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ON KUREPA'S HYPOTHESIS FOR THE LEFT FACTORIAL

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Abstract. In this paper sequences of integer numbers are defined and their properties are examined. The equivalent of Kurepa's hypothesis for the left factorial is given using the sequence $\{d_n\}$. It is shown that the sequence $\{d_n\}$ the base of papers of [5](G. V. Milovanović) and [8], [9] (Z. Šami). In view of the generalization of Wilson's theorem given in [1] (V. Kirin) it is shown that some results of papers [8] and [9] can be obtained by elementary methods. The prablem which is more general then the problem of Kurepa's hypothesis is considered.

1. Introduction

In [2] (Đ. Kurepa), it is defined left factorial $!n$ with $!n = 0! + 1! + 2! + \cdots + (n-2)! + (n-1)!$ Also, the hypothesis, which is called latter *Kurepa's hypothesis for left factorial* (KH), is formulated

$$(!n\ n!) = 2, \quad n \in N,\ n > 1\,, \tag{1}$$

where $(!n\ n!)$ is the greatest common divisor for $!n$ and $n!$.
In [2], it is proved that the equivalent assertion for (1) is the assertion that for any prime numbers p, $p > 2$ it applies:

$$!p \not\equiv 0\ (mod\, p). \tag{2}$$

The left factorial in complex domain is defined by

$$!z = \int_0^\infty e^{-x}\, \frac{x^z - 1}{x - 1} dx \tag{3}$$

where z is a complex number, $(Re\, z > 0)$.
It is proved that

$$!(z+1) = \Gamma(z+1) + !z\,,$$

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where $\Gamma(z)$ is a gamma function given by

$$\Gamma(z)=\int_0^{\infty} e^{-x}x^{z-1}dx\,.$$

Using the function

$$f(x)=\frac{e^{-x}}{1-x}\,,\; n\in N_0$$

Z. Šami defined in [8] and [9] the sequence y_n:

$$y_n=f^{(n)}(0)\,,$$

i.e.

$$y_n=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}k!\,. \tag{4}$$

The terms of sequence y_n are $y_0=1$, $y_1=0$, $y_2=1$, $y_3=2$, $y_4=9$, $y_5=44,\ldots$

In view of properties of sequence y_n it is proved that for any prime number the following hold:

$$\begin{aligned}&\left[\frac{(p-1)!}{e}\right]+1\equiv !p\;(mod\,p)\\&\left[\frac{p!}{e}\right]\equiv -1\;(mod\,p)\,,\\&\left[\frac{(p-1)!}{e}\right]+\left[\frac{(p-2)!}{e}\right]\equiv 0\;(mod\,p)\end{aligned} \tag{5}$$

wher $[x]$ is a finction defined by $[x]\in Z$ and $[x]\le x<[x]+1$.
The sequence $u_m(x)$, $m\in Z$:

$$u_m(x)=\begin{cases} e^x\int_0^x dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{m-1}}f(t_m)dt_m, & m>0\\ e^x f^{(-m)}(x), & m\le 0\end{cases}$$

is defined.
By the sequence $u_m(x)$ the sequence $u_{n,m}$, $n,m\in N$:

$$u_{n,m}(x)=u_m^{(n)}(0)\,,$$

is defined, and it is proved that following statements

$$(7)\qquad \begin{array}{r} (\exists k)(k \geq p \wedge u_{k,2} \not\equiv 1\ (mod\, p)),\ \text{for all primes}\ p \geq 3,\\ u_{p-1,2} \not\equiv 0\ (mod\, p),\ \text{for all primes}\ p \geq 3,\\ u_{p-2,2} \not\equiv 0\ (mod\, p),\ \text{for all primes}\ p \geq 3,\\ u_{p+1,2} \not\equiv p+1\ (mod\, p^2),\ \text{for all primes}\ p \geq 3 \end{array}$$

are equivalent to K. H.

The sequence

$$(8)\qquad S_t = t! \sum_{i=0}^{t} \frac{(-1)^i}{i!},\ (i \geq o),\ i.e.$$

$$(9)\qquad S_t = tS_{t-1} + (-1)^t,\ for S_0 = 1$$

is defined by G. V. Milovanović in [5]. The terms of sequence S_t are $S_0 = 1$, $S_1 = 0$, $S_2 = 1$, $S_3 = 2$, $S_4 = 9$, $S_5 = 44, \ldots$ In view of the sequence S_t the function

$$(10)\qquad K_m(n) = \sum_{t=0}^{n-1} \binom{m+n}{t+m+1} S_t\,,$$

is defined and its properties are examined.

Wilson's Theorem: The necessary and the sufficient condition for a number $p > 2$ to be a prime number is for

$$(11)\qquad (p-1)! + 1 \equiv 0\, (mod\, p)$$

to hold.

V. Kirin, [1], give the generalization of Wilson's Theorem:

Theorem 1. *The necessary and the sufficient condition for a number $n > 2$ to be prime is that for every $m \in N$ the following holds*

$$(12)\qquad (m-1)!(n-m)! + (-1)^{m-1} \equiv 0\, (mod\, n)$$

Two proves of this Theorem are given in [6].

2. Application of generalization of Wilson's Theorem

The statement (5) can be proved using the Theorem 1. Let p be a prime number and $!p \equiv t \ (mod\, p)$

$$\Leftrightarrow (p-1)! + (p-2)! + \cdots + 2! + !1 + 0! \equiv t \ (mod\, p)$$
$$\Leftrightarrow (p-3)! + (p-4)! + \cdots + 2! + !1 + 0! \equiv t \ (mod\, p) \ \ /2!$$
$$\Leftrightarrow 2!(p-3)! + 1 + 2!((p-4)! + \cdots + 2! + !1 + 0!) - 1 \equiv 2!t \ (mod)$$
$$\Leftrightarrow 2!((p-4)! + \cdots + 2! + !1 + 0!) - (\frac{1!}{1!}) \equiv 2!t \ (mod\, p) \ /3$$
$$\Leftrightarrow 3!(p-4)! - 1 + 3!((p-5)! + \cdots 2! + !1 + 0!) - 3 + 1 \equiv 3!t \ (mod\, p)$$
$$\Leftrightarrow 3!((p-5)! + \cdots + 2! + !1 + 0!) - \left(\frac{3!}{2!}\right) + \left(\frac{3!}{3!}\right) \equiv 3!t \ (mod\, p) \ /4$$
$$\Leftrightarrow 4!(p-5)! + 1 + 4!((p-6)! + \cdot + 2! + !1 + 0!) - \left(\frac{4!}{2!}\right) + \left(\frac{4!}{3!}\right) - \left(\frac{4!}{4!}\right) \equiv 4!t \ (mod\, p)$$
$$\Leftrightarrow 4!((p-6)! + \cdots + 2! + !1 + 0!) - \left(\frac{4!}{2!}\right) + \left(\frac{4!}{3!}\right) - \left(\frac{4!}{4!}\right) \equiv 4!t \ (mod\, p)$$

$\vdots$

after k steps we get

$$(13) \quad k!((p-k-2)! + \cdots + 0!) - \left(\frac{k!}{2!}\right) + \cdots + (-1)^{k+1}\left(\frac{k!}{k!}\right) \equiv k!t \ (mod\, p)$$

if in expression (13) we take $k = p-2$ we get:

$$(p-2)!\left(1 - \frac{1}{2!} + \frac{1}{3!} - \cdots - \frac{1}{(p-3)!} + \frac{1}{(p-2)!}\right) \equiv (p-2)!\, t \ (mod\, p)$$
$$\Leftrightarrow (p-2)!\left(1 - \frac{1}{2!} + \frac{1}{3!} - \cdots - \frac{1}{(p-3)!} + \frac{1}{(p-2)!}\right) \equiv (p-2)!t - t + t \ (mod\, p)$$
$$\Leftrightarrow (p-2)!\left(1 - \frac{1}{2!} + \frac{1}{3!} - \cdots - \frac{1}{(p-3)!} + \frac{1}{(p-2)!}\right) \equiv t \ (mod\, p)$$

$$(14) \qquad (p-2)!\left(1 - \frac{1}{2!} + \frac{1}{3!} - \cdots - \frac{1}{(p-3)!} + \frac{1}{(p-2)!}\right) \equiv p \ (mod\, p)$$

From expansion of the function e^{-1} into series we have

$$(15) \qquad 1 - \frac{1}{2!} + \frac{1}{3!} - \cdots - \frac{1}{(p-3)!} + \frac{1}{(p-2)!} = 1 - e^{-1} + \frac{e^{\alpha}}{(p-1)!}$$

where is $e^\alpha < 1$.
If we substitute expression (15) in (14) we have:

$$(14) \Leftrightarrow (p-2)!\left(1-e^{-1}+\frac{e^\alpha}{(p-1)!}\right) \equiv !p \ (mod\, p)$$

$$\Leftrightarrow (p-2)!-1+(p-2)!\left(-e^{-1}+\frac{e^\alpha}{(p-1)!}\right)+1 \equiv !p \ (mod\, p)$$

$$\Leftrightarrow (p-2)!\left(\frac{e^\alpha}{(p-1)!}-e^{-1}\right)+1 \equiv !p \ (mod\, p) \ / -(p-1)$$

$$\Leftrightarrow -(p-1)!\left(\frac{e^\alpha}{(p-1)!}-e^{-1}\right)-p+1 \equiv -!p(p-1) \ (mod\, p)$$

$$\Leftrightarrow -(p-1)!\cdot\frac{e^\alpha}{(p-1)!}+(p-1)! \ e^{-1}+1 \equiv !p \ (mod\, p)$$

$$\Leftrightarrow \frac{(p-1)!}{e}+1-e^\alpha \equiv !p \ (mod)$$

The Theorem has been proved.

Remark. The majority of results given in [8] and [9] can be proved using the above method.

3. The sequence $\{d_n\}$

Definition 1. The sequence of integers $\{d_n\}$ is defined by the following recurrent formula:

$$d_1 = -1\,,$$
$$d_n = -(n+1)d_{n-1} - 1\,,$$

for every natural number n.

The terms of sequence $\{d_n\}$ are $d_1 = -1$, $d_2 = 2$, $d_3 = -9$, $d_4 = 44$, $d_5 = -265, \ldots$. Sequence $\{d_n\}$ is the union of two disjuctive sub-sequences, a sub-sequence whose terms are negative numbers $\{d_n^-\}$ and a sub-sequence whose terms are positive numbers $\{d_n^+\}$.

In view of (4) and (8) we get that

$$y_n = S_n\,,\ n = 0, 1, 2, \ldots$$

The Definition 1 inplies that

$$y_{n+1} = S_{n+1} = d_n \text{ for } n = 2k,\quad k = 1, 2, 3, \ldots$$
$$y_{n+1} = S_{n+1} = -d_n \text{ for } n = 2k-1,\quad k = 1, 2, 3, \ldots$$

Definition 2. $d_j \in \{d_n^-\} \iff (d_j \in \{d_n\} \wedge d_j < 0)$.

Definition 3. $d_j \in \{d_n^+\} \iff (d_j \in \{d_n\} \wedge d_j > 0)$.

Consequence 1. *Sequence $\{d_n^-\}$ is given by the following recurrent formula:*

$$\begin{aligned} d_1^- &= -1 , \\ d_n^- &= (2n-1)(2nd_{n-1}^- + 1) , \end{aligned}$$

for every natural number n.

Consequence 2. *Sequence $\{d_n^+\}$ is given by the following recurrent formula:*

$$\begin{aligned} d_1^+ &= 2 , \\ d_n^+ &= 2n((2n+1)d_{n-1}^+ + 1) , \end{aligned}$$

for any natural number n.

Theorem 2. *For every term of sequence $\{d_n\}$, it applies:*
a) $d_n + 1 \equiv 0 \ (mod \ n-1)$
b) $d_n \equiv 0 \ (mod \ n)$
c) $d_n + 1 \equiv 0 \ (mod \ n+1)$

Cosequence 3. *For every term of sequence $\{d_n^-\}$, it applies:*
a) $d_n^- + 1 \equiv 0 \ (mod \ 2n-2)$
b) $d_n^- \equiv 0 \ (mod \ 2n-1)$
c) $d_n^- + 1 \equiv 0 \ (mod \ 2n)$

Theorem 3. *For every term of sequence $\{d_n\}$ it applies:*
a) $d_i > 0 \Rightarrow \frac{d_{i+2}}{(|d_i|+|d_{i+1}|)} = i+2$
b) $d_i < 0 \Rightarrow \frac{d_{i+2}}{(|d_i|+|d_{i+1}|)} = -(i+2)$

Consequence 4. *For every term of sequence $\{d_n\}$ it applies:*

$$d_{i+2} \equiv 0 \ (mod \ |d_i| + |d_{i+1}|) .$$

4. Equivalent to KH

Theorem 4. *Let p be a prime number. Then:*

$$!p \equiv -d_{p-2} \ (mod\, p) ,$$

where $d_{p-2} \in \{d_n\}$.

Proof.

$$(p-1)! + (p-2)! + (p-3)! + \cdots + 2! + 1! + 0! \equiv !p \ (mod\, p)$$
$$\Leftrightarrow (p-3)! + (p-4)! + \cdots + 2! + 1! + 0! \equiv !p \ (mod\, p) \ /(p-2) \equiv \mathbf{-2}$$
$$\Leftrightarrow (p-2)! + (p-2)(p-4)! + \cdots + (p-2)1! + (p-2)0! \equiv !p\,(-2!) \ (mod\, p)$$
$$\Leftrightarrow (p-2)(p-4)! + \cdots + (p-2)1! + (p-2)0! \equiv \mathbf{-1}+!p\,(-2!) \ (mod\, p)/(p-3) \equiv \mathbf{-3}$$
$$\Leftrightarrow (p-2)! + (p-2)(p-3)(p-5)! + \cdots + (p-2)(p-3)1! + (p-2)(p-3)0!$$
$$\equiv 3+!p(3!) \ (mod\, p)$$
$$\Leftrightarrow (p-2)(p-3)(p-5)! + \cdots + (p-2)(p-3)1! + (p-2)(p-3)0!$$
$$\equiv \mathbf{2}+!p(3!) \ (mod\, p)/(p-4) \equiv \mathbf{-4}$$
$$\vdots$$
$$\Leftrightarrow (p-2)\cdots(p-(p-4))2! + (p-2)\cdots(p-(p-4))1! + (p-2)\cdots(p-(p-4))0!$$
$$\equiv \mathbf{d}_{p-5}+!p\,((p-4)!) \ (mod\, p)/(p-(p-3)) \equiv -(p-3)$$
$$\Leftrightarrow (p-2)! + \frac{(p-2)!}{2} + \frac{(p-2)!}{2} \equiv -(p-3)d_{p-5}+!p\,(-(p-3)!)(mod\, p)$$
$$\Leftrightarrow \frac{(p-2)!}{2} + \frac{(p-2)!}{2} \equiv -(p-3)d_{p-5} - 1+!p\,(-(p-3)!) \ (mod\, p)$$
$$\Leftrightarrow (p-2)! \equiv -(p-3)d_{p-5} - 1+!p\,(-(p-3)!) \ (mod\, p)$$
$$\Leftrightarrow d_{p-4}+!p(-(p-3)!) \equiv (p-2)! \ (mod\, p)$$
$$\Leftrightarrow d_{p-4} - 1 \equiv !p\,(p-3)! \ (mod\, p)/-(p-2)$$
$$\Leftrightarrow d_{p-3} - 1 \equiv !p\,(-(p-2)!) \ (mod\, p)/-(p-1)$$
$$\Leftrightarrow d_{p-2} \equiv !p\,(p-1)! \ (mod\, p)$$
$$\Leftrightarrow -d_{p-2} \equiv !p \ (mod\ p)\ .$$

Definition 4. The sequence of integers $\{d_n\}$ is defined by the following recurrent formula:

$$a_1 = 0$$
$$a_{n+1} = -(n+2)a_n - d_n,$$

for every natural number n and $d_n \in \{d_n\}$.

The terms of sequence $\{a_n\}$ are $a_1 = 0, a_2 = 1, a_3 = -6, a_4 = 39, \ldots.$

Theorem 5. *For any term of sequence* $\{d_n\}$ *if applies:*

$$d_{j-2} \equiv t \ \ (mod\ j) \ \Rightarrow \ d_{j+k} \equiv \ (-1)^k j(k+1)!\,(t+1) + ja_k + d_k \ \ (mod\ j^2)$$

for any integer t and any natural number k, $j > 2$.

Proof. Let $d_{j-2} \equiv t \ (mod\, j) \ / -j$

$$\begin{aligned}
&\Leftrightarrow -jd_{j-2} - 1 \equiv -jt - 1 \ (mod\, j^2)\\
&\Leftrightarrow d_{j-1} \equiv -jt - 1 \ (mod\, j^2) \ / -(j+1)\\
&\Leftrightarrow -(j+1)d_{j-1} - 1 \equiv (-jt-1)(-(j+1)) - 1 \ (mod\, j^2)\\
&\Leftrightarrow d_j \equiv (j\,t+1)(j+1) - 1 \ (mod\, j^2)\\
&\Leftrightarrow d_j \equiv j(t+1) \ (mod\, j^2) \ / -(j+2)\\
&\Rightarrow -(j+2)d_j - 1 \equiv j(t+1)(-(j+2)) - 1 \ (mod\, j^2)\\
&\Rightarrow d_{j+1} \equiv -2!\,j(t+1) + \mathbf{0}j + (\mathbf{-1}) \ (mod\, j^2) \ / -(j+3)\\
&\Rightarrow -(j+3)d_{j+1} - 1 \equiv (j+3)2!\,j(t+1) + j + 3 - 1 \ (mod\, j^2)\\
&\Rightarrow d_{j+2} \equiv 3!j(t+1) + \mathbf{1}j + \mathbf{2} \ (mod\, j^2) \ / -(j+4)\\
&\Rightarrow d_{j+3} \equiv -4!j(t+1) + (\mathbf{-6})j + (\mathbf{-9}) \ (mod\, j^2) \ / -(j+5)\\
&\Rightarrow d_{j+4} \equiv 5!j(t+1) + \mathbf{39}j + \mathbf{44} \ (mod\, j^2) \ / -(j+6)\\
&\vdots\\
&\Rightarrow d_{j+k} \equiv ((-1)^k j(k+1)!(t+1) + ja_k + d_k \ (mod\, j^2).
\end{aligned}$$

Consequence 5. *For any term of sequence $\{d_n\}$ and natural number $r, j > 2$ is :*

$$d_{j-2} \equiv d_{rj-2} \ (mod\, j).$$

Proof. Based on Definition 4 and Theorem 5 taking for $k = j-1$, $k = 2j-1, ..., k = (r-1)j-1$, it is obtained in sequence $d_{j-2} \equiv d_{2j-2} \equiv d_{3j-2} \equiv \cdots \equiv d_{rj-2} \ (mod\, j)$.

5. The generalization of problem

Let p be a prime number. We denote

$$Ne(p) = (p-2)! + (p-4)! + (p-6)! + \cdots + 3! + 1!$$

$$Pa(p) = (p-1)! + (p-3)! + (p-5)! + \cdots + 4! + 2! + 0!$$

In view of the definition of the left factoriel the equality

$$!p = Ne(p) + Pa(p)\,.$$

If $Ne(p) \equiv 0 \ (mod\, p)$ and $Pa(p) \equiv 0\,(mod\, p)$, then $!p \equiv 0\,(mod\, p)$. Does the coverse statement hold ? i.e. if $!p \equiv 0\,(mod\, p)$, does equalities $Ne(p) \equiv 0 \ (mod\, p)$ and $Pa(p) \equiv 0 \ (mod\, p)$ hold ?

Let us define two sequences $\{f_n\}$ and $\{g_n\}$. The sequence $\{f_n\}$ is given by the following recurrent formula:

$$f_1 = -1$$
$$f_{n+1} = (2n-1)2nf_n - 1.$$

The terms of sequence $\{f_n\}$ are $f_1 = -1$, $f_2 = -3$, $f_3 = -37$, $f_4 = -1111, \ldots$ The sequence $\{g_n\}$ is given by the following recurrent formula:

$$g_1 = 1$$
$$g_{n+1} = (2n+1)2ng_n - 1.$$

The terms of sequence $\{g_n\}$ are $g_1 = 1$, $g_2 = 7$, $g_3 = 141$, $g_4 = 5923, \ldots$

In the same way as it done to prove (5), we derive

Theorem 6. *For all a prime number p is*

$$g_{\frac{p-1}{2}} \equiv Ne(p) \ (mod\, p) \wedge f_{\frac{p+1}{2}} \equiv -Pa(p)\,(mod\, p).$$

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