Solution to Monthly Problem 11567

Adam M. Goyt
Department of Mathematics
Minnesota State University Moorhead
Moorhead, MN 56563
goytadam@mnstate.edu

Lara K. Pudwell
Department of Mathematics and Computer Science
Valparaiso University
Valparaiso, IN 46383
Lara.Pudwell@valpo.edu

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How many arrangements $(a_1, a_2, \ldots, a_{2n})$ of the multiset $\{1, 1, 2, 2, \ldots, n, n\}$ satsify the following two conditions: (i) All entries between the two occurrences of any given value i exceed i, and (ii) No three entries increase from left to right with the last two adjacent?

Theorem 0.1. *The number of such arrangements is given by* $\displaystyle\sum_{k=1}^{n} 2^{n-k}\left\{ {n \atop k} \right\}$, *where* $\left\{ {n \atop k} \right\}$ *denotes the Stirling number of the second kind.*

Proof. Let A_n be the set of arrangements of $\{1, 1, 2, 2, \ldots, n, n\}$ satisfying (i) and (ii) from the problem. We will provide a bijective proof that $|A_n| = \displaystyle\sum_{k=1}^{n} 2^{n-k}\left\{ {n \atop k} \right\}$ by showing that these arrangements can be put into bijection with a set of colored set partitions which have the same enumeration.

A *partition* of the set $[n] = \{1, 2, \ldots, n\}$ is a family of pairwise disjoint subsets B_1, $B_2, \ldots B_k$ of $[n]$, whose union, $\uplus_{i=1}^{k} B_i = [n]$. We refer to the subsets as blocks, and write our set partitions with the blocks in canonical order $B_1/B_2/\ldots/B_k$ with $\min B_1 < \min B_2 < \cdots < \min B_k$. For example, $1378/25/46$ is the partition of $[7]$ with blocks $\{1, 3, 7, 8\}$, $\{2, 5\}$, and $\{4, 6\}$. We denote the set of all partitions of $[n]$ by Π_n.

Let $\Pi_n \wr C_2$ be the set of partitions of the set $[n]$ such that each element is colored either (r)ed or (b)lue. An example of one of these colored set partitions is $1^r 3^r 7^b 8^r/2^r 5^r/4^r 6^b$.

Let $P_n \subseteq \Pi_n \wr C_2$ be the set containing only those partitions whose first element in each block is colored red. Notice that this means that if a partition has k blocks then the $n-k$ elements that are not the smallest in any block can be colored either red or blue. Since there are $\left\{ n \atop k \right\}$ partitions of $[n]$ with k blocks, there are $2^{n-k}\left\{ n \atop k \right\}$ such partitions with k blocks in P_n. Summing over k shows that $|P_n| = \displaystyle\sum_{k=1}^{n} 2^{n-k}\left\{ n \atop k \right\}$.

To construct a bijection between A_n and P_n, we need to understand the structure of the elements of A_n. Let $\alpha \in A_n$ and locate the two copies of i. By condition (i), all numbers between these two must be larger than i and by (ii) these numbers must appear in decreasing order. If $j > i$ then both occurrences of j appear before, between or after both i's. Finally if k and j are larger than a left to right minimum i and appear after i, but k and j appear before the next left-to-right minimum then k and j appear in decreasing order otherwise we violate (ii).

Let $\phi : P_n \to A_n$ be defined as follows. Suppose $\pi = B_1/B_2/\dots/B_k$. Define $\phi(\pi) = \phi(B_k)\phi(B_{k-1})\dots\phi(B_1)$ be the arrangement constructed as follows. For any block B_i let $m_i = \min B_i$ and let $\phi(B_i) = m_i\beta_i m_i\rho_i$, where β_i is the decreasing sequence of the blue colored elements from B_i and ρ_i is the decreasing sequence of the red colored elements from B_i except m_i. For example, $\phi(1^r3^r7^b8^r) = 17718833$, and $\phi(1^r3^r7^b8^r/2^r5^r/4^r6^b) = \phi(4^r6^b)\phi(2^r5^r)\phi(1^r3^r7^b8^r) = 4664225517718833$.

ϕ has an obvious inverse, and thus ϕ must be a bijection. Hence, we must have that $|A_n| = \displaystyle\sum_{k=1}^{n} 2^{n-k}\left\{ n \atop k \right\}$.

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