About a New Kind of Ramanujan-Type Series

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CONTENTS

We propose a new kind of Ramanujan-type formula for $1/\pi^2$ and conjecture that it is related to the theory of modular functions.

1. INTRODUCTION

In my papers [Guillera 02, Guillera 03], I prove the identities

$$\sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}\right)_n^5}{n!^5 2^{10n}} (820n^2 + 180n + 13) = \frac{128}{\pi^2}, \qquad (1–1)$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3 \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{n!^5 2^{4n}} (120n^2 + 34n + 3) = \frac{32}{\pi^2}, \qquad (1–2)$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}\right)_n^5}{n!^5 2^{2n}} (20n^2 + 8n + 1) = \frac{8}{\pi^2}. \qquad (1–3)$$

Inspired by these results and by Ramanujan's formulas [Borwein and Borwein 87, Chudnovsky and Chudnovsky 88, Ramanujan 14], I had the feeling that more formulas of the same type could exist. So, I experimented in order to find them. I now describe that research.

2. RAMANUJAN-TYPE FORMULAS

The kind of formulas we are looking for have the form

$$\sum_{n=0}^{\infty} \frac{B(n)}{q^n} (an^2 + bn + c) = \frac{d\sqrt{k}}{\pi^2}, \qquad (2–1)$$

where d, k, a, b, c are integers, $B(n) = n!^{-5}\ C(n)$ or $B(n) = (-1)^n n!^{-5} C(n)$, and $C(n)$ is the product of 5 rising factorials of fractions smaller than unity satisfying the following condition: For every denominator in the fraction of a rising factorial, we must have rising factorials with all possible nonreducible fractions corresponding to that denominator. Taking this into account, we have the following cases for $C(n)$:

$$\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n,$$

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$$\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n,$$
$$\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n,$$
$$\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n,$$
$$\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n,$$
$$\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n,$$
$$\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n,$$
$$\left(\frac{1}{2}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n,$$
$$\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n,$$
$$\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n,$$
$$\left(\frac{1}{2}\right)_n \left(\frac{1}{8}\right)_n \left(\frac{3}{8}\right)_n \left(\frac{5}{8}\right)_n \left(\frac{7}{8}\right)_n,$$
$$\left(\frac{1}{2}\right)_n \left(\frac{1}{5}\right)_n \left(\frac{2}{5}\right)_n \left(\frac{3}{5}\right)_n \left(\frac{4}{5}\right)_n,$$
$$\left(\frac{1}{2}\right)_n \left(\frac{1}{12}\right)_n \left(\frac{5}{12}\right)_n \left(\frac{7}{12}\right)_n \left(\frac{11}{12}\right)_n,$$
$$\left(\frac{1}{2}\right)_n \left(\frac{1}{10}\right)_n \left(\frac{3}{10}\right)_n \left(\frac{7}{10}\right)_n \left(\frac{9}{10}\right)_n.$$

For q, we consider

$$q = j^2, \quad q = j^2 - 1, \quad q = j^3, \quad q = (j^2-1)^3, \quad q = j^4,$$

where j is also an integer. We will look for integer relations between

$$F_0 = \sum_{n=0}^{\infty} \frac{B(n)}{q^n}, \quad F_1 = \sum_{n=0}^{\infty} \frac{B(n)}{q^n} n,$$
$$F_2 = \sum_{n=0}^{\infty} \frac{B(n)}{q^n} n^2, \quad G = \frac{\sqrt{k}}{\pi^2}.$$

This means that we want to find integers a, b, c, and d such that $aF_0 + bF_1 + cF_2 + dG = 0, \quad d \neq 0$. The algorithms that solve this problem are called *integer relations algorithms*. The software we are using for this purpose is PARI-GP, because it is very fast at making numerical calculations and has the LINDEP function which looks for integer relations. To avoid the integer variable k, we also use a variant of this method and look for integer relations between

$$F_0^2, \quad F_1^2, \quad F_2^2, \quad F_0F_1, \quad F_0F_2, \quad F_1F_2, \quad \frac{1}{\pi^4}.$$

This variant is especially interesting if there exist formulas with large values of k. The new formulas my computer found using these numerical methods are

$$\sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{n!^5 2^{10n}} \times (1640n^2 + 278n + 15) = \frac{256\sqrt{3}}{3\pi^2}, \tag{2–2}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n}{n!^5 48^n} \times (252n^2 + 63n + 5) = \frac{48}{\pi^2}, \tag{2–3}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{n!^5 80^{3n}} \times (5418n^2 + 693n + 29) = \frac{128\sqrt{5}}{\pi^2}, \tag{2–4}$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{8}\right)_n \left(\frac{3}{8}\right)_n \left(\frac{5}{8}\right)_n \left(\frac{7}{8}\right)_n}{n!^5 7^{4n}} \times (1920n^2 + 304n + 15) = \frac{56\sqrt{7}}{\pi^2}. \tag{2–5}$$

Once the software PARI-GP found these series, I used Maple to check again if they were correct. The numerical results show that they are correct to hundreds of digits. Now examine the following Ramanujan-type formulas [Borwein and Borwein 87, Chudnovsky and Chudnovsky 88, Ramanujan 14]:

$$\sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{n!^3 48^n} (28n + 3) = \frac{16\sqrt{3}}{3\pi}, \tag{2–6}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{n!^3 80^{3n}} (5418n + 263) = \frac{640\sqrt{15}}{3\pi}, \tag{2–7}$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{n!^3 7^{4n}} (40n + 3) = \frac{49\sqrt{3}}{9\pi}. \tag{2–8}$$

It is interesting to observe that the numbers $48, 80^3, 7^4$ are repeated in the denominators. This leads me to think that formulas of type (2–1), such as (1–1), (1–2), (1–3), (2–2), (2–3), (2–4), and (2–5), can be proved using the theory of modular functions, as is the case with Ramanujan-like formulas, (2–6), (2–7), and (2–8).

3. SUPPORTING THE CONJECTURE

To support this conjecture, I will explain the origin of the number 80^3 in formula (2–7). We begin by considering Klein's absolute invariant [Borwein and Borwein 87, Chudnovsky and Chudnovsky 88]

$$J(q) = \frac{4}{27}\frac{[1-\lambda(q)+\lambda^2(q)]^3}{\lambda^2(q)[1-\lambda(q)]^2},$$

where

$$\lambda(q) = \left[\frac{\vartheta_2(q)}{\vartheta_3(q)}\right]^4.$$

It is known [Chudnovsky and Chudnovsky 88] that when d is an integer such that $Q(\sqrt{-d})$ has class number 1, then $J\left(\frac{1+\sqrt{-d}}{2}\right)$ is also an integer. For these singular values, there exist [Chudnovsky and Chudnovsky 88] integers a, b, c, k such that

$$\sum_{n=0}^{\infty} \frac{(12)^{3n}\left(\frac{1}{2}\right)_n\left(\frac{1}{6}\right)_n\left(\frac{5}{6}\right)_n}{n!^3 J^n\left(\frac{1+\sqrt{-d}}{2}\right)}(an+b) = \frac{c\sqrt{k}}{\pi}. \qquad (3–1)$$

There are not many numbers with that property: 2, 3, 7, 11, 19, 43, 67, and 163. For $d = 43$, we have $J\left(\frac{1+\sqrt{-43}}{2}\right) = -960^3$, and the corresponding formula is (2–7). Our new formula (2–4) is intriguing because of the repetition of the numbers 80^3 and 5418. I think that one can find a proof of this formula using the theory of modular functions.

In [Berggren et al. 00], one can find the references [Chudnovsky and Chudnovsky 88] and [Ramanujan 14] and many more fascinating papers. In addition, the paper, [Berndt and Chan 01], reinforces the hope that the theory of modular forms is the key to proving the formulas developed in this paper.

4. RELATED FORMULAS

Boris Gourevitch [Gourevitch 02] has sent me, by email, the formula below for $1/\pi^3$. He has found it by using *integer relations algorithms*:

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^7}{n!^7\, 2^{6n}}(168n^3+76n^2+14n+1) = \frac{32}{\pi^3}. \qquad (4–1)$$

On the other hand, we consider the functions

$$F(k) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}} \frac{\left(\frac{1}{2}+k\right)_n^5}{(1+k)_n^5}[20(n+k)^2+8(n+k)+1]$$

$$G(k) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{10n}} \frac{\left(\frac{1}{2}+k\right)_n^5}{(1+k)_n^5}[820(n+k)^2+180(n+k)+13]$$

$$H(k) = \sum_{n=0}^{\infty} \frac{1}{2^{6n}} \frac{\left(\frac{1}{2}+k\right)_n^7}{(1+k)_n^7}[168(n+k)^3+76(n+k)^2 + 14(n+k)+1].$$

We have seen in (1–3), (1–1), and (4–1) that

$$F(0) = \frac{8}{\pi^2}, \quad G(0) = \frac{128}{\pi^2}, \quad H(0) = \frac{32}{\pi^3}.$$

It is very curious that using the Simon Plouffe inverter [Plouffe], we find

$$F\left(\frac{1}{2}\right) = 7\cdot\zeta(3), \quad G\left(\frac{1}{2}\right) = 256\cdot\zeta(3), \quad H\left(\frac{1}{2}\right) = \frac{\pi^4}{2}.$$

The evaluation $G(1/2)$ has been proved by T. Amdeberhan [Amdeberhan 97].

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