non-increasing since

$$a_{n+1}^{(k+1)} - a_n^{(k+1)} = \frac{a_{n+1}^{(k)} - a_n^{(k)}}{2} - \frac{a_n^{(k)} - a_{n-1}^{(k)}}{2} = \frac{a_{n+1}^{(k)} - a_{n-1}^{(k)}}{2} \geq 0,$$

hence $\mathcal{P}^{k+1}\mathbf{s}$ is an alternating divergent series again. By induction, this holds for all $k \in \mathbb{N}_0$.

The oscillation for iteration $k$ is then bounded as follows:

$$\left| s_{n+1}^{(k)} - s_n^{(k)} \right| = a_{n+1}^{(k)} = \frac{a_{n+1}^{(k-1)} - a_n^{(k-1)}}{2} < \frac{a_{n+1}^{(k-1)}}{2},$$

so by induction

$$\left| s_{n+1}^{(k)} - s_n^{(k)} \right| < \frac{a_{n+1}^{(0)}}{2^k} = \frac{a_{n+1}}{2^k}.$$

□

The estimate is still quite rough and the following examples imply that the actual value lies somewhere between $\frac{a_{n+1}}{2^k}$ and $\frac{a_{n+1}}{2^{2k}}$.

By the proposition, the method should work (after a finite number of iterations) for those series with magnitude of oscillation growth smaller than $2^k$. What about the series with the magnitude of growth approximately the same as $2^k$?

**Example 1.20.** $\sum\limits_{n=0}^{\infty}(-1)^n 2^n = 1-2+4-8+16-\ldots \approx \frac{1}{3} \quad (\mathcal{P},2500)$

Clearly, no finite number of iterations will give a convergent sequence, since $(\mathcal{P},k)$ will reduce the terms $a_n(k)$ by factor $2^k$ and the terms greater than that will oscillate without a bound.

However, applying the method enough times can yield a good estimate of the sum. Using 5000 terms and approximately 2500 iterations resulted in an estimate close to $\frac{1}{3}$ with precision to 100 decimal places. This agrees with the results in Hardy (1949) and with the formal generalization of the formula for summing geometric series:
$\sum\limits_{n=0}^{\infty}(-2)^n \leftrightarrow \frac{1}{1-(-2)} = \frac{1}{3}$. ▲

**Example 1.21.** $\sum\limits_{n=0}^{\infty}(-1)^n n2^n = 0\times1-1\times2+2\times4-3\times8+4\times16-\ldots \approx -0.\overline{22} = -\frac{2}{9}$ $(\mathcal{P}, 1523)$
3000 terms used, precision 10 decimal places. This series is a term-by-term derivative of the geometric series $2\sum_{n=0}^{\infty}(-2)^n$, and the result agrees with the derivative of its formal sum, that is, $\frac{\mathrm{d}}{\mathrm{d}x}\frac{2}{1-x}\Big|_{x=-2} = -\frac{2}{9}$. ▲

**Example 1.22.** $\sum\limits_{n=0}^{\infty}(-1)^n 2^{2n} = 1-4+16-64+256-\ldots \approx \frac{1}{5}$ $(\mathcal{P}, 4252)$
The method still seems to work for this series (the oscillation growth being $2^{2n}$), with less precision than the previous example. Summing 3000 terms to a precision of 10 decimal places requires 4252 iterations and the result converges to $\frac{1}{5}$ in agreement with the geometric series formula. ▲

**Example 1.23.** $\sum\limits_{n=0}^{\infty}(-1)^n 2^{3n} = 1-16+256-\ldots$ not summable,
After a number of iterations the result converges to the first term of the series, instead of the expected result $\frac{1}{9}$. The same happens when the first term is replaced by an arbitrary number. This seems to imply that the growth of the oscillation should be of the magnitude less than $2^{2n}$ to allow summation or approximation by this method. ▲

**Example 1.24.** $\sum\limits_{n=0}^{\infty}(-n)^n = 1-1+4-27+256-\ldots$ not summable with the same result as in the previous example.
▲

What exactly is happening and why, for series with too steep a growth, the iterations converge to the value of the first term? To clarify this, let us take a closer look at matrices $\mathcal{P}^k$:

$$\mathcal{P} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \qquad \mathcal{P}^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 & \cdots \\ \frac{1}{4} & \frac{2}{4} & \frac{1}{4} & 0 & 0 & \cdots \\ 0 & \frac{1}{4} & \frac{2}{4} & \frac{1}{4} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

$$\mathcal{P}^3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{7}{8} & \frac{1}{8} & 0 & 0 & 0 & \cdots \\ \frac{4}{8} & \frac{3}{8} & \frac{1}{8} & 0 & 0 & \cdots \\ \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} & 0 & \cdots \\ 0 & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} & \cdots \\ 0 & 0 & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \qquad \mathcal{P}^4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{15}{16} & \frac{1}{16} & 0 & 0 & 0 & 0 & \cdots \\ \frac{11}{16} & \frac{4}{16} & \frac{1}{16} & 0 & 0 & 0 & \cdots \\ \frac{5}{16} & \frac{6}{16} & \frac{4}{16} & \frac{1}{16} & 0 & 0 & \cdots \\ \frac{1}{16} & \frac{4}{16} & \frac{6}{16} & \frac{4}{16} & \frac{1}{16} & 0 & \cdots \\ 0 & \frac{1}{16} & \frac{4}{16} & \frac{6}{16} & \frac{4}{16} & \frac{1}{16} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

The pattern is fairly simple and easy to prove by induction: fix $k$ and consider the binomial coefficients $\binom{k}{0}, \binom{k}{1}, \binom{k}{3}, \ldots, \binom{k}{k}$. Their sum is established by the binomial identity

$$\sum_{m=0}^{k} \binom{k}{m} = 2^k, \tag{1.8}$$

so divided by $2^k$ the new sum will be exactly 1. The matrix $\mathcal{P}^k$ is constructed as follows: in each row, distribute these numbers (divided by $2^k$) one by one starting from the diagonal and continuing to the left; when reaching the first column, add all the remaining coefficients and let this be the value of the first element in the row. The $(k+1)$-st row will be the first complete row listing all the coefficients separately, the following rows will then be identical but always shifted by one to the right. Identity (1.8) above guarantees that the sum of each row is 1.

In general, $\mathcal{P}^k =$

$$\frac{1}{2^k}\begin{bmatrix}
2^k & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots \\
2^k - \binom{k}{0} & \binom{k}{0} & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots \\
2^k - \binom{k}{1} - \binom{k}{0} & \binom{k}{1} & \binom{k}{0} & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots \\
2^k - \binom{k}{2} - \binom{k}{1} - \binom{k}{0} & \binom{k}{2} & \binom{k}{1} & \binom{k}{0} & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots \\
0 & \binom{k}{k} & \binom{k}{k-1} & \binom{k}{k-2} & \binom{k}{k-4} & \cdots & \binom{k}{m} & \cdots & \binom{k}{1} & \binom{k}{0} & 0 & \cdots \\
0 & 0 & \binom{k}{k} & \binom{k}{k-1} & \binom{k}{k-3} & \cdots & \binom{k}{m-1} & \cdots & \binom{k}{2} & \binom{k}{1} & \binom{k}{0} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}$$

Because of the accumulation of the coefficients in the first term of the first $k$ rows, each iteration puts more weight on $s_1$. If the oscillation growth of the series is much faster than that of $2^n$, it forces the summation to be iterated too many times, i.e. the number of iterations $k$ is significantly larger than that of terms ($n$) used. In the meantime the first term of the series will gradually take over all the other terms. The result is that the partial sums of the $k$-th iteration will all converge to the value of the first term faster than the oscillation error $a_n^{(k)}$ will converge to 0.

More precisely, notice the value of the $n$-th component of $\mathcal{P}^k\mathbf{s}$:

$$s_n^{(k)} = \frac{2^k - \binom{k}{n-1} - \ldots - \binom{k}{1} - \binom{k}{0}}{2^k} s_0 - \ldots = \left(1 - \frac{\sum_{m=0}^{n-1}\binom{k}{m}}{2^k}\right) s_0 + \ldots.$$

The term in the brackets can get arbitrarily close to 1 depending on choice of $k$. Because of the distribution of binomial numbers in each line of Pascal's triangle (increasing towards the middle) and the fact that there are $k+1$ numbers in $k$-th line summing to $2^k$, it is certainly true that (as long as $n < \frac{k}{2}$)

$$\frac{\sum_{m=0}^{n-1}\binom{k}{m}}{2^k} < \frac{\frac{n-1}{k+1}2^k}{2^k} < \frac{n}{k},$$

therefore a suitable choice of $k$ can make $s_n^{(k)}$ arbitrarily close to $s_0$. (In reality it converges a lot faster than our very crude estimate).

Now let's say we want to use 100 terms of the Wallis' hypergeometric series to approximate its sum using this method. As 100! has approximately the same magnitude as $2^{525}$, we will need at least 525 iterations to make the oscillation error small. After only 300 iterations, $s_{100}(300) \approx 0.999999996 s_0$ and subsequent iterations will decrease the influence of other terms drastically. By the 500-th iteration all the terms are roughly equal to 1 (value of $s_0$), with error at most $10^{-5}$. As the number of iterations needed will only increase with more terms added, the problem persists.

Notice in the previous examples that the number of iterations was lower than the number of terms used when the method worked, the borderline example being $\sum_{n=0}^{\infty}(-1)^n 2^{2n}$ where $k > n$ but still $k < 2n$ so the first term has a small influence on the result. All the preceding examples had oscillation growth less than $2^{2n}$ and those not summable at all had a greater growth.

*Remark* 5. The plausibility of these results is relying heavily on the fact that the series are given by an explicit and non-changing formula for all terms, without a sudden change later on. In other words, the approximation will be accurate if the "pattern" of the series will not change. This is an interesting result, as it indicates a property known to convergent series: in order to get arbitrarily close to the limit it is sufficient to sum up a finite number of terms (provided the series has an eventual pattern).

*Remark* 6. Recall Example 1.18. Based on the trials run for odd $p$ up to $p = 21$, it is proposed that the series $\sum_{n=0}^{\infty}(-1)^n(2n+1)^p$ is $\mathcal{P}^{p+1}$-summable to 0, with an additional property that the $(p+1)$-times transformed sequence of the partial sums is eventually constant, i.e.:

$$s_N^{(p+1)} = 0 \qquad \text{for } N > p+1.$$

If we write this result in the explicit form, that is, as the vector of the partial sums multiplied by the $N$-th line of the matrix $\mathcal{P}^{p+1}$, the proposition is as follows:

For any $p$ odd and all $N > p+1$

$$\sum_{i=0}^{p+1} \frac{\binom{p+1}{i}}{2^{p+1}} \sum_{n=N}^{N+i} (-1)^n (2n+1)^p = 0,$$

which is equivalent to

$$\sum_{j=0}^{p+1} (-1)^j (2j+2N+1)^p \sum_{i=j}^{p+1} \binom{p+1}{i} = 0. \tag{1.9}$$

# Chapter 2

# Euler's third method: ODE

As we defined earlier, Wallis' hypergeometric series that Euler got intrigued by is the series

$$\sum_{n=0}^{\infty}(-1)^n n! = 0! - 1! + 2! - 3! + 4! - 5! + \dots$$

which is the case $z = 1$ of the hypergeometric power series (1)

$$F(z) = \sum_{n=0}^{\infty}(-1)^n n!\, z^n = 1 - 1!\, z + 2!\, z^2 - 3!\, z^3 + 4!\, z^4 - 5!\, z^5 + \dots,$$

that converges only for $z = 0$. It is not summable by any of the methods mentioned so far due to its fast growth except for the Borel method, as demonstrated in Example 1.9. At the time of Euler's life this method was not yet invented, but Euler arrived at the "right" result in a few different ways. One of them was solving an ordinary differential equation that the series formally satisfies.

First, we will follow the process as it is outlined in Hardy (1949), section 2.4, filling in the details. This approach can hardly be considered rigorous as it relies heavily on formal operations with series and integrals, leaving out many details that need to be properly addressed. That might prove to be quite difficult though, so we will instead propose a slightly different solution in section 2.2 that utilises the results from the previous part but avoids most of its issues.

Throughout this section we will differentiate between a series that formally solves a given equation (denoted by a capital letter) and a well defined function that is a solution to the same equation (denoted by the corresponding small letter).

## 2.1 Outline of the method as described in Hardy (1949)

For $x > 0$ define formally a function

$$\Phi(x) = xF(x) = \sum_{n=0}^{\infty}(-1)^n n!\, x^{n+1} = x - 1!\, x^2 + 2!\, x^3 - 3!\, x^4 + \dots.$$

Term-by-term differentiation suggests that $\Phi(x)$ formally solves the equation

$$x^2\Phi'(x) + \Phi(x) \;=\; x^2(1! - 2!\, x + 3!\, x^2 - \dots) + x - 1!\, x^2 + 2!\, x^3 - 3!\, x^4 + \dots \;=\; x. \tag{2.1}$$

Let $\phi(x)$ be a solution to this equation, that is, $x^2\phi'(x) + \phi(x) = x$. It has an integrating factor $x^{-2}e^{-\frac{1}{x}}$ which transforms it into a separable equation

$$\phi'(x)e^{-\frac{1}{x}} + \frac{\phi(x)e^{-\frac{1}{x}}}{x^2} \;=\; \frac{e^{-\frac{1}{x}}}{x} \quad\iff\quad \left[\phi(x)e^{-\frac{1}{x}}\right]' \;=\; \frac{e^{-\frac{1}{x}}}{x}.$$

Integrating both sides yields

$$\phi(x)e^{-\frac{1}{x}} = \int_0^x \frac{e^{-\frac{1}{t}}}{t}\,\mathrm{d}t \quad \Longrightarrow \quad \phi(x) = e^{\frac{1}{x}}\int_0^x \frac{e^{-\frac{1}{t}}}{t}\,\mathrm{d}t.$$

In accordance with the original series $\phi(x)$ vanishes with $x$, as can be proven by integrating by parts (differentiating $t$ and integrating $e^{-\frac{1}{t}}t^{-2}$):

$$\begin{aligned}\left|e^{\frac{1}{x}}\int_0^x \frac{e^{-\frac{1}{t}}}{t}\,\mathrm{d}t\right| &= \left|e^{\frac{1}{x}}\left(\left[te^{-\frac{1}{t}}\right]_0^x - \int_0^x e^{-\frac{1}{t}}\,\mathrm{d}t\right)\right| = \left|x - e^{\frac{1}{x}}\int_0^x e^{-\frac{1}{t}}\,\mathrm{d}t\right| \\ &\le |x| + \left|e^{\frac{1}{x}}\int_0^x e^{-\frac{1}{t}}\,\mathrm{d}t\right| = x + e^{\frac{1}{x}}\int_0^x e^{-\frac{1}{t}}\,\mathrm{d}t,\end{aligned}$$

the last equality true because the exponential function is positive. Now since $t$ takes values between 0 and $x$ (positive), we have $t \le x$ thus $-\frac{1}{t} \le -\frac{1}{x}$. Hence the integrand can be bounded by $e^{-\frac{1}{x}}$ resulting in the estimate

$$|\phi(x)| \le x + \int_0^x e^{\frac{1}{x}}e^{-\frac{1}{x}}\,\mathrm{d}t = x + (x-0) = 2x.$$

This shows for small $x$ that $\phi(x) = O(x)$ and therefore vanishes with $x$.

For our original series $F(x) = \Phi(x)/x$ we then have a corresponding function

$$f(x) = \frac{\phi(x)}{x} = \frac{e^{\frac{1}{x}}}{x}\int_0^x \frac{e^{-\frac{1}{t}}}{t}\,\mathrm{d}t,$$

which we can rewrite as

$$f(x) = \int_0^x \frac{e^{\frac{t-x}{xt}}}{xt}\,\mathrm{d}t.$$

Using a somewhat unintuitive but valid substitution $t = \frac{x}{1+xw}$ (with $\mathrm{d}t = \frac{-x^2}{(1+xw)^2}\,\mathrm{d}w$ and $\frac{x-t}{xt} = w$, changing limits to $\infty$ and 0) we get

$$f(x) = \int_\infty^0 \frac{e^{-w}}{\frac{x^2}{1+xw}}\frac{-x^2}{(1+xw)^2}\,\mathrm{d}w = \int_0^\infty \frac{e^{-w}}{1+xw}\,\mathrm{d}w, \tag{2.2}$$

the very same function as the Borel sum of the series (1) in Example 1.9. This form is interesting at least for one reason - expanding the integrand as a geometric series (formally, since $|xw| < 1$ will not be always satisfied) and integrating term-by-term brings us back to the original series, as is shown next:

$$\begin{aligned}\int_0^\infty \frac{e^{-w}}{1+xw}\,\mathrm{d}w &= \int_0^\infty e^{-w}\sum_{n=0}^\infty (-xw)^n\,\mathrm{d}w = \sum_{n=0}^\infty \int_0^\infty e^{-w}(-xw)^n\,\mathrm{d}w = \sum_{n=0}^\infty (-x)^n \int_0^\infty e^{-w}w^n\,\mathrm{d}w \\ &= \sum_{n=0}^\infty (-x)^n n! = 1 - 1!\,x + 2!\,x^2 - 3!\,x^3 + 4!\,x^4 - \dots,\end{aligned}$$

where we utilized Lemma 1.1.

## 2.2 Rigorous approach to the ODE method

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Let us start by defining an ordinary differential equation for the original series $F(x) = 1 - x + 2!\,x^2 + 3!\,x^3 - 4!\,x^4 + \ldots$ with $x \geq 0$. Either by substituting $\Phi(x) = xF(x)$ in equation (2.1) or by direct computation with the series we find that $F(x)$ formally (term by term) satisfies the differential equation

$$(1+x)F(x) + x^2 F'(x) = 1 \tag{2.3}$$

$$F(0) = 1, \tag{2.4}$$

with the initial condition agreeing with the original series. First, the equation will be solved by the power series method and later by finding the general solution.

**Proposition 2.1.** *The power series solution to* (2.3)-(2.4) *is* $\sum\limits_{n=0}^{\infty} (-1)^n n!\,x^n$*.*

*Proof.* Assume there is a solution in the form of a power series $\sum\limits_{n=0}^{\infty} a_n x^n$ and plug it into (2.3). Then

$$(1+x)\sum_{n=0}^{\infty} a_n x^n + x^2 \sum_{n=1}^{\infty} a_n n x^{n-1} = \sum_{n=0}^{\infty} a_n x^n + \sum_{n=1}^{\infty} a_{n-1} x^n + \sum_{n=2}^{\infty} a_{n-1} n x^n$$

$$= a_0 + \sum_{n=1}^{\infty} \left[a_n + a_{n-1} + (n-1)a_{n-1}\right] x^n = a_0 + \sum_{n=0}^{\infty} \left(a_n + n a_{n-1}\right) x^n = 1$$

and matching the coefficients implies

$$a_0 = 1, \quad a_n = -n a_{n-1} \quad \text{for} \quad n \in \mathbb{N},$$

resulting, as expected, in the original series where $a_n = (-1)^n n!$ for $n \in \mathbb{N}_0$. □

Now let us find a general solution to (2.3)-(2.4) which we will call $f(x)$. First we solve the homogeneous equation: if $x = 0$ then $f_h(0) = 0$. For $x > 0$ the equation is separable:

$$\frac{f_h'}{f_h} = -\frac{1+x}{x^2}.$$

Integrating both sides w.r.t. $x$ we get

$$\ln|f_h| = -\frac{1}{2}\ln\left|x^2\right| + \frac{1}{x} + c = \ln\left(x^2\right)^{-\frac{1}{2}} + \frac{1}{x} + c = \ln\frac{1}{x} + \ln e^{\frac{1}{x}} + \ln D = \ln\frac{De^{\frac{1}{x}}}{x}$$

(where $c \in \mathbb{R}$ and $D = e^c > 0$), hence $f_h(x) = \frac{Ce^{\frac{1}{x}}}{x}$ for $C \in \mathbb{R}$ including the trivial solution.

As a particular solution we will conveniently use the improper integral form from (2.2), so the function $f(x)$ defined in (2):

$$f_p(x) = \int_0^{\infty} \frac{e^{-w}}{1+xw}\,\mathrm{d}w\,, \tag{2.5}$$

showing by direct substitution into (2.3) that it indeed gives the desired result (notably for all $x \geq 0$ since this integral is defined for all such $x$). Before that, however, we need to show that this function is well-behaved, allowing us to differentiate under the integral sign. Let us recall the following theorem for differentiating under the improper integral sign:

**Theorem 2.2.** *Let $g(w,x)$ be a function defined on $D=[a,\infty)\times[c,d]$ with $g$ and $g_x$ continuous on $D$. Suppose the improper integrals $\int_a^\infty g(w,x)\,\mathrm{d}w$ and $\int_a^\infty g_x(w,x)\,\mathrm{d}w$ are both absolutely convergent. Then $h(x)=\int_a^\infty g(w,x)\,\mathrm{d}w$ is differentiable and*

$$h'(x) \;=\; \int\limits_a^\infty g_x(w,x)\,\mathrm{d}w.$$

A proof can be found in Zorich (2002), Chapter 17.

**Corollary 2.3.** *Function $f(x)$ defined in (2) is infinitely many times differentiable and*

$$f^{(k)}(x) \;=\; \int\limits_0^\infty \frac{(-1)^k k!\,w^k e^{-w}}{(1+xw)^{k+1}}\,\mathrm{d}w.$$

*Proof.* Define $g(w,x)=\frac{e^{-w}}{1+xw}$ with $a=0$ and $d\geq c\geq 0$ arbitrary. The first two conditions of the theorem are clearly satisfied. The derivatives can be computed inductively as

$$\frac{\partial^k}{\partial x^k}\,g(w,x)=\frac{(-1)^k k!\,w^k e^{-w}}{(1+xw)^{k+1}}\,.$$

Each of these is bounded by $k!\,w^k e^{-w}$, which is an integrable majorant with a finite integral (Lemma 1.1), therefore all integrals $\int_0^\infty \frac{\partial^k}{\partial x^k}\,g(w,x)\,\mathrm{d}w$ converge uniformly and absolutely. Applying Theorem 2.2 inductively implies that $f(x)$ is infinitely many times differentiable on any closed interval $[c,d]$ with $c$, $d\geq 0$, therefore as well on $[0,\infty)$, which concludes the proof. □

Now we can verify equation (2.3) for $f_p(x)=f(x)$:

$$\begin{aligned}(1+x)f_p+x^2 f_p' &= (1+x)\int\limits_0^\infty \frac{e^{-w}}{1+xw}\,\mathrm{d}w \;+\; x^2\int\limits_0^\infty \frac{-e^{-w}w}{(1+xw)^2}\,\mathrm{d}w\\ &= \int\limits_0^\infty \frac{e^{-w}}{1+xw}\,\mathrm{d}w \;+\; \int\limits_0^\infty \frac{e^{-w}x}{(1+xw)^2}\,\mathrm{d}w.\end{aligned}$$

Integrating the second integral by parts (differentiating $e^{-w}$ and integrating $\frac{x}{(1+xw)^2}$) will result in

$$\begin{aligned}(1+x)f_p+x^2 f_p' &= \int\limits_0^\infty \frac{e^{-w}}{1+xw}\,\mathrm{d}w+\lim_{R\to\infty}\left[\frac{-e^{-w}}{1+xw}\right]_0^R-\int\limits_0^\infty \frac{e^{-w}}{1+xw}\,\mathrm{d}w \;=\; \lim_{R\to\infty}\frac{-e^{-R}}{1+xR}+1\\ &=1,\end{aligned}$$

proving that $f_p(x)$ is indeed a particular solution for (2.3).

The general solution of (2.3) is then defined as

$$f_h(x)+f_p(x)=\begin{cases}\int\limits_0^\infty e^{-w}=1 & \text{for } x=0,\\ C\frac{e^{\frac{1}{x}}}{x}+\int\limits_0^\infty \frac{e^{-w}}{1+xw}\,\mathrm{d}w,\quad C\in\mathbb{R} & \text{for } x>0.\end{cases}$$

The only choice of $C$ satisfying the auxiliary condition and making the solution continuous on $[0,\infty)$ is $C=0$, since $\frac{e^{\frac{1}{x}}}{x}$ is unbounded. Hence we have proved the following proposition:

**Proposition 2.4.** *The unique solution to* (2.3)-(2.4) *is*

$$f(x) = \int_0^\infty \frac{e^{-w}}{1+xw}\,\mathrm{d}w.$$

Since the power series solution was $\sum_{n=0}^{\infty}(-1)^n n!\,x^n$, these two solutions are, in a sense, equivalent. The actual link between the two solutions will be clarified in the next section, where we define asymptotic series and prove that $\sum_{n=0}^{\infty}(-1)^n n!\,x^n$ is an asymptotic series for $f(x)$ and solves the same equation not just by coincidence.

At $x=1$ the value of this integral $f(1)=\delta$ corresponds to WHS. An approximate value of $f(1)$ will be computed by multiple methods:

- Chapter 2, equation (2.7) below (4 decimal places),
- Chapter 3, Table 3.1 (2 decimal places),
- Chapter 5, Table 5.1 (272 decimal places) and Table 5.2 (8 683 decimal places).

The first method that we include here is one that Euler mentioned in a letter to Niklaus Bernoulli but did not include in his paper Euler (1760). It makes use of the following form of $f(x)$:

$$f(1) = \int_0^\infty \frac{e^{-w}}{1+w}\,\mathrm{d}w = \int_0^1 \frac{1}{1+\ln v}\,\mathrm{d}v.$$

Since $\ln v$ is analytic at $v=1$ and $\frac{1}{1+x}$ is analytic at $x=0$, the composition of the two functions is analytic at $v=1$, hence the Taylor series of the integrand at $v=1$ with coefficients

$$c_n = \left.\frac{\mathrm{d}^n}{\mathrm{d}v^n}\frac{1}{1-\ln v}\right|_{v=1}$$

will converge for $v=(0,1]$. Moreover the limit

$$\lim_{x\to 0}\int_x^1 \frac{1}{1+\ln v}\,\mathrm{d}v$$

is finite, so the Taylor series can be integrated term by term:

$$\begin{aligned}\int_0^1 \frac{1}{1+\ln v}\,\mathrm{d}v &= \int_0^1 \sum_{n=0}^{\infty}\frac{c_n(v-1)^n}{n!}\,\mathrm{d}v = \sum_{n=0}^{\infty} c_n \int_0^1 \frac{(v-1)^n}{n!}\,\mathrm{d}v = \left[\sum_{n=0}^{\infty}\frac{c_n(v-1)^{n+1}}{(n+1)!}\right]_0^1 \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n c_n}{(n+1)!}\,. \end{aligned} \tag{2.6}$$

Euler computed only the first few derivatives $c_n$ and since there seems to be no simple enough explicit pattern, we used Maxima to compute the first 1000 derivatives (see the source code in Appendix B, Example B.2). The series converges quite slowly and the approximate value of $\delta$ after adding the terms from $n=0$ to $n=1000$ is

$$\delta \sim 0.\underline{5963}58, \tag{2.7}$$

which agrees with the known decimal expansion of $\delta$ to 4 (underlined) decimal places.

## 2.3 Asymptotic series and $f(x)$

An asymptotic expansion describes the asymptotic behaviour of a function in terms of a sequence of gauge functions. It has the property that truncating the series after a finite number of terms provides an approximation to the given function as the argument of the function tends towards a particular point (as opposed to the usual concept of a limit of a series at a fixed point). A convergent Taylor series of a continuous function at $x=0$ fits this definition, and so is the convergent case of an asymptotic series, but the definition is more general as it also allows divergent series (which are usually meant by the name asymptotic series). Moreover, the same asymptotic series represents infinitely many functions, although there are some uniqueness theorems that depend on exact bounds of the error terms.

The definition was introduced by Poincaré and it is introduced here as the real case (the complex case is analogous). Let us first define the gauge functions:

**Definition 13. (Asymptotic scale):** If $\{\varphi_n\}_{n\in\mathbb{N}_0}$ is a sequence of continuous functions on some domain $D\subseteq\mathbb{R}$, $L$ a limit point of $D$ (possibly infinity), and for every $n\in\mathbb{N}_0$ we have $\varphi_{n+1}(x)=o(\varphi_n(x))$ as $x\to L$, we call the sequence $\{\varphi_n\}$ an *asymptotic scale* or *asymptotic sequence.* The functions $\varphi_n$ are called *gauge functions.*

An example of such a sequence would be $\varphi_n(x)=x^n$. Since $\lim_{x\to 0}\frac{\varphi_{n+1}(x)}{\varphi_n(x)}=\lim_{x\to 0}\frac{x^{n+1}}{x^n}=0$, it satisfies the condition $\varphi_{n+1}(x)=o(\varphi_n(x))$ as $x\to 0$. Similarly the functions $\{x^{-n}\}_{n\in\mathbb{N}}$ form an asymptotic scale for $x\to\infty$.

**Definition 14. (Asymptotic series):** If $\{\varphi_n\}_{n\in\mathbb{N}_0}$ is an asymptotic scale on domain $D$ and $g:D\to\mathbb{R}$ a function continuous on $D$, then we say $g$ has an asymptotic (series) expansion $\sum_{n=0}^{\infty}a_n\varphi_n(x)$ and write

$$g(x)\;\sim\;\sum_{n=0}^{\infty}a_n\varphi_n(x)$$

if

$$\begin{aligned} & g(x)-\sum_{n=0}^{N}a_n\varphi_n(x) \;=\; o(\varphi_N) \qquad \text{as}\quad x\to L\\ \text{or}\quad & g(x)-\sum_{n=0}^{N}a_n\varphi_n(x) \;=\; O(\varphi_{N+1}) \quad \text{as}\quad x\to L. \end{aligned}$$

We call $R_N(x)=g(x)-\sum_{n=0}^{N}a_n\varphi_n(x)$ the error term or the remainder.

**Proposition 2.5.** *If a function $g(x)$ has an asymptotic expansion (for a given asymptotic scale $\{\varphi_n\}_{n\in\mathbb{N}_0}$), this expansion is unique.*

*Proof.* We assume the gauge functions do not vanish in some punctured neighbourhood of $L$ (which is usually the case). Then the coefficients of the series are uniquely determined as follows:

$$\begin{aligned}
a_0 &= \lim_{x\to L} \frac{f(x)}{\varphi_0(x)}, \\
a_1 &= \lim_{x\to L} \frac{f(x) - a_0\varphi_0(x)}{\varphi_1(x)}, \\
&\vdots \\
a_n &= \lim_{x\to L} \frac{f(x) - \sum_{k=0}^{n-1} a_k\varphi_k(x)}{\varphi_n(x)}.
\end{aligned}$$

□

Asymptotic series have many desirable properties that make them a useful tool for solving ordinary differential equations: linearity is obvious from the definition, i.e.

$$g_1(x) \sim \sum_{n=0}^{\infty} a_n\varphi_n(x) \quad \text{and} \quad g_2(x) \sim \sum_{n=0}^{\infty} b_n\varphi_n(x) \quad \text{as } x \to L,$$

implies

$$\alpha g_1(x) + \beta g_2(x) \sim \sum_{n=0}^{\infty} (\alpha a_n + \beta b_n)\varphi_n(x) \quad \text{as } x \to L$$

for any $\alpha, \beta$ complex. Moreover, in case the gauge functions are (positive or negative) powers of $x$, the product of the two functions is asymptotically represented by the Cauchy product of their respective asymptotic expansions. Similar results hold for a composition of two functions $g_2(g_1(x))$ and a reciprocal of a function $\frac{1}{g(x)}$, with the necessary conditions so that the functions and their expansions are well defined.

Asymptotic expansions of a (complex) function $g(x)$ that is analytic in a sector $S = \{x \in \mathbb{C} : 0 < |x| \leq M, \alpha \leq \arg x \leq \beta\}$ where $M > 0$ and $\beta > \alpha$, can be integrated term by term to get an asymptotic expansion of $\int_0^x g(t)\mathrm{d}t$. Similarly, term-wise differentiation is possible and the resulting expansion represents $g'(x)$ in every proper subsector $S^*$ of $S$, that is, for $S^* = \{x \in \mathbb{C} : 0 < |x| \leq M, \alpha^* \leq \arg x \leq \beta^*\}$, where $\alpha < \alpha^* \leq \beta^* < \beta$.

More interesting theory about asymptotic solutions to ODEs can be found in Wasow (1987). Proofs of the above results are in Section 8 and the following important theorem can be found in Section 12.

**Theorem 2.6. (Main Asymptotic Existence Theorem):** *Let $S$ be an open sector of the complex plane with vertex at the origin and a positive central angle not exceeding $\frac{\pi}{q+1}$ ($q$ a non-negative integer). Let $g(x,y)$ ($x$ and $y$ both complex) be a function with the following properties:*

*(i) $g(x,y)$ is a polynomial in $y$ with coefficients that are analytic in the region*

$$0 < |x| \leq M, \quad x \in S \quad (M \ \textit{constant});$$

*(ii) the coefficients of the polynomial $g(x,y)$ have asymptotic series in powers of $x$ as $x \to 0$, in $S$;*

*(iii) the limit*

$$\lim_{\substack{x\to 0 \\ x\in S}} \left( \left.\frac{\partial g}{\partial y}\right|_{y=0} \right)$$

*is different from zero;*

*(iv) The differential equation*

$$x^q y' = g(x,y) \tag{2.8}$$

*is formally satisfied by a power series of the form $\sum_{n=0}^{\infty} a_n x^n$.*

*Then there exists, for sufficiently small $x \in S$, a solution $y = \phi(x)$ of (2.8) such that in every proper subsector of $S$ $\phi(x) \sim \sum_{n=0}^{\infty} a_n x^n$ as $x \to 0$.*

If we examine differential equation (2.3) from the previous section, it is easy to see it can be written as $x^2 y' = 1-(1+x)y$. Then according to the theorem above, $q = 2$ and so $S$ can be taken as $S = \{x \in \mathbb{C} : 0 < |x| \leq M, -\frac{\pi}{6} < \arg x < \frac{\pi}{6}\}$ for any $M > 0$. The right side of the equation, $g(x,y) = 1-(1+x)y$, is a polynomial in $y$ with coefficients 1 and $1+x$, which are both analytic in $S$ and trivially are their own asymptotic series, since they are polynomials. The third condition is satisfied as well:

$$\lim_{\substack{x\to 0\\ x\in S}} 1+x \;=\; 1 \;\neq\; 0.$$

Lastly, we know that the equation is formally satisfied by the series (1). The theorem then implies this series is an asymptotic expansion to the known solution of the equation, which, together with the initial condition (2.4), is the function $f(x) = \int_0^{\infty} \frac{e^{-w}}{1+xw}\,\mathrm{d}w$.

*Remark* 7. Although we have solved (2.3)-(2.4) only for nonnegative $x$, Theorem 2.2 of Wasow (1987) implies there is a unique analytic solution to the equation in the above defined sector $S$. Since $f(x)$ is analytic in this sector and solves the equation for $x > 0$, it must be the unique solution in $S$.

The consequence of the above theorem can be also shown directly from the definition of asymptotic series:

**Proposition 2.7.** *The asymptotic series for $f(x) = \int_0^{\infty} \frac{e^{-w}}{1+xw}\,\mathrm{d}w$ at $x = 0$ is $\sum_{n=0}^{\infty}(-1)^n n!\,x^n$, that is,*

$$f(x) \;\sim\; \sum_{n=0}^{\infty}(-1)^n n!\,x^n.$$

*Proof.* We will consider only real positive $x$, the proof for complex $x$ ($x \in \mathbb{C}\setminus\mathbb{R}^-$) is similar. The integrand can be expanded as follows:

$$\begin{aligned}
f(x) \;&=\; \int_0^{\infty} e^{-w}\left(1-xw+(xw)^2-\cdots+(-xw)^n+\frac{(-xw)^{n+1}}{1+xw}\right)\mathrm{d}w \\
&=\; \int_0^{\infty} e^{-w}\left(1-xw+(xw)^2-\cdots+(-xw)^n\right)\mathrm{d}w \;+\; \int_0^{\infty} e^{-w}\frac{(-xw)^{n+1}}{1+xw}\,\mathrm{d}w \\
&=\; 1-1!\,x+2!\,x^2-3!\,x^3+\cdots+(-1)^n n!\,x^n+R_n(x),
\end{aligned}$$

where $R_n(x)$ can be bounded since $x, w$ are positive, hence by Lemma 1.1

$$|R_n(x)| \;\leq\; \int_0^{\infty}\left|e^{-w}\frac{(-xw)^{n+1}}{1+xw}\right|\mathrm{d}w \;\leq\; x^{n+1}\int_0^{\infty} e^{-w}w^{n+1}\mathrm{d}w \;=\; x^{n+1}(n+1)! \;=\; o(x^n).$$

This concludes the proof. □

*Remark* 8. One can easily verify that the Taylor series of $f(x)$ at $x = 0$ is again the series (1), which is convergent only for $x = 0$.

Now the relation between the two solutions of the ODE, a formal divergent power series and an analytic function, is explained. As an interesting fact we note that asymptotic expansions can be used to find, or at least approximate solutions to many linear and nonlinear differential equations and systems of differential equations, including boundary value problems with small parameters.

The main difference between convergent series and asymptotic series is the parameter in the limit; while the convergence is inspected as $n \to \infty$ for a fixed $x$, asymptoticity at $x = L$ inspects the behaviour of a (fixed) partial sum as $x \to L$. With asymptotic series, after adding finitely many terms (as a rule of thumb truncating the series after the smallest term) we get the best possible approximation, as opposed to increasing precision with increasing number of terms added from a convergent series.

This means that the leading term in the series $\sum_{n=0}^{\infty}(-1)^n n! x^n$ is the best approximation for $f(x)$ in a neighbourhood of 0, but applying some acceleration method to the series could improve this. One example, Euler series transform, is described in Chapter 3 (see also Table 3.1 for approximation of $f(1)$ by this method).

### 2.3.1 Borel's summation method and asymptotic series

Recall Example 1.9, where the Borel sum of $F(z)$ was found to be $f(z) = \int_0^\infty \frac{e^{-t}}{1+zt}\mathrm{d}t$. It is the same function as in the ODE method, whose asymptotic series is exactly the original series to which the method was applied. This is not a coincidence, as Borel's methods can, under certain conditions, give a function that has the original series as its asymptotic expansion. Watson's recovery theorem, which is a consequence of Watson' uniqueness theorem, describes this result.

**Theorem 2.8. (Watson's uniqueness theorem):** *Let $\{a_0, a_1, a_2, \ldots\}$ be a sequence of complex numbers and let $h(z)$ be a function satisfying the conditions*

*(i) $h(z)$ is analytic and single-valued in the sector $S(\alpha,\beta) = \{z \in \mathbb{C} : 0 < |z| < \infty, \alpha < \arg z < \beta\}$ with $\beta - \alpha > \pi$,*

*(ii) for all $z \in S(\alpha,\beta)$ and every $n \in \mathbb{N}$*

$$|R_{n-1}(z)| = \left| h(z) - \sum_{i=0}^{n-1} a_i z^i \right| \leq c^{n+1} n! |z|^n,$$

*where the positive constant $c$ does not depend on $z$ and $n$ but may depend on $h(z)$.*

*Then the function $h(z)$ is uniquely determined on $S(\alpha,\beta)$.*

**Theorem 2.9. (Watson's recovery theorem):** *Assume that the function $h(z)$ satisfies conditions (i) and (ii) of Watson's uniqueness theorem in a sector $S(-\frac{\pi}{2} - \varepsilon, \frac{\pi}{2} + \varepsilon)$ for some $\varepsilon \in (0, \frac{\pi}{2})$. Then*

*(i) the Borel transform of the (formal) series $A = \sum_{n=0}^{\infty} a_n$ given as $\mathcal{B}A(t) = \sum_{n=0}^{\infty} \frac{a_n t^n}{n!}$ is absolutely convergent and represents an analytic function $H(t)$ in the disk $D_c$ with radius a and centre at the origin;*

*(ii) the function $H(t)$ can be continued analytically from the disk $D_c$ to the region $D_c \cup \{t \in \mathbb{C} : |\arg t| < \varepsilon\}$;*

*(iii) the function $h(z)$ can be expressed as the Borel sum of its asymptotic series*

$$h(z) = \int_0^\infty e^{-t} H(zt)\,\mathrm{d}t,$$

*where the integral is absolutely convergent for $z \in S(-\frac{\pi}{2}, \frac{\pi}{2})$.*

The details and proofs of these theorems can be found in Watson (1912), Sections 8 and 9. In this sense, a function satisfying the properties in this theorem is the most suitable function for its asymptotic series among infinitely many functions with the same asymptotic series.

Since the function $f(z) = \int_0^\infty \frac{e^{-t}}{1+zt}\,\mathrm{d}t$ satisfies the conditions of the theorem, as a consequence it is equal to the Borel sum of its asymptotic series $\sum_{n=0}^\infty (-1)^n z^n n!$, which was already clear from the previous section.

# Chapter 3

# Euler's first method: Euler series transform

In a convergent alternating series, the partial sums after adding a positive term bound the sum from above, while those where the last term added was negative bound the sum from below. If we extend this notion to divergent alternating series as well, the sequence of partial sums again gives alternating lower and upper bounds for the value we wish to assign to the series. These will not get more accurate as the series progresses, however, and can possibly grow without bound. The best bounds are then those pairs of partial sums closest to each other. This correlates with the rule of truncating an asymptotic series after adding the smallest term, which was addressed in Section 2.3.

Of course, in case of WHS, these bounds are always increasing, so to begin with we can only tell that the value will be between 0 and 1. Provided we can define a transformation that will result again in an alternating series equivalent to the original one under this definition but slower in its divergence, we may be able to improve these bounds. In other words, we accelerate the series. One such transformation is motivated by the Euler summation method defined by Definition 5; called Euler transform, it is defined in the very same way, only now dropping the requirement of the series being convergent for small values of $x$. It can be therefore considered a weaker version, but it will be shown that it is totally regular nevertheless and even obeys rules (I)-(III). Its repeated application to WHS results in approximations of $\delta$, listed in Table 3.1.

In Section 3.2 we define the generalised Euler's summation $(E, q)$ for $q > 0$ and its corresponding generalised Euler transform and prove regularity and consistency with rules (I)-(III) and the connection to repeated Euler transform. In the last section the connection to Borel's summation method is explained.

## 3.1 Euler transform and its application on WHS

We begin with definition of the Euler transform $\mathcal{E}$ that corresponds to Euler's summation $(E, 1)$.

**Definition 15. (Euler series transform):** For a series $\sum_{n=0}^{\infty} a_n$ define its Euler transform $\mathcal{E}$ as

$$\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} b_n, \quad \text{where} \quad b_n = \sum_{i=0}^{n} \binom{n}{i} a_i. \tag{3.1}$$

*Remark* 9. In an alternative definition, reserved for alternating series $\sum_{n=0}^{\infty}(-1)^n a_n$ with $a_n \geq 0$, the transform is given as

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} b_n, \quad \text{with} \quad b_n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} a_{n-i}.$$

Using the difference operator the coefficients can be expressed as

$$b_n = \Delta^n a_0,$$

which is a consequence of the following Lemma.

**Lemma 3.1.** *For any sequence $\{a_m\}_{m\in\mathbb{N}_0}$ with non-negative terms it is true for any $n \in \mathbb{N}_0$ and any $m \in \mathbb{N}_0$ that*

$$\Delta^n a_m = \sum_{i=0}^{n} (-1)^i \binom{n}{i} a_{n+m-i}.$$

*Proof.* For $n = 0$ trivially $a_m = \Delta^0 a_m$ for any $m$ and for $n = 1$ $a_{m+1} - a_m = \Delta^1 a_m$. Assume that for some $n \in \mathbb{N}_0$ the following holds:

$$\forall m \in \mathbb{N}_0 : \quad \Delta^n a_m = \sum_{i=0}^{n} (-1)^i \binom{n}{i} a_{n+m-i},$$

then for $n+1$ and an arbitrary $m$ we get

$$\begin{aligned} \Delta^{n+1} a_m &= \Delta^n a_{m+1} - \Delta^n a_m = \sum_{i=0}^{n} (-1)^i \binom{n}{i} a_{n+m+1-i} - \sum_{i=0}^{n} (-1)^i \binom{n}{i} a_{n+m-i} \\ &= \sum_{i=0}^{n} (-1)^i \binom{n}{i} a_{n+m+1-i} + \sum_{i=1}^{n+1} (-1)^i \binom{n}{i-1} a_{n+m+1-i} = \sum_{i=0}^{n+1} (-1)^i \binom{n+1}{i} a_{n+1+m-i}, \end{aligned}$$

as a consequence of the binomial identity $\binom{n}{i} + \binom{n}{i-1} = \binom{n+1}{i}$. By induction, the assertion holds for all $n \in \mathbb{N}_0$. □

It is possible to represent the Euler transform by a matrix. It is derived in Hardy (1949), Section 8.2, but here we will instead use a simpler approach to prove the relation. Another proof will be given in Section 3.2 as a consequence of the matrix representation of the generalised Euler transform.

**Proposition 3.2.** *The matrix representation of the Euler series transform is given as $\mathcal{E} = (c_{m,n})$ with*

$$c_{m,n} = \begin{cases} \frac{1}{2^{m+1}} \binom{m+1}{n+1} & \text{if } n \leq m, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* Denote $s_n = \sum_{k=0}^{n} a_k$ the partial sums of the original series and $t_m = \sum_{k=0}^{m} \frac{1}{2^{k+1}} b_k$ the partial sums of the transformed series. For $m = 0$ it is trivially true that $t_0 = \frac{1}{2} b_0 = \frac{1}{2} a_0 = \frac{1}{2} s_0$ and so $c_{0,0} = \frac{1}{2}$ and $c_{0,n} = 0$ for all $n > 0$.

Assume that for some $m$ it is true that $t_m = \sum_{n=0}^{m} \frac{1}{2^{m+1}} \binom{m+1}{n+1} s_n$, thus $c_{m,n} = \frac{1}{2^{m+1}} \binom{m+1}{n+1}$ if $n \leq m$ and 0 otherwise. Then using this assumption,

$$t_{m+1} = \sum_{n=0}^{m+1} \frac{1}{2^{n+1}} b_n = \sum_{n=0}^{m} \frac{1}{2^{n+1}} b_n + \frac{1}{2^{m+2}} b_{m+1} = \sum_{n=0}^{m} \frac{1}{2^{m+1}} \binom{m+1}{n+1} s_n + \frac{1}{2^{m+2}} b_{m+1}.$$

We expand $b_{m+1}$ and divide the first sum by 2 to lay out the formula $s_n = s_{n-1} + a_n$:

$$\begin{aligned} t_{m+1} &= \sum_{n=0}^{m} \frac{1}{2^{m+1}} \binom{m+1}{n+1} s_n + \sum_{n=0}^{m+1} \frac{1}{2^{m+2}} \binom{m+1}{n} a_n \\ &= \sum_{n=0}^{m} \frac{1}{2^{m+2}} \binom{m+1}{n+1} s_n + \sum_{n=0}^{m} \frac{1}{2^{m+2}} \binom{m+1}{n+1} s_n + \sum_{n=0}^{m+1} \frac{1}{2^{m+2}} \binom{m+1}{n} a_n \\ &= \sum_{n=0}^{m} \frac{1}{2^{m+2}} \binom{m+1}{n+1} s_n + \sum_{n=1}^{m+1} \frac{1}{2^{m+2}} \binom{m+1}{n} s_{n-1} + \sum_{n=0}^{m+1} \frac{1}{2^{m+2}} \binom{m+1}{n} a_n \end{aligned}$$

and after adding the last two terms we can use the binomial identity $\binom{k}{l} + \binom{k}{l+1} = \binom{k+1}{l+1}$:

$$\begin{aligned} t_{m+1} &= \sum_{n=0}^{m} \frac{1}{2^{m+2}} \binom{m+1}{n+1} s_n + \sum_{n=0}^{m+1} \frac{1}{2^{m+2}} \binom{m+1}{n} s_n \\ &= \sum_{n=0}^{m+1} \frac{1}{2^{m+2}} \binom{m+2}{n+1} s_n, \end{aligned}$$

implying that $c_{m+1,n} = \frac{1}{2^{m+2}} \binom{m+2}{n+1}$ for $n \leq m+1$ and 0 otherwise. By induction, this is true for all $m \in \mathbb{N}_0$, concluding the proof. □

The following theorems will be used to show that this method is totally regular.

**Theorem 3.3. (Toeplitz):** *A summation method represented by matrix $\mathcal{T} = (c_{m,n})$ is regular if and only if:*

*(i) there is a number $H \geq 0$ such that $\sum_{n=0}^{\infty} |c_{m,n}| < H$ for all $m$ in $N_0$,*

*(ii) $\lim_{m\to\infty} c_{m,n} = 0$ for all $n$ in $N_0$ and*

*(iii) $\lim_{m\to\infty} \sum_{n=0}^{\infty} c_{m,n} = 1$.*

A proof can be found in Hardy (1949), Section 3.3.

**Definition 16.** We call a transformation $\mathcal{T} = (c_{m,n})$ *positive* if there is $n_0 \in \mathbb{N}_0$ such that $c_{m,n} \geq 0$ for all $m \in \mathbb{N}_0$ and $n \geq n_0$.

**Theorem 3.4.** *A transformation $\mathcal{T} = (c_{m,n})$ is totally regular if it is positive, regular and lower triangular, i.e. $c_{m,n} = 0$ for $n > m$.*

This theorem was proved by W.A. Hurwitz, in PLMS (1926), pages 231-248.

**Corollary 3.5.** *The Euler series transform and its iterations are totally regular summation methods.*

*Proof.* Notice that $c_{m,n} \geq 0 \quad \forall m,n \in \mathbb{N}_0$ and $c_{m,n} = 0$ for $n > m$, hence the matrix is positive, lower triangular and

(i) $\sum_{n=0}^{\infty} |c_{m,n}| = \sum_{n=0}^{m} \frac{1}{2^{m+1}} \binom{m+1}{n+1} < \frac{1}{2^{m+1}} \sum_{n=0}^{m+1} \binom{m+1}{n+1} = 1$ for all $m$,

(ii) $\lim\limits_{m\to\infty} c_{m,n} = \lim\limits_{m\to\infty} \dfrac{(m+1)!}{2^{m+1}(m-n)!(n+1)!} < \lim\limits_{m\to\infty} \dfrac{(m+1)!}{2^{m+1}(m-n)!} < \lim\limits_{m\to\infty} \dfrac{(m+1)^{n+1}}{2^{m+1}}$

$= \lim\limits_{m\to\infty} \dfrac{(n+1)!}{2^{m+1}(\ln 2)^{n+1}} = 0$

after applying L'Hospital's rule $(n+1)$-times, and

(iii) $\lim\limits_{m\to\infty} \sum\limits_{n=0}^{\infty} c_{m,n} = \lim\limits_{m\to\infty} \frac{1}{2^{m+1}} \sum\limits_{n=0}^{m} \binom{m+1}{n+1} = \lim\limits_{m\to\infty} \left(1 - \frac{1}{2^{m+1}}\right) = 1.$

The conditions of both theorems are satisfied, hence the method is totally regular.

Next we show that these properties are preserved for all powers of $\mathcal{E}$. Since $\mathcal{E}$ has only positive terms, all its powers will trivially be positive as well. Also, due to Theorem 1.12, $\mathcal{E}^k$ will be lower triangular for all $k$. It remains to prove properties (i), (ii) and (iii) for $\mathcal{E}^k = (c_{m,n}^{(k)})$. All three parts are proved by induction.

Case $k = 1$ is true and if we assume $\mathcal{E}, \mathcal{E}^2, \ldots, \mathcal{E}^k$ have the said properties, then for $k+1$ we have $\mathcal{E}^{k+1} = \mathcal{E} \times \mathcal{E}^k$, thus $c_{m,n}^{(k+1)} = \sum_{i=0}^{m} c_{m,i} c_{i,n}^{(k)}$.

(i) Fix $m \in \mathbb{N}_0$.

$$\sum_{n=0}^{\infty} c_{m,n}^{(k+1)} = \sum_{n=0}^{m} \sum_{i=0}^{m} c_{m,i} c_{i,n}^{(k)} = \sum_{i=0}^{m} c_{m,i} \sum_{n=0}^{m} c_{i,n}^{(k)},$$

where the second sum is less than 1 based on the assumption, hence

$$\sum_{n=0}^{\infty} c_{m,n}^{(k+1)} < \sum_{i=0}^{m} c_{m,i} < 1$$

for all $m \in \mathbb{N}_0$.

(ii) Fix $n \in \mathbb{N}_0$ and $\varepsilon > 0$. From the assumption for $\mathcal{E}^k$ there is an $N \in \mathbb{N}_0$ such that $c_{i,n}^{(k)} < \frac{\varepsilon}{2}$ for all $i \geq N$, so

$$c_{m,n}^{(k+1)} = \sum_{i=0}^{m} c_{m,i} c_{i,n}^{(k)} < \sum_{i=0}^{N-1} c_{m,i} c_{i,n}^{(k)} + \sum_{i=N}^{m} c_{m,i} \frac{\varepsilon}{2} < \sum_{i=0}^{N-1} c_{m,i} c_{i,n}^{(k)} + \frac{\varepsilon}{2}$$

since the sum of each row of $\mathcal{E}$ is less than 1 from (i).

Now for each $i = 0, 1, \ldots, N-1$ we can choose $m_i$ so that $c_{m,i} < \frac{\varepsilon}{2N}$ for all $m \geq m_i$ and take $M = \max\{m_0, m_1, \ldots, m_{N-1}\}$. Then, taking into account that all terms $c_{m,n}^{(k)}$ are less than 1, we have

$$c_{m,n}^{(k+1)} < \sum_{i=0}^{N-1} \frac{\varepsilon}{2N} + \frac{\varepsilon}{2} = \varepsilon \quad \text{for } m \geq M,$$

hence the limit for $m \to \infty$ is 0 for any $n \in \mathbb{N}_0$, as required.

(iii) Define a sequence

$$\{r_m\}_{m\in\mathbb{N}_0} = \left\{ \sum_{n=0}^{m} c_{m,n}^{(k+1)} \right\}_{m\in\mathbb{N}_0}.$$

From (i) we have for all $m \in \mathbb{N}_0$

$$r_m < 1. \tag{3.2}$$

Fix an arbitrary $\varepsilon > 0$. From the assumption (iii) for $\mathcal{E}^k$ there is $N \in \mathbb{N}_0$ s.t. $\sum_{n=0}^{\infty} c_{i,n}^{(k)} > 1-\varepsilon$ whenever $i > N$. Then

$$\sum_{n=0}^{\infty} c_{m,n}^{(k+1)} = \sum_{n=0}^{m}\sum_{i=0}^{m} c_{m,i}c_{i,n}^{(k)} = \sum_{n=0}^{m}\sum_{i=0}^{N} c_{m,i}c_{i,n}^{(k)} + \sum_{i=N+1}^{m} c_{m,i}\sum_{n=0}^{m} c_{i,n}^{(k)} > \sum_{i=N+1}^{m} c_{m,i}(1-\varepsilon)$$

where the first finite sum was neglected since all terms are positive. Since $c_{m,i}$ are just (scaled) binomial coefficients $\frac{1}{2^{m+1}}\binom{m+1}{n+1}$, we can use an argument similar to that in Subsection 1.4.1 to choose $m$ big enough so that the missing first $N+1$ coefficients sum up to less than $\varepsilon$. Thus for this $m$ (and all $m$ greater than that)

$$r_m = \sum_{n=0}^{\infty} c_{m,n}^{(k+1)} > (1-\varepsilon)(1-\varepsilon) > 1-3\varepsilon. \tag{3.3}$$

Inequalities (3.2) and (3.3) together show that for all $\varepsilon > 0$ and sufficiently large $m$

$$1-3\varepsilon < r_m < 1,$$

hence the limit $\lim\limits_{m\to\infty} r_m$ exists and is equal to 1, concluding the proof.

□

Euler transform applied to Wallis' hypergeometric series (repeatedly) gives a rough approximation of $\delta$. With each iteration, we will truncate the sum after the smallest term, resulting in the closest approximation of $f(1)$ (with $f(x)$ as defined in (2)) in the sense of asymptotic series. Table 3.1 lists the results computed by Maxima after each iteration ($k$), with the number of terms of the series used ($m$) in the second column and the truncated sum in the third ($\mathcal{E}^k\mathbf{s}(m)$). The decimal places that agree with known decimals of $\delta$ are underlined. (The source code of the script can be found in Appendix B, Example B.3.)

**Table 3.1:** Iterations of Euler transform applied to WHS

| $k$ | $m$ | $\mathcal{E}^k\mathbf{s}(m)$ |
|---|---|---|
| 1 | 1 | $0.\underline{5}$ |
| 2 | 7 | $0.\underline{5}726\ldots$ |
| 3 | 21 | $0.\underline{5}854\ldots$ |
| 4 | 49 | $0.\underline{5}8867\ldots$ |
| 5 | 105 | $0.\underline{5}8981\ldots$ |
| 6 | 219 | $0.\underline{59}051\ldots$ |
| 7 | 447 | $0.\underline{59}082\ldots$ |
| 8 | 907 | $0.\underline{59}107\ldots$ |
| 9 | 1825 | $0.\underline{59}116\ldots$ |

As can be seen from the table, subsequent iterations do not improve the value much even though the number of terms they require grows quite fast and makes computations very time consuming. Even with a computer, Euler's estimate computed by hand ($\approx 0.58$) was only improved by one decimal place.

## 3.2 Generalised Euler's summation[†] $(E,q)$

As in Definition 5, we start with the motivation behind the definition of the method. Assume the series $g(x)=\sum_{n=0}^{\infty}a_nx^{n+1}$ converges for small $x$ and let

$$x = \frac{y}{1-qy}, \qquad y = \frac{x}{1+qx}$$

for a $q>0$. Then for small $x$ and corresponding $y$ we have

$$g(x) = \sum_{n=0}^{\infty}a_nx^{n+1} = \sum_{n=0}^{\infty}a_ny^{n+1}\left(\frac{1}{1-qy}\right)^{n+1} = \sum_{n=0}^{\infty}a_ny^{n+1}\sum_{i=0}^{\infty}\binom{n+i}{i}(qy)^i,$$

where in the last equality we used the Taylor series for $\frac{1}{(1-z)^{n+1}}$. Substituting $m$ for $n+i$ and subsequently changing the order of summation yields

$$\begin{aligned} g(x) &= \sum_{n=0}^{\infty}a_n\sum_{m=n}^{\infty}\binom{m}{n}q^{m-n}y^{m+1} = \sum_{m=0}^{\infty}y^{m+1}\sum_{n=0}^{m}\binom{m}{n}q^{m-n}a_n \\ &= \sum_{m=0}^{\infty}\underbrace{[y(1+q)]^{m+1}}_{=:z^{m+1}}\underbrace{\frac{1}{(1+q)^{m+1}}\sum_{n=0}^{m}\binom{m}{n}q^{m-n}a_n}_{=:a_m^{(q)}} = \sum_{m=0}^{\infty}a_n^{(q)}z^{m+1}, \end{aligned}$$

with $z=y(1+q)=\frac{x+xq}{1+xq}$ and so $x=\frac{z}{1+q-qz}$. If $x$ is small, so is $z$ no matter the choice of $q$.

**Definition 17. (Generalised Euler's summation):** Let $q>0$ and assume that the series $\sum_{n=0}^{\infty}a_nx^{n+1}$ converges for $x$ in some neighbourhood of 0. If the series

$$\sum_{m=0}^{\infty}a_m^{(q)} = \sum_{m=0}^{\infty}\frac{1}{(1+q)^{m+1}}\sum_{n=0}^{m}\binom{m}{n}q^{m-n}a_n$$

converges to a value $A$, we call $A$ the $(E,q)$-sum of the series $\sum_{n=0}^{\infty}a_n$ and write

$$\sum_{n=0}^{\infty}a_n = A \ (E,q).$$

*Remark* 10. For $q=1$ this is the previously defined Euler's summation. Taking $q=0$ yields the regular summation that works only for convergent series.

*Remark* 11. The same way as Euler transform $\mathfrak{E}$ was defined as $(E,1)$ only without the requirement of convergence for small $x$, we can define the **Generalised Euler transform** of $\sum_{n=0}^{\infty}a_n$ for any $q>0$ ($q$-th Euler transform) as $\sum_{m=0}^{\infty}a_m^{(q)}$ and write $\sum_{n=0}^{\infty}a_n=A\ \ (E,q)$ whenever $\sum_{m=0}^{\infty}a_m^{(q)}=A$.

**Example 3.6.** Let $a_n=z^n$ with $z$ complex. The $(E,q)$-sum of the series $\sum_{n=0}^{\infty}z^n$ is $\sum_{m=0}^{\infty}a_m^{(q)}$ with

$$a_m^{(q)} = \frac{\sum_{n=0}^{m}\binom{m}{n}q^{m-n}z^n}{(1+q)^{m+1}} = \frac{1}{1+q}\frac{(z+q)^m}{(1+q)^m}$$

[†]The outline of this section follows Hardy (1949), Sections 8.2 and 8.3, filling in the details.

and so for $|z+q|<1+q$ the series converges to the $(E,q)$-sum

$$\sum_{m=0}^{\infty} a_m^{(q)} = \sum_{m=0}^{\infty} \frac{1}{1+q}\left(\frac{z+q}{1+q}\right)^m = \frac{1}{1+q}\frac{1}{1-\frac{z+q}{1+q}} = \frac{1}{1-z},$$

valid for $z$ in the circle with centre at $-q$ and radius $1+q$. For $q \to \infty$ this region approaches the half-plane $\{z : \mathrm{Re} z < 1\}$, which is the same as the region of summability of this series by Borel methods $B$ and $wB$, that also assign to it the same sum (see Example 1.8). As will be explained in Section 3.3, this is not a coincidence, since the Borel methods can be considered the limiting case of $(E,q)$ when $q \to \infty$. ▲

Next we will find the matrix representation of $(E,q)$ as a first step to prove the regularity of $q$-th Euler transform.

**Proposition 3.7.** *The matrix representation of $(E,q)$ is given as $E^{(q)} = (c_{m,n})$ with*

$$c_{m,n} = \begin{cases} \frac{1}{(1+q)^{m+1}}\binom{m+1}{n+1}q^{m-n} & \text{if } n \le m, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* Denote $s_m^{(q)} = a_0^{(q)} + a_1^{(q)} + \ldots + a_m^{(q)}$ the partial sums of the transformed series. Our aim is to find the representation of these partial sums in the form

$$s_m^{(q)} = \sum_{n=0}^{\infty} c_{m,n} s_n,$$

with $s_n = a_0 + a_1 + \ldots + a_n$ the partial sums of the original series. For this purpose we define the shift operator $U$ as $Ua_n = a_{n+1}$ for any $n \in \mathbb{N}_0$, therefore $a_n = U^n a_0$ and so, keeping in mind linearity of $U$,

$$\begin{aligned} s_m^{(q)} &= \sum_{n=0}^{m} a_n^{(q)} = \sum_{n=0}^{m} \frac{1}{(1+q)^{n+1}} \sum_{i=0}^{n} \binom{n}{i} q^{n-i} a_i = \sum_{n=0}^{m} \frac{1}{(1+q)^{n+1}} \sum_{i=0}^{n} \binom{n}{i} q^{n-i} U^i a_0 \\ &= \sum_{n=0}^{m} \frac{1}{(1+q)^{n+1}} (q+U)^n a_0 = \frac{1}{(1+q)^{m+1}} \sum_{n=0}^{m} \left((1+q)^{m-n}(q+U)^n\right) a_0, \end{aligned}$$

where in the second line we used the binomial formula $\sum_{n=0}^{m}\binom{m}{n}x^{m-n}y^n = (x+y)^m$. The last expression is in the form of $a^m b^0 + a^{m-1}b^1 + \ldots + a^0 b^m$ which is equal to $\frac{a^{m+1}-b^{m+1}}{a-b}$. Thus using these two formulas a number of times we derive

$$\begin{aligned} s_m^{(q)} &= \frac{1}{(1+q)^{m+1}} \left(\frac{(1+q)^{m+1} - (q+U)^{m+1}}{(1+q)-(q+U)}\right) a_0 \\ &= \frac{1}{(1+q)^{m+1}} \left(\frac{\sum_{i=0}^{m+1}\binom{m+1}{i} q^{m+1-i} 1^i - \sum_{i=0}^{m+1}\binom{m+1}{i} q^{m+1-i} U^i}{1-U}\right) a_0 \\ &= \frac{1}{(1+q)^{m+1}} \sum_{i=0}^{m+1} \binom{m+1}{i} q^{m+1-i} \left(\frac{1-U^i}{1-U}\right) a_0 \\ &= \frac{1}{(1+q)^{m+1}} \sum_{i=1}^{m+1} \binom{m+1}{i} q^{m+1-i} \sum_{k=0}^{i-1} U^k a_0 = \frac{1}{(1+q)^{m+1}} \sum_{i=1}^{m+1} \binom{m+1}{i} q^{m+1-i} \sum_{k=0}^{i-1} a_k \\ &= \frac{1}{(1+q)^{m+1}} \sum_{i=1}^{m+1} \binom{m+1}{i} q^{m+1-i} s_{i-1} = \frac{1}{(1+q)^{m+1}} \sum_{n=0}^{m} \binom{m+1}{n+1} q^{m-n} s_n, \end{aligned} \tag{3.4}$$

proving that $c_{m,n} = \frac{1}{(1+q)^{m+1}} \sum_{n=0}^{m} \binom{m+1}{n+1} q^{m-n}$ for $n \le m$ and 0 otherwise.

Since in the above expression we used the inverse $(q-U)^{-1}$ in the form $\frac{1}{(q-U)}$, it should be shown that this operation was valid. That can be justified by proving that the expressions in the numerators (in particular $U$) can be multiplied by $(q-U)^{-1}$ from left or from right, giving the same result in both cases. Indeed, for any $m \in \mathbb{N}_0$ we have $(q-U)(q-U)^{-1}Ua_m = Ua_m = a_{m+1}$ and $(q-U)U(q-U)^{-1}a_m = U(q-U)(q-U)^{-1}a_m = Ua_m = a_{m+1}$, hence also

$$(q-U)^{-1}Ua_m = U(q-U)^{-1}a_m \quad \forall m \in \mathbb{N}_0,$$

proving the assertion. □

**Theorem 3.8.** *Methods $(E,q)$ are totally regular for all $q>0$.*

*Proof.* Since the matrix $E^{(q)} = (c_{m,n})$ is positive and lower triangular, we only need to show properties (i)-(iii) of Theorem 3.3. For (i) we have

$$\begin{aligned}\sum_{n=0}^{\infty} |c_{m,n}| &= \sum_{n=0}^{m} \frac{1}{(1+q)^{m+1}} \binom{m+1}{n+1} q^{m-n} = \frac{1}{(1+q)^{m+1}} \sum_{i=1}^{m+1} \binom{m+1}{i} q^{m+1-i} 1^i \\ &= \frac{1}{(1+q)^{m+1}} \left((1+q)^{m+1} - q^{m+1}\right) = 1 - \left(\frac{q}{1+q}\right)^{m+1} < 1\end{aligned}$$

and since the limit $\lim_{m\to\infty} 1 - \left(\frac{q}{1+q}\right)^{m+1} = 1$, (iii) is also satisfied. Lastly,

$$\begin{aligned}\lim_{m\to\infty} c_{m,n} &= \lim_{m\to\infty} \frac{1}{(1+q)^{m+1}} \binom{m+1}{n+1} \frac{q^m}{q^n} = \lim_{m\to\infty} \frac{(m+1)m\dots(m+1-n)}{(n+1)!\,q^n(1+q)} \left(\frac{q}{1+q}\right)^m \\ &< \lim_{m\to\infty} C \left(\frac{q}{1+q}\right)^m (m+1)^{n+1} = 0,\end{aligned}$$

because $C$ is a constant independent of $m$ and an exponential function $x^m$ with $|x|<1$ converges to 0 faster than the polynomial $(m+1)^{n+1}$ grows to infinity, proving (ii). □

The following theorem is an important result that will be the key to proving the connection between Euler's summations and Borel methods.

**Theorem 3.9.** *A composition of two Euler's summations $(E,q)$ and $(E,r)$ is again an Euler's summation $(E,q+r+qr)$. In particular for the transforms, for any sequence of partial sums $\mathbf{s} = \{s_0, s_1, \dots\}$ and any $q,r>0$*

$$E^{(r)}\left(E^{(q)}\mathbf{s}\right) = E^{(q+r+qr)}\mathbf{s}.$$

*Proof.* If the original series is denoted as $\mathbf{s} = \sum_{n=0}^{\infty} a_n x^{n+1}$, its $(E,q)$-summation as $\mathbf{s}^{(q)} = \sum_{m=0}^{\infty} a_m^{(q)} z^{m+1}$ as in the Definition 17 and the $(E,r)$-summation of that as

$$\left(\mathbf{s}^{(q)}\right)^{(r)} = \sum_{m=0}^{\infty} b_m^{(r)} w^{m+1}, \quad \text{with } b_m^{(r)} = \frac{1}{(1+r)^{m+1}} \sum_{n=0}^{m} \binom{m}{n} r^{m-n} a_n^{(q)} \quad \text{and } z = \frac{w}{1+r-rw},$$

then it needs to be shown that $b_m^{(r)} = a_m^{(q+r+qr)}$ for all $m \in \mathbb{N}_0$ and $x = \frac{w}{1+(q+r+qr)-(q+r+qr)w}$.

The second assertion is fairly simple, for

$$x = \frac{z}{1+q-qz} = \frac{\frac{w}{1+r-rw}}{1+q-\frac{qw}{1+r-rw}} = \frac{w}{1+(q+r+qr)-(q+r+qr)w},$$

as required.

For the first assertion, notice that for any $n,m,i \in \mathbb{N}_0$ such that $m \geq n \geq i$ it follows that

$$\binom{m}{n}\binom{n}{i} = \frac{m!\,n!}{(m-n)!\,n!\,(n-i)!\,i!} = \frac{m!\,(m-i)!}{(m-n)!\,(m-i)!\,(n-i)!\,i!} = \binom{m}{i}\binom{m-i}{n-i}$$

and therefore for any $a,b \in \mathbb{R}$

$$\begin{aligned}\sum_{n=i}^{m}\binom{m}{n}\binom{n}{i}a^{m-n}b^{n-i} &= \sum_{n=i}^{m}\binom{m}{i}\binom{m-i}{n-i}a^{m-n}b^{n-i} = \binom{m}{i}\sum_{k=0}^{m-i}\binom{m-i}{k}a^{m-i-k}b^{k}\\ &= \binom{m}{i}(a+b)^{m-i},\end{aligned}$$

where $k = n-i$ was substituted. Using this equality and after some rearranging we will find that

$$\begin{aligned}b_m^{(r)} &= \frac{1}{(1+r)^{m+1}}\sum_{n=0}^{m}\binom{m}{n}r^{m-n}a_n^{(q)} = \frac{1}{(1+r)^{m+1}}\sum_{n=0}^{m}\binom{m}{n}r^{m-n}\frac{1}{(1+q)^{n+1}}\sum_{i=0}^{n}\binom{n}{i}q^{n-i}a_i\\ &= \sum_{i=0}^{m}\sum_{n=i}^{m}\frac{1}{(1+r)^{m+1}}\frac{(1+q)^{m-n}}{(1+q)^{m+1}}\binom{m}{n}\binom{n}{i}r^{m-n}q^{n-i}a_i\\ &= \sum_{i=0}^{m}\frac{1}{[(1+r)(1+q)]^{m+1}}\,a_i\sum_{n=i}^{m}\binom{m}{n}\binom{n}{i}\,[(1+q)r]^{m-n}\,q^{n-i}\\ &= \sum_{i=0}^{m}\frac{1}{(q+r+qr+1)^{m+1}}\binom{m}{i}(q+r+qr)^{m-i}a_i = a_m^{(q+r+qr)},\end{aligned}$$

as expected. □

As a consequence, repeated application of $q$-th transform is again totally regular and, in general, $(E,q)$ form a group of methods increasing in strength with $q$:

**Corollary 3.10.** *If a series is $(E,q')$-summable and $q > q'$, then it is also $(E,q)$-summable to the same number.*

*Proof.* Follows from Theorem 3.9 and regularity of $(E,q)$ for any $q > 0$. □

**Corollary 3.11.** *For Euler transform $(E,1)$ with its matrix representation $\mathcal{E}$, for any $k \in \mathbb{N}$*

$$\mathcal{E}^k = E^{(2^k-1)}$$

*and all properties proved for $(E,q)$ in this section trivially hold for $\mathcal{E}$ and all its iterations.*

Euler transforms are even more well-behaved than just totally regular - they obey the rules we described in the first chapter:

**Theorem 3.12.** *For any $q > 0$, the summation method $(E,q)$ is consistent with rules (I)-(III).*

*Proof.* Linearity is trivial so we only need to show that stability is satisfied. If we denote $b_n = a_{n+1}$, then the assertion is as follows:

$$\sum_{n=0}^{\infty} a_n^{(q)} = A \iff \sum_{n=0}^{\infty} b_n^{(q)} = A - a_0.$$

Since $(E,q)$ is linear, we can assume without the loss of generality that $a_0 = 0$. Denote as usual $s_n$ the partial sums $\sum_{i=0}^{n} a_i$ and $t_n$ the partial sums $\sum_{i=0}^{n} b_i$, then $t_n = s_{n+1}$. Hence by (3.4)

$$t_m^{(q)} = \frac{1}{(1+q)^{m+1}} \sum_{n=1}^{m+1} \binom{m+1}{n} q^{m+1-n} t_{n-1} = \frac{1}{(1+q)^{m+1}} \sum_{n=1}^{m+1} \binom{m+1}{n} q^{m+1-n} s_n,$$

and it follows that

$$t_m^{(q)} - s_m^{(q)} = \frac{1}{(1+q)^{m+1}} \sum_{n=1}^{m+1} \binom{m+1}{n} q^{m+1-n} (s_n - s_{n-1}) = \frac{1}{(1+q)^{m+1}} \sum_{n=1}^{m+1} \binom{m+1}{n} q^{m+1-n} a_n.$$

Since $a_0 = 0$, this is the same as

$$t_m^{(q)} - s_m^{(q)} = (1+q) \frac{1}{(1+q)^{m+2}} \sum_{n=0}^{m+1} \binom{m+1}{n} q^{m+1-n} a_n = (1+q) a_{m+1}^{(q)}. \tag{3.5}$$

Now assume $\sum_{n=0}^{\infty} a_n^{(q)} = A$, then $a_m^{(q)}$ must converge to 0 (and $s_m^{(q)}$ to $A$) and so

$$\lim_{m\to\infty} t_m^{(q)} = \lim_{m\to\infty} s_m^{(q)} + \lim_{m\to\infty} (1+q) a_{m+1}^{(q)} = A,$$

proving the first part.

For the other direction, rewrite (3.5) as $t_m^{(q)} = s_m^{(q)} + (1+q)\left(s_{m+1}^{(q)} - s_m^{(q)}\right) = (1+q)s_{m+1}^{(q)} - q s_m^{(q)}$, so that

$$s_{m+1}^{(q)} = \frac{1}{1+q} t_m^{(q)} + \frac{q}{1+q} s_m^{(q)}.$$

Remembering that $s_0^{(q)} = a_0^{(q)} = a_0 = 0$, this means that

$$s_1^{(q)} = \frac{t_1^{(q)}}{1+q}, \quad s_2^{(q)} = \frac{t_2^{(q)}}{1+q} + \frac{q t_1^{(q)}}{(1+q)^2}, \quad \ldots, \quad s_{m+1}^{(q)} = \frac{t_m^{(q)}}{1+q} + \frac{q t_{m-1}^{(q)}}{(1+q)^2} + \ldots + \frac{q^m t_0^{(q)}}{(1+q)^{m+1}}.$$

This can be treated as a transformation with matrix representation $\mathcal{M} = (d_{m,n})$, where

$$d_{m,n} = \begin{cases} \frac{q^{m-n}}{(1+q)^{m-n+1}} \binom{m+1}{n+1} & \text{if } n \le m, \\ \\ 0 & \text{otherwise.} \end{cases}$$

If this transformation is regular and $t_m^{(q)}$ converges to $A$, so does $s_m^{(q)}$, therefore it remains to prove properties (i)-(iii) of Theorem 3.3 for $d_{m,n}$. For (i) we have

$$\begin{aligned} \sum_{n=0}^{\infty} |d_{m,n}| &= \sum_{n=0}^{m} \frac{q^{m-n}}{(1+q)^{m-n+1}} = \frac{1}{1+q} \sum_{n=0}^{m} \left(\frac{q}{1+q}\right)^{m-n} = \frac{1}{1+q} \sum_{n=0}^{m} \left(\frac{q}{1+q}\right)^{n} \\ &= \frac{1}{1+q} \frac{1 - \left(\frac{q}{1+q}\right)^m}{1 - \frac{q}{1+q}} = 1 - \left(\frac{q}{1+q}\right)^m < 1 \end{aligned}$$

for all $m \in \mathbb{N}_0$, and also

$$\lim_{m\to\infty} \sum_{n=0}^{\infty} d_{m,n} = \lim_{m\to\infty} 1 - \left(\frac{q}{1+q}\right)^m = 1,$$

proving (iii). Lastly, (ii) is satisfied as well since

$$\lim_{m\to\infty} d_{m,n} = \lim_{m\to\infty} \frac{1}{1+q}\left(\frac{q}{1+q}\right)^{m-n} = 0,$$

which concludes the proof. □

As a consequence, $s_n \to A\ (E,q)$ is equivalent to $s_{n+1} \to A\ (E,q)$, so from (3.4) the following are also equivalent:

$$\begin{aligned} s_{m-1}^{(q)} &= \frac{1}{(1+q)^m}\sum_{n=1}^{m}\binom{m}{n} q^{m-n} s_{n-1} \to A \\ \iff & \frac{1}{(1+q)^m}\sum_{n=1}^{m}\binom{m}{n} q^{m-n} s_n \to A \\ \iff & \frac{1}{(1+q)^m}\sum_{n=0}^{m}\binom{m}{n} q^{m-n} s_n \to A, \end{aligned}$$

since the 0-th term $\left(\frac{q}{1+q}\right)^m s_0$ vanishes with $m \to \infty$. Owing to this equivalence we can modify the partial sums to a more symmetric formula

$$\hat{s}_m^{(q)} = \frac{1}{(1+q)^m}\sum_{n=0}^{m}\binom{m}{n} q^{m-n} s_n = \frac{1}{(1+q)^m}\sum_{n=0}^{m}\binom{m}{n} q^{m-n} U^n s_0 = \left(\frac{q+U}{1+q}\right)^m s_0, \tag{3.6}$$

while preserving the convergence, i.e.

$$\sum_{n=0}^{\infty} a_n = A\ (E,q) \qquad \iff \qquad \lim_{n\to\infty} \hat{s}_n^{(q)} = A. \tag{3.7}$$

## 3.3 Connection to Borel methods

In the previous section we have built up all the necessary tools needed to show that Borel methods are consistent with but stronger than Euler methods $(E,q)$, and can therefore be considered a limiting case of $(E,q)$ as $q \to \infty$. Apart from that we will introduce the necessary condition for a series to be $(E,q)$-summable and thus clarify why it is not applicable to the hypergeometric series (1) for any $q > 0$. A formal connection between summing (1) by both methods concludes the chapter.

**Theorem 3.13.** *If $\sum_{n=0}^{\infty} a_n$ is $(E,q)$-summable for some $q > 0$, then it is wB-summable (and therefore also B-summable) to the same number.*

*Proof.* Recall the Cauchy product of two power series:

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{m=0}^{\infty} b_m x^m\right) = \sum_{k=0}^{\infty} c_k x^k \quad \text{with} \quad c_k = \sum_{i=0}^{k} a_i b_{k-i}.$$

Taking $s_n$ the partial sums of $\sum_{n=0}^{\infty} a_n$, the following product of two sums can then be expressed as

$$\begin{aligned} e^{qx} \sum_{m=0}^{\infty} \frac{s_m x^m}{m!} &= \left(\sum_{n=0}^{\infty} \frac{(qx)^n}{n!}\right)\left(\sum_{m=0}^{\infty} \frac{s_m x^m}{m!}\right) = \sum_{k=0}^{\infty} x^k \sum_{i=0}^{k} \frac{q^i s_{k-i}}{i!\,(k-i)!} = \sum_{k-0}^{\infty} \frac{x^k}{k!} \sum_{i=0}^{k} \binom{k}{i} q^i s_{k-i} \\ &= \sum_{k-0}^{\infty} \frac{x^k}{k!} (1+q)^k \hat{s}_k^{(q)} \end{aligned}$$

with $\hat{s}_k^{(q)}$ as defined in (3.6). Then the weak Borel sum can be expressed as

$$\begin{aligned} \lim_{x\to\infty} e^{-x} \sum_{n=0}^{\infty} \frac{s_n x^n}{n!} &= \lim_{x\to\infty} e^{-x(1+q)} e^{qx} \sum_{n=0}^{\infty} \frac{s_n x^n}{n!} = \lim_{x\to\infty} e^{-x(1+q)} \sum_{n=0}^{\infty} \frac{\hat{s}_n^{(q)} [x(1+q)]^n}{n!} \\ &= \lim_{y\to\infty} e^{-y} \sum_{n=0}^{\infty} \frac{\hat{s}_n^{(q)} y^n}{n!}, \end{aligned}$$

with $y = x(1+q)$. If $\sum_{n=0}^{\infty} a_n = A\ (E,q)$, then by (3.7) $\hat{s}_n^{(q)} \to A$ and so by regularity of $wB$ the above limit is equal to $A$ as well, implying that the series is $wB$-summable (and by Theorem 1.5(i) also $B$-summable) to $A$. □

We have seen examples of series that are summable by a Borel method but not by Euler methods for any $q$, one of them being series (1). Moreover, methods $(E,q)$ increase in strength with increasing $q$, as stated in Corollary 3.10 and demonstrated on power series $\sum_{n=0}^{\infty} z^n$ in Example 3.6. While Borel methods retain total regularity and other properties essential to Euler methods, there is one property that is lost as a price for a stronger use - recall that while $(E,q)$ is consistent with rules (I)-(III) (Theorem 3.12) for all $q>0$, only part of rule (III) is satisfied by Borel methods (Corollary 1.7). Also note that Borel methods are not totally regular, but this might not be considered a bad property, since assigning a finite value to a series that diverges to infinity has useful applications in physics.

The following proposition describes the necessary condition for the terms of a series summable by $(E,q)$.

**Proposition 3.14.** *If a series $\sum_{n=0}^{\infty} a_n$ is $(E,q)$-summable for some $q>0$, then*

$$a_n = o((2q+1)^n).$$

*Proof.* If $\sum_{n=0}^{\infty} a_n = A\ (E,q)$, then $a_m^{(q)} \to 0$ and so $(1+q)a_m^{(q)} \to 0$ as $n\to\infty$, or alternatively put,

$$(1+q)a_m^{(q)} = o(1).$$

From the definition of $a_m^{(q)}$ then we get

$$(1+q)a_m^{(q)} = \frac{1}{(1+q)^m} \sum_{n=0}^{m} \binom{m}{n} q^{m-n} a_n = \frac{1}{(1+q)^m} \sum_{n=0}^{m} \binom{m}{n} q^{m-n} U^n a_0 = \frac{(q+U)^m a_0}{(1+q)^m} \to 0,$$

thus $(q+U)^m a_0 = o((1+q)^m)$. For the terms $a_n$ we can now derive the following estimate:

$$\begin{aligned} a_n &= U^n a_0 = (U-q+q)^n a_0 = \sum_{i=0}^{m} \binom{m}{i} (q+U)^{m-i} a_0 (-q)^i = o\left(\sum_{i=0}^{m} \binom{m}{i} (1+q)^{m-i} q^i\right) \\ &= o((2q+1)^n), \end{aligned}$$

as required. Example 3.6 shows that the series $\sum_{n=0}^{\infty} z^n$ is summable for $z$ such that $|z+q| < q+1$, which for real $z$ implies $-(2q+1) < z < 1$, showing that $a_n = o((2q+1)^n)$ is the lowest possible estimate. □

It is obvious from this proposition that WHS cannot be summed by Euler method $\mathcal{E}^k = (E, 2^k - 1)$ for any (finite) $k$. Nevertheless, by applying it enough times, a good approximation of $\delta$ can be obtained. Since WHS is $B^*$-summable, it implies that applying Euler transform an infinitely number of times should work. The following procedure shows (formally!) why it should be true.

Write WHS as follows:

$$\sum_{n=0}^{\infty}(-1)^n n! \;=\; \sum_{n=0}^{\infty}(-1)^n \int_0^{\infty} e^{-t}t^n \mathrm{d}t \;=\; \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}(-1)^n \int_{\alpha_p}^{\alpha_{p+1}} e^{-t}t^n \mathrm{d}t,$$

where $\alpha_0 = 0$ and $\alpha_p = 2^p + 2^{p-1} - 1$ for $p > 0$. Formally switch the two sums and consider the resulting series

$$\sum_{p=0}^{\infty}\sum_{n=0}^{\infty}(-1)^n a_{n,p} \qquad \text{with} \qquad a_{n,p} \;=\; \int_{\alpha_p}^{\alpha_{p+1}} e^{-t}t^n \mathrm{d}t. \tag{3.8}$$

**Proposition 3.15.** *For each $p \in \mathbb{N}_0$ the series $\mathbf{s_p} = \sum_{n=0}^{\infty}(-1)^n a_{n,p}$ is $(E, 2^{p+1}-1)$-summable and*

$$\sum_{n=0}^{\infty}(-1)^n a_{n,p} \;=\; \int_{\alpha_p}^{\alpha_{p+1}} \frac{e^{-t}}{1+t}\mathrm{d}t \;\; (E, 2^{p+1}-1).$$

*Remark* 12. Recall from Corollary 3.11 that $(E, 2^{p+1}-1)$ is $(E,1)$ applied $p+1$ times, so we can write

$$\mathcal{E}^{p+1}\mathbf{s_p} \;=\; \int_{\alpha_p}^{\alpha_{p+1}} \frac{e^{-t}}{1+t}\mathrm{d}t.$$

*Proof.* The $(2^{p+1}-1)$-th Euler transform of $\mathbf{s_p}$ is

$$\begin{aligned}
\mathcal{E}^{p+1}\mathbf{s_p} \;&=\; \sum_{m=0}^{\infty} \frac{1}{(2^{p+1})^{m+1}} \sum_{k=0}^{m}\binom{m}{k}\left(2^{p+1}-1\right)^{m-k}(-1)^k \int_{\alpha_p}^{\alpha_{p+1}} e^{-t}t^k \mathrm{d}t \\
&=\; \sum_{m=0}^{\infty} \frac{1}{(2^{p+1})^{m+1}} \int_{\alpha_p}^{\alpha_{p+1}} e^{-t} \sum_{k=0}^{m}\binom{m}{k}(-t)^k\left(2^{p+1}-1\right)^{m-k}\mathrm{d}t \\
&=\; \sum_{m=0}^{\infty} \frac{1}{(2^{p+1})^{m+1}} \int_{\alpha_p}^{\alpha_{p+1}} e^{-t}\left(2^{p+1}-1-t\right)^m \mathrm{d}t \\
&=\; \sum_{m=0}^{\infty} \int_{\alpha_p}^{\alpha_{p+1}} \frac{e^{-t}}{2^{p+1}}\left(\frac{2^{p+1}-1-t}{2^{p+1}}\right)^m \mathrm{d}t.
\end{aligned} \tag{3.9}$$

For a fixed $p \in \mathbb{N}_0$, $t \in (\alpha_p, \alpha_{p+1})$ implies that $(2^{p+1}-1-t) \in (-2^p, 2^{p-1})$, therefore

$$\int_{\alpha_p}^{\alpha_{p+1}} \left|\frac{e^{-t}}{2^{p+1}}\left(\frac{2^{p+1}-1-t}{2^{p+1}}\right)^m\right| \mathrm{d}t \;\le\; C_p M^m$$

where $C_p$ is a constant independent of $m$ and $M<1$. Since the series $\sum_{m=0}^{\infty} C_p M^m$ converges, we can interchange the sum an integral above to get

$$\mathcal{E}^{p+1}\mathbf{s_p} = \int_{\alpha_p}^{\alpha_{p+1}} \frac{e^{-t}}{2^{p+1}} \sum_{m=0}^{\infty} \left(\frac{2^{p+1}-1-t}{2^{p+1}}\right)^m \mathrm{d}t = \int_{\alpha_p}^{\alpha_{p+1}} \frac{e^{-t}}{2^{p+1}} \frac{1}{1-\frac{2^{p+1}-1-t}{2^{p+1}}} \mathrm{d}t = \int_{\alpha_p}^{\alpha_{p+1}} \frac{e^{-t}}{1+t} \mathrm{d}t,$$

which concludes the proof. □

As a consequence, the sum of the series (3.8) after applying Euler transform an infinite number of times is

$$\sum_{p=0}^{\infty} \mathcal{E}^{p+1}\mathbf{s_p} = \int_0^{\infty} \frac{e^{-t}}{1+t} \mathrm{d}t = \delta,$$

consistent with the $B^*$-sum of WHS.

We can also show that applying Euler transform only a finite number of times and taking only finitely many terms of the first few resulting series gives a good approximation of $\delta$.

**Proposition 3.16.** *The error after adding the first $N$ terms of the first $P$ transformed series $\mathbf{s_p}$ can be estimated as*

$$\delta - \sum_{p=0}^{P-1}\sum_{n=0}^{N-1} \mathcal{E}^{p+1}\mathbf{s_p}(n) = O\left(2^{-N}\right) + O\left(2^{-P} e^{-2^P}\right).$$

*Proof.* Fix a $p \in \mathbb{N}_0$. The missing terms of the transformed series are then (from (3.9))

$$\begin{aligned}\sum_{n=N}^{\infty} \mathcal{E}^{p+1}\mathbf{s_p}(n) &= \int_{\alpha_p}^{\alpha_{p+1}} \frac{e^{-t}}{2^{p+1}} \sum_{n=N}^{\infty} \left(\frac{2^{p+1}-1-t}{2^{p+1}}\right)^n \mathrm{d}t = \int_{\alpha_p}^{\alpha_{p+1}} \frac{e^{-t}}{2^{p+1}} \frac{\left(\frac{2^{p+1}-1-t}{2^{p+1}}\right)^N}{1+t} \mathrm{d}t \\ &< \int_{\alpha_p}^{\alpha_{p+1}} e^{-t} \left(\frac{2^{p+1}-1-t}{2^{p+1}}\right)^N \mathrm{d}t,\end{aligned}$$

Since both $2^{p+1}$ and $1+t$ are greater than 1. For $t \in (\alpha_p, \alpha_{p+1})$ the numerator $\left|2^{p+1}-1-t\right|$ is bounded by $2^p$, hence

$$\left|\sum_{n=N}^{\infty} \mathcal{E}^{p+1}\mathbf{s_p}(n)\right| \le \frac{1}{2^N} \int_{\alpha_p}^{\alpha_{p+1}} e^{-t}\mathrm{d}t = \frac{1}{2^N} e^{1-2^p} \underbrace{\left(e^{-2^{p-1}} - e^{-2^{p+1}}\right)}_{<1} = O\left(2^{-N} e^{-2^p}\right). \tag{3.10}$$

Furthermore, the Euler sums of $\mathbf{s_p}$ for $p>P$ that are not accounted for can be bounded as well (since $\frac{1}{1+t} \le \frac{1}{2^p}$ for $t \in (\alpha_p, \alpha_{p+1})$):

$$\left|\mathcal{E}^{p+1}\mathbf{s_p}\right| = \int_{\alpha_p}^{\alpha_{p+1}} \frac{e^{-t}}{1+t} \mathrm{d}t \le \frac{1}{2^p} \int_{\alpha_p}^{\alpha_{p+1}} e^{-t}\mathrm{d}t = O\left(2^{-p} e^{-2^p}\right). \tag{3.11}$$

Combining (3.10) and (3.11) the bound for the total error is

$$\begin{aligned}\delta - \sum_{p=0}^{P-1}\sum_{n=0}^{N-1} \mathcal{E}^{p+1}\mathbf{s_p}(n) &= \sum_{p=0}^{P-1}\sum_{n=N}^{\infty} \mathcal{E}^{p+1}\mathbf{s_p}(n) + \sum_{p=P}^{\infty} \mathcal{E}^{p+1}\mathbf{s_p} \\ &= \sum_{p=0}^{P-1} O\left(2^{-N} e^{-2^p}\right) + \sum_{p=P}^{\infty} O\left(2^{-p} e^{-2^p}\right) \\ &= O\left(2^{-N}\right) + O\left(2^{-P} e^{-2^P}\right),\end{aligned}$$

as desired. □

This result, although the procedure above is not exactly the same process as applying Euler transform repeatedly to WHS, explains why it can still approximate the sum using only finitely many terms of the $p$-th transform.

# Chapter 4

# Euler's second method: Extrapolation of a polynomial

In a sense, this method is perhaps the most interesting, since Euler's approach was not entirely justified and yet it yielded a sufficiently convincing result. He defined an infinite polynomial

$$P(n) \;=\; 1+(n-1)+(n-1)(n-2)+(n-1)(n-2)(n-3)+\dots$$

which has the property that formally

$$P(0) \;=\; 1-1!+2!-3!+4!\dots,$$

and therefore he tried to extrapolate it at $n=0$ to get an estimate on the value of $\delta$. We will first describe his approach and give evidence that it most likely does not work, then we will introduce a different method of extrapolation through Borel summation (introduced in Section 1.3), following an outline in Barbeau (1979). As a preparation we briefly introduce Newton's extrapolation formula and factorial series.

For any sequence $\{a_n\}_{n\in\mathbb{N}}$ the successor is given as $a_{n+1}=(1+\Delta)a_n$, hence the relation

$$a_n \;=\; (1+\Delta)^{n-1}a_1 \;=\; \sum_{k=0}^{n-1}\binom{n-1}{k}\Delta^k a_1$$

holds for any $n\in\mathbb{N}$, and so we can define a function for $z$ complex (where defined)

$$A(z) \;=\; a_1+(z-1)\Delta a_1+\frac{(z-1)(z-2)}{2!}\Delta^2 a_1+\frac{(z-1)(z-2)(z-3)}{3!}\Delta^3 a_1+\cdots \tag{4.1}$$

with the property $A(n)=a_n$ for $n\in\mathbb{N}$. This is called Newton's extrapolation formula and the series above is in the form of a factorial series of the second type, defined as follows:

$$S(z) \;=\; u_0+\frac{(z-1)}{1!}u_1+\frac{(z-1)(z-2)}{2!}u_2+\cdots \;=\; \sum_{n=0}^{\infty}\frac{\Gamma(z)}{\Gamma(z-n-1)n!}u_n$$

where $u_n$ are real or complex coefficients.

Such a series has some useful properties, provided it converges on some interval:

(i) it converges for $\operatorname{Re} z>\theta_0$ for some finite $\theta_0$, absolutely for $\operatorname{Re} z>\theta_1$ where $0\le\theta_1-\theta_0\le 1$ and uniformly to an analytic function on any compact subset of this domain;

(ii) if $A(z)$, $B(z)$ are factorial series extrapolating sequences $\{a_n\}$, $\{b_n\}$ respectively, then $cA(z)$, $A(z)+B(z)$, $A(z)B(z)$ (expanded in ascending factorial powers) extrapolate sequences $\{ca_n\}$, $\{a_n+b_n\}$, $\{a_nb_n\}$, respectively;

(iii) if $\Phi$ is a rational function and $A(z)$ extrapolates $\{a_n\}$, then $\Phi\circ A(z)$ extrapolates $\{\Phi(a_n)\}$. In particular, $\frac{1}{A(z)}$ extrapolates $\left\{\frac{1}{a_n}\right\}$.

Properties (ii) and (iii) naturally depend on domains of convergence specified in (i). A detailed theory for factorial series of the second type can be found in Nörlund (1926).

## 4.1 Euler's approach

Euler defined a sequence $P_n$ as

$$P_1 = 1 \quad \text{and} \quad P_{n+1} = nP_n + 1,$$

so that $\{P_1, P_2, P_3, \ldots\} = \{1, 2, 5, 16, 65, 326, \ldots\}$. This sequence is related to WHS owing to the following property:

**Lemma 4.1.** *For the sequence $\{P_n\}_{n\in\mathbb{N}}$ defined above we have for any $n \in \mathbb{N}_0$*

$$\Delta^n P_1 \; = \; n!\,.$$

*Proof.* For $n=0$ trivially $\Delta^0 P_1 = P_1 = 1 = 0!$. Assume that $\Delta^n P_1 \; = \; n!$. Using Lemma 3.1, the recurrent relation for $P_n$ and then the binomial identity $0 = (1-1)^{n+1} = \sum_{i=0}^{n+1}\binom{n+1}{i}(-1)^i$, we derive

$$\begin{aligned}
\Delta^{n+1}P_1 \; &= \; \sum_{i=0}^{n+1}(-1)^i\binom{n+1}{i}P_{n+2-i} \; = \; \sum_{i=0}^{n}(-1)^i\binom{n+1}{i}(n+1-i)P_{n+1-i} + \sum_{i=0}^{n+1}(-1)^i\binom{n+1}{i} \\
&= \; \sum_{i=0}^{n}(-1)^i\frac{(n+1)!\,(n+1-i)}{(n+1-i)!\,i!}P_{n+1-i} \; = \; (n+1)\sum_{i=0}^{n}(-1)^i\binom{n}{i}P_{n+1-i} \; = \; (n+1)\Delta^n P_1 \\
&= \; (n+1)!,
\end{aligned}$$

concluding the proof by induction. □

Hence by Newton's extrapolation formula a factorial series $P(z)$ extrapolating the sequence $\{P_n\}$ is given as

$$\begin{aligned}
P(z) \; &= \; P_1 + (z-1)1! + \frac{(z-1)(z-2)}{2!}2! + \frac{(z-1)(z-2)(z-3)}{3!}3! + \cdots \\
&= \; 1 + (z-1) + (z-1)(z-2) + (z-1)(z-2)(z-3) + \ldots,
\end{aligned}$$

with $P(0)$ formally equal to WHS. Unfortunately, it is convergent only for positive integer values, for which it truncates after finitely many terms, and divergent everywhere else, hence the properties (i)-(iii) might not apply.

Euler was interested in the extrapolated value at $z=0$ and tried to obtain it by implicitly using properties like the one described in (iii). His first attempt was taking a new sequence $\{a_n\}$ with $a_n = 1/P_n$ and computing the extrapolated term $a_0$ from (4.1) with $n=-1$:

$$a_0 \; = \; a_{-1+1} \; = \; a_1 - \Delta a_1 + \Delta^2 a_1 - \Delta^3 a_1 + \ldots . \tag{4.2}$$

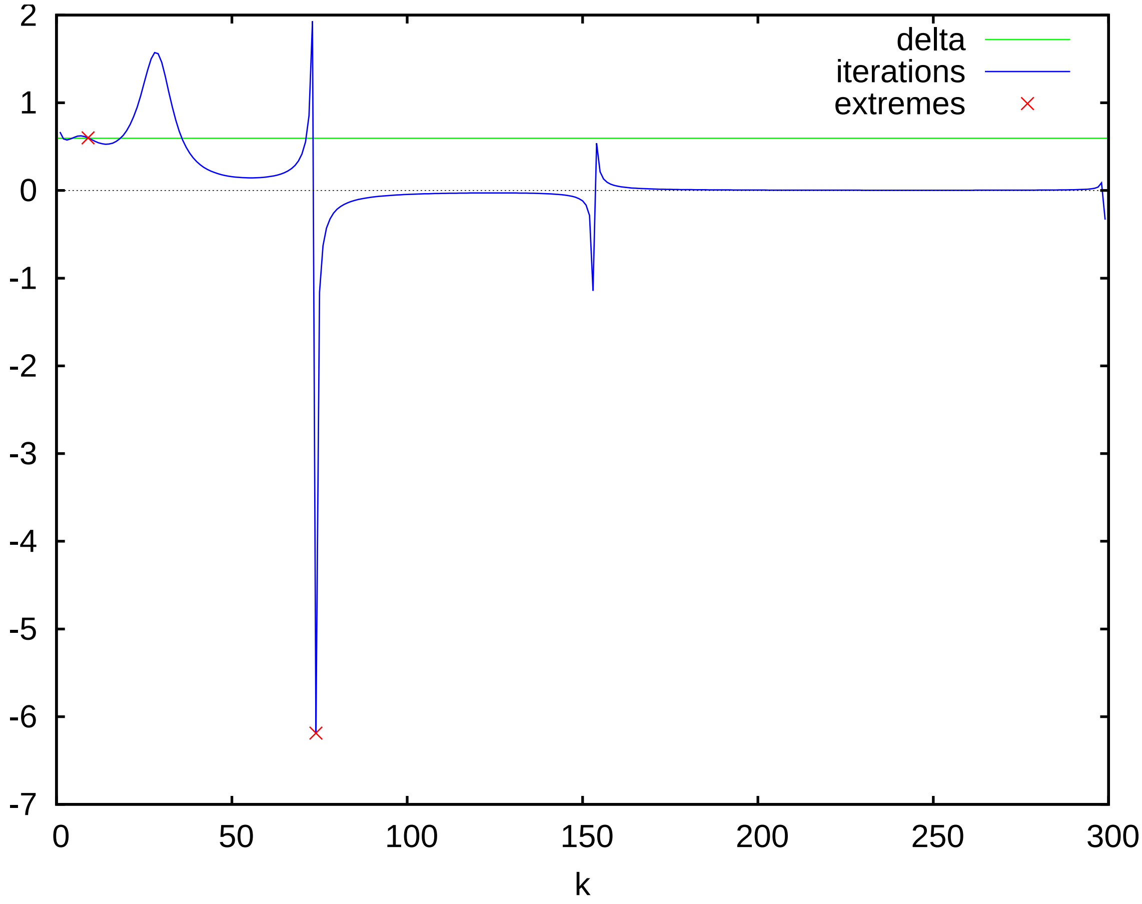


**Figure 4.1:** Iterations of $P_0(k)$ (on the vertical axis) for $k \in \{0, 1, ..., 300\}$ using Newton's extrapolation of the sequence $\{a_n\} = \{1/P_n\}$

After adding 6 terms, his estimate for $a_0$ was 1.65174, implying that the value of $P_0$ is $1/a_0 \approx 0.6$ (the exact value after adding 6 terms is $\frac{169520}{280003} \approx 0.605422$). However, computing more terms with the help of a computer (the source code of the Maxima script can be found in Appendix B, Example B.4) implies this was rather a lucky coincidence; if $k$ is the number of terms used in (4.2) yielding a result $a_0(k)$, and $P_0(k)$ is its inverted value, then the data for $k = \{1, 2, \ldots, 10\,000\}$ imply the following:

- $P_0(9)$ is the result closest to the actual value of $\delta$, with the difference $\approx 0.004$;
- $P_0(74)$ is the result that differs from $\delta$ the most, with the error $\approx 6.78$. These two extremes can be seen in Figure 4.1 where the first 300 iterations are plotted;
- the sequence $\{a_0(k)\}_{k=1}^{10\,000}$ forms blocks of numbers with the same sign which are similar in size within the same block. These blocks oscillate and grow steadily with small pertubations where the sign changes, implying that the sequence $\{a_0(k)\}$ does not converge but $P_0(k)\}$ converges to 0. Iterations 1000 to 10 000 can be seen in Figure 4.2.

Similarly, Euler tried to use $\{a_n\} = \{\log_{10} P_n\}$ and computed 6 terms to approximate $a_0$, with the result $a_0(6) \approx 1.7779089$ and thus $P_0(6) \approx 0.59966$ (the actual figure should be $P_0(6) \approx 0.586636$). Computing the first 1000 terms, we see that again, this is more of a coincidence. The term closest to $\delta$ is $P_0(7) \approx 0.59448758$ with the difference $\approx 0.00186$. After iteration 190

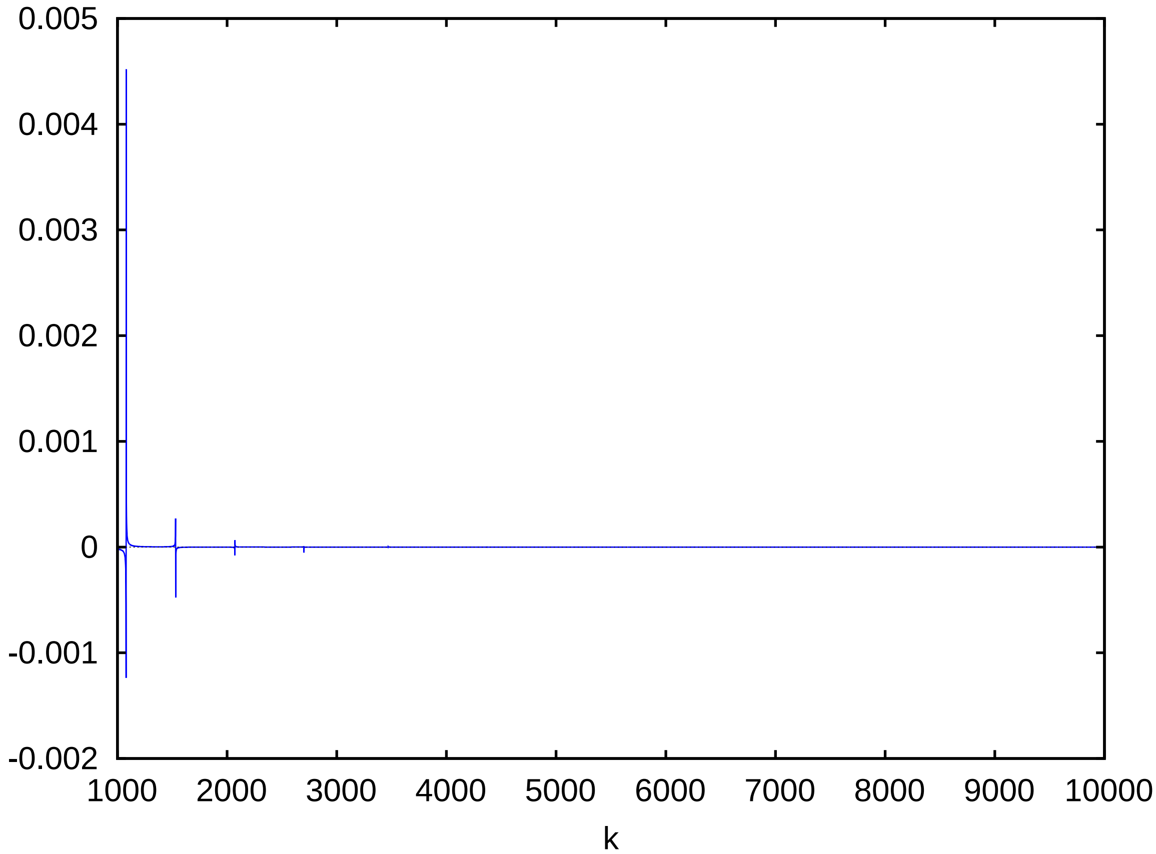


**Figure 4.2:** Iterations of $P_0(k)$ (on the vertical axis) for $k \in \{1000, 1001, ..., 10000\}$ using Newton's extrapolation of the sequence $\{a_n\} = \{1/P_n\}$

the terms start growing rapidly and are grouped in blocks of alternatively extremely small or extremely large numbers (magnitudes greater than $10^{10^{100}}$ or smaller than $10^{-10^{100}}$), indicating no convergence.

Iterations for $k$ from 1 to 100 can be seen in Figure 4.3 and iterations up to $k = 180$ in Figure 4.4.

## 4.2 Borel sum of $P(z)$

It is obvious that Euler's approach is not successful and another way of extrapolating the polynomial should be found, preferably an at least continuous function $h(z)$ with the properties $h(n) = P(n)$ for $n \in \mathbb{N}$ and $h(z+1) = zh(z)+1$ for all complex $z$ in some region containing the real positive line. It turns out that a suitable function is given by the Borel sum of the series $P(z)$: the Borel transform of $P(z)$ is

$$\mathcal{B}P(z)(t) \;=\; \sum_{n=0}^{\infty} \frac{(z-1)(z-2)\dots(z-n)t^n}{n!},$$

which, as can be easily verified, is the Taylor series of $g(t) = (1+t)^{z-1}$, convergent for $t \in (-1,1)$ for any $z \in \mathbb{C}$ and its analytic continuation to $t \geq 0$ is the same function. Then the Borel sum

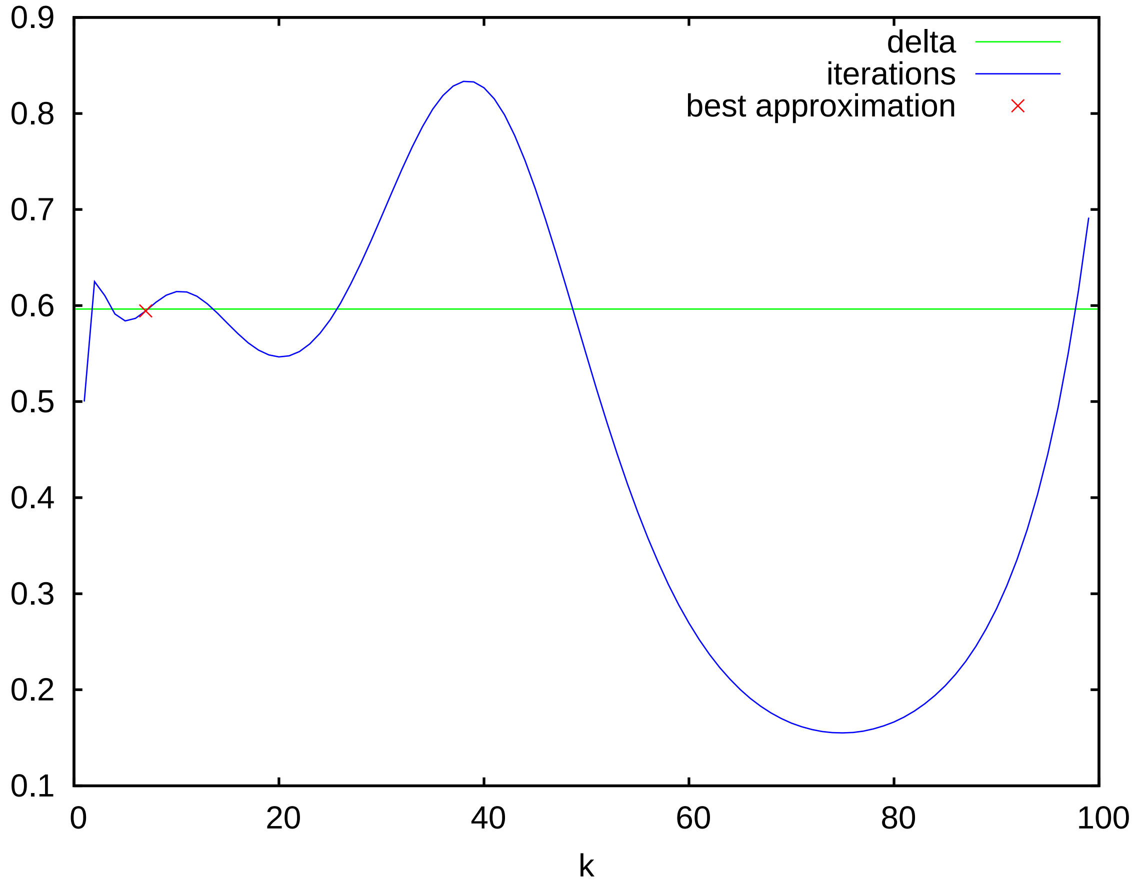


**Figure 4.3:** Iterations of $P_0(k)$ (on the vertical axis) for $k \in \{0,1,...,100\}$ using Newton's extrapolation of the sequence $\{a_n\} = \{\log_{10} P_n\}$

of $P(z)$ is given as

$$h(z) = \int_0^\infty e^{-t}(1+t)^{z-1}\mathrm{d}t,$$

convergent for all $z$ complex to an entire function. We can confirm that $h(z)$ is a suitable extrapolation of $P(z)$:

**Proposition 4.2.** *The function $h(z)$ defined as above has the following properties:*

*(i)* $h(z+1) = zh(z)+1$ *for all* $z \in \mathbb{C}$*;*

*(ii)* $h(n) = P(n)$ *for all* $n \in \mathbb{N}$*;*

*(iii) formal expansion of* $h(z)$ *yields the original series* $P(z)$*.*

*Proof.* Integration by parts shows

$$\begin{aligned} h(z+1) &= \int_0^\infty e^{-t}(1+t)^z\mathrm{d}t = \left[-e^{-t}(1+t)^z\right]_0^\infty + \int_0^\infty ze^{-t}(1+t)^{z-1}\mathrm{d}t = 1+z\int_0^\infty e^{-t}(1+t)^{z-1} \\ &= zh(z)+1, \end{aligned}$$

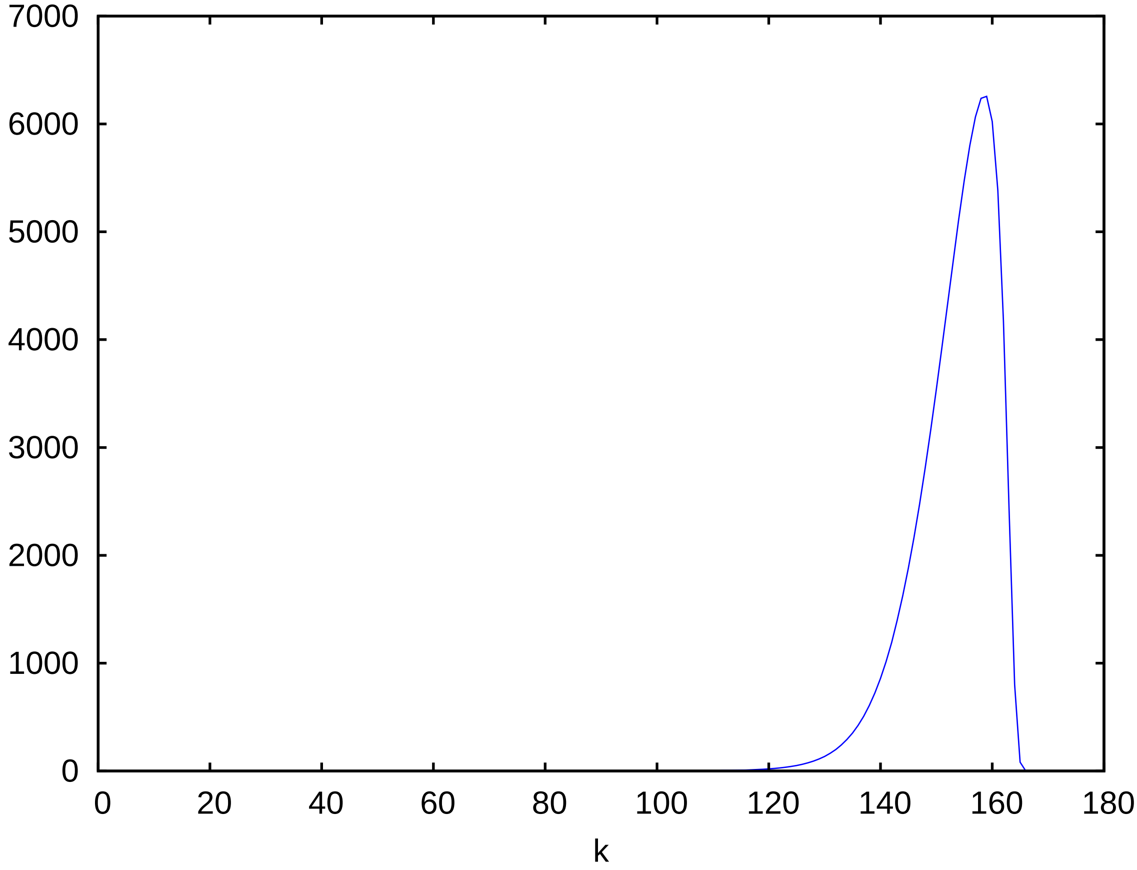


**Figure 4.4:** Iterations of $P_0(k)$ (on the vertical axis) for $k \in \{0,1,...,190\}$ using Newton's extrapolation of the sequence $\{a_n\} = \{\log_{10} P_n\}$

proving (i). Evaluating $h(z)$ at one we find that

$$h(1) = \int_0^\infty e^{-t} \mathrm{d}t = 1 = P(1),$$

so together with (i) this proves (ii). For (iii), we note that the process is simply reversing the Borel summation, which involves integrating term by term a series that is not convergent for $z \notin \mathbb{N}$. $\square$

Of most interest is the value of $h(z)$ at $z = 0$, as an extrapolated value of $P(0)$. Again the value is the same as the one acquired in the previous chapters, that is,

$$h(0) = f(1) = \int_0^\infty \frac{e^{-t}}{1+t} \mathrm{d}t = \delta$$

(with $f(z)$ as in (2)), establishing yet another connection between WHS and $\delta$, and also between the methods of summation used for series (1) so far.

A more sophisticated relation between the series and the function in terms of asymptotic series can be found as a consequence of the following lemma.

**Lemma 4.3.** *Define the incomplete Gamma function as*

$$\Gamma(z,x) = \int_x^\infty e^{-t} t^{z-1} \mathrm{d}t.$$

*Then for any fixed complex $z$, $e\Gamma(z,x)$ has an asymptotic expansion with gauge functions $\varphi_n(x) = x^{-(n+1-z)}$ at $x = \infty$:*

$$e\Gamma(z,x) \sim \sum_{n=0}^{\infty} \frac{e^{1-x}(z-1)(z-2)\dots(z-n)}{x^{n+1-z}} \qquad \text{as } x \to \infty.$$

*Proof.* The integral expansion for $e\Gamma(z,x)$ can be rewritten as

$$e\Gamma(z,x) = \int_{x-1}^\infty e^{-t}(1+t)^{z-1} \mathrm{d}t.$$

Integrating repeatedly by parts we derive the relation

$$\begin{aligned} e\Gamma(z,x) &= \left[-e^{-t}(1+t)^{z-1}\right]_{x-1}^{\infty} + \int_{x-1}^\infty e^{-t}(z-1)(1+t)^{z-2}\mathrm{d}t \\ &= e^{1-x}x^{z-1} + \left[-(z-1)e^{-t}(1+t)^{z-2}\right]_{x-1}^{\infty} + \int_{x-1}^\infty e^{-t}(z-1)(z-2)(1+t)^{z-2}\mathrm{d}t = \dots \\ &= e^{1-x}\sum_{k=0}^{n}\frac{(z-1)\dots(z-n)}{x^{n+1-z}} + (z-1)\dots(z-n-1)\int_{x-1}^\infty e^{-t}(1+t)^{z-n-2}\mathrm{d}t, \end{aligned}$$

thus the remainder $R_n(x)$ (with $n \geq z$) vanishes as $x \to \infty$:

$$\begin{aligned} \lim_{x\to\infty} |R_n(x)| &= \lim_{x\to\infty} \left|(z-1)\dots(z-n-1)\int_{x-1}^\infty e^{-t}(1+t)^{z-n-2}\mathrm{d}t\right| \\ &\leq \lim_{x\to\infty} C\int_{x-1}^\infty \frac{1}{(1+t)^{n+2-z}}\,\mathrm{d}t = \frac{C}{(n+1-z)x^{n+1-z}} = o(\varphi_n(x)) \end{aligned}$$

(with $C$ some positive constant), for any fixed $z \in \mathbb{C}$ and any $n \geq z$, concluding the proof. □

Notice that $h(z) = e\Gamma(z,1)$, so each value of $h(z)$ can roughly be estimated by the asymptotic expansion above with $x = 1$. In particular, for $z = 0$ we have the expansion

$$\int_{x-1}^\infty \frac{e^{-t}}{1+t}\,\mathrm{d}t \sim \sum_{n=0}^{\infty}\frac{e^{1-x}(-1)^n n!}{x^{n+1}} \qquad \text{as } x \to \infty,$$

where for $x = 1$ the right hand side yields WHS and the left hand side is equal to $\delta$, as expected.

# Chapter 5

# Euler's fourth method: Continued fraction

In his paper On Divergent Series (Euler (1760)) Euler first attempts to express series (1), i.e.

$$F(x) = \sum_{n=0}^{\infty} (-1)^n n!\,x^n = 0! - 1!\,x + 2!\,x^2 - 3!\,x^3 + 4!\,x^4 - 5!\,x^5 + \dots$$

with $x \geq 0$ as a continued fraction by setting it equal to a fraction $\frac{1}{1+A}$ and expressing $A$ formally by comparing the coefficients of powers of $x$. Then he sets $A = \frac{1}{1+B}$ and repeats the process with $B = \frac{x}{1+C}$, $C = \frac{x}{1+D}$ etc., revealing one by one the coefficients of the continued fraction and assuming the pattern continues.

He later derives a more general formula for a whole class of series (including (1)) using the same approach, shows that a related class of functions and solutions to a class of ODEs have the same continued fraction expansion and also computes the value of the continued fraction at $x = 1$ to a precision of 9 decimal places, improving his earlier estimate of $\delta$ resulting from repeated Euler's transformation.

Instead of this somewhat vague approach used by Euler, we will use the techniques described in Wall (1967), Chapter XVIII, but in terms more specific to our case and leaving out theorems with complicated proofs that would require a lot of additional theory for the sake of being more general. Therefore we restrict our attention to results specific to a class of series including (1). It will turn out that both the series (1) and the function $f(x)$ have the same convergent continued fraction expansion (in case of the series only formal, since it is divergent).

In Section 5.3, a proper summation method by continued fractions attributed to Stieltjes will be introduced that also leads to another continued fraction representation of $\delta$. Both representations are used to approximate $\delta$ and the results can be found in Table 5.1 and Table 5.2.

For a small introduction into notation and basics of continued fraction theory see Appendix A.

## 5.1 Continued fraction representation of (1)

We start by deriving the continued fraction representation for a class of divergent series including (1). First, define a class of power series as follows:

**Definition 18.** For any pair $a,b$ of real (or complex) numbers define a formal series

$$\Omega(a,b;x) := 1 - abx + \frac{a(a+1)b(b+1)}{2!}x^2 - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(a+n)\Gamma(b+n)}{\Gamma(a)\Gamma(b)} x^n \tag{5.1}$$

where $x$ is a real (or complex) variable.

Notice that for $a$ or $b$ equal to 0 or any negative integer the series becomes a polynomial. For any other pair of parameters $a,b$ the series is divergent everywhere except for $x=0$. Also the series is symmetric in $a,b$, i.e. for any pair $a,b \in \mathbb{C}$,

$$\Omega(a,b;x) = \Omega(b,a;x). \tag{5.2}$$

This will be an important tool to derive the continued fraction representation of the series. We start by deriving the following identities:

**Proposition 5.1.** *For any pair $a,b \in \mathbb{C}$ and all $n \in \mathbb{N}_0$ the identities*

$$\frac{\Omega(a+n,b+n+1;x)}{\Omega(a+n,b+n;x)} = \cfrac{1}{1+(a+n)x\cfrac{\Omega(a+n+1,b+n+1;x)}{\Omega(a+n,b+n+1;x)}} \tag{5.3}$$

*and*

$$\frac{\Omega(a+n+1,b+n+1;x)}{\Omega(a+n,b+n+1;x)} = \cfrac{1}{1+(b+n+1)x\cfrac{\Omega(a+n+1,b+n+2;x)}{\Omega(a+n+1,b+n+1;x)}} \tag{5.4}$$

*hold for all $x$ real (or complex) (provided that the denominators are not 0).*

*Proof.* For $n=0$ we have from the definition of $\Omega(a,b;x)$ the following power series identity (found by comparing the coefficients by each power of $x$):

$$\Omega(a,b+1;x) - \Omega(a,b;x) = -ax\Omega(a+1,b+1;x),$$

or, equivalently, provided that $\Omega(a,b;x) \neq 0$,

$$\begin{aligned}\frac{\Omega(a,b+1;x)}{\Omega(a,b;x)} &= \frac{\Omega(a,b;x) - ax\Omega(a+1,b+1;x)}{\Omega(a,b;x)} = \cfrac{1}{1+\cfrac{ax\Omega(a+1,b+1;x)}{\Omega(a,b;x)-ax\Omega(a+1,b+1;x)}} \\ &= \cfrac{1}{1+\cfrac{ax\Omega(a+1,b+1;x)}{\Omega(a,b+1;x)}},\end{aligned}$$

which is true for any $a,b \in \mathbb{C}$. Thus replacing $a$ with $a+n$ and $b+1$ with $b+n+1$ yields

$$\frac{\Omega(a+n,b+n+1;x)}{\Omega(a+n,b+n;x)} = \cfrac{1}{1+\cfrac{(a+n)x\Omega(a+n+1,b+n+1;x)}{\Omega(a+n,b+n+1;x)}},$$

proving identity (5.3) for all $n \in \mathbb{N}_0$. Replacing again $a+n$ with $a+n+1$ implies

$$\frac{\Omega(a+n+1,b+n+1;x)}{\Omega(a+n+1,b+n;x)} = \cfrac{1}{1+\cfrac{(a+n+1)x\Omega(a+n+2,b+n+1;x)}{\Omega(a+n+1,b+n+1;x)}}.$$

Thanks to the symmetry of $\Omega(a,b;x)$ (5.2), we can switch the parameters in the above equality and then rename them again ($b$ to $a$ and vice versa) to get

$$\frac{\Omega(a+n+1,b+n+1;x)}{\Omega(a+n,b+n+1;x)} = \cfrac{1}{1+\cfrac{(b+n+1)x\Omega(a+n+1,b+n+2;x)}{\Omega(a+n+1,b+n+1;x)}},$$

proving the second identity (5.4) for all $n \in \mathbb{N}_0$ and concluding the proof. □

Taking a closer look at the identities (5.3) and (5.4), we see that the last fraction in the RHS of each identity is in the from of the LHS of the other identity. Hence these, used alternatively, give rise to a formula unfolding the continued fraction expansion for $\frac{\Omega(a,b+1;x)}{\Omega(a,b;x)}$:

$$\frac{\Omega(a,b+1;x)}{\Omega(a,b;x)} = \cfrac{1}{1+\cfrac{ax\Omega(a+1,b+1;x)}{\Omega(a,b+1;x)}} = \cfrac{1}{1+\cfrac{ax}{1+\cfrac{(b+1)x\Omega(a+1,b+2;x)}{\Omega(a+1,b+1;x)}}} = \cdots,$$

so, formally (as the series is almost nowhere convergent),

$$\frac{\Omega(a,b+1;x)}{\Omega(a,b;x)} \leftrightarrow \frac{1}{1+}\,\frac{ax}{1+}\,\frac{(b+1)x}{1+}\,\frac{(a+1)x}{1+}\,\frac{(b+2)x}{1+}\cdots\frac{(a+n)x}{1+}\,\frac{(b+n+1)x}{1+}\cdots \tag{5.5}$$

for any $a,b \in \mathbb{C}$ and all $x$ real (or complex) such that $\Omega(a,b;x) \neq 0$. Notice that taking a negative integer for $a$ or $b$ or $a=0$ would produce a rational function on the LHS and the process would halt after finitely many steps, resulting in a finite continued fraction, equal to the LHS.

We will further inspect convergence properties of this continued fraction. First we state a theorem describing convergence in general terms for complex coefficients $a,b$ and variable $x$, referring to a proof in Wall (1967). For our more specific real case with some restrictions we will prove a slightly stronger result.

**Theorem 5.2. (Wall)** *Let $a,b$ be arbitrary complex constants. Let $G$ be any closed bounded region in $\mathbb{C}\setminus\{(-\infty,0)\}$. Then the continued fraction in (5.5) converges on $G$ except possibly at certain isolated points, and uniformly on the region obtained from $G$ by removing the interiors of small discs with centres at these points. The value of the continued fraction is an analytic function having these points as poles.*

The theorem with a proof can be found in Wall (1967) on page 351 (Theorem 92.2).

**Proposition 5.3.** *Let $a \geq 0$ and $b \geq -1$. Then the continued fraction in (5.5) converges uniformly on $[0,\infty)$ if $a \geq b+1$. For $a < b+1$ it converges uniformly on any closed bounded interval $[0,K]$.*

*Proof.* Cases $a=0$ and $b=-1$ are trivial, therefore we assume $a>0$ and $b>-1$. Notice for this choice of coefficient that for all $x>0$ the continued fraction has positive partial numerators and denominators, hence by Lemma A.6 its series representation will be an alternating series, for which convergence is easy to check and, if convergent, the limit is a positive number.

To start, we will transform the continued fraction by an equivalence transformation defined by sequence

$$\{c_n\}_{n\in\mathbb{N}} = \left\{1, \frac{1}{ax}, \frac{a}{b+1}, \frac{b+1}{a(a+1)x}, \frac{a(a+1)}{(b+1)(b+2)}, \dots\right\},$$

resulting in

$$\frac{1}{1+}\frac{ax}{1+}\frac{(b+1)x}{1+}\frac{(a+1)x}{1+}\frac{(b+2)x}{1+}\cdots = \frac{1}{1+}\frac{1}{\frac{1}{ax}+}\frac{1}{\frac{a}{b+1}+}\frac{1}{\frac{b+1}{a(a+1)x}+}\frac{1}{\frac{a(a+1)}{(b+1)(b+2)}+}\cdots$$

From this form it is easy to compute the series representation

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}a_1(x)a_2(x)\ldots a_n(x)}{B_n(x)B_{n-1}(x)}$$

of the continued fraction, since the partial numerators $a_n(x)$ are all equal to 1 for all $x$ and for $B_n(x)$ we utilize the recursive formula $B_n(x) = b_n(x)B_{n-1}(x) + a_n(x)B_{n-2}(x)$ with $B_{-1}(x) = 0$, $B_0(x) = 1$ and $b_n(x)$ the partial denominators given by

$$b_n(x) = \begin{cases} \frac{a(a+1)\ldots(a+k-1)}{(b+1)(b+2)\ldots(b+k)} & \text{if } n = 2k+1, \\ \\ \frac{(b+1)(b+2)\ldots(b+k-1)}{a(a+1)\ldots(a+k-1)x} & \text{if } n = 2k. \end{cases}$$

We will treat the two cases in the theorem separately.

First assume that $a \geq b+1$. Inspecting the coefficients $b_n(x)$ we see that $b_n(x) > 1$ for odd $n$ regardless of the choice of $x$. Computing the first few coefficients yields

$$\begin{array}{rclcll} B_1(x) & = & 1\times 1+0 & & & = 1, \\ B_2(x) & = & b_2(x)\times B_1(x)+B_0(x) & > & B_0(x) & = 1, \\ B_3(x) & = & b_3(x)\times B_2(x)+B_1(x) & > & 1\times 1+1 & = 2, \\ B_4(x) & = & b_4(x)\times B_3(x)+B_2(x) & > & B_2(x) & > 1, \\ B_5(x) & = & b_5(x)\times B_4(x)+B_3(x) & > & 1\times 1+2 & = 3, \\ B_6(x) & = & b_6(x)\times B_5(x)+B_4(x) & > & B_4(x) & > 1, \\ \vdots & & & & & \end{array}$$

implying that $B_n(x) > 1$ for $n$ even and $B_n(x) > \frac{n+1}{2}$ for $n$ odd, which is easily verifiable by induction on $n$. Hence the series representation of the continued fraction is given as an alternating series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{B_n(x)B_{n-1}(x)}$ with $B_n(x)B_{n-1}(x) > \frac{n}{2}$ for all $n \in \mathbb{N}$ and all $x$ positive, supplying the conditions for uniform convergence on $(0,\infty)$: given arbitrary $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that

$$\left|\sum_{n=N}^{\infty}\frac{(-1)^{n-1}}{B_n(x)B_{n-1}(x)}\right| \leq \frac{1}{B_N(x)B_{N-1}(x)} \leq \frac{2}{N} < \varepsilon$$

for all $x \in (0,\infty)$.

As for $x = 0$ the fraction is finite, so trivially convergent. In conclusion, the continued fraction converges uniformly on the union of the two sets, that is, on $[0,\infty)$.

For the second case where $a < b$ we can notice that $b_n(x) > \frac{1}{aK}$ for $n$ even and all $x \in (0,K]$.

Similarly then the coefficients $B_n(x)$ can be bounded:

$$\begin{aligned}
B_1(x) &= 1\times 1+0 && && = 1,\\
B_2(x) &= b_2(x)\times B_1(x)+B_0(x) &&> \frac{1}{aK}\times 1+1 && = 1+\frac{1}{aK},\\
B_3(x) &= b_3(x)\times B_2(x)+B_1(x) &&> B_1(x) && = 1,\\
B_4(x) &= b_4(x)\times B_3(x)+B_2(x) &&> \frac{1}{aK}\times 1+1+\frac{1}{aK} && = 1+\frac{2}{aK},\\
B_5(x) &= b_5(x)\times B_4(x)+B_3(x) &&> B_3(x) && > 1,\\
B_6(x) &= b_6(x)\times B_5(x)+B_4(x) &&> \frac{1}{aK}\times 1+1+\frac{2}{aK} && = 1+\frac{3}{aK},\\
&\vdots
\end{aligned}$$

Again, inductively we can prove that $B_n(x) > 1$ for $n$ odd and $B_n(x) > 1+\frac{n}{2aK}$ for $n$ even on $(0,K]$, which makes $B_n(x)B_{n-1}(x)$ unbounded and proves uniform convergence of the series on $(0,K]$ by the same argument as in the first case. As before, the convergence extends trivially to $[0,K]$. □

Setting $a=1$ and $b=0$ in (5.5) yields series (1) and its continued fraction expansion, which converges uniformly on $[0,\infty)$ by the previous proposition. In particular, for $x=1$ we can get an estimate for the series representation of the continued fraction of WHS, namely

$$\sum_{n=0}^{\infty}(-1)^n n! \leftrightarrow \frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{2}{1+}\frac{2}{1+}\frac{3}{1+}\cdots = \frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{\frac{1}{2}+}\frac{1}{1+}\frac{1}{\frac{1}{3}+}\cdots = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{B_nB_{n-1}} \tag{5.6}$$

with $B_n > n$ for $n$ odd and $B_n > 2n$ for $n$ even, thus $\frac{1}{B_nB_{n-1}} < \frac{1}{2n^2-2n}$, which gives us some bound on the error when approximating the sum, although in reality the convergence is faster. The results of computing the convergents of (5.6) in Maxima and a description of the method can be found at the end of Section 5.2 (Table 5.1).

Notice that by formally assigning power series (1) to its continued fraction representation and expressing the latter in a series form, we have linked the two alternating series, one divergent and the other convergent, and assigned the sum of the convergent one to its divergent counterpart. As such, this way we defined a summation method that can be applied to the class of divergent series $\Omega(a,1;x)$. This can be generalized, since $\Omega(a,b;x)$ is a variation of a bigger class of series called the Gauss hypergeometric series and defined as

$$F(a,b,c;z) = \sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+n)n!}z^n$$

for $a,b,c\in\mathbb{C}$, $z$ real or complex variable. Using the very same approach for finding the continued fraction representation of the quotient $\frac{F(a,b+1,c+1;z)}{F(a,b,c;z)}$ leads to a similar result, called the continued fraction of Gauss. Class $\Omega(a,b;x)$ can be derived from $F(a,b,c;z)$ by substituting $cx$ for $z$ and letting $c\to\infty$. For details we refer to Chapter XVIII of Wall (1967). In Section 5.3 we define Stieltjes summability, which includes the techniques used in this section and defines the method properly.

A fair question is about the regularity of this method. It is obviously linear, thus rules (I) and (II) are obeyed. Moreover, Theorem 89.1 of Wall (1967) states that the continued fraction representing the quotient $\frac{F(a,b+1,c+1;z)}{F(a,b,c;z)}$ defines its analytic continuation throughout the complex plane (excluding a cut along $(1,\infty)$ and possibly some isolated points) and converges to the quotient uniformly in some neighbourhood of the origin. Although this does not imply regularity, a fair number of convergent series is covered in this class with their sum equal to the value of their corresponding continued fraction.

## 5.2 Continued fraction expansion of $f(x)$

We will now prove that the same continued fraction that represents $\Omega(a,b;x)$ is also an expansion of a related function $f(a,b;x)$, defined as follows:

**Definition 19.** Let $a>0$ and $b\geq 0$, then we define for $x\geq 0$ a function

$$f(a,b;x) \;=\; \frac{1}{\Gamma(a)}\int_0^\infty \frac{e^{-w}w^{a-1}}{(1+xw)^b}\,\mathrm{d}w.$$

Notice that for any choice of $a,b$ the function is positive and well-defined, as the integral is finite for any $x\geq 0$. Moreover, $f(1,0;x)\equiv 1$.

As in the previous section, we will prove the following two identities:

**Proposition 5.4.** *For any pair $a>0$, $b\geq 0$ and any $n\in\mathbb{N}_0$ the identities*

$$\frac{f(a+n,b+n+1;x)}{f(a+n,b+n;x)} \;=\; \cfrac{1}{1+\cfrac{(a+n)xf(a+n+1,b+n+1;x)}{f(a+n,b+n+1;x)}} \tag{5.7}$$

*and*

$$\frac{f(a+n+1,b+n+1;x)}{f(a+n,b+n+1;x)} \;=\; \cfrac{1}{1+\cfrac{(b+n+1)xf(a+n+1,b+n+2;x)}{f(a+n+1,b+n+1;x)}} \tag{5.8}$$

*hold for all $x\geq 0$.*

*Proof.* For $x=0$ both equalities are trivial, so let $x>0$. Recall that $\Gamma(a+1)=a\Gamma(a)$ for any real positive number $a$. Then we have

$$\begin{aligned}
&f(a+n,b+n+1;x)+(a+n)xf(a+n+1,b+n+1;x)\\
&\qquad = \frac{1}{\Gamma(a+n)}\int_0^\infty \frac{e^{-w}w^{a+n-1}}{(1+xw)^{b+n+1}}\,\mathrm{d}w+\frac{a+n}{\Gamma(a+n+1)}\int_0^\infty \frac{e^{-w}w^{a+n}}{(1+xw)^{b+n+1}}\,\mathrm{d}w\\
&\qquad = \frac{1}{\Gamma(a+n)}\left[\int_0^\infty \frac{e^{-w}w^{a+n-1}}{(1+xw)^{b+n+1}}\,\mathrm{d}w+xw\int_0^\infty \frac{e^{-w}w^{a+n-1}}{(1+xw)^{b+n+1}}\,\mathrm{d}w\right]\\
&\qquad = \frac{1}{\Gamma(a+n)}\int_0^\infty \frac{e^{-w}w^{a+n-1}}{(1+xw)^{b+n}}\,\mathrm{d}w \;=\; f(a+n,b+n;x),
\end{aligned}$$

which, after a bit of rearranging, gives the first identity. For the second one we employ integration by parts to get

$$f(a+n+1,b+n+2;x) = \frac{1}{\Gamma(a+n+1)}\int_0^\infty \frac{e^{-w}w^{a+n}}{(1+xw)^{b+n+2}}\,\mathrm{d}w$$

$$= \frac{1}{\Gamma(a+n+1)}\left[-\int_0^\infty \frac{e^{-w}w^{a+n}}{(b+n+1)x(1+xw)^{b+n+1}}\,\mathrm{d}w + (a+n)\int_0^\infty \frac{e^{-w}w^{a+n-1}}{(b+n+1)x(1+xw)^{b+n+1}}\,\mathrm{d}w\right]$$

$$= \frac{-1}{(b+n+1)x}f(a+n+1,b+n+1;x) + \frac{1}{(b+n+1)x}f(a+n,b+n+1;x),$$

and again we rearrange to get the desired form. □

It is obvious that identities (5.3) and (5.4) are analogous with the new identities (5.7) and (5.8) respectively, and therefore generate the same continued fraction, representing both $\frac{\Omega(a,b+1;x)}{\Omega(a,b;x)}$ and $\frac{f(a,b+1;x)}{f(a,b;x)}$. We do know that the series defined by the first expression is divergent and therefore only equal to its continued fraction representation formally, however the second expression is well-defined and finite, prompting the question of its true equality to the continued fraction. This is indeed the case.

**Proposition 5.5.** *The continued fraction*

$$\frac{1}{1+}\,\frac{ax}{1+}\,\frac{(b+1)x}{1+}\,\frac{(a+1)x}{1+}\,\frac{(b+2)x}{1+}\cdots\frac{(a+n)x}{1+}\,\frac{(b+n+1)x}{1+}\cdots \tag{5.9}$$

*converges to the function $\frac{f(a,b+1;x)}{f(a,b;x)}$ uniformly on the regions defined in Proposition 5.3.*

*Proof.* Uniform convergence has already been proven in Proposition 5.3, it remains to show that the convergents approximate $\frac{f(a,b+1;x)}{f(a,b;x)}$. For $x = 0$ this is trivially true, so we assume $x > 0$.

Unfolding identities (5.7) and (5.8) alternatively $n$-times yields a finite continued fraction that is equal to $\frac{f(a,b+1;x)}{f(a,b;x)}$ and can be expressed using the recurrent relation for the convergents:

$$\frac{f(a,b+1;x)}{f(a,b;x)} = \frac{A_n(x)+k_{n+1}(x)A_{n-1}(x)}{B_n(x)+k_{n+1}(x)B_{n-1}(x)},$$

where $A_n(x), B_n(x)$ are the numerators and denominators of the convergents of (5.9) and $k_{n+1}(x)$ are the last terms in these finite continued fractions, given as

$$k_{n+1}(x) = \begin{cases} \frac{(a+n)xf(a+n+1,b+n+1;x)}{f(a+n,b+n+1;x)} & \text{for } n \text{ odd,} \\ \\ \frac{(b+n+1)xf(a+n+1,b+n+2;x)}{f(a+n+1,b+n+1;x)} & \text{for } n \text{ even.} \end{cases}$$

Since the convergents $\frac{A_n(x)}{B_n(x)}$ form an oscillating sequence, assume $\frac{A_n(x)}{B_n(x)} < \frac{A_{n-1}(x)}{B_{n-1}(x)}$, then, since all terms $A_n(x), B_n(x), k_n(x)$ are positive (for any $x$ on a corresponding region), we have

$$\frac{f(a,b+1;x)}{f(a,b;x)} - \frac{A_n(x)}{B_n(x)} = \frac{k_{n+1}(x)\left[A_{n-1}(x)B_n(x)-A_n(x)B_{n-1}(x)\right]}{B_n^2(x)+k_{n+1}(x)B_n(x)B_{n-1}(x)} > 0$$

and

$$\frac{f(a,b+1;x)}{f(a,b;x)} - \frac{A_{n-1}(x)}{B_{n-1}(x)} = \frac{A_n(x)B_{n-1}(x)-A_{n-1}(x)B_n(x)}{B_n(x)B_{n-1}(x)+k_{n+1}(x)B_{n-1}^2(x)} < 0$$

(or analogously for the case $\frac{A_n(x)}{B_n(x)} > \frac{A_{n-1}(x)}{B_{n-1}(x)}$), implying that $\frac{f(a,b+1;x)}{f(a,b;x)}$ always lies between two subsequent convergents of (5.9). This concludes the proof. □

For the particular choice $a=1$ and $b=0$ the continued fraction (5.9) converges uniformly to the function $f(x)$ from (2), since

$$\frac{f(1,1;x)}{f(1,0;x)} = \int_0^\infty \frac{e^{-w}}{(1+xw)}\,\mathrm{d}w = f(x),$$

strengthening yet again the connection between the hypergeometric series (1) and $f(x)$ and, in particular, their value at $x=1$. The corresponding continued fraction (5.6) at this point represents $\delta$:

$$\delta = \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{2}{1+}\,\frac{2}{1+}\,\frac{3}{1+}\,\frac{3}{1+}\,\frac{4}{1+}\,\frac{4}{1+}\,\frac{5}{1+}\,\frac{5}{1+}\,\frac{6}{1+}\cdots$$

See the table below for results of approximating $\delta$ by the convergents $A_n/B_n$ of (5.6). Number $n$ indicates the index of the convergent $A_n/B_n$ used, the second and fourth column indicate the number of correct digits. Computations were done in Maxima (the source code can be found

**Table 5.1:** Precision of approximating $\delta$ by convergents of (5.6)

| $n$ | # of digits | $n$ | # of digits |
|---:|---|---:|---|
| 5 | 0 | 15 000 | 148 |
| 10 | 2 | 20 000 | 172 |
| 50 | 7 | 25 000 | 192 |
| 100 | 10 | 30 000 | 211 |
| 500 | 24 | 35 000 | 227 |
| 1 000 | 37 | 40 000 | 243 |
| 5 000 | 85 | 45 000 | 258 |
| 10 000 | 120 | 50 000 | 272 |

in Appendix B, Example B.5) and the numerators $A_n$ and the denominators $B_n$ were taken to precision of 100 000 digits, resulting in small errors that did not influence the precision of the results.

$$\begin{aligned}\delta \approx \frac{A_{49755}}{B_{49755}} \approx 0.&\underline{5963473623}\ \underline{2319407434}\ \underline{1078499369}\ \underline{2793760741}\ \underline{7786015254}\ \underline{8781573484}\\ &\underline{9104823272}\ \underline{1911487441}\ \underline{7470430497}\ \underline{0936127603}\ \underline{4423703474}\ \underline{8428623689}\\ &\underline{8120782995}\ \underline{2905719661}\ \underline{7369222665}\ \underline{8940243185}\ \underline{1351436829}\ \underline{3763296254}\\ &\underline{7711879740}\ \underline{2524323020}\ \underline{5211788573}\ \underline{7856177283}\ \underline{6523651378}\ \underline{5594867425}\\ &\underline{3562181300}\ \underline{8120833784}\ \underline{2384485959}\ \underline{80}84,\end{aligned}$$

correct to 272 decimal places which are underlined.

## 5.3 Stieltjes continued fraction of $\delta$

Recall again the continued fraction expansion of $f(a,b;x)$ from the previous section with $a>0$ arbitrary and $b=1$:

$$f(a,1;x) = \frac{1}{1+}\,\frac{ax}{1+}\,\frac{1x}{1+}\,\frac{(a+1)x}{1+}\,\frac{2x}{1+}\,\frac{(a+2)x}{1+}\cdots,$$

valid for $x\in[0,\infty)$ and converging uniformly on this set. For $x\neq 0$ it is then possible to substitute $x=\frac{1}{z}$ and divide both sides by $z$ while maintaining the equality, defining a new function that will be denoted as $g(a;z)$ with the following expansion:

$$\begin{aligned} g(a;z) \;:=\; \frac{1}{\Gamma(a)}\int_0^\infty \frac{e^{-w}w^{a-1}}{z+w}\,\mathrm{d}w \;&=\; \frac{1}{z}\,\frac{1}{1+}\,\frac{a/z}{1+}\,\frac{1/z}{1+}\,\frac{(a+1)/z}{1+}\,\frac{2/z}{1+}\,\frac{(a+2)/z}{1+}\,\frac{3/z}{1+}\cdots \\ &=\; \frac{1}{z+}\,\frac{a}{1+}\,\frac{1/z}{1+}\,\frac{(a+1)/z}{1+}\,\frac{2/z}{1+}\,\frac{(a+2)/z}{1+}\,\frac{3/z}{1+}\cdots \\ &=\; \frac{1}{z+}\,\frac{a}{1+}\,\frac{1}{z+}\,\frac{a+1}{1+}\,\frac{2}{z+}\,\frac{a+2}{1+}\,\frac{3}{z+}\cdots \end{aligned} \tag{5.10}$$

for $z \in (0,\infty)$, the last line resulting from the equivalence transformation given by $\{c_n\}_{n\in\mathbb{N}} = \{1,1,z,1,z,1,z,1\ldots\}$. The corresponding power series is attained by the same procedure from $\Omega(a,1;x)$:

$$\Phi(a;z) \;:=\; \frac{1}{z}\Omega(a,1;1/z) \;=\; \sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(a+n)}{\Gamma(a)z^{n+1}} \;=\; \frac{1}{z} - \frac{a}{z^2} + \frac{a(a+1)}{z^3} - \frac{a(a+1)(a+2)}{z^4} + \cdots$$

As in the original continued fraction, for any $z>0$ the convergents of (5.10) will oscillate, so by taking only even convergents we create an increasing converging sequence. This is possible to do within the continued fraction itself:

**Proposition 5.6.** *Given two continued fractions*

$$\begin{aligned} &\frac{1}{z+}\,\frac{a}{1+}\,\frac{1}{z+}\,\frac{a+1}{1+}\,\frac{2}{z+}\,\frac{a+2}{1+}\,\frac{3}{z+}\cdots \\ \textit{and}\qquad &\frac{1}{z+a-}\,\frac{1a}{z+a+2-}\,\frac{2(a+1)}{z+a+4-}\,\frac{3(a+2)}{z+a+6-}\cdots \end{aligned} \tag{5.11}$$

*with convergents* $\frac{A_n(z)}{B_n(z)}$ *and* $\frac{C_n(z)}{D_n(z)}$ *respectively, for all* $z>0$ *it is true that* $\frac{A_{2n}(z)}{B_{2n}(z)} = \frac{C_n(z)}{D_n(z)}$ *for* $n\in\mathbb{N}$.

*Remark* 13. This means the two continued fractions converge to the same number for each $z>0$.

*Proof.* For $n=1$ it is easy to see that

$$\frac{A_2(z)}{B_2(z)} \;=\; \frac{1}{z+a} \;=\; \frac{C_1(z)}{D_1(z)}$$

and for $n=2$ similarly

$$\frac{A_4(z)}{B_4(z)} = \cfrac{1}{z+\cfrac{a}{1+\cfrac{1}{z+a+1}}} = \cfrac{1}{z+\cfrac{a(z+a+1)}{z+a+2}} = \cfrac{1}{z+a-\cfrac{1a}{z+a+2}} = \frac{C_2(z)}{D_2(z)}.$$

We will prove by induction that $A_{2n}(z) = C_n(z)$, the proof for $B_{2n}(z)$ and $D_n(z)$ is analogous. For better readability we omit the argument $z$ for the rest of the proof.

Assume $A_{2n} = C_n$ and $A_{2n-2} = C_{n-1}$. From the recurrent formulas we have $A_{2n+1} = zA_{2n} + nA_{2n-1}$, $A_{2n+2} = A_{2n+1} + (a+n)A_{2n}$ and $C_{n+1} = (z+a+2n)C_n - n(a+n-1)C_{n-1}$, hence using the assumption

$$
\begin{aligned}
C_{n+1} &= (z+a+2n)A_{2n} - n(a+n-1)A_{2n-2} \\
&= \underbrace{zA_{2n} + nA_{2n-1}} - nA_{2n-1} + (a+2n)A_{2n} - n(a+n-1)A_{2n-2} \\
&= \underbrace{A_{2n+1} \qquad + (a+n)A_{2n}}_{A_{2n+2}} + nA_{2n} \underbrace{- nA_{2n-1} - n(a+n-1)A_{2n-2}}_{-nA_{2n}} \\
&= A_{2n+2},
\end{aligned}
$$

as desired. □

Hence (5.11) is a continued fraction expansion for $g(a;z)$ and in particular for $a=1$, $x=1$ we have a new continued fraction representation of $\delta$:

$$\delta = g(1;1) = \int_0^\infty \frac{e^{-w}}{1+w}\,\mathrm{d}w = \frac{1}{2-}\,\frac{1^2}{4-}\,\frac{2^2}{6-}\,\frac{3^2}{8-}\,\frac{4^2}{10-}\,\frac{5^2}{12-}\cdots \tag{5.12}$$

This continued fraction was found by Stieltjes in 1895 and the approach can be generalized to a summability method using J-fractions. It is defined below. Approximation of $\delta$ by Stieltjes's continued fraction can be found at the end of this section in Table 5.2.

*Remark* 14. Stieltjes continued fraction (5.12) gives another sequence of rational approximations $\frac{C_n(1)}{D_n(1)}$ of $\delta$, converging twice as fast as those of (5.6). In Aptekarev (2009) the asymptotic behaviour of the coefficients $C_n(1)$, $D_n(1)$ is mentioned, namely

$$
\begin{aligned}
D_n(1) &= n!\,\frac{e^{2\sqrt{n}}}{\sqrt[4]{n}}\left(\frac{1}{2\sqrt{\pi e}} + O\left(n^{-1/2}\right)\right), \\
C_n(1) - \delta D_n(1) &= O\left(n!\,\frac{e^{-2\sqrt{n}}}{\sqrt[4]{n}}\right)
\end{aligned}
$$

as $n \to \infty$, which gives us the asymptotic bound for the approximations:

$$\frac{C_n(1)}{D_n(1)} - \delta = O\left(e^{-4\sqrt{n}}\right).$$

*Remark* 15. Another interesting result mentioned in Aptekarev (2009) stems from a new integral representation of $\delta$:

$$\delta = \int_0^\infty \frac{e^{-w}}{1+w}\,\mathrm{d}w = \int_1^\infty \frac{e^{1-v}}{v}\,\mathrm{d}v = e\int_1^\infty \frac{e^{-t}}{t}\,\mathrm{d}t = -e\mathrm{Ei}(-1), \tag{5.13}$$

where $\mathrm{Ei}(x)$ is the exponential integral defined for $x \in \mathbb{R}\setminus\{0\}$ as $\mathrm{Ei}(x) = -\int_{-x}^\infty \frac{e^{-t}}{t}\,\mathrm{d}t$ and has a series representation involving Euler-Mascheroni constant $\gamma$:

$$\mathrm{Ei}(x) = \gamma + \ln|x| + \sum_{n=1}^\infty \frac{x^n}{nn!}.$$

Together with (5.13) this leads to the following identity at $x=-1$:

$$\delta = -e\gamma - e\sum_{n=1}^\infty \frac{(-1)^n}{nn!}. \tag{5.14}$$

If $\delta$ and $\gamma$ were both rational, then $e$ and $\sum_{n=1}^{\infty} \frac{(-1)^n}{nn!}$ would solve a polynomial in two variables with rational coefficients, therefore they would be algebraically dependent. However, it was shown that these two numbers are algebraically independent (see Shidlovskii (1989), Chapter 7, Theorem 1) and so at least one of the constants $\delta$, $\gamma$ has to be irrational. Irrationality (and also transcendentality) of each of the constants separately is still an open problem.

**Definition 20.** A continued fraction fraction of the form

$$\frac{a_1}{b_1 + z-}\,\frac{a_2}{b_2 + z-}\,\frac{a_3}{b_3 + z-}\,\frac{a_4}{b_4 + z-}\cdots$$

with $a_n, b_n$ constants and $z$ a complex variable is called a *J-fraction*.

For every J-fraction there is a uniquely determined power series

$$P(1/z) \;=\; \sum_{i=0}^{\infty} \frac{c_i}{z^{i+1}} \;=\; \frac{c_0}{z} + \frac{c_1}{z^2} + \frac{c_2}{z^3} + \frac{c_3}{z^4}\cdots$$

such that it agrees term-by-term with the expansion of each convergent $\frac{A_n(z)}{B_n(z)}$ in powers of $1/z$ for the first $2n$ terms, i.e. if

$$\frac{A_n(z)}{B_n(z)} \;=\; \sum_{i=0}^{\infty} \frac{d_i^{(n)}}{z^{i+1}},$$

then $c_i = d_i^{(n)}$ for $i \in \{0, 1, 2, \ldots, 2n\}$. This is called the *equivalent power series of a J-fraction*. The exact algorithm to find the corresponding power series $P(1/z)$ to a J-fraction and vice versa can be found in §51 of Wall (1967). By this algorithm (5.11) is the J-fraction of the power series $\Phi(a;z)$.

The J-fraction might converge even if its corresponding power series is totally divergent (i.e. its radius of convergence is 0), furnishing a generalized sum of the divergent power series. This process of summing a divergent series by means of its J-fraction (provided the series has such representation) is called **Stieltjes summability** (see Wall (1967), Chapter XIX). Naturally we then ask about the properties of the function represented by the converging J-fraction.

Below we examine the properties of $g(a;z)$; from Section 5.2 we already know that $g(a;z)$ is the limit (or at least the point-wise limit) of the J-fraction (5.11) and that $\Phi(a;z)$ is the unique power series corresponding to J-fraction (5.11). It is expected that there is also a connection between $g(a;z)$ and $\Phi(a;z)$:

**Proposition 5.7.** *The series $\Phi(a;z)$ is the asymptotic series of $g(a;z)$ at $z = \infty$ for the sequence of gauge functions $\varphi_n(z) = 1/z^{n+1}$, i.e.*

$$\frac{1}{\Gamma(a)} \int_0^{\infty} \frac{e^{-w} w^{a-1}}{z + w}\,\mathrm{d}w \sim \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(a+n)}{\Gamma(a) z^{n+1}} \qquad as \quad z \to \infty.$$

*Proof.* According to Definition 14 we need to show that the remainder

$$R_n(z) \;=\; g(a;z) - \sum_{i=0}^{n} \frac{(-1)^i \Gamma(a+i)}{\Gamma(a) z^{i+1}} \;=\; o(\varphi_n(z))$$

as $z \to \infty$. Using the formula for the Gamma function $\Gamma(a) = \int_0^\infty e^{-w} w^{a-1} \mathrm{d}w$ for $a > 0$, we can expand $g(a;z)$ as follows:

$$\begin{aligned}\frac{1}{\Gamma(a)} \int_0^\infty \frac{e^{-w} w^{a-1}}{z+w} \mathrm{d}w &= \frac{1}{\Gamma(a)} \int_0^\infty e^{-w} \left( \frac{w^{a-1}}{z} - \frac{w^a}{z^2} + \cdots + \frac{(-1)^{n-1} w^{a+n-1}}{z^{n+1}} + \frac{(-1)^n w^{a+n}}{z^{n+2}\left(1+\frac{w}{z}\right)} \right) \mathrm{d}w \\ &= \frac{1}{z} - \frac{\Gamma(a+1)}{\Gamma(a) z^2} + \cdots + \frac{(-1)^{n-1}\Gamma(a+n)}{\Gamma(a) z^{n+1}} + \frac{(-1)^n}{\Gamma(a)} \int_0^\infty \frac{e^{-w} w^{a+n}}{z^{n+2}\left(1+\frac{w}{z}\right)},\end{aligned}$$

thus, since $z > 0$ and so $w > 0$ as well,

$$|R_n(z)| = \frac{1}{\Gamma(a)} \int_0^\infty \frac{e^{-w} w^{a+n}}{z^{n+2}\left(1+\frac{w}{z}\right)} \le \frac{1}{\Gamma(a) z^{n+2}} \int_0^\infty e^{-w} w^{a+n} \mathrm{d}w = \frac{\Gamma(a+n+1)}{\Gamma(a) z^{n+2}} = o(\varphi_n(z))$$

as $z \to \infty$, concluding the proof. □

In particular we have

$$g(1;z) = \int_0^\infty \frac{e^{-w}}{z+w} \mathrm{d}w \sim \sum_{n=0}^\infty \frac{(-1)^n n!}{z^{n+1}} \qquad \text{as } z \to \infty,$$

agreeing formally with the asymptotic series of $f(x)$ at $x = 0$ from the relation $f(x) = x g(1; 1/x)$. Moreover, as it was the case with $F(z)$ and $f(z)$, the series $\Phi(1;z)$ is the Borel sum of $g(1;z)$:

**Proposition 5.8.** *The series $\Phi(1;z)$ is Borel-summable for $z \in \mathbb{C} \setminus \{(-\infty, 0]\}$ and*

$$\Phi(1;z) = \sum_{n=0}^\infty \frac{(-1)^n n!}{z^{n+1}} = \int_0^\infty \frac{e^{-t}}{z+t} \mathrm{d}t \ (B^*) = g(1;z) \ (B^*).$$

*Proof.* The Borel transform of $\Phi(1;z)$

$$\mathcal{B}\Phi(1;z)(t) = \sum_{n=0}^\infty \frac{(-1)^n t^n}{z^{n+1}} = \frac{1}{z} \sum_{n=0}^\infty \left( \frac{-t}{z} \right)^n$$

converges for any $z \neq 0$ in the neighbourhood $|t| < |z|$ to an analytic function $\frac{1}{z} \frac{1}{1+\frac{t}{z}} = \frac{1}{z+t}$, which can be analytically extended to $t > 0$. Then the Borel sum is equal to

$$\int_0^\infty \frac{e^{-t}}{z+t} \mathrm{d}t,$$

which is convergent for all $z$ not real and negative or zero. □

In the following table approximations of $\delta$ are obtained from the convergents $A_n/B_n$ of the Stieltjes continued fraction (5.12), where $A_n, B_n$ were taken to a precision of 400 000 digits. The last computed approximation computed from the 25 000 000-th convergent was still not influenced by the rounding and is correct to 8 683 decimal places. The decimal expansion can be found in Appendix C.

**Table 5.2:** Precision of approximating $\delta$ by convergents of (5.12)

| $n$ | # of digits | $n$ | # of digits | $n$ | # of digits |
|---|---|---|---|---|---|
| 5 | 2 | 15 000 | 211 | 2 500 000 | 2 745 |
| 10 | 4 | 20 000 | 243 | 5 000 000 | 3 882 |
| 50 | 10 | 25 000 | 273 | 10 000 000 | 5 492 |
| 100 | 16 | 50 000 | 386 | 15 000 000 | 6 725 |
| 500 | 37 | 100 000 | 547 | 20 000 000 | 7 767 |
| 1 000 | 52 | 250 000 | 867 | 25 000 000 | 8 683 |
| 5 000 | 120 | 500 000 | 1226 | | |
| 10 000 | 172 | 1 000 000 | 1 735 | | |

*Remark* 16. Notice that the numbers of correct decimal places at $n$ in Table 5.2 correlate to those in Table 5.1 at $2n$. This is because the convergents of Stieltjes continued fraction (5.12) are the even convergents of the original continued fraction (5.6).

On the whole, this section describes another example of a well defined summation method that assigns the value $\delta$ to WHS and in general a well defined function $g(1;z)$ to its asymptotic series at infinity, that is formally related to the hypergeometric series (1) the same way as the function $g(1;z)$ is related to $f(x)$. Moreover, this summation method includes the approach used in the previous sections of this chapter to find the continued fraction expansion of $f(x)$ and the series (1), in the following sense:

Let $a_n(x)$ be the convergents of the formal continued fraction expansion of a series $A(x)$ obtained by the technique described in Section 5.1, that converges to a function $a(x)$. Then it is possible to convert this expansion to a J-fraction with convergents $b_n(y)$, a corresponding series $B(y)$ and the limit $b(x)$, such that the relation between $a(x)$ and $b(y)$ is the same as the relation between (possibly a subsequence of) the convergents $a_n(x)$ and convergents $b_n(y)$ and also the same as the formal relation between series $A(x)$ and $B(y)$. (More information can be found in Wall (1967), Chapter XIX.)

As a conclusion, the summation method described in Section 5.1 is also well defined and thus it is yet another example of a method that assigns the function $f(x)$ to the hypergeometric series (1).

# Conclusion

Throughout the thesis we introduced 4 different summation methods used by Euler and one newer method, all assigning the same number $\delta$ to Wallis' hypergeometric series $0! - 1! + 2! - 3! + 4! - 5! + \ldots$ in various forms.

## Relations between $f(z)$ and $F(z)$

A formally defined power series

$$F(z) = \sum_{n=0}^{\infty} (-1)^n n!\, z^n = 0! - 1!\, z + 2!\, z^2 - 3!\, z^3 + 4!\, z^4 - 5!\, z^5 + \ldots$$

that is totally divergent, and the function

$$f(z) = \int_0^{\infty} \frac{e^{-t}}{1+zt}\, \mathrm{d}t,$$

analytic for $z \in D = \mathbb{C} \setminus \mathbb{R}^-$, are connected in several ways:

- The Borel sum of $F(z)$ for $z \in D$ is $f(z)$. The method is totally regular and consistent with rules (I) and (II), and partially with rule (III).
- The asymptotic series of $f(z)$ in $D$ at $z = 0$ is $F(z)$. This has connection to the Borel sum of $F(z)$ - the function $f(z)$ behaves well enough (in accordance with conditions of Watson's recovery Theorem), so that it is equal to the Borel sum of its own asymptotic series $F(z)$. Since the same series represents infinitely many functions, such function can be considered the most natural choice for said asymptotic series.
- Although $F(1)$ is not $(E,q)$-summable for any finite $q > 0$, repeated Euler transform applied to $F(1)$ converges to the value of $f(z)$ at $z = 1$. The reason is that $(E,q)$ form a chain of increasingly stronger methods and the Borel methods are consistent with but stronger than all of them, and formally can be considered the limiting case of $(E,q)$ as $q \to \infty$.

  Repeated Euler transform is another totally regular method consistent with rules (I)-(III). It accelerates series $F(1)$, and thus decreases the error when estimating $f(1)$ by its asymptotic series $F(z)$.
- A linear ODE for $x \geq 0$ whose power series solution is $F(x)$ has a general solution $f(x)$. Both the series and the function vanish at $x = 0$, which is the initial condition of the ODE. Since the equation satisfies the properties of Main Asymptotic Existence Theorem, the formal power series solution $F(x)$ approximates the actual solution $f(x)$ in the asymptotic sense as $x \to 0$.

- A formal continued fraction expansion of $F(x)$ converges to $f(x)$ uniformly on $[0,\infty)$. This is a particular case of a bigger class of series with convergent continued fractions that furnish an analytic continuation of the corresponding series. The approach, although we have not proved it is regular, can be defined properly as a summation method attributed to Stieltjes.

# Relations between $g(1;z)$ and $\Phi(1;z)$

A series formally defined by substituting $\frac{1}{z}$ in $F(z)$ and dividing by $z$, i.e.

$$\Phi(1;z) = \frac{0!}{z} - \frac{1!}{z^2} + \frac{2!}{z^3} - \frac{3!}{z^4} + \frac{4!}{z^5} - \frac{5!}{z^5} + \cdots,$$

again totally divergent, and the function obtained from $f(z)$ in the same way,

$$g(1;z) = \int_0^\infty \frac{e^{-t}}{z+t}\,\mathrm{d}t,$$

analytic for $z$ not negative or zero, are also connected in similar ways (and their value at $z=1$ is again WHS and $\delta$, respectively):

- The Borel sum of $\Phi(1;z)$ for $z \in \mathbb{C} \setminus \{(-\infty,0]\}$ is $g(1;z)$.
- The asymptotic series of $g(1;z)$ at $z=\infty$ is $\Phi(1;z)$.
- The series $\Phi(1;z)$ is Stieltjes summable by means of J-fraction, to $g(1;z)$. The corresponding continued fraction is obtained from the continued fraction of $f(x)$ by the same relation as above (substituting $\frac{1}{z}$ and dividing by $z$), then transformed into an equivalent J-fraction.

# Relations between $h(z)$ and $P(z)$

An infinite polynomial defined as

$$P(z) = 1 + (z-1) + (z-1)(z-2) + (z-1)(z-2)(z-3) + \dots$$

convergent only for positive integers, and an entire function

$$h(z) = \int_0^\infty e^{-t}(1+t)^{z-1}\mathrm{d}t$$

correspond to WHS and $\delta$ respectively for $z=0$. Again, there are connections between the two:

- The Borel sum of $P(z)$ converging for all $z \in \mathbb{C}$ is $h(z)$. This function extrapolates $P(z)$, agreeing with the values at $z=n$, $n \in \mathbb{N}$ and preserving the recurrent relation of the terms in the sequence $\{P(n)\}_{n\in\mathbb{N}}$.
- $h(z)$ can be expressed as $h(z) = e\Gamma(z,1)$, where $e\Gamma(z,x)$ has an asymptotic expansion at $x=\infty$ related to series $P(z)$. In particular for $z=0$,

$$e\Gamma(0,x) = \int_{x-1}^\infty \frac{e^{-t}}{1+t}\,\mathrm{d}t \sim \sum_{n=0}^\infty \frac{e^{1-x}(-1)^n n!}{x^{n+1}} \qquad \text{as } x \to \infty,$$

which yields $\delta$ and WHS at $x=1$.

# Expressions representing $\delta$

All four methods used by Euler are therefore connected and even follow one from another, with their strongest links being the Borel summation method and asymptotic expansions. Through these methods we came to various functions and series representing delta in some way. To summarise, $\delta$ was expressed as

- the value of $f(1)$, $g(1;1)$ or $h(0)$ in the integral forms:

$$\delta = \int_0^\infty \frac{e^{-t}}{1+t}\,\mathrm{d}t = \int_0^1 \frac{e^{-\frac{1}{t}}}{t}\,\mathrm{d}t = \int_0^1 \frac{1}{1+\ln t}\,\mathrm{d}t = e\int_1^\infty \frac{e^{-t}}{t}\,\mathrm{d}t,$$

  where the last form involves the exponential integral $\mathrm{Ei}(t)$

- a convergent series representation of the third integral above obtained from the Taylor series of $\frac{1}{1+\ln v}$:

$$\delta = \sum_{n=0}^{\infty} \frac{(-1)^n c_n}{(n+1)!} \qquad \text{with } c_n = \frac{\mathrm{d}^n}{\mathrm{d}v^n} \frac{1}{1-\ln v}\bigg|_{v=1}$$

- a convergent sequence of Euler transforms of WHS, or equivalently, the limit

$$\delta = \lim_{q\to\infty} a_q, \qquad \text{where } a_q = \sum_{m=0}^{M_q} \frac{1}{(1+q)^{m+1}} \sum_{n=0}^{m} \binom{m}{n} q^{m-n}(-1)^n n!\,,$$

  i.e. the partial sum cut off after the smallest term with coefficient $M_q$

- a convergent continued fraction

$$\delta = \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{2}{1+}\,\frac{2}{1+}\,\frac{3}{1+}\,\frac{3}{1+}\,\frac{4}{1+}\,\frac{4}{1+}\,\frac{5}{1+}\,\frac{5}{1+}\,\frac{6}{1+}\cdots$$

- the sequence of convergents of the above continued fraction

$$\delta = \lim_{n\to\infty} \frac{A_n}{B_n} \qquad \text{where } A_0 = 0,\ A_1 = 1,\ A_n = A_{n-1} + \left\lfloor \frac{n}{2} \right\rfloor A_{n-2},$$
$$B_0 = 1,\ B_1 = 1,\ B_n = B_{n-1} + \left\lfloor \frac{n}{2} \right\rfloor B_{n-2},$$

  which are the partial sums of the series

$$\delta = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{B_n B_{n-1}}$$

- the Stieltjes continued fraction

$$\delta = \frac{1}{2-}\,\frac{1^2}{4-}\,\frac{2^2}{6-}\,\frac{3^2}{8-}\,\frac{4^2}{10-}\,\frac{5^2}{12-}\,\frac{6^2}{14-}\,\frac{7^2}{16-}\,\frac{8^2}{18-}\cdots$$

- the sequence of convergents of the Stieltjes continued fraction

$$\delta = \lim_{n\to\infty} \frac{A_n}{B_n} \qquad \text{where } A_0 = 0,\ A_1 = 1,\ A_n = 2^n A_{n-1} - (n-1)^2 A_{n-2},$$
$$B_0 = 1,\ B_1 = 2,\ B_n = 2^n B_{n-1} - (n-1)^2 B_{n-2},$$

which are the partial sums of the series

$$\delta = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \prod_{i=1}^{n} (i-1)^2}{B_n B_{n-1}}$$

All of the above expressions represent WHS in a well defined manner. In particular, each expression is either a Borel sum of a series at the particular value that represents WHS, or the particular value of a function with an asymptotic expansion that represents WHS at that value. Both of these concepts are well defined and consistent with the theory of convergent series. Asymptotic series in particular are an important tool for solving various differential equations. Using these tools, we have formed a well defined and strong link between Wallis' hypergeometric series and $\delta$.

In addition, we used some of the representations to compute the digits of $\delta$. The most useful for this purpose proved to be the Stieltjes continued fraction, whose convergents allowed us to approximate to a precision of 8 683 decimal places.

# Appendix A

# Continued Fractions

We will define simple and generalised continued fractions, some basic operations and theorems.

**Definition A.1.** A continued fraction is an expression of the form

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cfrac{a_4}{b_4 + \ddots}}}}$$

where $a_n, b_n$ are rational, real or complex numbers. The leading term $b_0$ is called the *integer part* of the continued fraction, $a_n$ are the *partial numerators* and $b_n$ $(n \in \mathbb{N})$ are the *partial denominators*.

As a compact form of a continued fraction we will use one of the following notations:

$$b_0 + \operatorname*{K}_{n=1}^{\infty} \frac{a_n}{b_n}, \qquad b_0 + \frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\,\frac{a_3}{b_3+}\,\frac{a_4}{b_4+}\cdots$$

**Definition A.2.** A *simple continued fraction* has $b_0 \in \mathbb{Z}$ and $a_n \in \{0,1\}$, $b_n \in \mathbb{N}$ $\forall n \in \mathbb{N}$.

Notice that if $a_n = 0$ for some $n$, the continued fraction is *finite*. This form in case of a simple continued fraction is very important as it characterises rational and irrational numbers in a way superior to that of fractions. The following theorem explains why.

**Theorem A.3.** *Every number $x \in \mathbb{R}$ has a unique simple continued fraction expansion which is finite if and only if $x$ is rational.*

*Proof.* If a simple continued fraction is finite, it is trivially a rational number since all the coefficients are integers. The other direction utilises the Euclidean algorithm of finding the greatest common divisor to find the unique simple continued fraction for any given rational number $x = \frac{p}{q}$:
First assume $\frac{p}{q}$ is positive and smaller than 1 (otherwise set $b_0 = \lfloor x \rfloor$, then continue with $\frac{p}{q} = x - b_0$). Since $p < q$, we can rewrite it as $\frac{1}{\frac{q}{p}}$. For $q > p$ there are unique $b_1 \in \mathbb{N}$, $r_1 \in \mathbb{N}_0$ s.t. $q = b_1 p + r_1$ with $r_1 < p$, or equivalently $\frac{q}{p} = b_1 + \frac{r_1}{p}$. If $r_1 \neq 0$, then $\frac{p}{q} = \frac{1}{b_1 + \frac{1}{\frac{p}{r_1}}}$ and the procedure can be repeated again with $r_1$ and $p$ replacing $p$ and $q$:

$$\text{find the unique } b_2 \in \mathbb{N},\ r_2 \in \mathbb{N}_0,\ r_2 < r_1 \text{ s.t. } p = b_2 r_1 + r_2, \text{ hence } \frac{p}{r_1} = b_2 + \frac{r_2}{r_1},$$

find the unique $b_3 \in \mathbb{N}$, $r_3 \in \mathbb{N}_0$, $r_3 < r_2$ s.t. $r_1 = b_3 r_2 + r_3$, hence $\dfrac{r_1}{r_2} = b_3 + \dfrac{r_3}{r_2}$,

$$\vdots$$

Since $r_1 > r_2 > r_3 > \dots$ are whole non-negative numbers, the algorithm will halt with $r_n = 0$ for some $n$. The resulting simple finite continued fraction is then

$$\frac{p}{q} = \frac{1}{b_1+}\,\frac{1}{b_2+}\cdots\frac{1}{b_{n-2}+}\,\frac{1}{b_{n-1}+\frac{r_n}{r_{n-1}}} = \frac{1}{b_1+}\,\frac{1}{b_2+}\cdots\frac{1}{b_{n-2}+}\,\frac{1}{b_{n-1}}.$$

For an irrational number $x$ the algorithm is analogous: $x$ can be expressed as its nearest smaller whole part and the remainder in a form of a reciprocal, i.e. $x = \lfloor x \rfloor + \frac{1}{y_1}$, $y_1 > 1$. Repeat with $y_1$, i.e. $y_1 = \lfloor y_1 \rfloor + \frac{1}{y_2}$, $y_2 > 1$ etc.

In case $x$ was rational, this is exactly the same as the Euclidian algorithm, so the process would halt after finitely many steps. For irrational $x$ the continued fraction in this form is unique since in each step the floor function and the remainder determines the coefficients uniquely: if $\lfloor y_n \rfloor$ is replaced by a smaller natural number for some $n$, the remainder $\frac{1}{y_{n+1}}$ will be smaller than 1 and so $b_{n+1} = 0$, which is undesirable. On the other hand, if $\lfloor y_n \rfloor$ is replaced by a greater number, the following remainder will be negative, which is again undesirable. Therefore the continued fraction representation in this form is unique for each $x \in \mathbb{R}$. □

The simple continued fraction representation of a real number is even more remarkable when we look at the fractions resulting from truncating the continued fraction after each step. These are called convergents (defined below) and they are the best rational approximations for the given number, that is: $\frac{p}{q}$ is the best rational approximation for $x$ if for any other rational number $\frac{r}{s}$ with $s \le q$ the distance $|x - \frac{r}{s}|$ is greater than $|x - \frac{p}{q}|$. This makes approximation of irrational numbers by their continued fractions very convenient.

We will not need this result as the continued fractions we use unfortunately are not simple. A proof can be easily found in many texts on continued fractions.

**Definition A.4.** The *convergents* $\{x_n\}_{n \in \mathbb{N}_0}$ of a continued fraction are the numbers resulting from truncating the continued fraction after $n$ steps:

$$x_n = b_0 + \mathop{\mathrm{K}}_{i=1}^{n} \frac{a_i}{b_i}.$$

Computing the convergents from this formula would be very time consuming. Fortunately it is possible to derive a recursive formula that simplifies the process. Let us express the first few convergents as simple fractions:

$$x_0 = b_0, \qquad x_1 = \frac{b_1 b_0 + a_1}{b_1}, \qquad x_2 = \frac{b_2(b_1 b_0 + a_1) + a_2 b_0}{b_2 b_1 + a_2}, \qquad \dots$$

The pattern is easy to spot and is proved as a following lemma.

**Lemma A.5.** *The convergents $\{x_n\}_{n \in \mathbb{N}_0}$ of a given continued fraction $b_0 + \mathop{\mathrm{K}}_{n=1}^{\infty} \frac{a_n}{b_n}$ can be computed recursively as follows:*

$$x_n = \frac{A_n}{B_n} \quad \textit{with} \quad \begin{aligned} A_{-1} &= 1, & A_0 &= b_0, & A_n &= b_n A_{n-1} + a_n A_{n-2}, \\ B_{-1} &= 0, & B_0 &= 1, & B_n &= b_n B_{n-1} + a_n B_{n-2} \quad \textit{for } n \in \mathbb{N}. \end{aligned}$$

*Proof.* Cases $n=0$ and $n=1$ are trivially true. Assume that for $\overline{x_n} = b_0 + \frac{a_1}{b_1+}\frac{a_2}{b_2+}\cdots\frac{a_{n-1}}{b_{n-1}+}\frac{\overline{a_n}}{\overline{b_n}}$ it is true that $\overline{x_n} = \frac{\overline{b_n}A_{n-1}+\overline{a_n}A_{n-2}}{\overline{b_n}B_{n-1}+\overline{a_n}B_{n-2}}$ for any choice of $\overline{a_n}$, $\overline{b_n}$, in particular, $x_n = \frac{b_nA_{n-1}+a_nA_{n-2}}{b_nB_{n-1}+a_nB_{n-2}} = \frac{A_n}{B_n}$. Then $x_{n+1}$ can be rewritten as follows:

$$x_{n+1} = b_0 + \frac{a_1}{b_1+}\frac{a_2}{b_2+}\cdots\frac{a_{n-1}}{b_{n-1}+}\frac{a_n}{b_n+}\frac{a_{n+1}}{b_{n+1}} = b_0 + \frac{a_1}{b_1+}\frac{a_2}{b_2+}\cdots\frac{a_{n-1}}{b_{n-1}+}\frac{b_{n+1}a_n}{b_{n+1}b_n+a_{n+1}}.$$

The last expression is in the form of $\overline{x_n}$ with coefficients $\overline{a_n} = b_{n+1}a_n$ and $\overline{b_n} = b_{n+1}b_n + a_{n+1}$, so by the assumption on $\overline{x_n}$ it can be expressed as

$$\begin{aligned} x_{n+1} &= \frac{(b_{n+1}b_n+a_{n+1})A_{n-1}+b_{n+1}a_nA_{n-2}}{(b_{n+1}b_n+a_{n+1})B_{n-1}+b_{n+1}a_nB_{n-2}} = \frac{b_{n+1}(b_nA_{n-1}+a_nA_{n-2})+a_{n+1}A_{n-1}}{b_{n+1}(b_nB_{n-1}+a_nB_{n-2})+a_{n+1}B_{n-1}} \\ &= \frac{b_{n+1}A_n+a_{n+1}A_{n-1}}{b_{n+1}B_n+a_{n+1}B_{n-1}} = \frac{A_{n+1}}{B_{n+1}}, \end{aligned}$$

which is the desired result for $n+1$. By induction the assertion is true for all $n \in \mathbb{N}_0$. $\square$

If a continued fraction converges to a (real or complex) number $x$, the sequence of convergents $\{x_n\}_{n\in\mathbb{N}_0}$ converges to $x$ and, in case the coefficients $a_n, b_n$ are all rational, is a sequence of rational approximations of $x$.

By computing the differences between consecutive convergents, it is also possible to express $x$ as a series:

**Lemma A.6.** *Given a continued fraction $b_0 + \mathrm{K}_{n=1}^{\infty}\frac{a_n}{b_n}$, the sequence of its convergents $\frac{A_n}{B_n}$ is the sequence of partial sums of the series $b_0 + \sum\limits_{n=1}^{\infty}\frac{(-1)^{n-1}a_1a_2\ldots a_n}{B_nB_{n-1}} = b_0 + \sum\limits_{n=1}^{\infty}\frac{(-1)^{n-1}\prod_{i=1}^{n}a_i}{B_nB_{n-1}}$.*

*Proof.* To show: $\frac{A_n}{B_n} - \frac{A_{n-1}}{B_{n-1}} = \frac{(-1)^{n-1}a_1a_2\ldots a_n}{B_nB_{n-1}}$ for all $n \in \mathbb{N}$. Since $\frac{A_n}{B_n} - \frac{A_{n-1}}{B_{n-1}} = \frac{A_nB_{n-1}-A_{n-1}B_n}{B_nB_{n-1}}$, it suffices to show the numerators are equal, which will be done by induction.

Case $n=1$: $A_1B_0 - A_0B_1 = (b_1b_0+a_1) - b_0b_1 = a_1 = (-1)^0a_1$, as desired. Now assume $A_nB_{n-1} - A_{n-1}B_n = (-1)^{n-1}\prod_{i=1}^{n}a_i$ for some $n \in \mathbb{N}$, then for $n+1$ we utilise the recursive formula for $A_{n+1}$, $B_{n+1}$:

$$\begin{aligned} A_{n+1}B_n - A_nB_{n+1} &= (b_{n+1}A_n + a_{n+1}A_{n-1})B_n - A_n(b_{n+1}B_n + a_{n+1}B_{n-1}) \\ &= -a_{n+1}(A_nB_{n-1} - A_{n-1}B_n) = -a_{n+1}(-1)^{n-1}\prod_{i=1}^{n}a_i = (-1)^n\prod_{i=1}^{n+1}a_i, \end{aligned}$$

which proves the assertion for $n+1$. By induction the formula holds for all $n \in \mathbb{N}$. $\square$

From this form a few things can be seen immediately: if the coefficients $a_n, b_n$ are positive, the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}a_1a_2\ldots a_n}{B_nB_{n-1}}$ is an **alternating series** and, in case it converges, the continued fraction converges to a positive number $x$. Moreover, the even convergents create an increasing sequence converging to $x$ from below and the odd convergents create a decreasing sequence converging to $x$ from above. The terms of the series also provide an estimate on the error when approximating $x$ by its convergents. This is always true and especially convenient in case of a simple continued fraction.

As an important tool for working with continued fractions let us introduce the equivalence transformation:

**Definition A.7. (Equivalence transformation):** Given any sequence $\{c_n\}_{n\in\mathbb{N}}$ of non-zero real (complex) numbers, define for a continued fraction $b_0 + \frac{a_1}{b_1+} \frac{a_2}{b_2+} \frac{a_3}{b_3+} \cdots \frac{a_n}{b_n+} \cdots$ its transformation as

$$b_0 + \frac{c_1 a_1}{c_1 b_1 +} \frac{c_1 c_2 a_2}{c_2 b_2 +} \frac{c_2 c_3 a_3}{c_3 b_3 +} \cdots \frac{c_{n-1} c_n a_n}{c_n b_n +} \cdots .$$

**Lemma A.8.** *The transformation defined above is an equivalence transformation, in the sense that all convergents of the transformed continued fraction are equal to the convergents of the original continued fraction.*

The proof is easily done by induction on $n$.

At last we introduce an interesting theorem stating sufficient conditions for irrationality of a convergent continued fraction. Unfortunately the theorem or any of its variations are not applicable to our case, however they can be used to prove irrationality of $\pi$, $e$ and related constants.

**Theorem A.9.** *If $a_n, b_n$ are positive integers and there is $N \in \mathbb{N}$ such that $a_k \leq b_k$ for all $k \geq N$, then the continued fraction $b_0 + \mathop{\mathrm{K}}_{n=1}^{\infty} \frac{a_n}{b_n}$ converges to an irrational number $x$.*

The proof is fairly simple and can be found together with some variations and corollaries in Angell (2007), Chapter 7.

# Appendix B

# Maxima scripts' source codes

**Example B.1. Midpoint method:** The script for applying the midpoint method on a given series until a given precision of the results is obtained. In the script, the series from Example 1.18 is used.

```
midptmethod():=(
/* asks for d,n,m and k, then echoes them back */
        d:read("enter # of decimal places"),
        n:read("enter # of elements for the computation"),
/*      m:read("enter # of iterations"), */ /* for an alternative cycle
   with a fixed number if iterations */
        k:read("enter precision as k in 10^(-k)"),
        print("d =",d," n =",n," k =",k),
        fpprec:d, /* sets precision of bigfloats */
        p:10^(-k),
        it:0,
        array(a,flonum,n), /* elements of the series */
        array(s,flonum,n), /* partial sums */
        a[0]:1,
        for i from 1 thru n do (a[i]:(-1)^i (2*i+1)^7),
        save("series.txt",a),
        s[0]:a[0],
        for i from 1 thru n do (s[i]:s[i-1]+a[i]),
        /* cycle of iterations conditioned by the given precision,
           returns the number of performed iterations */
        for j:1 while (abs(s[n]-s[n-1])>p) do (
/*      for j:1 thru m do ( */ /* alternative cycle with a fixed number
   of iterations m */
                for i from 1 thru n do (a[i]:(s[i-1]+s[i])/2),
                fillarray(s,a),
                save("nr_of_iters.txt",j)   ),
        save("midptseries.txt",s), /* saves the iterated series */
/*      listarray(s) */ /* output on screen, whole array */
        print("last result: ",bfloat(s[n])) /* output on screen, only the
            last result in given precision d */
)$
```

▲

**Example B.2. Computing decimals of $\delta$ from the series** (2.6)**:**

```
derivs():=(
        n:read("enter # of elements for the computation"),
        d:read("enter the decimal precision"),
        print("n= ",n,"d= ",d),
        fpprec:d,
        define(f(x),1/(1-log(x))),
        c[0]:f(1),
        s:c[0],
        for i from 1 thru n do (
                c[i]:bfloat(at(diff(f(x),x,i),x=1)), /* compute ith
                   derivative at 1 to a given precision d */
                s:s+((-1)^i*c[i]/(i+1)!)    ), /* add the ith term of the
                   Taylor series */
        save("derivs.txt",c),
        print("s(n)= ",s)
)$
```

▲

**Example B.3. Computing decimals of $\delta$ by repeated application of Euler transform:** The script computes a given number of terms of the series and applies Euler transform until the last term is also the smallest.

```
eulertransform():=(
/* asks for d,n and m, then echoes them back */
        d:read("enter # of decimal places"), /* precision of bigfloats */
        n:read("enter # of elements for the computation"),
/*      m:read("enter # of iterations"), */ /* for an alternative cycle
   with a fixed # of iterations */
        print("d =",d," n =",n),
        fpprec:d,
/*      p:10^(-k), */
        iterate:true,
        array(a,flonum,n),
        array(b,flonum,n),
        array(minterm,fixnum,n),
        s[0]:0,
        minterm[0]:1,
        for i from 0 thru n do (a[i]:(-1)^i * i!), /* inputs the original
            series */
/*      for k from 1 thru m do ( */ /* alternative cycle with a fixed #
   of iterations */
        for k:1 while iterate do ( /* makes the transform until it uses
           up the available elements */
                b[0]:a[0]/2,
                for i from 1 thru n do ( /* computes the values of the
                   transformed series b */
                        b[i]:a[0],
                        for j from 1 thru i do
                                b[i]:b[i]+((a[j]*i!)/((i-j)!*j!)),
                        b[i]:b[i]/2^(i+1)    ),
                if (abs(b[n])<=abs(b[n-1]) and iterate) then (iterate:
```

```
                    false), /* checks if the last elements is the smallest
                     and therefore no more iterations are needed */
                fillarray(a,b), /* rewrites the old series with the new
                    one so it can be used again in the next iteration */
                s[k]:bfloat(b[0]),
                for l:1 while (l<=n and abs(b[l])<=abs(b[l-1])) do ( /*
                    sums the transformed series until the smallest term,
                    in given precision d */
                        s[k]:bfloat(s[k]+b[l]),
                        minterm[k]:l /* saves index of the smallest term
                           after each iteration */
                        )   ),
        save("transform.txt",b),
        save("sums.txt",s),
        save("nr_elmnts_usd.txt",minterm),
        listarray(s) /* output on screen */
)$
```

▲

**Example B.4. Newton's extrapolation of the sequence $\{P_n\}_{n\in\mathbb{N}}$:** The script computes approximations of $P_0$ using extrapolation of $a_n = \log_{10} P_n$ to $a_0$. The last part of the script provides an output file with a plot of the results.

```
extrpol_log():=(
        /* asks for d,n, then echoes them back */
        d:read("enter # of decimal places"), /* sets the precision of
           bigfloats */
        n:read("enter # of elements for the computation"),
        print("d =",d," n =",n),
        fpprec:d,
        declare(delta,constant),
        delta:0.5963473623231940743410784993692793760741778601525 48),
        /* initial values and control values: */
        P:1,
        a[1]:bfloat(log(1)/log(10)),
        da1[0]:a[1],
        sum[0]:da1[0],
        s[0]:10^(sum[0]),
        dmin:1, kmin:0,
        dmax:0, kmax:0,
        prec:1, kprec:0,
        dprec:1, kdprec:0,
        err:0, kerr:0,
        for i from 2 thru n do (P:(i-1)*P+1, a[i]:bfloat(log(P)/log(10)))
            , /* computes sequence  a_i = log_10 P_i
             in given precision d */
        for i from 1 thru (n-1) do (
                for j from 1 thru (n-i) do (
                        b[j]:a[j+1]-a[j],
                        a[j]:b[j]           ), /* computes i-th differences
                           Δ^i a_j */
```

```
        da1[i]:a[1],                    /* Δ^i a_1 */
        sum[i]:sum[i-1]+(-1)^i*da1[i],       /* sum of the terms a_i
           from 0 to i, gives an approximation of log(P_0) */
        s[i]:bfloat(10^(sum[i])), /* approximation of P_0 */
        if (abs(da1[i])<dmin) then (dmin:abs(da1[i]), kmin:i), /*
           check for the smallest term added */
        if (abs(da1[i])>dmax and i>5) then (dmax:abs(da1[i]),
          kmax:i), /* check for the biggest term added */
        if (prec>abs(s[i]-s[i-1])) then (prec:abs(s[i]-s[i-1]),
          kprec:i), /* check for the most precise term until i-
          th sum (in case of comvergence) */
        if (dprec>abs(s[i]-delta)) then (dprec:abs(s[i]-delta),
          kdprec:i), /* check for the term closest to δ
           until i-th sum */
        if (err<abs(s[i]-s[i-1])) then (err:abs(s[i]-s[i-1]),
          kerr:i)    ), /*check for the biggest error from δ
           until i-th sum */
    with_stdout ("deltak_log.txt", for i:1 thru (n-1) do print (i,
      da1[i])),
    with_stdout ("sk_log.txt", for i:1 thru (n-1) do print (i, s[i]))
      ,
    with_stdout ("sumk_log.txt", for i:1 thru (n-1) do print (i, sum[
      i])),
    print(" min =",dmin," at index ",kmin," max=",dmax," at index ",
      kmax," precision=",prec," at index ",kprec," delta closest=",
      dprec," at index ",kdprec," error=",err," at index ",kerr,"sum
       =",s[n-1]),
    data: read_matrix ("sk_log.txt"),
    xy:[[kdprec,s[kdprec]],[kerr,s[kerr]]],
    plot2d([[discrete,[0,n],[delta,delta]],[discrete, transpose(data)
      [1],transpose(data)[2]],[discrete,xy]],[style,lines,lines,
      points],[color,green,blue,red],[point_type,times],[legend,"
      delta","results","extremes"],[xlabel,"number of terms used"],[
      ylabel,"result (s)"],[ps_file,"graph.ps"]) /* plots a graph */
)$
```

▲

**Example B.5. Computing decimals of $\delta$ from a continued fraction:** This particular script uses the Stieltjes continued fraction. First script starts from the first convergent and saves the coefficients of the last two computed convergents in output files. Those are used as input for the second script that continues from there and can be repeated in the same way as necessary.

```
contfr():=(
    /* asks for d,dest,n then echoes them back: */
    d:read("enter # of digits for coefficients"), /* the numerators
      and denominators of the convergents will be kept with this
      precision */
    dest:read("enter upper estimate on the precision of results"), /*
       the convergents will be computed with this precision */
    n:read("enter the total # of elements to be used"),
```

```
        print("d =",d,"dest = ",dest," n =",n),
        fpprec:d,
        a1:0, a2:1,
        b1:1, b2:2,
        s[0]:0, s[1]:1/2,
        for i from 2 thru n do (
                a3:bfloat(2*i*a2-a1*(i-1)^2), /* numerators of the
                    Stieltjes con. fraction */
                b3:bfloat(2*i*b2-b1*(i-1)^2), /* denominators -||- */
                fpprec:dest,
                s[i]:bfloat(a3/b3),
                fpprec:d,
                a1:a2, a2:a3,
                b1:b2, b2:b3,
                k:i,
                save("step_stielt.txt",k)    ),
        p:abs(s[n]-s[n-1]), /* precision of the last convergent */
        j:5,
        while (p<1) do (
                j:j+1,
                p:p*10    ),
        fpprec:j,
        with_stdout ("confr_stielt_results0.txt", for i:1 thru n do print
            (i, bfloat(s[i]))),
        save("num1_stielt.txt",a1),
        save("num2_stielt.txt",a2),
        save("denom1_stielt.txt",b1),
        save("/denom2_stielt.txt",b2),
        print("s =",bfloat(s[n])," precision =",j)
)$


contfr_cont():=(
        /* asks for d,dest,n then echoes them back: */
        d:read("enter # of digits for coefficients"), /* the numerators
           and denominators of the convergents will be kept with this
           precision */
        dest:read("enter the upper estimate on the precision of results")
           , /* the convergents will be computed with this precision */
        n:read("enter the total # of elements to be used"),
        print("d =",d,"dest = ",dest," n =",n),
        fpprec:d, /* sets precision of bigfloats */
        fail:false,
        /* loads coefficients of the last 2 convergents a1,b1,a2,b2 and
           the index k of the last one */
        load("num1_stielt.txt"),
        load("num2_stielt.txt"),
        load("denom1_stielt.txt"),
        load("denom2_stielt.txt"),
        load("step_stielt.txt"),
        for i from (k+1) thru n do (
```

```
                    a3:bfloat(2*i*a2-a1*(i-1)^2), /* numerators of the
                        Stieltjes con. fraction */
                    b3:bfloat(2*i*b2-b1*(i-1)^2), /* denominators -||- */
                    a1:a2, a2:a3,
                    b1:b2, b2:b3,
                    k:i,
                    if integerp(k/100) then save("step_stielt.txt",k),
                    if (integerp(k/100) and not fail) then ( /* check if
                        rounding the coefficients influences the results */
                            acut:bfloat(2*i*a3-a3*(i-1)^2), /* estimate on
                                rounding */
                            bcut:bfloat(2*i*b3-b3*(i-1)^2),
                            fpprec:dest,
                            p:abs(bfloat(a1/b1)-bfloat(a2/b2)), /* computes
                                precision of the last result */
                            scut:bfloat(acut/bcut),
                            er:abs(bfloat(a2/b2)-scut), /* computes the
                                difference between the last result and the
                                rounded last result */
                            if (er*(10^10)>p) then ( /* compares the two */
                                    fail:true,
                                    save("failat_stielt.txt",k),
                                    save("numfail_stielt.txt",a3),
                                    save("denomfail_stielt.txt",b3)    ),
                            fpprec:d    )    ),
            fpprec:dest,
            s:bfloat(a2/b2),
            p:abs(bfloat(a1/b1)-s), /* precision of the last result */
            j:5,
            while (p<1) do (
                    j:j+1,
                    p:p*10    ),
            fpprec:j,
            with_stdout("confr_stielt_results01.txt", print (n, bfloat(s))),
                /* rename after every new session */
            save("num1_stielt.txt",a1),
            save("num2_stielt.txt",a2),
            save("denom1_stielt.txt",b1),
            save("denom2_stielt.txt",b2),
            print("s =",bfloat(s)," precision =",j-8)
)$
```

▲

# Appendix C

# Decimal Expansion of $\delta$

Decimal expansion of delta obtained from the 25 000 000-th convergent of the Stieltjes continued fraction, correct to 8 683 decimal places (the incorrect digits are underlined). The small numbers at the end of each line indicate the number of digits listed so far.

$\delta \approx$ 0.5963473623 2319407434 1078499369 2793760741 7786015254 8781573484 9104823272 $_{70}$
1911487441 7470430497 0936127603 4423703474 8428623689 8120782995 2905719661 $_{140}$
7369222665 8940243185 1351436829 3763296254 7711879740 2524323020 5211788573 $_{210}$
7856177283 6523651378 5594867425 3562181300 8120833784 2384485959 8066698359 $_{280}$
3217826489 6860472310 9996451085 5581415383 5206162575 0083188741 8701758151 $_{350}$
8579310050 6116043552 9456710340 1503666363 5029755807 1419646592 0537060256 $_{420}$
3858754392 2397638393 2709618635 5595420814 1117245933 8654652495 5277108782 $_{490}$
9990958035 0929917916 2163896356 9135506973 1255489979 5693719307 1784387014 $_{560}$
6967280775 1781700499 1066054484 7225494624 4137072561 3792849019 7549983003 $_{630}$
7495298303 8426547682 4531113896 6510460616 0569870635 0683471618 9312449123 $_{700}$
0526414991 8184343827 7456488042 8194626569 1438208018 6774444601 7483136989 $_{770}$
5915267564 7833695487 1867400992 5960221310 7786153781 8589021632 2629566420 $_{840}$
7851298732 5163348487 5883402568 4438975074 7943861531 4792993932 8077843998 $_{910}$
8176958921 9826357740 6237721682 2805716991 6069633006 6837801738 2783396325 $_{980}$
4442620979 9414229337 3856284907 9664290058 4404467939 4103198964 1229605940 $_{1050}$
2188104681 3200793890 9285851600 9893724636 3442959532 5880103042 4171949405 $_{1120}$
9628245482 4845791789 3316493231 2014080888 4897296704 2503970136 4868161454 $_{1190}$
4244863208 9912951968 8768323599 1546987126 4650070157 8596184914 6474855159 $_{1260}$
7300740389 1357349352 4087701290 4840348891 1297313193 0354162042 9817394774 $_{1330}$
7522854785 3151009041 4721686651 6435383892 5806468199 6549154712 4910777086 $_{1400}$
9596251886 3064364330 8475384344 3776568445 6857595015 8071119575 8812339849 $_{1470}$
3684566043 5670823252 1556970704 7170587946 0986929923 9983515885 1763646468 $_{1540}$
4626732195 3042921057 4912631465 8757803341 5479274265 7682394695 9522147835 $_{1610}$
7682775355 8220257758 3297219115 5493205311 3617131688 5370672920 9200350672 $_{1680}$
4618108994 8021533146 2574019993 8906476942 4836888926 9551372630 3264341436 $_{1750}$
9373631123 6836867005 2138550537 2046761735 3145794883 2369049764 2944018711 $_{1820}$

2212427989 9942534175 6279416685 1447989648 2817985999 5269467279 3688814865 1890
5348171017 9361591503 4707793259 2921832770 6191285391 9701755059 0894948585 1960
4382490303 0218858807 8814277749 4203468132 0864965990 6462235396 0569276564 2030
8578076625 7918124411 7296126281 1837227238 1809811451 8608364338 8784014815 2100
2408582073 5771588554 8103115831 7045870758 0852370603 7026700738 8706559964 2170
5493589086 2965177931 1383546351 3282466680 6648762443 7400197621 4014436567 2240
8894673770 6291586690 5944363721 4055274253 3688765324 0520153727 9995394903 2310
4234736717 7111642130 0701668938 3413312914 3522989797 1633891280 7325202941 2380
0979782410 1475725699 0947516316 5742286788 4438684050 3336242993 1053732413 2450
1791669847 5200785900 6217158324 1113071797 9955671483 8789378797 2174930211 2520
2778002250 3270996144 7974296851 3727272482 5628939533 9580416242 3846270737 2590
0444043691 3475547699 9593394158 7031931047 4882201040 3461340993 1851583023 2660
3642364247 3156566511 4572633593 8714353777 4024981849 5745727876 3691496163 2730
2313940900 1071439528 0160135939 0970587860 2806115009 8760385727 8685574094 2800
2119688086 8181078365 2376108062 4395711236 8282149987 8271239485 1935368520 2870
1121801652 1607376178 8354169160 6672712916 2768878433 0612941901 1363629989 2940
4677446344 2646132959 6802631405 9497618435 1257473768 6229882899 6919429870 3010
6552509315 4308727709 0351147958 6932177993 6409225376 5616414593 8929657054 3080
0661981703 9514545740 9313628790 9292879192 6770013077 0866978175 8653251327 3150
8287128354 0209700702 2936574675 5878877776 2146010404 3226958557 4890140204 3220
9381630073 8192301521 5832356070 5193088075 3698575339 1071829764 3109144648 3290
5021876307 9810286918 0169094485 6080472522 7277485567 2921775189 2324210772 3360
4107655843 8606623977 0988222874 9293646009 6613258195 9712957886 8090043022 3430
5621140282 7492198642 5329362829 9484775280 7360682021 5825394680 9553107076 3500
2496687421 0801088812 7328032334 9096623899 3204248389 5768720442 8416481618 3570
6444817230 8180237359 2613761732 1137798896 9651156543 2883519380 1024483349 3640
6600422046 4182473189 0045066729 6069855356 4926820495 0756874810 2155989784 3710
4995223268 5428554527 3257648335 5868387722 3728041533 1602800970 9031027241 3780
3887845093 7413496323 1672453899 4891096490 4529255057 3760361233 6926694218 3850
0470305086 3748479510 9938736908 8373329438 6143941569 6919418257 4585707022 3920
3288019094 1956949150 1645177458 2883857379 4557065100 9413988918 4524244095 3990
2328349471 3606768112 8615660538 2880483052 4535669931 9567558548 5212911551 4060
7108258580 7001008155 4386421114 5261960608 4905894639 0276337137 7225934523 4130
1001206580 9145133430 9639018523 2188769766 4771156525 5911251226 6755813718 4200
0828050177 1168537702 1754044634 8194465555 9324672032 8533825315 9090846153 4270
0410935602 1872594830 6518737197 5511988380 1553709784 0372480323 7367994244 4340
0685194440 7810349703 7884504251 8153220796 2660028141 2943495656 1819035867 4410
8000604887 4596777107 8923323243 5746562718 6410546804 9241467465 3235183500 4480
2776955393 0816469226 9671402900 4530635211 3480060519 0854064733 7429513415 4550
5881111001 4015779919 4375528077 5441386894 5103890029 2042635502 1610762885 4620

4353589467 8231453892 0302777155 2766201963 6705612171 2780199472 5507788386 4690
2228729896 6763252833 3772144833 5093175909 3294812417 9348200903 0031105559 4760
9224129569 1256147843 8881601701 6681973442 9910827310 3141072557 7463594960 4830
6853411472 5131113696 9764619894 2696549121 5163892426 8952121045 5276781036 4900
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