Geombinatorics 4(1994), 49 - 56

UNIFORM TILINGS OF 3-SPACE

by Branko Grünbaum[1]
University of Washington GN-50, Seattle, WA 98195
e-mail: grunbaum@math.washington.edu

Early in 1992, while preparing material for a course on polyhedra and related topics, I happened to look again at an often-mentioned paper by Andreini (published in 1905) in which the uniform tilings of the 3-dimensional space are enumerated. I have seen that paper several times in the past — but this time I noticed that one of the 25 tilings listed as uniform is not a uniform tiling at all! Becoming suspicious, I decided to enumerate the uniform tilings on my own. As it turned out, Andreini was wrong on several counts: not only did he include one non-uniform tiling (#13' in his list), he also missed four uniform ones. As often happens, within a very short time after completing my enumeration I had two other occasions to think about uniform tilings of 3-space. First, I discovered that N. W. Johnson, in the manuscript of a book he was preparing to publish, also enumerated the uniform tilings, arriving at the same 28 tilings as I did -- and he did it well before I did so. Second, shortly after that I received a letter from I. Alexeyev (from Pskov in Russia) saying that he has enumerated the uniform tilings and asking whether this has been done before. Alexeyev did not say what the results of his enumeration are, nor when did he carry it out. I communicated to him what I knew, but did not receive any further word from him. Seeing that the publication of Johnson's book appears to have been delayed, and that the accounts of uniform tilings in other publications (Critchlow (1970), Williams (1972)) are even more deficient than that of Andreini, I decided to publish my list, with the explicit acknowledgement that the priority of carrying out the enumeration belongs to Johnson.

To begin with, precise definitions of several concepts which are relevant to the enumeration will be given. This is particularly important because of the varying usage of some of the terms.

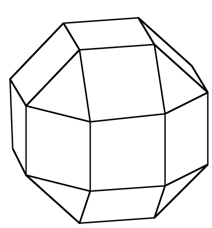
In this note by *polyhedron* we mean a 3-dimensional compact convex polyhedron. The *neighborhood* of a vertex V of a polyhedron P is the set of faces of P that contain V. A polyhedron P is *Archimedean* if all its faces are regular polygons and the neighborhoods

---

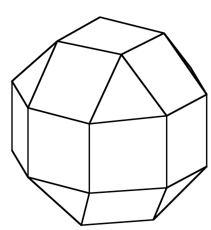
1 Research supported in part by NSF grant DMS-9300657.

of all vertices are mutually congruent; P is *uniform* if its faces are regular polygons and its vertices form one orbit under the symmetries of P. For the Archimedean and the uniform polyhedra it is customary to describe them by the sequence of faces that form the neighborhood of a vertex. It is well known that the uniform polyhedra can be enumerated as follows: There are two infinite families (prisms (4.4.n) and antiprisms (3.3.3.n)), five regular (Platonic) polyhedra (with symbols (3.3.3), (3.3.3.3), (3.3.3.3.3), (4.4.4), (5.5.5)), and 13 other polyhedra (with symbols (3.3.3.3.4), (3.3.3.3.5), (3.4.3.4), (3.4.4.4), (3.4.5.4), (3.5.3.5), (3.6.6), (3.8.8), (3.10.10), (4.6.6), (4.6.8), (4.6.10), (5.6.6)). Clearly, every uniform polyhedron is Archimedean, but there is one nonuniform Archimedean polyhedron, the "pseudorhombicuboctahedron" (3.4.4.4)*; it is shown in Figure 1, together with the uniform polyhedron (3.4.4.4) that has the same neighborhoods of vertices. By a *tiling* of the 3-dimensional space we shall understand a collection of polyhedra covering the space without gaps or overlaps, such that the polyhedra used meet face-to-face. The *neighborhood* of a vertex of a tiling is the collection of all polyhedra of the tiling that contain that vertex. We shall consider only tilings by uniform polyhedra, and we shall say that a tilings is *Archimedean* if all vertices have congruent neighborhoods, and that it is *uniform* if all the vertices form one orbit under symmetries of the tiling. (One could make analogous definitions for tilings in which Archimedean polyhedra are admitted; however, it turns out that no additional tilings would result.) To denote a uniform or Archimedean tiling we shall list the types of polyhedra that occur in the neighborhood of each vertex, with an exponent to denote the number of polyhedra of each kind in such a neighborhood.

I do not know of any elegant method of enumerating uniform tilings. In my enumeration I first compiled lists of possibilities of uniform polyhedra fitting around an edge, then lists of uniform polyhedra



(3.4.4.4)



(3.4.4.4)*

Figure 1. A uniform polyhedron and a nonuniform Archimedean polyhedron with the same vertex neighborhoods.

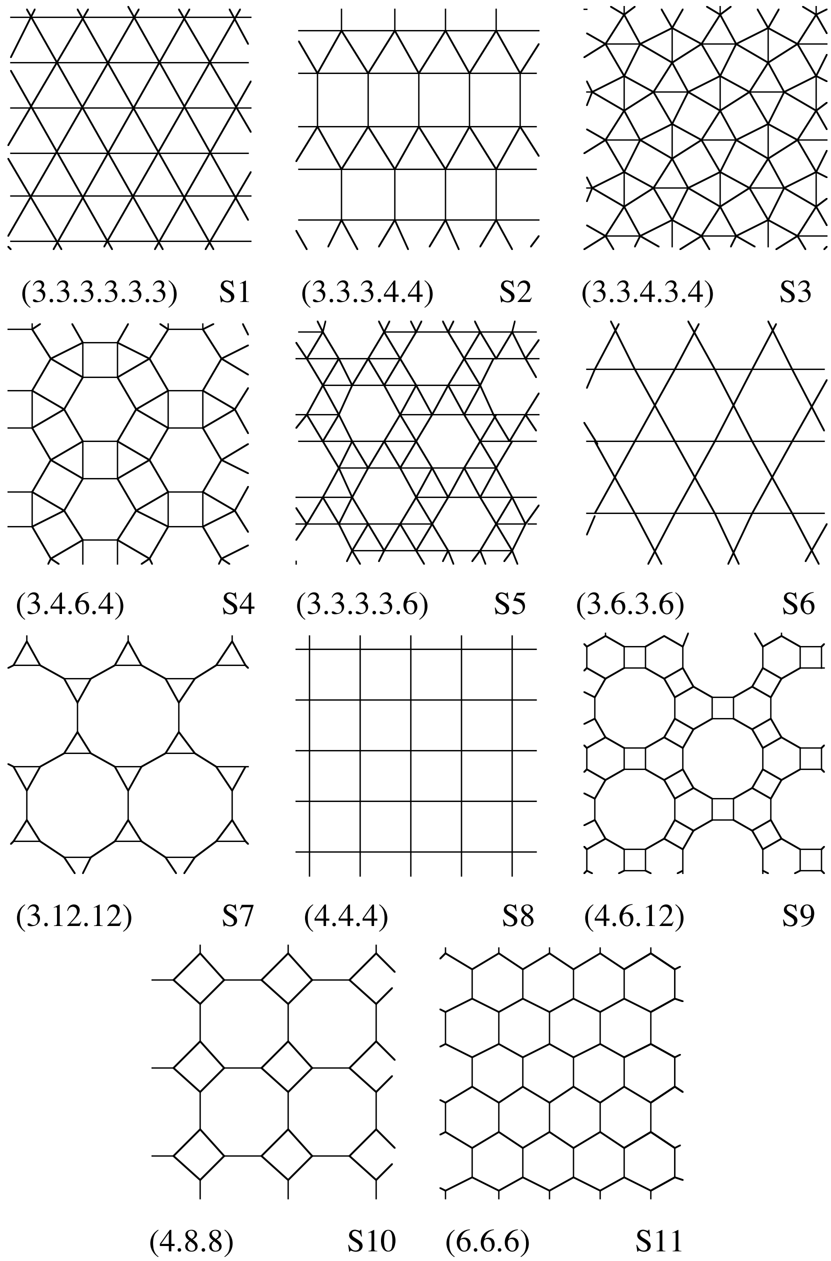
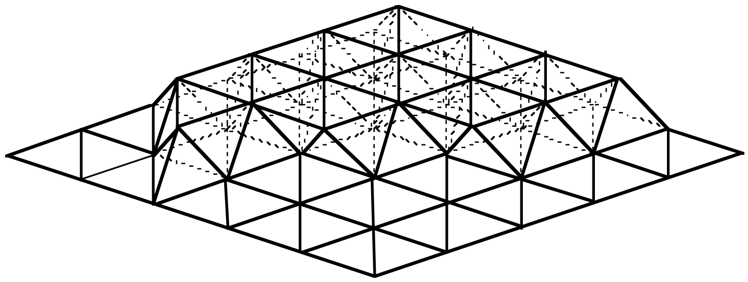


Figure 2. The eleven uniform tilings of the plane, which give rise to the slabs S1 to S11. By simple stacking, these slabs yield uniform tilings listed as ##11, 13, 14, 16, 17, 18, 19, 22, 23, 24, and 26 in Table 1.

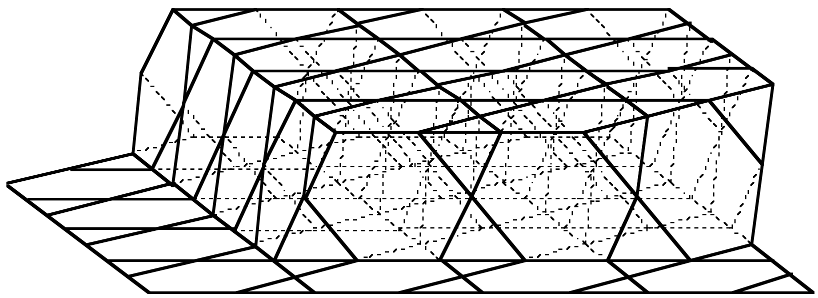
that could fill up the space around a vertex, and finally considering for each case whether a "candidate" arrangement can actually be uniformly extended from the neighborhood of a vertex to a tiling of the whole space. This seems to be close to the method followed by Andreini. Clearly, such a procedure leaves ample space for errors, and I was very happy when I found that my enumeration coincides with Johnson's. The result of the enumeration is given in List 1. To save space in the list, the following remarks are useful.

A *slab* is the part of space between a pair of parallel planes. Many of the uniform tilings can be viewed as consisting of stacks of slabs. For example, for each of the eleven uniform tilings of the plane shown in Figure 2 (see Grünbaum & Shephard (1987, Chapter 2) for a more detailed discussion) we can form a slab consisting of prisms with squares as faces of the mantle; we denote these slabs by S1 to S11. Tilings ##11, 13, 14, 16, 17, 18, 19, 22, 23, 24, and 26 in Table 1 are obtained by stacks of slabs of one kind. Three other slabs are shown in Figure 3. Both tilings ##1 and 2 are formed by the same slabs S12 consisting of tetrahedra and truncated tetrahedra; the difference is that in tiling #1 each of the triangles that form the boundary of the slabs is adjacent to a tetrahedron in one slab and an octahedron in the other, while in tiling #2 each such triangle is adjacent either to two tetrahedra, or to two octahedra. Tiling #6 consists of slabs S13 of tetrahedra and truncated tetrahedra, while tiling #12 consists of slabs S14 made from 3-sided prisms arranged so that the boundary of the slab is formed by squares; in adjacent slabs the orientations of the prisms differ by 90°. Tilings ##3 and 4 result from tilings ##1 and 2 by intercalation of slabs S1 of 3-sided prisms (with triangles on the slab boundary), while tiling #15 is obtained from tiling #12 by intercalating slabs of cubes. Thus 18 of the 28 tilings are best understood as consisting of slabs; the other ten have a more intricate structure. Lack of space prevents us from showing illustrations of these ten tilings; however, they all can be found in both Critchlow (1970) and Williams (1972).

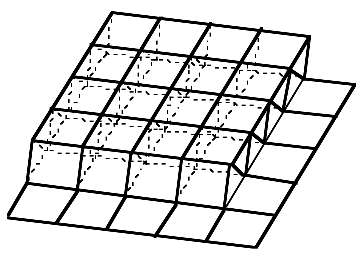
It must be noted that Critchlow and Williams are not explicit in the characterization of the kind of tilings they show; they neither claim, nor achieve, completeness in listing uniform tilings. Moreover, although Archimedean tilings of 3-space fit the class of tilings they discuss, they both miss the fact that there are uncountably many such tilings. Nonuniform Archimedean tilings of 3-space can be obtained simply by alternating slabs S14 and S8, and utilizing the freedom to choose one of the two possible orientations of S14 at each level (the



Slab S12 consisting of tetrahedra and octahedra.



Slab S13 consisting of tetrahedra and truncated tetrahedra.



Slab S14 consisting of 3-sided pyramids.

Figure 3. Three slabs, not generated by prisms over uniform tilings of the plane, which are used in the construction of uniform tilings. To facilitate the viewing, the "bottom" of each slab is shown in front of the polyhedra that form the slab.

uniform tilings ##13 and 15 correspond to consistent choices for positions of S14 throughout the stack; all other choices yield nonuniform tilings). Another such family results from alternating slabs
uniform tilings ##13 and 15 correspond to consistent choices for positions of S14 throughout the stack; all other choices yield nonuniform tilings). Another such family results from alternating slabs

S12 and S1 (here the consistent choices of positions for the S12 slabs yields uniform tilings ## 3 and 4). I do not know whether there exist other nonuniform Archimedean tilings.

List 1. UNIFORM TILINGS OF EUCLIDEAN 3-SPACE.

These are tilings by uniform polyhedra, in which the polyhedra meet face-to-face, and all vertices are equivalent under symmetries of the tiling. The polyhedra are indicated by their "Schläfli" symbol, and the superscripts show how many polyhedra of the given kind meet at each vertex of the tiling. "Ratio" indicates the ratio of the quantity of the different polyhedra in the tiling. The references are as follows: A stands for Andreini (1905); C stands for Critchlow (1970); W stands for Williams (1972); J stands for Johnson (1991), and the two digits xy that follow stand for Johnson's expression 4.51xy.

#1. $(3.3.3)^8.(3.3.3.3)^6$ -A; ratio 1 : 2.
Tetrahedra and octahedra; stacks of S12, with tetrahedra and octahedra meeting at boundaries of slabs. [A-2, C, W-9, J-21 = J-31 = J-51]

#2. $(3.3.3)^8.(3.3.3.3)^6$ -B; ratio 1 : 2.
Reflected layers of octahedra and tetrahedra; stacks of S12, with tetrahedra meeting tetrahedra and octahedra meeting octahedra at boundaries of slabs. [A-2', J-52]

#3. $(3.3.3)^4.(3.3.3.3)^3.(3.4.4)^6$ -A; ratio 2 : 1 : 3.
Alternating layers of 3-sided prisms and layers of tetrahedra and octahedra; slabs S12 (as in #1) intercalated by slabs of S1. [J-61]

#4. $(3.3.3)^4.(3.3.3.3)^3.(3.4.4)^6$ -B; ratio 2 : 1 : 3.
Alternating layers of 3-sided prisms and reflected layers of octahedra and tetrahedra; slabs S12 (as in #2) intercalated by slabs of S1. [J-62]

#5. $(3.3.3).(3.4.4.4)^3.(4.4.4)$; ratio 2 : 1 : 1.
Tetrahedra, rhombicuboctahedra and cubes. [A-16, C, W-11, J-23]

#6. $(3.3.3)^2.(3.6.6)^6$; ratio 1 : 1.
Tetrahedra and truncated tetrahedra; stacks of slabs S13. [A-13, C, W-10, J-25 = J-33]

#7. $(3.3.3.3)^2 (3.4.3.4)^4$; ratio 1 : 1.
Octahedra and cuboctahedra. [A-15, C, W-14, J-12 = J-32]

#8. $(3.3.3.3).(3.8.8)^4$; ratio 1 : 1.

Octahedra and truncated cubes. [A-14, C, W-15, J-13]

#9. $(3.4.3.4).(3.4.4.4)^2.(4.4.4)^2$; ratio 1 : 1 : 3.
Cuboctahedra, rhombicuboctahedra and cubes. [A-17, C, W-12, J-14]

#10. $(3.4.3.4).(3.6.6)^2.(4.6.6)^2$; ratio 1 : 2 : 1.
Cuboctahedra, truncated tetrahedra and truncated octahedra. [A-21, C, W-17, J-22 = J-34]

#11. $(3.4.4)^{12}$ -A
Layers of three-sided prisms; stacks of S1. [A-4, C, J-41]

#12. $(3.4.4)^{12}$ -B
Square-faced layers of three-sided prisms, rotated 90°; stacks of slabs S14. [J-63]

#13. $(3.4.4)^6.(4.4.4)^4$ -A; ratio 2 : 1.
Layers of prisms over (3.3.3.4.4); stacks of S2. [A-11', C, J-65]

#14. $(3.4.4)^6.(4.4.4)^4$ -B; ratio 2 : 1.
Layers of prisms over (3.3.4.3.4); stacks of S3. [A-11, C, J-44]

#15. $(3.4.4)^6.(4.4.4)^4$ -C; ratio 2 : 1.
Alternating layers of square-faced layers of three-sided prisms and cubes; the layers of prisms related by rotations; slabs S14 (as in #12) intercalated by slabs of S8. [J-64]

#16. $(3.4.4)^2.(4.4.4)^4.(4.4.6)^2$; ratio 2 : 3 : 1.
Layers of prisms over (3.4.6.4); stacks of S4. [A-9, C, J-47]

#17. $(3.4.4)^8.(4.4.6)^2$; ratio 8 : 1.
Layers of prisms over (3.3.3.3.6); stacks of S5. [A-12, C, J-48]

#18. $(3.4.4)^4.(4.4.6)^4$; ratio 2 : 1.
Layers of prisms over (3.6.3.6); stacks of S6. [A-8, C, J-43]

#19. $(3.4.4)^2.(4.4.12)^4$; ratio 2 : 1.
Layers of prisms over (3.12.12); stacks of S7. [A-7, J-46]
(Both C and W list this tiling, but show incorrect diagrams.)

#20. $(3.4.4.4).(3.8.8).(4.4.4).(4.4.8)^2$; ratio 1 : 1 : 3 : 3.
Rhombicuboctahedra, truncated cubes, cubes and octagonal prisms. [A-19, W-19, J-18]

#21. $(3.6.6).(3.8.8).(4.6.8)^2$; ratio 2 : 1 : 1.
Truncated tetrahedra, truncated cubes and truncated cuboctahedra. [A-20, C, W-16, J-24]

#22. $(4.4.4)^8$
Cubes; stacks of S8. [A-1, C, W-1, J-11 = J-15]

#23. $(4.4.4)^2.(4.4.6)^2.(4.4.12)^2$; ratio 3 : 2 : 1.
Layers of prisms over (4.6.12); stacks of S9. [A-10, C, W-8, J-49]

#24. $(4.4.4)^2.(4.4.8)^4$; ratio 1 : 1.
Layers of prisms over (4.8.8); stacks of S10. [A-6, C, J-45]

#25. $(4.4.4).(4.6.6).(4.6.8)^2$; ratio 3 : 1 : 1.
Cubes, truncated octahedra and truncated cuboctahedra. [A-18, C, W-13, J-17]

#26. $(4.4.6)^6$
Layers of hexagonal prisms; stacks of S11. [A-5, C, W-3, J-42]

#27. $(4.4.8)^2.(4.6.8)^2$; ratio 3 : 1.
Octagonal prisms and truncated cuboctahedra. [A-22, C, W-18, J-19]

#28. $(4.6.6)^4$
Truncated octahedra. [A-3, C, W-2, J-16 = J-35]

References:

Angelo Andreini, Sulle reti di poliedri regolari e semiregolari e sulle corrispondenti reti correlative. Mem. Societa Italiana delle Scienze, Ser. 3, Vol.14 (1905), pp. 75 - 129.

Keith Critchlow, Order in Space. Viking Press, New York 1970.

Branko Grünbaum and G. C. Shephard, Tilings and Patterns. Freeman, New York 1987.

Norman W. Johnson, Uniform Polytopes. Manuscript, 1991.

Robert Williams, Natural Structure. Eudaemon Press, Moorpark, CA, 1972. Reprinted as: The Geometrical Foundation of Natural Structure, Dover, New York, 1979.