# 3

# Topology and topological spaces

*Topology* is a major area of mathematics. In topology we study the properties of objects which are not sensitive to continuous deformations, i.e., deformations where it is not allowed to cut objects and glue them together. These properties are called *topological properties*. In this chapter we give an introduction to some of basic topological notions. We mostly try to avoid rigorous definitions and replace them by geometrically intuitive ones. There are many good textbooks on topology including the ones oriented to physics audience. We refer the reader to these textbooks for proofs and more thorough justifications of statements we make. Our main goal is to equip the reader with basic tools necessary to work with topological defects and textures and topological terms.

## 3.1 Examples of topological properties

Let us consider subsets of a Euclidian space $R^n$ of some dimension $n$. The topological properties are the properties not sensitive to continuous deformation of those subsets. The following is the list of some topological properties.

- Dimensionality
- Existence of boundary (open disk vs. closed disk)
- Orientability (e.g., Möbius band vs. sphere)
- Connectedness (consisting of several "connected components")
- Connectivity (simply connected vs multiply connected)
- Compactedness (2d sphere vs. 2d plane)
- ...

### 3.2 Topology and topological spaces

<u>Definition</u>: Let $X$ be a set. A family $\mathcal{F}$ of subsets of $X$ is a *topology* for $X$ if $\mathcal{F}$ has the following three properties:

(i) Both $X$ and the empty set $\emptyset$ belong to $\mathcal{F}$,
(ii) Any union of sets in $\mathcal{F}$ belongs to $\mathcal{F}$,
(iii) Any finite intersection of sets in $\mathcal{F}$ belongs to $\mathcal{F}$.

A *topological space* is a pair $(X, \mathcal{F})$, where $X$ is a set and $\mathcal{F}$ is a topology for $X$.

The sets in $\mathcal{F}$ are called *open sets*.

**Example 1: inseparable 2-point set** Consider a set consisting of just two elements (points) $X = \{x, y\}$. We endow this set with topology $\mathcal{F} = \Big\{\emptyset, \{x, y\}, \{x\}\Big\}$. One can easily check that this is correctly defined topology.

This example is somewhat pathological example of inseparable topological space (lacking Hausdorff property).

**Example 2: metric-induced topology** If $X$ is a *metric* space, then the open (in terms of metric) subsets of $X$ form a topology for $X$ which is called *metric topology* for $X$.

Here open subset of metric space is the subset containing the vicinity of its every point.

Metric spaces are the spaces which are used the most in physics. There are exceptions though. For example the phase space of classical mechanics does not have a natural metric. In the following we restrict ourselves to topological spaces which are

(i) metric spaces (with metric topology)
(ii) $n$-dimensional manifolds. Roughly, one can think of $n$-dimensional manifold as of s set which reminds $R^n$ locally with some way to interpolate between points.
(iii) spaces that can be embedded into $R^m$ (for some $m \geq n$) with induced metric and topology.

### 3.3 Continuity and homeomorphism

The notion of *continuity* is the fundamental notion in topology and can be defined for a mapping between topological spaces.

<u>Definition</u>: The mapping $f : X \to Y$ between two topological spaces is

*continuous* if for every open (in topology on $Y$) set $U \subset Y$ the inverse image $f^{-1}(U) \subset X$ is open in topology on $X$.

If both the mapping $f : X \to Y$ and its inverse $f : Y \to X$ are continuous, the mapping is called *homeomorphism*. More precisely we have:

Definition: The *bijection* (one-to-one and onto mapping) $f : X \to Y$ between two topological spaces is called *homeomorphism* if it is true that $f^{-1}(U)$ is open in $X$ iff $U$ is open in $Y$.

Definition: Two topological spaces $X$ and $Y$ are called *homeomorphic* if there exists a homeomorphism between $X$ and $Y$. We denote homeomorphic spaces as $X \sim Y$.

**Example:** An open interval $X = (-\pi/2, \pi/2)$ is homeomorphic to a line $Y = R^1$. As a homeomorphism one can take $y = \tan(x)$.

**Example:** $n$-dimensional sphere $S^n$ with punched point is homeomorphic to $n$-dimensional Euclidian space $R^n$. As a homeomorphism one can take a stereographic projection of a sphere to $R^n$.

Definition: A property of a topological space is a *topological property* if it is preserved under homeomorphism.

## 3.4 Examples of topological spaces

Here we give examples of topological spaces that we use and introduce corresponding notations. We also mention some ways to construct topological spaces from the existent ones.

### *3.4.1 Euclidean spaces, spheres, and balls ($R^n$, $C^n$, $S^n$, $D^n$)*

The sets of points

$$R^n = \{\{x_1, \ldots, x_n\}, x_i \in R\}, \tag{3.1}$$
$$C^n = \{\{z_1, \ldots, z_n\}, x_i \in C\} \tag{3.2}$$

are called *real Euclidian space* and *complex space* respectively. Here $z_i = x_i + iy_i$. These spaces are assumed to be endowed with conventional metric $d(r_1, r_2) = |r_1 - r_2|$ and with topology induced by this metric.

It is easy to see that $C^n \sim R^{2n}$ with homeomorphism obtained by $z \to x, y$.

The sets of points

$$S^n = \left\{ \{x_1, \ldots, x_{n+1}\}, x_1^2 + \ldots + x_{n+1}^2 = 1 \right\}, \quad (3.3)$$
$$D^n = \left\{ \{x_1, \ldots, x_n\}, x_1^2 + \ldots + x_n^2 < 1 \right\} \quad (3.4)$$

are called $n$-dimensional sphere and open $n$-dimensional ball respectively. We defined them as embedded into $R^{n+1}$ and $R^n$ respectively and assume that they have the metric and topology induced by the embedding.

The boundary of the ball is sphere $\partial D^{n+1} \sim S^n$. The ball $D^n$ is homeomorphic to $R^n$ with homeomorphism given explicitly, e.g., by $R = \tan(\pi r/2)$. Here $r$ is the radial coordinate of a point in a ball and $R$ is the one in $R^n$.

The sphere $S^n$ is a *compact* topological space while $R^n, C^n, D^n$ are non-compact.†

### *3.4.2 Direct product of topological spaces* ($T^n = (S^1)^{\times n}$, *etc*)

Given two topological spaces $X$ and $Y$ one can form a topological space $X \times Y$ which consists of pairs of points $\{x, y\}$. The topology on $X \times Y$ is defined in a following way. We make a basis of open sets making direct products of all pairs $U \times V$ of open sets from $X$ and $Y$ respectively. Then we call open all sets obtained by arbitrary unions and finite intersections of basis sets.

For example, one can think of $R^n$ as of direct product $R \times R \times \ldots \times R$ or shortly $R^n = (R^1)^{\times n}$.

The product of $n$ one-dimensional spheres (circles) is called $n$-dimensional torus $T^n = \overbrace{S^1 \times S^1 \times \ldots \times S^1}^{\mathrm{n}} = (S^1)^{\times n}$.

### *3.4.3 Real projective space* $RP^n$

Real projective space $RP^n$ is defined as a set of straight lines through the origin in $R^{n+1}$ with metric topology. The metric defines the distance as an angle between lines. If $x \in R^{n+1}$, i.e., $x = (x_0, x_1, \ldots, x_n)$ one can write

$$RP^n = \{x, x \sim \lambda x\}. \quad (3.5)$$

Here $\lambda$ is any real number $\lambda \in R$, $\lambda \neq 0$ and the points $x$ and $\lambda x$ should be identified. The coordinates $x_0, \ldots x_n$ defined up to multiplication by $\lambda \neq 0$ are called homogeneous coordinates of $RP^n$.

† The topological space is called *compact* if from any covering of the space by open sets one can choose finite covering. If the space is a subset of Euclidian space this topological definition is equivalent to the conventional one from analysis. That is, the space is compact if it is closed and bounded.

One can think of $RP^n \sim S^n/Z_2$, i.e., as of $n$-sphere with pairwise identification of points connected by any diameter of the sphere. Similarly $RP^n$ can be thought of as $D^n$ with boundary and with opposite points of the boundary identified.

$RP^1 \sim S^1$ (prove).

$RP^2$ is a real projective plane. This is non-orientable surface which can not be embedded into $R^3$ without self-intersections. $RP^2$ is one of classic surfaces widely used in geometry.

Its polygon representation.

### *3.4.4 Complex projective space* $CP^n$

If we replace $R \to C$ in the previous section we obtain the definition of a complex projective space $CP^n$. It is defined by its complex homogeneous coordinates $z = (z_0, \dots, z_n)$ with identification $z \sim wz$ with $w \in C$, $w \neq 0$.

Let us take $z = (z_0, \dots, z_n)$ restricted by $|z_0|^2 + \dots + |z_n|^2 = 1$ (this means that we have chosen $|w|$ in a proper way). Obviously the set of such points $z$ forms $(2n+1)$-sphere. Then $CP^n$ is obtained by identification $z \sim e^{i\alpha} z$. In other words $CP^n \sim S^{2n+1}/S^1$.

Let us consider a complex projective space $CP^1$ in more detail. It is parameterized by a complex 2-component vector $z = (z_1, z_2)^t$ defined up to multiplication by non-zero complex number. Or, similarly, by normalized complex vector $z$ so that $z^\dagger z = 1$ defined up to a multiplication by a phase $e^{i\alpha}$. Let us consider a mapping of $CP^1$ onto $S^2$ given by

$$\begin{aligned} \vec{n} &= z^\dagger \vec{\sigma} z \\ &= (z_1^* z_2 + z_2^* z_1, -i z_1^* z_2 + i z_2^* z_1, z_1^* z_1 + z_2^* z_2), \end{aligned} \tag{3.6}$$

where $\vec{\sigma}$ is a set of Pauli matrices.

It is easy to check that (3.6) is a well-defined mapping $CP^1 \to S^2$. Indeed, check that $\vec{n}^2 = 1$ and that $\vec{n}$ does not change if $z \to e^{i\alpha} z$. This mapping is homeomorphism and its existence proves that $CP^1 \sim S^2$. It is often called Hopf map or Hopf fibration. In physics the mapping (3.6) is sometimes called $CP^1$ representation of $\vec{n}$-field.

### *3.4.5 Grassmann manifolds* $G(n,k)$ *and* $CG(n,k)$

Real Grassmann manifold $G(n,k)$ is defined as set of $k$-dimensional subspaces of $n$-dimensional space $R^n$ going through the origin. It is obvious that $G(n,k) = G(n, n-k)$ and that $G(n,1) = RP^n$.

Similarly we define complex Grassmann manifold $CG(n,k)$ and similarly $CG(n,k) = CG(n,n-k)$ and $CG(n,1) = CP^n$.

### 3.4.6 Compact classic groups $O(n)$, $U(n)$, $SO(n)$, $SU(n)$, $Sp(n)$

**Orthogonal groups** The orthogonal group $O(n)$ is defined as a set of real $n \times n$ matrices subject to a condition $O^T O = 1$. This definition embeds the group into $R^{n^2}$ and we use an induced topology in $O(n)$. The condition $O^T O = 1$ is equivalent to $(n^2+n)/2$ real conditions (the matrix $O^T O$ is symmetric). Therefore, the dimension of $O(n)$ is $\dim(O(n)) = \frac{n^2-n}{2}$.

One can think of $O(n)$ as of the group of orthogonal transformations of $R^n$.

For the orthogonal matrix $M \in O(n)$ we have $\det(M) = \pm 1$. The group $O(n)$ is not connected but consists of two connected components. These components can be labeled by a sign of the determinant of corresponding matrices. It is obvious that two matrices having different sign of the determinant can not be connected within $O(n)$.

The connected component of $O(n)$ containing unit matrix forms a subgroup called $SO(n)$. It is specified by a condition $\det(M) = +1$. It has the same dimension $\dim(SO(n)) = \frac{n^2-n}{2}$.

It is easy to check that $SO(2) \sim S^1$ and that $SO(3) \sim RP^3$.

**Unitary groups** The unitary group $U(n)$ is a group of unitary transformations of $C^n$. It can be parameterized by $n \times n$ complex matrices subject to $U^\dagger U = 1$. The dimension of $U(n)$ is $\dim(U(n)) = n^2$. It is connected. The matrices of $U(n)$ has determinants which are pure phases.

The subgroup of $U(n)$ specified by a condition $\det(M) = 1$ is called $SU(n)$. It has a dimension $\dim(SU(n)) = n^2 - 1$.

It is easy to check that $SU(2) \sim S^3$. $SU(2)$ gives a two fold covering of $SO(3)$.

**Symplectic groups** The symplectic group $Sp(n)$ is a group of unitary transformation of $n$-dimensional quaternionic space. One can think of it as of a set of $2n \times 2n$ complex matrices $M$ subject to conditions $M^\dagger M = 1$ and $M^T \Omega M = \Omega$. Here $\Omega = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$ is the $2n \times 2n$ skew-symmetric matrix. Matrices from $Sp(n)$ have unit determinants.

The dimension of $Sp(n)$ is $\dim(Sp(n)) = n(2n+1)$. We have $Sp(1) \sim SU(2)$.

#### *3.4.7 Action of groups on spaces. Coset spaces.*

Let us suppose that there exists an *action* of the group $G$ on a space $X$, i.e. there for every $g \in G$ we have a mapping (homeomorphism) $g : X \to X$. Then, one can construct topological spaces identifying orbits of the action as points.

The natural action of the group $O(n)$ on $R^n$ induces its actions on $S^{n-1}$, $G(n,k)$ etc. The latter actions are *transitive*. The latter means that for any two points $x, y \in X$ there is $g \in G$ such that $g : x \to y$. The spaces $S^{n-1}$, $G(n,k)$ are then *homogeneous* spaces (spaces with transitive group actions) of the group $O(n)$ (and similarly of $SO(n)$).

One can think, therefore, of many classical spaces as of coset spaces of classical groups. Explicitly:

$$S^{n-1} = O(n)/O(n-1) = SO(n)/SO(n-1), \tag{3.7}$$

$$S^{2n-1} = U(n)/U(n-1) = SU(n)/SU(n-1), \tag{3.8}$$

$$S^{4n-1} = Sp(n)/Sp(n-1), \tag{3.9}$$

$$G(n,k) = O(n)\Big/O(k) \times O(n-k), \tag{3.10}$$

$$CG(n,k) = U(n)\Big/U(k) \times U(n-k). \tag{3.11}$$

#### *3.4.8 Classic surfaces*

A *surface* is a two-dimensional manifold. Surfaces (up to a homeomorphism) can be classified in the following way. First of all we distinguish *sphere* $S^2$, *real projective plane* $RP^2$ and *Klein bottle* $K$. Other classic surfaces can be obtained from the above three by attaching handles and drilling holes (only by attaching handles if we are interested in closed surfaces only).

Closed surface can be constructed from an oriented *fundamental polygon* of the surface by pairwise identification of its edges. The simplest polygons are shown in Figure 3.1. They can be represented by words (in obvious notations), e.g, sphere $ABB^{-1}A^{-1}$ (or simply $AA^{-1}$, projective plane $ABAB$ (or simply $AA$), Klein bottle $ABAB^{-1}$, torus $ABA^{-1}B^{-1}$ etc.

Inserting the following word $ABA^{-1}B^{-1}$ into any polygon word means attaching an additional handle to the surface. Similarly, inserting $AA$ attaches $RP^2$ (through a pipe) to the original surface.

Any closed surface is homeomorphic to one of the following (i) sphere (Euler characteristics 2), (ii) chain of $g$ tori connected by pipes (Euler characteristics $2-2g$), (iii) chain of $k$ projective planes connected by pipes (Euler characteristics $2-k$).

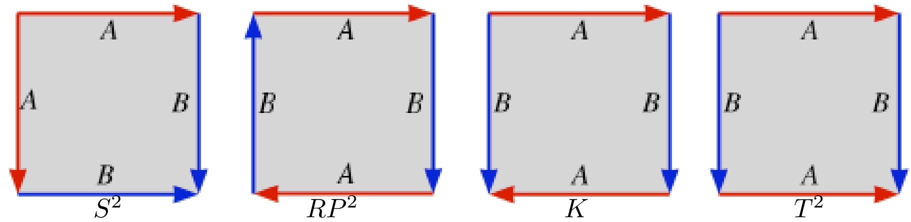


Fig. 3.1. Fundamental polygons (squares) for sphere, projective plane, Klein bottle, and torus, respectively (from Wikipedia).

A closed surface is determined, up to homeomorphism, by two pieces of information: its Euler characteristic, and whether it is orientable or not.

#### *3.4.9 Space of mappings*

The space of continuous mappings of topological spaces $X \to Y$ is denoted by $C(X, Y)$ and is a topological space itself (there is a canonic way to define a topology on this space †).

The spaces of mappings play a very important role in topology and in homotopy theory introduced in the next section.

If $X$ consists of only one point then $C(X, Y) = Y$. If $X$ is a set of $n$ isolated points $C(X, Y) = Y \times Y \times \ldots \times Y$ ($n$ times).

If $I = [0, 1]$ is a closed interval, the continuous mapping $f : I \to X$ is called a *path* in $X$ with the beginning $x_0 = f(0)$ and the end $x_1 = f(1)$. The space $C(I, X)$ with additional restriction that $x_0$ and $x_1$ are fixed is a space $E(X; x_0, x_1)$ of paths in $X$ with the beginning at $x_0$ and the end at $x_1$. The space $\Omega(X; x_0) = E(X; x_0, x_0)$ is the space of loops in $X$ beginning and ending at the same point $x_0$.

### 3.5 Topological invariants

One of the main problems of topology is to classify topological spaces up to homeomorphisms. In particular, given two topological spaces $X$ and $Y$ one should be able to say whether they are homeomorphic or not. There is no complete solution of this problem yet. However, if $X$ and $Y$ have

† The base of open sets in $C(X, Y)$ can be constructed in the following way. Take any compact subset $K$ of $X$ and any open subset $O$ of $Y$. Then the set of mappings $f : X \to Y$ such that $f(K) \subset O$ is called open. Taking arbitrary unions and finite intersections of such basis sets one obtains the topology on $C(X, Y)$.

different *topological invariants* we can be sure that $X$ and $Y$ are not homeomorphic to each other. Here *topological invariants* are the quantities which are conserved under homeomorphisms.

As examples of topological invariants one can consider the number of connected components, compactness, Euler characteristic etc.

An open interval is not homeomorphic to a closed one as the latter is compact while the former is not. A torus is not homeomorphic to a sphere as they have different Euler characteristic. Let us consider Euler characteristic as an example.

**Euler characteristic** Here we confine ourself to the Euler characteristic of two-dimensional surfaces. Suppose that a surface $X$ is homeomorphic to a *polyhedron* $K$ (a geometrical object surrounded by faces). Then the Euler characteristics $\chi(X)$ is defined as

$$\chi(K) \quad = \quad V - E + F, \tag{3.12}$$

where $V, E, F$ are the numbers of vertices, edges, and faces in $K$ respectively. The remarkable fact that $\chi(X)$ does not depend on the choice of $K$ but only on $X$ itself.

Given a two dimensional surface one performs *triangulation* of that surface and calculates $\chi(X)$. If we consider fundamental polygons of sphere, projective plane, Klein bottle, and torus we calculate the triplet $(V, E, F)$ to be $(3, 2, 1)$, $(2, 2, 1)$, $(1, 2, 1)$, and $(1, 2, 1)$ respectively. This gives $\chi(S^2) = 2$, $\chi(RP^2) = 1$, $\chi(K) = 0$, and $\chi(T^2) = 0$.

One can see that $\chi(X)$ is not enough to distinguish between closed two-dimensional surfaces. For example, the Klein bottle and the torus have the same $K$. However, in case of surfaces two invariants: Euler charactistic and orientability fully define the surface up to homeomorphism.

It would be nice to have a full set of topological invariants to distinguish between any two non-homeomorphic topological spaces. This set is not found yet. In the next section we consider a way to construct a very rich set of topological invariants which are called *homotopy groups*.

## 3.6 Exercises

**Exercise 3.1: Topology of configurational spaces** What is the configuration space of

a) Double spherical pendulum with suspension point which is allowed to move along straight line.

b) Quantum diatomic molecule made out of identical atoms (e.g, $N_2$). Assume that at the relevant energy scale one can neglect the change of the distance between atoms.

c) Rigid body.

**Exercise 3.2: Classic surfaces** Show that

a) The projective plane $RP^2$ with a hole is homeomorphic to the Möbius band.

b) Two Klein bottles connected by a pipe are homeomorphic to the Klein bottle with handle.

c) Two projective planes $RP^2$ connected by a pipe are homeomorphic to the Klein bottle.

**Exercise 3.3: Euler characteristic of sphere** Calculate $(V, E, F)$ for tetrahedron, cube, and octahedron and obtain from them the Euler characteristic of $S^2$.

**Exercise 3.4: Euler characteristic** Show that
a) Attaching handle to a surface decreases its Euler characteristic by 2.
b) Attaching $RP^2$ to a surface by pipe decreases its Euler characteristic by 1.

**Exercise 3.5: Stereographic projection** Stereographic projection maps a sphere parameterized by a unit vector $\vec{n} = (n_1, n_2, n_3)$ ($\vec{n}^2 = 1$) onto a plane tangent to a sphere at south pole ($\vec{n} = (0, 0, -1)$). The points of the plane are parameterized by Cartesian coordinates $(x, y)$.
a) Find an explicit relation between $x, y$ and $\vec{n}$.
b) The same but use polar coordinates $\rho, \phi$ instead of $x, y$.
c) The same but use complex coordinates $w = x + iy$.

**Exercise 3.6: Topological invariant: $S^3 \to S^3$** Consider a three-dimensional unit vector field $\vec{\pi} \in S^3$ on a three-dimensional space $\vec{\pi}(x, y, z)$ with constant boundary conditions $\vec{\pi}(x, y, z) \to (0, 0, 0, 1)$ as $(x, y, z) \to \infty$. Show that

$$Q = A \int d^2x\, \epsilon^{\mu\nu\lambda} \epsilon^{abcd} \pi^a \partial_\mu \pi^b \partial_\nu \pi^c \partial_\lambda \pi^d \tag{3.13}$$

is an integer-valued topological invariant with properly chosen normalization constant $A$. Namely,
a) Show that under small variation $\delta\vec{\pi}$ of a vector field the corresponding variation $\delta Q = 0$.
b) Show that the integrand in (3.13) is a Jacobian of the change of variables from $x, y, z$ to a sphere $\vec{\pi}$ up to normalization.
c) Choose $A$ so that it is normalized in such a way that the area of the 3-sphere is 1. Therefore, $Q$ is an integer degree of mapping of a space (with constant boundary conditions) onto a 3-sphere.
*Hint:* In b) consider the vicinity of the northern pole of the 3-sphere only and then extend your result to the whole 3-sphere by symmetry.