Fractional Differential Calculus
Volume 3, Number 1 (2013), 21–29 doi:10.7153/fdc-03-02

ANALYTICAL SOLUTION FOR THE GENERALIZED TIME–FRACTIONAL TELEGRAPH EQUATION

V. B. L. CHAURASIA AND RAVI SHANKER DUBEY

Abstract. We discuss and derive the analytical solution for the generalized time-fractional telegraph equation. These problems are solved by taking the Laplace and Fourier transforms in variable t and x respectively. Here we use Green function also to derive the solution of the given differential equation.

1. Introduction

Fractional differential equations have attracted in the recent years a considerable interest due to their frequent appearance in various fields and their more accurate models of systems under consideration provided by fractional derivatives. Application of fractional derivatives have been used successfully to model frequency dependent damping behaviour of many viscoelastic materials, modeling of many chemical processed, mathematical biology and many other problems in engineering. The history and a comprehensive treatment of fractional differential equations are provided by Podlubny [1] and a review of some applications of fractional differential equations are given by Mainardi [2].

The fractional telegraph equation have been considered by many authors, namely Cascaval, Eckstein, Frota and Goldstein [3], Orsingher and Beghin [4], Chen, Liu and Anh [5], Orsingher and Zhao [6], Camargo, Chiacchio and Oliveira [8], Momani [7], Mainardi [9, 11]. Many author have been discussed the time-fractional telegraph equations, dealing with well-posedness and presenting a study involving asymptotic by using the Riemann-Liouville approach. The time fractional telegraph equation with Brownian time,was studied by Orsingher and Beghin [4]. The solution of the time fractional telegraph equation with three kinds of non-homogeneous boundary conditions using the separating variables method was studied and obtained by Chan, Liu and Anh [5]. Orsingher and Zhao [6] considered the space-fractional telegraph equations, obtaining the Fourier transform of its fundamental solution and presenting a symmetric process with discontinuous trajectories, whose transition function satisfies the space-fractional telegraph equation.The analytic and approximate solutions of the space and time fractional telegraph differential equations by means of the so called Adomian decomposition method, discussed by Momani [7], Camargo et al. [8] discussed the so-called

Mathematics subject classification (2010): Primary 26A33, 33C20, 33E12; secondary 47B38, 47G10.
Keywords and phrases: Green function, M function (of the Wright type), Mittag-Leffler function, Laplace transform, Fourier transform method, Fox H-function.

Paper FDC-03-02

general space-time fractional telegraph equations by the methods of differential and integral calculus, discussing the solution by means of the Laplace and Fourier transforms in variables t and x, respectively.

An attempt has been made to study the following generalized time-fractional telegraphic equation

$$a_1 D_t^{\alpha} v(x,t) + a_2 D_t^{2\alpha} v(x,t) + \ldots + a_n D_t^{n\alpha} v(x,t) = d \frac{\partial^2 v(x,t)}{\partial x^2} + f(x,t), \quad t \in R^+, \tag{1.1}$$

where $a_1, a_2, \ldots, a_n$ are positive constants, $1/n < \alpha \leqslant 1$, D_t^{β} is the fractional derivative defined in the Caputo sense:

$$D_t^{\beta} f(t) = \begin{cases} \frac{d^n f(t)}{dt^n}, & \beta = n \in N, \\ \frac{1}{\Gamma(n-\beta)} \int\limits_0^t (t-\tau)^{n-\beta-1} \frac{d^n f(\tau)}{d\tau^n} d\tau, & n-1 < \beta < n, \end{cases} \tag{1.2}$$

where $f(t)$ is a continuous function. Properties and more details about the Caputo's fractional derivative can also be found in [1, 2].

The Laplace transform of this derivative is given in [14] in the form;

$$L\{D_t^{\alpha} f(x,t); s\} = s^{\alpha} F(x,s) - \sum_{r=0}^{m-1} s^{\alpha-r-1} f^{(r)}(x,0), \quad (m-1 < \alpha \leqslant m). \tag{1.3}$$

The above formula is useful in deriving the solution of differential and integral equations of fractional order governing certain physical problems.

By the definition of Fourier transform

$$F\left\{ \frac{\partial^{\alpha}}{\partial x^{\alpha}} f(x,t) \right\}(k) = (-ik)^{\alpha} F[f(x,t)](k). \tag{1.4}$$

For the generalized TFTE (1.1), we will consider three basic problems with the following three kinds of initial and boundary conditions, respectively.

PROBLEM 1. Generalized TFTE in a whole-space domain (Cauchy problem)

$$v(x,0) = \phi(x), \quad \frac{\partial^n v(x,0)}{\partial t^n} = 0, \quad \forall n \in [1,n], \quad x \in R, \tag{1.5}$$

$$v(\mp\infty, t) = 0, \quad t > 0.$$

PROBLEM 2. Generalized TFTE in a half-space domain (Signaling problem)

$$v(x,0) = \frac{\partial^n}{\partial t^n} v(x,0) = 0, \quad \forall n \in [1,n], \quad x \in R^+, \tag{1.6}$$

$$v(0,t) = \xi(t), \quad v(+\infty,t) = 0, \quad t > 0, \tag{1.7}$$

and we set $f(x,t) = 0$ in (1.1).

PROBLEM 3. TFTE in a bounded-space domain

$$v(x,0) = \phi(x), \quad \frac{\partial}{\partial t} v(x,0) = \Psi(x), \quad 0 < x \leqslant M, \tag{1.8}$$

$$v(0,t) = v(M,t) = 0, \quad t > 0, \tag{1.9}$$

here we also set $f(x,t) = 0$ in (1.1).

In this paper, we derive the analytical solutions of the previous three problems for the generalized TFTE. The structure of the paper is as follows. In Section 2, by using the method of Laplace and Fourier transforms, the fundamental solution of Problem 1 is derived. In section 3, by investigating the explicit relationships of the Laplace Transforms to the Green functions between Problem 1 and 2 the fundamental solution of the Problem 2 is also derived. To solve the Cauchy problem we use Laplace Fourier transform which is defined in equation (1.3, 1.4). Similarly we can solve Problem 3 in the same manner.

$E_{\alpha,\beta}(z)$ is Mittag-Leffler function of two parameters is defined below:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad \alpha, \beta > 0 \quad \text{and} \quad E_{\alpha,1} = E_{\alpha}. \tag{1.10}$$

2. The Cauchy problem for the generalized TFTE

We first focus our attention on (1.1) in a whole-space domain, that is to say, Problem 1 will to be considered, which we refer to as the so-called Cauchy problem.

First of all we solve (1.1) by taking Laplace transform with the help of (1.3) and using boundary conditions.

$$a_1 s^{\alpha} \tilde{v}(x,s) + a_2 s^{2\alpha} \tilde{v}(x,s) + \ldots + a_n s^{n\alpha} \tilde{v}(x,s) - \left(a_1 s^{\alpha-1} + a_2 s^{2\alpha-1} + \ldots + a_n s^{n\alpha-1}\right) \phi(x) = d \frac{\partial^2 \tilde{v}(x,s)}{\partial x^2} + \tilde{f}(x,s), \tag{2.1}$$

on taking Fourier transform of equation (2.1) with the help of (1.4),

$$\sum_{i=1}^{n} a_i s^{i\alpha} \tilde{\tilde{v}}(k,s) - s^{\alpha-1}\left(a_1 + a_2 s^{\alpha} + \ldots + a_n s^{(n-1)}\alpha\right) \tilde{\phi}(k) = -k^2 d \tilde{\tilde{v}}(k,s) + \tilde{\tilde{f}}(k,s). \tag{2.2}$$

Then we get

$$\begin{aligned} \tilde{\tilde{v}}(k,s) &= \frac{\left(a_1 s^{\alpha-1} + a_2 s^{2\alpha-1} + \ldots + a_n s^{n\alpha-1}\right)}{\sum_{i=1}^{n} a_i s^{i\alpha} + k^2 d} \tilde{\phi}(k) + \frac{\tilde{\tilde{f}}(k,s)}{\sum_{i=1}^{n} a_i s^{i\alpha} + k^2 d} \\ &:= \tilde{\tilde{G}}_1(k,s)\tilde{\phi}(k) + \tilde{\tilde{G}}_2(k,s)\tilde{\tilde{f}}(k,s), \end{aligned} \tag{2.3}$$

where

$$\tilde{\tilde{G}}_1(k,s) = \tilde{\tilde{G}}_{1,1}(k,s) + \tilde{\tilde{G}}_{1,2}(k,s) + \ldots + \tilde{\tilde{G}}_{1,n}(k,s), \tag{2.4}$$

$$\tilde{\tilde{G}}_{1,1} = \frac{a_1 s^{\alpha-1}}{\sum_{i=1}^{n} a_i s^{i\alpha} + k^2 d}, \; \tilde{\tilde{G}}_{1,2} = \frac{a_2 s^{2\alpha-1}}{\sum_{i=1}^{n} a_i s^{i\alpha} + k^2 d}, \; \ldots, \; \tilde{\tilde{G}}_{1,n} = \frac{a_n s^{n\alpha-1}}{\sum_{i=1}^{n} a_i s^{i\alpha} + k^2 d}, \tag{2.5}$$

and

$$\tilde{\tilde{G}}_2 = \frac{1}{\sum\limits_{i=1}^{n} a_i s^{i\alpha} + k^2 d}. \tag{2.6}$$

By the Fourier transform pair

$$\left[e^{-c|x|}\right] \overset{F}{\longleftrightarrow} \frac{2c}{c^2 + k^2}, \tag{2.7}$$

we also have

$$\left.\begin{aligned}
&(2.8.1) \qquad \tilde{G}_{1,1}(x,s) = \frac{a_1 s^{\alpha-1}}{2\sqrt{d\left(\sum\limits_{i=1}^{n} a_i s^{i\alpha}\right)}} e^{-\sqrt{\sum\limits_{i=1}^{n} \frac{a_i s^{i\alpha}}{d}}|x|}, \\
&(2.8.2) \qquad \tilde{G}_{1,2}(x,s) = \frac{a_2 s^{2\alpha-1}}{2\sqrt{d\left(\sum\limits_{i=1}^{n} a_i s^{i\alpha}\right)}} e^{-\sqrt{\sum\limits_{i=1}^{n} \frac{a_i s^{i\alpha}}{d}}|x|}, \\
&\qquad\qquad \vdots \\
&(2.8.n) \qquad \tilde{G}_{1,n}(x,s) = \frac{a_n s^{n\alpha-1}}{2\sqrt{d\left(\sum\limits_{i=1}^{n} a_i s^{i\alpha}\right)}} e^{-\sqrt{\sum\limits_{i=1}^{n} \frac{a_i s^{i\alpha}}{d}}|x|}.
\end{aligned}\right\} \tag{2.8}$$

On solving (2.3) we get

$$v(x,t) = \int\limits_{-\infty}^{+\infty} G_1(x-y,t)\,\phi(y)dy + \int\limits_{-\infty}^{+\infty} dy \int\limits_{0}^{t} G_2(x-y,t-\tau)\, f(y,\tau)d\tau, \tag{2.9}$$

where $G_1(x,t)$, $G_2(x,t)$ is the corresponding Green function or fundamental solution obtained when $\phi(x) = \delta(x)$, $f(x) = 0$ and $\phi(x) = 0$, $f(x,t) = \delta(x)\delta(t)$ respectively, which is characterized by (2.5) or (2.6).

To express the Green function, we recall two Laplace transform pairs and one Fourier transform pair,

$$\begin{aligned}
&F_1^{(\beta)}(ct) := t^{-\beta} M_\beta\left(ct^{-\beta}\right) \overset{L}{\longleftrightarrow} s^{\beta-1} e^{-cs^\beta}, \\
&F_2^{(\beta)}(ct) := c w_\beta(ct) \overset{L}{\longleftrightarrow} e^{-(s/c)^\beta}, \\
&F_3^{(\beta)}(cx) := \tfrac{1}{2\sqrt{\pi}} c^{-1/2} e^{-x^2/4c} \overset{F}{\longleftrightarrow} e^{-ck^2},
\end{aligned} \tag{2.10}$$

where M_β denotes the so-called M function (of the Wright type) of order β, which is defined

$$M_\beta(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma[-\beta n + (1-\beta)]}, \quad 0 < \beta < 1. \tag{2.11}$$

Mainardi [9] has showed that $M_\beta(z)$ is positive for $z > 0$, the other general properties can be found in some references (see [1, 9–11] e.g.).

w_β $(0<\beta<1)$ denotes the one-sided stable probability density which can be explicitly expressed by Fox function [12]

$$w_\beta(t)=\beta^{-1}t^{-2}H_{11}^{10}\left(t^{-1}\left|\begin{array}{l}(-1,1)\\(-1/\beta,1/\beta)\end{array}\right.\right). \tag{2.12}$$

Then the Fourier-Laplace transform of the Green function (2.4) can be rewritten in integral form

$$\begin{aligned}
\tilde{\tilde{G}}_1(k,s) &= \left(a_1s^{\alpha-1}+a_2s^{2\alpha-1}+a_3s^{3\alpha-1}+...+a_ns^{n\alpha-1}\right)\int_0^\infty e^{-v\left(\sum\limits_{i=1}^n a_is^{i\alpha}+k^2d\right)}dv\\
\tilde{\tilde{G}}_1(k,s) &= a_1\int_0^\infty L\left\{F_1^{(\alpha)}(va_1t)\right\}\left[L\left\{F_2^{(2\alpha)}(va_2)^{-1/2\alpha}t\right\}L\left\{F_2^{(3\alpha)}(va_3)^{-1/3\alpha}t\right\}...\right.\\
&\quad\left....L\left\{F_2^{(n\alpha)}(va_n)^{-1/n\alpha}t\right\}\right]F\left\{F_3(dvx)\right\}dv+...+a_n\int_0^\infty L\left\{F_1^{(n\alpha)}(va_nt)\right\}\\
&\quad\times\left[L\left\{F_2^{(\alpha)}(va_1)^{-1/\alpha}t\right\}L\left\{F_2^{(2\alpha)}(va_2)^{-1/2\alpha}t\right\}...\right.\\
&\quad\left....L\left\{F_2^{(\overline{n-1}\alpha)}(va_{n-1})^{-1/(n-1)\alpha}t\right\}\right]F\left\{F_3(dvx)\right\}dv.
\end{aligned} \tag{2.13}$$

Going back to the space-time domain, we obtain the relation

$$\begin{aligned}
G_1(x,t) &= a_1\int_0^\infty\left\{F_1^{(\alpha)}(va_1t)\right\}*\left[\left\{F_2^{(2\alpha)}(va_2)^{-1/2\alpha}t\right\}\left\{F_2^{(3\alpha)}(va_3)^{-1/3\alpha}t\right\}...\right.\\
&\qquad\left....\left\{F_2^{(n\alpha)}(va_n)^{-1/n\alpha}t\right\}\right]F_3(dvx)dv+...\\
&\quad+a_n\int_0^\infty\left\{F_1^{(\alpha)}(va_nt)\right\}*\left[\left\{F_2^{(\alpha)}(va_1)^{-1/\alpha}t\right\}\left\{F_2^{(2\alpha)}(va_2)^{-1/2\alpha}t\right\}...\right.\\
&\qquad\left....\left\{F_2^{(\overline{n-1}\alpha)}(va_{n-1})^{-1/(n-1)\alpha}t\right\}\right]F_3(dvx)dv\\
&= a_1\int_0^\infty\left\{F_1^{(\alpha)}(va_1t)\right\}*\left\{\prod_{r=2}^n F_2^{(r\alpha)}(va_r)^{-1/r\alpha}t\right\}F_3(dvx)dv\\
&\quad+a_2\int_0^\infty\left\{F_1^{(2\alpha)}*(va_2t)\right\}\left\{F_2^{(\alpha)}(va_1)^{-1/\alpha}t\prod_{r=3}^n F_2^{(r\alpha)}(va_r)^{-1/r\alpha}t\right\}F_3(dvx)dv\\
&\quad+...+a_n\int_0^\infty\left\{F_1^{(n\alpha)}(va_nt)*\prod_{r=1}^{n-1}F_2^{(r\alpha)}(va_r)^{-1/r\alpha}t\right\}F_3(dvx)dv\\
&= a_1\int_0^\infty F_3(dvx)\left(\int_0^t F_1^{(\alpha)}\left[va_1(t-\tau)\right]\prod_{r=2}^n F_2^{(r\alpha)}(va_r)^{-1/r\alpha}\tau d\tau\right)dv\\
&\quad+a_2\int_0^\infty F_3(dvx)\left(\int_0^t F_1^{(2\alpha)}\left[va_2(t-\tau)\right]\left\{F_2^{(\alpha)}(va_1)^{-1/\alpha}\tau\right.\right.\\
&\quad\left.\left.\times\prod_{r=3}^n F_2^{(r\alpha)}(va_r)^{-1/r\alpha}\tau\right\}d\tau\right)dv\\
&\quad+...+a_n\int_0^\infty F_3(dvx)\left(\int_0^t F_1^{(n\alpha)}\left[va_n(t-\tau)\right]\left\{\prod_{r=1}^{n-1}F_2^{(r\alpha)}(va_r)^{-1/r\alpha}\tau\right\}d\tau\right)dv\\
&= G_{1,1}(x,t)+G_{1,2}(x,t)+G_{1,3}(x,t)+...+G_{1,n}(x,t).
\end{aligned} \tag{2.14}$$

By the same technique, we can obtain the expression for $G_2\ (x,t)$:

$$\begin{aligned}\tilde{\tilde{G}}_2(x,p) &= \int_0^\infty e^{-v\left(a_1s^\alpha+a_2s^{2\alpha}+\ldots+a_ns^{n\alpha}+k^2d\right)}dv\\ &= \int_0^\infty e^{-va_1s^\alpha}e^{-va_2s^{2\alpha}}\ldots e^{-va_ns^{n\alpha}}e^{-k^2dv}dv\\ &= \int_0^\infty L\left\{\left(F_2^{(\alpha)}(va_1)^{-1/\alpha}t\right)\left[\left\{F_2^{(2\alpha)}(a_2v)^{-1/2\alpha}t\right\}\ldots\left\{F_2^{(n\alpha)}(a_nv)^{-1/n\alpha}t\right\}\right]\right\}\\ &\quad\times FF_3(dvx)dv.\end{aligned} \tag{2.15}$$

Taking Fourier-Laplace inverse transform, we obtain the following relation

$$G_2(x,t) = \int_0^\infty F_3(dvx)\left(\int_0^t F_2^\alpha(a_1v)^{-1/\alpha}(t-\tau)\left\{\prod_{r=2}^n F_2^{r\alpha}(a_rv)^{-1/r\alpha}\tau\right\}d\tau\right)dv. \tag{2.16}$$

We can ensure that the green functions are non-negative by the non-negative prosperities of $F_1^{(\beta)}$, $F_2^{(\beta)}$, $F_3^{(\beta)}$.

3. The solution for the generalized TFTE in half-space domain (signaling problem)

In the section, we considered Problem 2, defined in a half-space domain, which is known as signaling problem.

On taking the Laplace transform to (1.1) and (1.6) using (1.7) with $f\equiv 0$ and the initial condition (1.6), we get

$$\frac{\partial^2\tilde{v}(x,s)}{\partial x^2} = \sum_{r=1}^n \frac{a_rs^{r\alpha}}{d}\tilde{v}(x,p) \tag{3.1}$$

$$\tilde{v}(0,s) = \tilde{\xi}(s),\tilde{v}(+\infty,s) = 0.$$

On solving the above equation

$$\tilde{v}(x,s) = \tilde{\xi}(s)e^{-\sqrt{\sum_{r=1}^n \frac{a_rs^{r\alpha}}{d}}x} \tag{3.2}$$

$$\tilde{v}(x,s) = L\{G_u(x,t)*\xi(t)\} \tag{3.3}$$

where $G_{\mathrm{u}}(x,t)$ is the Green function or fundamental solution of the Signaling problem obtained when $\xi(x)=\delta(x)$, which is characterized by

$$\tilde{G}_u(x,s) = e^{-\sqrt{\sum_{r=1}^n \frac{a_rs^{r\alpha}}{d}}x}. \tag{3.4}$$

By taking inverse Laplace transform of (3.2) gives the solution of signaling problem

$$v(x,t) = G_u(x,t)*g(t) = \int_0^t g(\tau)G(x,t-\tau)d\tau. \tag{3.5}$$

Taking partial differentiation of equation (3.2) and using equation (2.8.1), (2.8.2), ellipsis , (2.8.n) we get

$$\frac{\partial}{\partial s}\tilde{G}_u(x,s) = -a_1\alpha x\tilde{G}_{1,1}(x,s) - 2a_2\alpha x\tilde{G}_{1,2}(x,s) - \ldots - na_n\alpha x\tilde{G}_{1,n}(x,s), \quad x>0. \tag{3.6}$$

On solving we get the answer in space-time domain

$$tG_u(x,t) = a_1\alpha xG_{1,1}(x,t) + 2a_2\alpha xG_{1,2}(x,t) + \ldots + na_n\alpha xG_{1,n}(x,t), \quad x>0. \tag{3.7}$$

4. The solution of the generalized TFTE in the bounded space domain

In this section, we find the solution of the generalized TFTE in bounded space domain. Taking the Fourier sine transform of (1.1) with f = 0 and applying the boundary condition (1.9), we get

$$a_1D_t^{\alpha}\bar{v}(p,t) + a_2D_t^{2\alpha}\bar{v}(p,t) + \ldots + a_nD_t^{n\alpha}\bar{v}(p,t) = -\left(\frac{pd\pi}{M}\right)^2\bar{v}(p,t), \quad t>0, \tag{4.1}$$

where p is the wave number, and

$$\bar{v}(p,t) = \int_0^M v(u,t)\sin\left(\frac{p\pi u}{M}\right)du, \tag{4.2}$$

is the finite sine transform of $v(x,t)$.

Applying the Laplace transform to (4.1) and using the initial boundary condition (1.8), we get

$$\tilde{\bar{v}}(p,s) = \left\{\frac{\sum\limits_{i=1}^{n} a_i s^{i\alpha-1}}{\sum\limits_{i=1}^{n} a_i s^{i\alpha} + \left(\frac{pd\pi}{M}\right)^2}\right\}\bar{v}(p,0) + \left\{\frac{\sum\limits_{i=2}^{n} a_i s^{i\alpha-2}}{\sum\limits_{i=1}^{n} a_i s^{i\alpha} + \left(\frac{pd\pi}{M}\right)^2}\right\}\bar{v}_t(p,0) \tag{4.3}$$

where

$$\bar{v}(p,0) = \int_0^M \phi(u)\sin\left(\frac{p\pi u}{M}\right)du, \tag{4.4}$$

We consider the roots of the polynomial is defined in below:

$$\sum_{i=1}^{n} a_i s^{i\alpha} + \left(\frac{pd\pi}{M}\right)^2 = (s^{\alpha}-\lambda_1)(s^{\alpha}-\lambda_2)\ldots(s^{\alpha}-\lambda_n). \tag{4.5}$$

We use the Laplace transform pair which is defined below on solving (4.3)

$$t^{\beta-1}E_{\alpha,\beta}(ct^{\alpha}) \overset{L}{\longleftrightarrow} \frac{s^{\alpha-\beta}}{s^{\alpha}-c}, \tag{4.6}$$

where $E_{\alpha,\beta}(z)$ is Mittag-Leffler function of two parameters is defined in (1.10).

$$\frac{\sum_{i=1}^{n} a_i s^{i\alpha-1}}{\sum_{i=1}^{n} a_i s^{i\alpha}+\left(\frac{pd\pi}{L}\right)^2} = s^{\alpha-1}\left[\frac{c_1}{(s^\alpha-\lambda_1)}+\frac{c_2}{(s^\alpha-\lambda_2)}+\ldots+\frac{c_n}{(s^\alpha-\lambda_n)}\right] \xleftrightarrow{L} \left[c_1E_\alpha(\lambda_1t^\alpha)+c_2E_\alpha(\lambda_2t^\alpha)+\ldots+c_nE_\alpha(\lambda_nt^\alpha)\right] \tag{4.7}$$

where

$$c_j = \frac{\sum_{i=1}^{n} a_i\lambda_j^i}{\prod_{\substack{i,j=1\\ j\neq i}}^{n}(\lambda_j-\lambda_i)}, \quad j=1,2,\ldots,n \tag{4.8}$$

and similarly

$$\frac{\sum_{i=2}^{n} a_i s^{i\alpha-2}}{\sum_{i=1}^{n} a_i s^{i\alpha}+\left(\frac{pd\pi}{L}\right)^2} = s^{\alpha-2}\left[\frac{d_1}{(s^\alpha-\lambda_1)}+\frac{d_2}{(s^\alpha-\lambda_2)}+\ldots+\frac{d_n}{(s^\alpha-\lambda_n)}\right] \xleftrightarrow{L} t^{\alpha-2}\left[d_1E_{\alpha,2}(\lambda_1t^\alpha)+d_2E_{\alpha,2}(\lambda_2t^\alpha)+\ldots+d_nE_{\alpha,n}(\lambda_nt^\alpha)\right] \tag{4.9}$$

where

$$d_j = \frac{\sum_{i=2}^{n} a_i\lambda_j^{i-1}}{\prod_{\substack{i,j=1\\ j\neq i}}^{n}(\lambda_j-\lambda_i)}, \quad j=1,2,\ldots,n. \tag{4.10}$$

Now on taking inverse finite sine and Laplace transform for (4.3), we get

$$\begin{aligned} v(x,t) = {} & \frac{2}{M}\sum_{m=1}^{\infty} c_1E_\alpha(\lambda_1t^\alpha)+c_2E_\alpha(\lambda_2t^\alpha)+\ldots \\ & +c_nE_\alpha(\lambda_nt^\alpha)\sin\left(\frac{p\pi x}{M}\right)\int\phi(u)\sin\left(\frac{p\pi u}{M}\right)du \\ & +\frac{2}{M}\sum_{m=1}^{\infty} d_1E_{\alpha,2}(\lambda_1t^\alpha)+c_2E_{\alpha,2}(\lambda_2t^\alpha)+\ldots \\ & +d_nE_{\alpha,2}(\lambda_nt^\alpha)\sin\left(\frac{p\pi x}{M}\right)\int\Psi(u)\sin\left(\frac{p\pi u}{M}\right)du \end{aligned} \tag{4.11}$$

Special cases

If we put $a_1 = 2a$, $a_2 = 1$, $a_3 = a_4$, ellipsis, $a_n = 0$ then the result of Problem 2, 3 and 4 will reduced in [13].

REFERENCES

[1] I. PODLUBNY, *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*, Mathematics in Science and Engineering, vol. **198**, Academic Press, San Diego, Calif, USA, 1999.

[2] F. MAINARDI, *Fractional calculus: some basic problems in continuum and statistical mechanics*, in *Fractals and Fractional Calculus in Continuum Mechanics*, A. Carpinteri and F. Mainardi, Eds., CISM Courses and Lectures, vol. **378**, pp. 291–348, Springer, Vienna, Austria, 1997.

[3] R. C. CASCAVAL, E. C. ECKSTEIN, C. L. FROTA, AND J. A. GOLDSTEIN, *Fractional telegraph equations*, Journal of Mathematical Analysis and Applications **276**, 1 (2002), 145–159.

[4] E. ORSINGHER AND L. BEGHIN, *Time-fractional telegraph equations and telegraph processes with brownian time*, Probability Theory and Related Fields **128**, 1 (2004), 141–160.

[5] J. CHEN, F. LIU, AND V. ANH, *Analytical solution for the time-fractional telegraph equation by the method of separating variables*, Journal of Mathematical Analysis and Applications **338**, 2 (2008), 1364–1377.

[6] E. ORSINGHER AND X. ZHAO, *The space-fractional telegraph equation and the related fractional telegraph process*, Chinese Annals of Mathematics Series B **24**, 1 (2003), 45–56.

[7] S. MOMANI, *Analytic and approximate solutions of the space- and time-fractional telegraph equations*, Applied Mathematics and Computation **170**, 2 (2005), 1126–1134.

[8] R. FIGUEIREDO CAMARGO, A. O. CHIACCHIO, AND E. CAPELAS DE OLIVEIRA, *Differentiation to fractional orders and the fractional telegraph equation*, Journal of Mathematical Physics **49**, 3 (2008), Article ID 033505, 12 pages.

[9] F. MAINARDI, *Fractional relaxation-oscillation and fractional diffusion-wave phenomena*, Chaos, Solitons & Fractals **7**, 9 (1996), 1461–1477.

[10] R. GORENFLO, Y. LUCHKO, AND F. MAINARDI, *Wright functions as scale-invariant solutions of the diffusion-wave equation*, Journal of Computational and Applied Mathematics **118**, 1–2 (2000), 175–191.

[11] F. MAINARDI, Y. LUCHKO, AND G. PAGNINI, *The fundamental solution of the space-time fractional diffusion equation*, Fractional Calculus & Applied Analysis **4**, 2 (2001), 153–192.

[12] W. R. SCHNEIDER AND W. WYSS, *Fractional diffusion and wave equations*, Journal of Mathematical Physics **30**, 1 (1989), 134–144.

[13] F. HUANG, *Analytical Solution for the Time-Fractional Telegraph Equation*, Hindawi Publishing Corporation, Journal of Applied Mathematics Vol. **2009**, Article ID 890158, 9 pages.

[14] MOUSTAFA EL-SHAHED, *MHD of a Fractional Viscoelasticity Fluid in a CircularTube*, Mechanics Research Communications **33** (2006), 261–268.

(Received September 18, 2010)

Dr. V. B. L. Chaurasia
Department of mathematics
University of Rajasthan
Jaipur – 302055, India

Ravi Shanker Dubey
Department of mathematics
YIT Sitapura
Jaipur – 302022, India
e-mail: ravishankerdubey@indiatimes.com