" A Multiobjective Optimization Approach to Smart Growth in Land Development,", S. A. Gabriel, J.A. Faria, G. E. Moglen, October 10, 2002
IN REVIEW AT *EUROPEAN JOURNAL OF OPERATIONAL RESEARCH*

A Multiobjective Optimization Approach to Smart Growth in Land Development

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Abstract

In this paper we describe a multiobjective optimization model of "Smart Growth" applied to land development in Montgomery Country, Maryland. The term "Smart Growth" is generally meant to describe those land development strategies which do not result in urban sprawl, however the term is somewhat open to interpretation. The multiobjective aspects arise when considering the conflicting interests of the various stakeholders involved: the government planner, the environmentalist, the conservationist, and the land developer. We present a formulation, which employs linear and convex quadratic objective functions for the stakeholders that are subject to polyhedral and binary constraints. As such, the resulting optimization problems are convex, quadratic mixed integer programs which are known to be NP-complete (Mansini and Speranza, 1999). We report numerical results with this model and present these results using a geographic information system (GIS).

Keywords: Multiple objective programming; Integer programming; Quadratic programming; Smart Growth; Large Scale Optimiation; Geographic Information System; Land use planning.

1. Introduction

Currently in land development, there is a move towards intelligent stewardship of the natural resources so as to avoid urban sprawl. Such development schemes are often called "Smart Growth". However, this term can be a bit nebulous and what constitutes Smart Growth for one stakeholder may not necessarily be intelligent management of the resources for another stakeholder. One needs to consider land development with all the main stakeholders' interests taken together. As presented in (Moglen *et al*., 2002), we consider four main classes of stakeholders whose interests need to be simultaneously considered:

1. The government planner,
2. The environmentalist,
3. The conservationist, and
4. The land developer.

The resulting mathematical formulation is a multiobjective optimization problem whose objectives correspond to each of the stakeholders interests restricted by general constraints such as land growth rates and zoning. Together with the work in (Moglen *et al*., 2002), this multiobjective approach as applied to Smart Growth is novel and allows regional planners and other interested parties to balance the tradeoffs between the competing stakeholders.

Unlike the case of single objective optimization in which the “total system cost” or other system attribute is being optimized, a different notion, that of “Pareto optimality” is needed. A Pareto optimal solution to a multiobjective optimization problem is such that an improvement in

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one of the objectives must come at the expense of at least one of the other objectives (Cohon, 1978; Steuer, 1986). In other words, if at least one of the objectives can be improved and the others do "no worse", then the current point is not Pareto optimal. In the current Smart Growth setting, a Pareto optimal point corresponds to a particular development plan for the land parcels under consideration.

Over the years, various authors have considered general land development problems from the multiobjective optimization perspective. What differs between these works is the specific problem formulations that were being studied as well as the solution methodologies that were employed. Indeed, some formulations involving integer restrictions and other nonconvexities have often been approached with heuristic methods due to the computational complexities involved. In these cases, enumeration of the entire Pareto optimal set, while possibly desirable, is computationally challenging. This is the perspective of the current work since for each set of "weights" used to induce a Pareto optimal point, one needs to solve a quadratic, mixed integer program with about 3,500 variables (mostly binary) and over 23,000 constraints. Our computational experience has indicated that in some cases, solving just one of these problems can take more than 20 hours on a fast desktop computer using state of the art software. [1] Moreover, exhibiting a selection of Pareto optimal points rather than the entire set of solutions is meant to illustrate the significant tradeoffs between the various stakeholders involving conservation of the environment, protection against urban sprawl, and economic benefits. However, we are considering specialized heuristic approaches to speed up these computations in (Faria and Gabriel, 2003). In what follows we briefly review some selected multiobjective optimization works related to land development; other works have been left out only for purposes of brevity.

Two of the early papers in multiobjective land development were by Bammi and Bammi (1975, 1979) in which they presented a multiobjective optimization model for land use planning in DuPage County, Illinois. A weighted objectives approach was used considering adjacent land uses, travel time, tax costs, negative environmental impacts, and costs borne by the community. Using a linear programming model for each of 147 planning regions, they computed acreage totals by land use type which were then allocated by planners on a parcel-by-parcel basis. Later, Wright *et al.* (1983) considered a multiobjective integer programming model for land acquisition. These integer restrictions greatly complicate the solution methodology and the authors developed a specialized approach given the possibility of "gap points" (Cohon, 1978), i.e., solutions which could be missed when using a weighting method. Their model considered three objectives: area of a cell, acquisition cost, and compactness of the developed cells. The largest problem they considered involved 30 cells which had 146 binary variables and 69 constraints and at that time, was at the limit of general-purpose multiobjective integer programming algorithms. Various parts of this work were extended in the later work by Benabdallah and Wright (1992). Gilbert *et al.* (1985) developed a four-objective optimization model which also contained integer restrictions on the variables. Their objectives included: the acquisition and development cost, the so-called "amenity" and "detractor" distances, and the shape objective. Due to their formulation's computational complexity, they developed an interactive, partial enumeration scheme. This method was applied to solve land development plans for Norris, Tennessee represented by 900 cells of approximately 2.5 acres each. More recently, the book edited by Beinat and Nijkamp (1998) describes a good collection of multiobjective land use papers with GIS components. Lastly, the recent work by Mogen *et al.* (2002) considered a multiobjective integer programming problem using 810 parcels in Montgomery County, Maryland. The positions of four stakeholders—environmentalist, conservationist, government planner, and land developer were considered in combination with certain global constraints such as growth rates by each of the five land use zones. This work

[1] MPL by Maximal Software and XPRESS-MP by Dash Optimization, Pentium III computer, 933 MHz clock speed.

provided a framework for analyzing Smart Growth via this multiobjective perspective, describing some of the key environmental and development tradeoffs while also providing selected numerical and geographical results for illustrative purposes.

The current work extends (Moglen *et al*., 2002) in several important ways and provides more of a mathematical perspective than this other work. First, using a different database of 913 undeveloped and 4,837 developed parcels for Montgomery County, Maryland, the current work includes specific integer constraints to classify unassigned parcels into one of the five zonal types: residential low density, residential medium density, residential high density, commercial or industrial. The previous work used a heuristic to assign unclassified parcels—so-called "Rural Density Transfer"-- to one of these zones *prior* to running the optimization. In (Moglen *et al*., 2002) the "rural density transfer" zoning code was revised on a parcel by parcel basis so that all parcels with this zoning code were amended to have one of the following zoning codes: low density residential, medium density residential, high density residential, commercial, or industrial. Assignments were done by assigning to all parcels 500 meters from main roads an industrial land use (consistent with land use elsewhere in this part of the county). Remaining patches of rural density transfer zoned areas were assigned one of the other land use categories in an ad hoc manner such that re-assigned parcels took on the same zoning category as nearby parcels already zoned in that category.

By contrast, in the current work, having the model choose which zone is appropriate for each parcel is more efficient from the land use perspective but represents a huge computational challenge in that for each of these 512 unassigned parcels, five additional binary variables (one for each of the zonal types) needs to be included. Secondly, the current work, unlike (Moglen *et al.,* 2002), also includes a set of constraints to insure that these unassigned parcels are only selected when necessary, the preference given to parcels already classified into one of the five zonal types. Third, the current work, also unlike (Moglen *et al.,* 2002), considers the "compactnes" of the developed area as an objective for the government planner. All else being equal, a more compact area is better from the perspective of the government planner since it means that less infrastructure (e.g., roads, water distribution network) is needed. Compactness *per se*, has been considered in a variety of ways by other authors, for example, Wright *et al*. (1983), and Gilbert *et al*. (1985). Thus, the current work also represent an extension of these works relative to compactness. The current approach uses the notion of minimizing the "outer rectangle" of the developed parcels by considering the diagonal of this rectangle and results in a (convex) quadratic objective function and an additional 23,000 constraints involving binary variables and is described below. As such the resulting optimizations using the weighting method (Cohon, 1978; Steuer, 1986) are instances of large-scale, quadratic mixed integer programs (QMIPs) with 23,551 constraints and 3,508 variables of which 3,483 are binary. It is known that the class of QMIPs is NP-complete (Mansini and Speranza, 1999). However, the relaxed version of these QMIPS are simply convex, quadratic programs with linear constraints and thus represent a reasonable computational burden given the state of the art in optimization solvers. As such, our approach represents a reasonable balance between representing "compactness" and computational considerations.

The rest of this paper is organized as follows: in Section 2 we briefly describe the positions of the four stakeholders and refer the reader to (Moglen *et al.*, 2002) for more details; Section 3 presents the multiobjective optimization formulation with several choices for stakeholder objectives; Section 4 presents several theoretical results for the multiobjective formulation that is adopted; Section 5 describes selected numerical results of solving the quadratic, mixed integer multiobjective optimizations described in Sections 3 and 4; and in Section 6 we summarize our findings.

2. The land development stakeholders

To fairly represent the land development process, we model the objectives of four main stakeholder groups: government planners, environmentalists, conservationists, and land developers. It is clear that these groups have competing objectives with in some cases, diametrically opposed viewpoints (e.g., conservation of the land vs. development of the land).

It is instructive to compare the current approach for Smart Growth with the one that is analyzed in this paper, that of using an explicit multiobjective optimization model. At present, competing objectives are not considered in most Smart Growth designs. Instead, a range of "best management practices" might be used such as incorporating porous pavement, rain gardens, or grassed swales in an effort to minimize the impact of development; for example see Schueler and Holland (2000) for a number of examples. Rigorous comparisons of multiple alternative development patterns are generally not considered. In fact, Smart Growth may be more complicated that was originally thought by the advocates of this strategy. Balancing the interests of the diverse stakeholders from a multiobjective optimization perspective involves some sort of compromise strategy which can be analyzed over many time periods or for one single one. However, it is important to note that the optimization as presented in this paper, is not done over a series of time periods but rather reflects a five-year "snapshot" at a given time.

3. The Smart Growth multiobjective optimization problem

As described above, we consider the perspective of four major stakeholders in smart growth: the government planner, the environmentalist, the conservationist, and the land developer. It is important to note that we are not interested in actually generating the complete set of Pareto optimal solutions. Indeed, for this problem, due to various computational reasons, generation of this set can be quite complex and is not covered in the present work. Rather, we describe a model which reflects the objectives of each of the stakeholders and generate typical compromise solutions. These solutions are then presented in a GIS format to illustrate their outcomes.

We begin by describing the objective functions and related constraints for each of the four stakeholders. Notationally, S is the set of indices for each of the parcels of land that might be developed. For a typical parcel $i \in S$ we have

$$d_i = \begin{cases} 1 \text{ if parcel } i \text{ is developed} \\ 0 \text{ otherwise} \end{cases}$$

Thus, we do not allow fractional development of a parcel.

3.1 The government planner

The government planner has several key goals in land allocation consistent with Smart Growth. First, the planner is interested in developing key “priority funding areas”. These areas have been targeted by the state for development to promote redevelopment of decaying urban areas and maximize existing capacity for facilities (e.g., water, sewer). Second, the planner is interested in minimizing the low density zone land parcels to minimize sprawl. Third, the planner would, all things being equal, prefer to keep the land that is developed in as compact an area as possible. This strategy would tend to allow larger patches of undeveloped land consistent with Smart Growth. There are many other possible considerations for the planners such as: minimizing traffic congestion, promoting urban renewal, etc. We consider only the first three of these in this work.

3.1.1 Priority funding areas

For the priority funding areas (PFAs) objective we define the following notation.

S_{PFA} = the set of parcels that have been designated as priority funding areas with $S_{PFA} \subseteq S$
a_1, a_2 = Minimum, maximum number of total PFAs to be developed, respectively, with $a_1 \le a_2$.

The priority funding areas constraint forced on the developer is to develop those PFA parcels so that the total number of them is between a given upper and lower bounds. Thus, mathematically, we have the following:

$$a_1 \le \sum_{i \in S_{PFA}} d_i \le a_2 \text{ or equivalently} \tag{1}$$

$$\begin{cases} -a_2 + \sum_{i \in S_{PFA}} d_i \le 0 \\ a_1 - \sum_{i \in S_{PFA}} d_i \le 0 \end{cases} \tag{2}$$

If desirable, these constraints can be converted to a "goal programming" form (Winston, 1994) by noting that there is no penalty if the total number of PFA parcels developed is in the range $[a_1, a_2]$ but a penalty of g_1 for each parcel that is below the lower bound and g_2 for each parcel above the upper bound, where $g_1 > 0$ and $g_2 > 0$; note that both g_1, g_2 are data to be specified. This relationship can be succinctly represented as

$$g_1 \max\left\{ a_1 - \sum_{i \in S_{PFA}} d_i, 0 \right\} + g_2 \max\left\{ -a_2 + \sum_{i \in S_{PFA}} d_i, 0 \right\} \tag{3}$$

which the planner would be trying to minimize. This construction is consistent with using penalties for both over-achieving and under-achieving the constraint. The "max" function is of course nonsmooth but can be converted to a more computationally attractive form as follows.

$$\begin{aligned} &\min g_1 \text{under}_{\text{PFA}} + g_2 \text{over}_{\text{PFA}} \\ &s.t. \\ &\begin{cases} \text{under}_{\text{PFA}} \ge a_1 - \sum_{i \in S_{PFA}} d_i \\ \text{under}_{\text{PFA}} \ge 0 \\ \text{over}_{\text{PFA}} \ge -a_2 + \sum_{i \in S_{PFA}} d_i \\ \text{over}_{\text{PFA}} \ge 0 \end{cases} \end{aligned} \tag{4}$$

Alternatively, we could also use as an objective for priority funding areas

$$\max \sum_{i \in S_{PFA}} area_i d_i \tag{5}$$

to maximize the total area of PFA parcels that are to be selected for development where $area_i$ is the area of parcel i.

3.1.2 Low density zones

Another objective for the government planner is to minimize the number of low density zoned parcels that are developed. One motivation for this objective is to minimize sprawl. Such a phenomenon can result if there are an inordinately large number of low density parcels. We let S_{LD} equals the set of low density zoned parcels with $S_{LD} \subseteq S$. Consequently, another objective is to

$$\min \sum_{i \in S_{LD}} d_i \tag{6}$$

or a version of this objective which weights each parcel by its area. Additionally, realistic bounds on the number of total low density areas to be developed could be applied and a goal programming strategy similar to (4) could be used.

3.1.3 Minimize maximum distance between developed areas

There have been several mathematical approaches to minimizing the "spread" of development or maximizing the compactness of the development area, see for example Wright *et al*. (1983), and Gilbert *et al*. (1985). We choose to measure the spread of the development area as the Euclidean distance between the ends of the development areas. Having the developed area more compact is consistent with Smart Growth and is equivalent to minimizing the maximum distance between two points in the developed areas.

First we suppose that the set of parcels fits into a rectangular grid with "rows" and "columns" in this grid assigned to each parcel. This is not to say that each of the land parcels is rectangular or even regularly shaped, just that there is a rectangular “outer envelope” surrounding the parcels in questions. The rows and columns of this rectangular grid can relate to longitude and latitude for example or some other geographical designation. Consider the following depiction of this scheme.

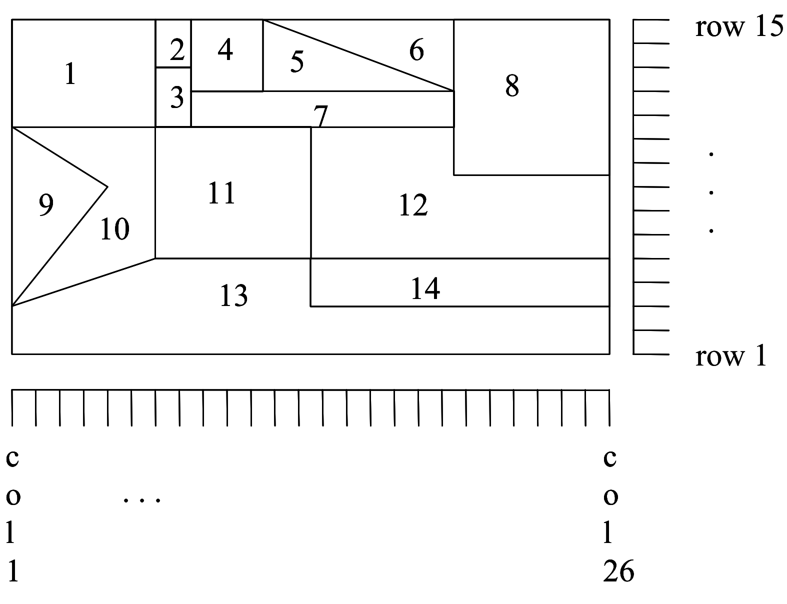

Fig. 1. Depiction of Rectangular Grid Around Parcels.

For each parcel *i* we define:

- $row^S(i)$ = The row number south of all points in parcel *i* and closest to the southernmost point in parcel *i*
- $row^N(i)$ = The row number north of all points in parcel *i* and closest to the northernmost point in parcel *i*
- $col^E(i)$ = The column number east of all points in parcel *i* and closest to the easternmost point in parcel *i*
- $col^W(i)$ = The column number west of all points in parcel *i* and closest to the westernmost point in parcel *i*

For example for parcel 6 in Figure 1 we have $row^S(6) = 12, row^N(6) = 15$, $col^E(6) = 20$, $col^W(6) = 12$, noting that parcel 6's southern, northern, and eastern borders exactly coincide with these values.

In order to describe the smallest box containing all the developed parcels, we first define the corner points of the box with the new variables r_0, r_1, c_0, c_1. We designate the value of the variable r_1 as the row index, which is north of all developed points but closest to the northernmost parcel. Also r_0 refers to the row index, which is south of all developed points but closest to the southernmost parcel. In a similar way, c_1, c_0 refer to the eastern and westernmost column indices (respectively) for this box. Thus, (r_0, c_0) is the southwestern corner of the box, (r_0, c_1), the southeastern corner, and $(r_1, c_0), (r_1, c_1)$ are respectively, the northwestern and northeastern corners.

Formally, these relationships for r_0, r_1, c_0, c_1 are given as follows.

$$r_1 = \max\{row^N(i) \mid d_i = 1\} \tag{7a}$$
$$r_0 = \min\{row^S(i) \mid d_i = 1\} \tag{7b}$$
$$c_1 = \max\{col^E(i) \mid d_i = 1\} \tag{7c}$$
$$c_0 = \min\{col^W(i) \mid d_i = 1\} \tag{7d}$$

These relationships can be encoded in terms of linear constraints in the following way

$$r_0 - row^S(i) \le (1 - d_i)M \tag{8a}$$
$$row^N(i) - r_1 \le (1 - d_i)M \tag{8b}$$
$$c_0 - col^W(i) \le (1 - d_i)M \tag{8c}$$
$$col^E(i) - c_1 \le (1 - d_i)M \tag{8d}$$

where M is a suitably large positive constant. It is easy to see that for example, that for each developed parcel, r_0 does in fact represent the southernmost row index (or just above it) since $d_i = 1 \Rightarrow r_0 \le row^S(i)$. When the parcel is undeveloped, we have $d_i = 0 \Rightarrow r_0 \le row^S(i) + M$ which, based on the value of M, provides no restriction on r_0. Of course, we want equality holding for at least one index i for a developed parcel in (8a) ; this is a natural consequence as shown in a later section. Lastly, the logic for the other three variables follows similarly. We can also add the realistic row-column bounds on the variables r_0, r_1, c_0, c_1,

$$0 \le r_0, r_1, c_0, c_1 \tag{9}$$

When considering a database of both undeveloped parcels and already developed ones, we do not actually need the variables d_i for the previously developed parcels. Clearly, all such variables would have a value of one and we can simply incorporate this value into (8a)-(8d). For example, (8a) reduces to $r_0 - row^S(i) \leq 0$ if parcel i is already developed; similar reasoning holds for (8b)-(8d). Since there were a huge number of parcels already developed in our database, this step represents a huge savings in the number of binary variable that would be needed and makes the computations more reasonable.

The Euclidean distance of the "box" containing all the parcels selected for development is thus given by

$$\max\_dist = \sqrt{(r_1 - r_0)^2 + (c_1 - c_0)^2} \tag{10}$$

which is the diagonal of this box; see Figure 2. Without loss of generality, we can consider minimizing the squared distance as

$$\min \max\_dist\_sq = (r_1 - r_0)^2 + (c_1 - c_0)^2 \tag{11}$$

Alternatively, we can minimize the " L_1 " distance, i.e.,

$$\min L_1\_dist = |r_1 - r_0| + |c_1 - c_0| = (r_1 - r_0) + (c_1 - c_0) \tag{12}$$

in light of the fact that $r_1 \geq r_0$ and $c_1 \geq c_0$. When (11) is used we end up with a convex, quadratic mixed integer program as the optimization problem solved as part of the “weighting method”. When (12) is used, this problem becomes a mixed integer linear program. The preference of (11) over (12) is due to the more natural notion of “distance” of the box in terms of its diagonal as compared to the sum of the two sides, i.e., half the perimeter when (12) is used. Also, using the diagonal relates to, in some sense, the maximum distance that infrastructure (e.g., roads, pipes, power lines) would need to be installed. Minimizing this distance would clearly be advantageous from the planner’s point of view. Additionally, one might ask why not minimize the area of the parcel, i.e., $(r_1 - r_0)(c_1 - c_0)$? As will be shown in Section 4, another advantage of using (11) is that the weighted subproblems that are solved to generate Pareto optimal solutions are convex, quadratic mixed integer programs which is not the case if area of the rectangle is used instead. Indeed, when area is used, the problem has a non-convex objective function in addition to the nonconvexities that arise from the binary constraints. Thus, (11) has advantages over these alternative formulations in terms of computations as well as from the planning perspective.

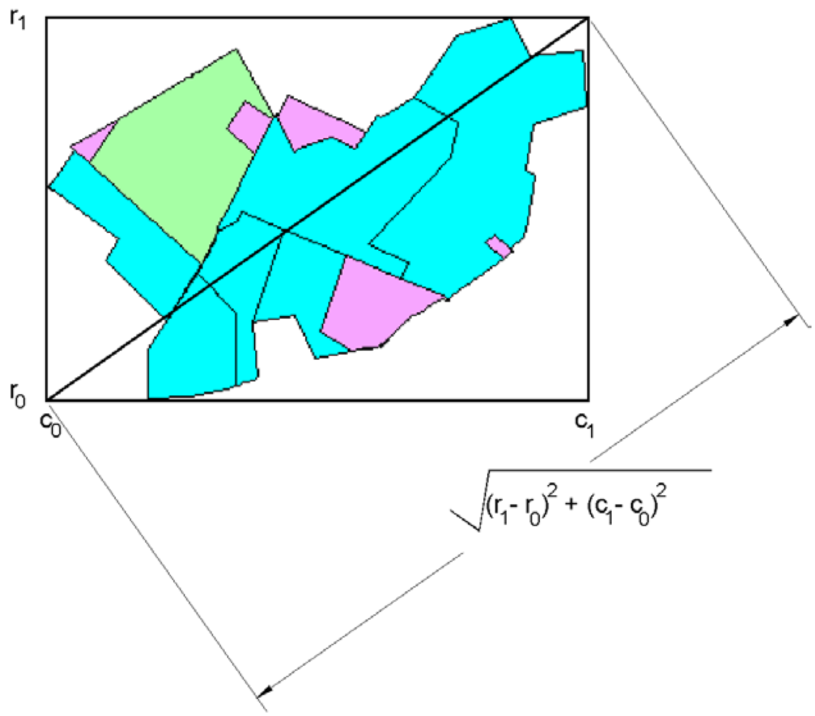

Fig. 2. Depiction of the Diagonal of the Outer Rectangle.

3.2 The environmentalist

The environmentalist has several objectives to be optimized which we describe below.

3.2.1 Direct development to parcels not containing a stream

This particular objective involves maximizing the distance to a stream, i.e.,

$$\max \sum_{i \in S} d\_stream_i d_i \tag{13}$$

where d_stream_i is the distance to a stream for parcel *i*. Equation (13) has the effect of selecting those parcels for development where $d\_stream_i > 0$, i.e., not containing a stream and thus the environmental impact of development should not be as great as when the parcel included a stream.

3.2.2. Minimize global change in imperviousness

Another objective (used in (Mogen *et al.,* 2002)) concerns the change in the level of the imperviousness, that is, the tendency to not allow infiltration of water. We have the following objective.

$$\min \sum_{i \in S} \Delta\_imperv_i \, area_i \, d_i \tag{14}$$

where $\Delta\_imperv_i d_i$ is the change in imperviousness and $area_i$ is the total area of the parcel. When the level of imperviousness is too high, this can tend to cause harm to various life forms that depend on the water.

3.2.3. Continue development in areas already "hit hard"

The next possible objective for the environmentalist favors development of those hydrological unit codes (HUCs) which already have had a good deal of development. Since a HUC represents one or more parcels, this objective would support finishing up development of parcels in a HUC that has already seen a substantial level of development. This objective is given as

$$\max \sum_{i \in \bar{S}} d_i \tag{15}$$

where $\bar{S}$ is a select set of HUCS that have been "hit hard" by development (an area-weighted version of this is also possible). This objective should be contrasted with the one described above in (14). The one in (14) favors spreading the pollution or damage (for example to streams) around, the so-called "dilution of pollution" strategy. By contrast, the objective in (15) favors an approach that concentrates development in certain areas.

3.3 Conservationist

The conservationist occupies the most environmentally-friendly position on the spectrum of interests of the four stakeholders being considered. Such a stakeholder is adamant about protection from development for certain key parcels denoted by the set $\tilde{S}$. In terms of an objective function, this leads to

$$\min \sum_{i \in \tilde{S}} area_i d_i \, , \tag{16}$$

i.e., minimize the total area of environmentally sensitive parcels to be developed to protect the flora and fauna in these areas.

3.4 Land developer

The developer is modeled so as to maximize the total values of the developed parcels where the value is calculated as shown below for a parcel *i*. [2]

$$\text{value}_\text{i} = 0.20 \begin{cases} \text{avg\_sales}_\text{LD} * \left(\dfrac{\text{area}_\text{i}}{\text{density}_\text{LD}} \right) \text{if parcel } i \text{ is low density, residential} \\ \text{avg\_sales}_\text{MD} * \left(\dfrac{\text{area}_\text{i}}{\text{density}_\text{MD}} \right) \text{if parcel } i \text{ is medium density, residential} \\ \text{avg\_sales}_\text{HD} * \left(\dfrac{\text{area}_\text{i}}{\text{density}_\text{HD}} \right) \text{if parcel } i \text{ is high density, residential} \\ \text{avg\_sales\_sq\_area}_\text{COM} * \left(\text{a}_\text{COM} + \text{b}_\text{COM} \text{area}_\text{i} \right) \text{if parcel } i \text{ is commercial} \\ \text{avg\_sales\_sq\_area}_\text{IND} * \left(\text{a}_\text{IND} + \text{b}_\text{IND} \text{area}_\text{i} \right) \text{if parcel } i \text{ is industrial} \end{cases} \tag{17}$$

[2] Note that the average sales per square area value (avg_sales_sq_area) used square feet as the square area in question given the original form of the data with one square foot equal to 0.0929 square meters.

where

- $value_i$ is the value of parcel *i* if developed ($), consistent with industry, we take 80% of this as costs so the net value is 20% of the right-hand side of (17) enclosed in the braces, we have assumed that this 80% cost is already taken out in what follows;
- $avg\_sales_{LD}, avg\_sales_{MD}, avg\_sales_{HD}$ represent the average sales dollars/unit for low density, medium density, and high density residential parcels taken in the recent years, respectively
- $\frac{area_i}{density_{LD}}, \frac{area_i}{density_{MD}}, \frac{area_i}{density_{HD}}$ are the estimates for the maximum number of units possible on the parcel if it's a low density, medium density, or high density residential parcel, respectively
- $avg\_sales\_sq\_area_{COM}t, avg\_sales\_sq\_area\_{}_{IND}$ are the average ratio of sales dollars for a unit to square area of the structure for commercial and industrial parcels, respectively
- $a_{COM} + b_{COM} area_i, a_{IND} + b_{IND} area_i$ are statistically estimated relationships between the area of the parcel and the square area for commercial and industrial parcels, respectively, useful for predicting the typical area of structures on yet undeveloped parcels

Based on our data set of residential parcels, we estimated the following parameters per unit for our Montgomery Country, Maryland database:

Table 1
Average Sales by Residential Zone

avg_sales_{LD}	avg_sales_{MD}	avg_sales_{HD}
$449,540	$291,366	$256,658

The densities of the residential areas consistent with definitions used by both the Maryland Department of Planning (Maryland Department of Planning, 2000) and the Natural Resources Conservation Service (Soil Conservation Service, 1985), were taken as follows where "du" means dwelling unit, "ha" is hectare" and "ac" is acre.

Table 2
Land Densities by Residential Zone

Low Density	Medium Density	High Density
2.47 du/ha (1 du/ac)	9.88 du/ha (4 du/ac)	19.8 du/ha (8 du/ac)

Based on our data set of commercial and industrial parcels, we estimated the following parameters:

Table 3
Commercial and Industrial Estimated Parameters

Zoning Category	avg_sales_sq_area	a	b
Commercial	315.6	15,553	9,736.9
Industrial	192.8	9,242.2	11,604

We note that parcels are grouped into the following designations.

$$\text{if zoning code}_i = \begin{cases} \text{"11" then parcel } i \text{ is designated as a 1 acre low density residential lot, the set of parcels is } S_{11} \\ \text{"12" then parcel } i \text{ is designated as a } 1/4 \text{ acre medium density residential lot,} \\ \text{the set of parcels is } S_{12} \\ \text{"13" then parcel } i \text{ is designated as a } 1/8 \text{ acre high density residential lot, the set of parcels is } S_{13} \\ \text{"14" then parcel } i \text{ is designated as a commercial lot, the set of parcels is } S_{14} \\ \text{"15" then parcel } i \text{ is designated as an industrial lot, the set of parcels is } S_{15} \\ \text{otherwise then parcel } i\text{'s designation is unassigned and is to be decided upon} \\ \text{by the model, the set of parcels is } S_{99} \end{cases} \quad (18)$$

For each parcel $i \in S_{99}$, hereafter called an "unassigned" parcel,

$$d_i = RLD_i + RMD_i + RHD_i + COMi + IND_i \text{ for all } i \in S_{99} \quad (19)$$

where

$$RLD_i, RMD_i, RHD_i, COM_i, IND_i \in \{0,1\} \text{ for all } i \in S_{99}$$

and these variables represent respectively, whether the unassigned parcel is selected to be residential low density, residential medium density, residential high density, commercial, or industrial, with exactly one of these choices made if the parcel is developed. Consequently, we see that the objective function for the developer becomes

$$\begin{aligned} \max \sum_{i\in S_{11}} \text{value}_i d_i + \sum_{i\in S_{12}} \text{value}_i d_i + \sum_{i\in S_{13}} \text{value}_i d_i + \sum_{i\in S_{14}} \text{value}_i d_i + \sum_{i\in S_{15}} \text{value}_i d_i \\ + \sum_{i\in S_{99}} \left(\text{value}_i RLD_i + \text{value}_i RMD_i + \text{value}_i RHD_i + \text{value}_i COM_i + \text{value}_i IND_i\right) \end{aligned} \quad (20)$$

3.5 Additional constraints

We next describe the set of system constraints incorporated in the model.

3.5.1 Growth rates on number of units and acres by zone

Based on five-year projections for growth rates of number of housing units for residential areas and hectares (acres) for commercial and industrial sites, we add constraints that provide lower and upper bounds for these target values. The lower and upper bounds represent –20% and +20% of these rates. We note that each of these designations takes parcels from a fixed set (i.e., for RLD it's 11) as well as potentially from the set of undecided designations (i.e., code equal to 99). Consequently, realistic bounds on new development for each of the zoning designations are given as follows.

$$a_3 \le \sum_{i\in S_{11}} units_i d_i + \sum_{i\in S_{99}} units_i RLD_i \le a_4 \quad (21a)$$

$$a_5 \le \sum_{i\in S_{12}} units_i d_i + \sum_{i\in S_{99}} units_i RMD_i \le a_6 \quad (21b)$$

$$a_7 \le \sum_{i\in S_{13}} units_i d_i + \sum_{i\in S_{99}} units_i RHD_i \le a_8 \quad (21c)$$

$$a_9 \le \sum_{i \in S_{14}} acres_i d_i + \sum_{i \in S_{99}} acres_i COM_i \le a_{10} \tag{21d}$$

$$a_{11} \le \sum_{i \in S_{15}} acres_i d_i + \sum_{i \in S_{99}} acres_i IND_i \le a_{12} \tag{21e}$$

with the values $a_3, a_4, \ldots, a_{12}$ representing minimum and maximum number of units (for residential) or minimum and maximum number of hectares (acres) (for commercial and industrial) to be developed, and $units_i, acres_i$ representing respectively, the positive number of units, acres that can be developed for parcel i. The following very reasonable assumption is made.

Assumption 1
$0 < a_3 < a_4, \ldots, 0 < a_{11} < a_{12}$.

3.5.2 Forcing zone 99 choices to be zero if sufficient number of units, hectares (acres) in the other zone types

Another set of constraints involving the classification of the unassigned parcels S_{99} is meant to insure that these parcels are not developed when there is a sufficient number of units in the existing pool of parcels, i.e., S_{11} for the residential low density ones, S_{12} for the residential medium density ones, etc. The rationale is that the bureaucratic effort needed to subdivide and rezone large essentially unzoned, i.e. "rural density transfer" land represents a significant impediment to development and so we're treating this impediment as strong enough that all undeveloped but acceptably zoned parcels will undergo development before significant efforts will be made to re-zone the parcels in the unzoned designation. This logic is consistent with minimizing the bureaucratic burden of establishing zonal types for unassigned parcels. These restrictions can be enforced with the following constraints. Let M be a suitably large positive number and y_{RLD}, y_{RMD}, y_{RHD}, y_{COM}, y_{IND} be binary variables. Then, we have the following:

$$\sum_{i \in S_{99}} RLD_i units_i \le My_{RLD}, \sum_{i \in S_{11}} units_i - a_3 \le M(1 - y_{RLD}) \tag{22a}$$

$$\sum_{i \in S_{99}} RMD_i units_i \le My_{RMD}, \sum_{i \in S_{12}} units_i - a_5 \le M(1 - y_{RMD}) \tag{22b}$$

$$\sum_{i \in S_{99}} RHD_i units_i \le My_{RHD}, \sum_{i \in S_{13}} units_i - a_7 \le M(1 - y_{RHD}) \tag{22c}$$

$$\sum_{i \in S_{99}} COM_i units_i \le My_{COM}, \sum_{i \in S_{14}} acres_i - a_9 \le M(1 - y_{COM}) \tag{22d}$$

$$\sum_{i \in S_{99}} IND_i units_i \le My_{IND}, \sum_{i \in S_{15}} acres_i - a_{11} \le M(1 - y_{IND}) \tag{22e}$$

We see that for example, if $\sum_{i \in S_{11}} units_i > a_3$ so that there are enough parcels in the "assigned pool" for residential low density, then necessarily $Y_{RLD} = 0$, which forces $RLD_i = 0$, for each $i \in S_{99}$ or that no units from unassigned parcels get converted to residential low density and none are developed. Conversely, if $\sum_{i \in S_{11}} units_i \le a_3$, then the binary variable Y_{RLD} can have a value of either 0 or 1. When a value of 1 is chosen, since M was chosen sufficiently large, this poses no restrictions on the potential residential low density parcels coming from the unassigned group.

Otherwise, when a value of 0 is selected, none of these other parcels are converted to residential low density. Presumably, the former case, all things being equal, will be selected when $\sum_{i \in S_{11}} units_i \le a_3$, since it allows for a large feasible region and hence a better (no worse) objective function value. In any event, the constraints are only meant to insure that if there is a sufficient number of assigned parcels, then no unassigned ones should be used. Similar reasoning holds for the other four zonal types.

3.5.3 Forcing the number of units, hectares (acres) in zone designations 11, ..,15 to be all used if insufficient capacity

The next set of constraints involving the unassigned parcels is meant to make sure that all the assigned ones i.e., those in $S_{11}, S_{12}, S_{13}, S_{14}, S_{15}$ are used completely if there is an insufficient number relative to the lower bounds in (21a)-(21e). The rationale is similar to what was stated in the previous section. For the constraints below, we let N be a suitably large positive number and w_{RLD}, w_{RMD}, w_{RHD}, w_{COM}, w_{IND} be binary variables. Then, we have the following:

$$a_3 - \sum_{i \in S_{11}} units_i \le N(1 - w_{RLD}), \sum_{i \in S_{11}} units_i - \sum_{i \in S_{11}} units_i d_i \le N w_{RLD} \tag{23a}$$

$$a_5 - \sum_{i \in S_{12}} units_i \le N(1 - w_{RMD}), \sum_{i \in S_{12}} units_i - \sum_{i \in S_{12}} units_i d_i \le N w_{RMD} \tag{23b}$$

$$a_7 - \sum_{i \in S_{13}} units_i \le N(1 - w_{RHD}), \sum_{i \in S_{13}} units_i - \sum_{i \in S_{13}} units_i d_i \le N w_{RHD} \tag{23c}$$

$$a_9 - \sum_{i \in S_{14}} acres_i \le N(1 - w_{COM}), \sum_{i \in S_{14}} acres_i - \sum_{i \in S_{14}} acres_i d_i \le N w_{COM} \tag{23d}$$

$$a_{11} - \sum_{i \in S_{15}} acres_i \le N(1 - w_{IND}), \sum_{i \in S_{15}} acres_i - \sum_{i \in S_{15}} acres_i d_i \le N w_{IND} \tag{23e}$$

We see that if for example, the existing residential low density units S_{11} are insufficient to meet even the minimum growth goal of a_3, that is if $a_3 > \sum_{i \in S_{11}} units_i$, then the binary variable w_{RLD} must equal 0 as shown in (23a). This in turn causes $\sum_{i \in S_{11}} units_i \le \sum_{i \in S_{11}} units_i d_i$ also by (23a). In combination with the fact that the inequality $\sum_{i \in S_{11}} units_i d_i \le \sum_{i \in S_{11}} units_i$ is always true, the desired result of $\sum_{i \in S_{11}} units_i d_i = \sum_{i \in S_{11}} units_i$ or that $d_i = 1 \ \forall i \in S_{11}$ follows since there are always a positive number of units on each parcel. This is the logic that is desired and is similar for the other four zonal types as well. Conversely, we also see that when there is a sufficient number of residential low density units, i.e., $a_3 \le \sum_{i \in S_{11}} units_i$, that w_{RLD} can equal either 0 or 1. A value of 0 will force $d_i = 1, \forall i \in S_{11}$ since $\sum_{i \in S_{11}} units_i \le \sum_{i \in S_{11}} units_i d_i$, a value of w_{RLD} equal to 1 will place no constraints on these units. Thus, presumably, all things being equal, a value of w_{RLD} equal to 1 will be preferred by the model since it allows for a larger feasible region. Similar logic also holds for the other four zonal types.

3.6 The multiobjective optimization model

It is possible to have multiple objective functions for each stakeholder as spelled out in previous sections and to take a weighted combination of these objectives to arrive at a "composite" objective function for each stakeholder. However, it is sometimes difficult to analyze such weighted objectives and so for clarity, we have chosen to select just one objective from the ones described above for each stakeholder.

For the government planner, we use a slightly modified notion of compactness compared to what was presented before. We suppose that the area in question is divided into Q quadrants where a typical quadrant is indexed by $q \in \{1,2,\ldots,Q\}$. The particular application will determine the appropriate number of quadrants but the idea is that the planner may want to promote a compactness measure separately for each quadrant. This is important if for example, there are already developed parcels in the corners of the overall land area to be considered. Such a situation would make the compactness objective, as described above by use of a bounding rectangle, less meaningful. To circumvent this to some extent, one can subdivide the area to allow for compactness to be determined within the quadrants. Clearly, though, even with multiple quadrants, it may not be possible to avoid one division that has less flexibility in maximizing compactness with the given metric. We explore these issues further in Section 5 when we report numerical results using four quadrants in our database.

The resulting modified notion of compactness is thus as follows. First, we let $\max\_dist\_sq^q = \left(r_1^q - r_0^q\right)^2 + \left(c_1^q - c_0^q\right)^2$. Then, we enforce the following constraints:

$$r_0^q - row^S(i) \le (1-d_i)M, \text{ for all } i \text{ in quadrant } q \tag{24a}$$

$$row^N(i) - r_1^q \le (1-d_i)M, \text{ for all } i \text{ in quadrant } q \tag{24b}$$

$$c_0^q - col^W(i) \le (1-d_i)M, \text{ for all } i \text{ in quadrant } q \tag{24c}$$

$$col^E(i) - c_1^q \le (1-d_i)M, \text{ for all } i \text{ in quadrant } q \tag{24d}$$

$$0 \le r_0^q, r_1^q, c_0^q, c_1^q, \text{ for all } i \text{ in quadrant } q \tag{24e}$$

Then, (24a)-(24e) replaces (8a)-(8d),(9) and the objective function (11) is replaced by

$$\sum_{q=1}^{Q} \max\_dist\_sq^q = \sum_{q=1}^{Q} \left(r_1^q - r_0^q\right)^2 + \left(c_1^q - c_0^q\right)^2 \tag{24f}$$

which is to be minimized.

For the environmentalist, we select the goal of minimizing the total change in imperviousness, (14). The goals for the conservationist and the developer are respectively, minimizing the area-weighted sum of environmentally sensitive parcels (16) and maximizing the total economic value of the developed parcels (20). The resulting multiobjective optimization problem for Smart Growth is given as follows.

$$\begin{cases} \min z_1 = \sum_{q=1}^{Q} \left(r_1^q - r_0^q\right)^2 + \left(c_1^q - c_0^q\right)^2, \ (planner) \\ \min z_2 = \sum_{i\in S} (\Delta\_imperv_i)(area_i)d_i, \ (environmentalist) \\ \min z_3 = \sum_{i\in \bar{S}} area_i\, d_i, (conservationist) \\ \max \mathrm{z}_4 = \\ \sum_{\mathrm{i}\in \mathrm{S}_{11}} \mathrm{value_i d_i} + \sum_{\mathrm{i}\in \mathrm{S}_{12}} \mathrm{value_i d_i} + \sum_{\mathrm{i}\in \mathrm{S}_{13}} \mathrm{value_i d_i} + \sum_{\mathrm{i}\in \mathrm{S}_{14}} \mathrm{value_i d_i} + \sum_{\mathrm{i}\in \mathrm{S}_{15}} \mathrm{value_i d_i} \\ + \sum_{\mathrm{i}\in \mathrm{S}_{99}} \left(\mathrm{value_i RLD_i} + \mathrm{value_i RMD_i} + \mathrm{value_i RHD_i} + \mathrm{value_i COM_i} + \mathrm{value_i IND_i}\right) \\ (planner) \\ s.t. \\ (19), (21\mathrm{a}) - (21\mathrm{e}), (22\mathrm{a}) - (22\mathrm{e}), (23\mathrm{a}) - (23\mathrm{e}), (24\mathrm{a}) - (24\mathrm{e}) \\ d_i \in \{0,1\}, \forall i \in S \\ RLD_i, RMD_i, RHD_i, COM_i, IND_i \in \{0,1\}, \forall i \in S_{99} \\ y_{RLD}, y_{RMD}, y_{RHD}, y_{COM}, y_{IND}, w_{RLD}, w_{RMD}, w_{RHD}, w_{COM}, w_{IND} \in \{0,1\} \end{cases} \tag{25}$$

More generally, to solve a multiple objective optimization problem of the form

$$\min\{z_1(x), \ldots, z_k(x)\}, s.t.\, x \in F \tag{26}$$

where $z_i(x), i = 1, \ldots, k$ are the objective functions, and F is the feasible region, one can use the "weighting method" (Cohon, 1978). If we let $w_1, \ldots, w_k$ be positive weights, then solving the following single objective problem will produce a Pareto optimal solution to (26).

$$\min \sum_{i=1}^{k} w_i z_i(x), \ s.t.\, x \in F \tag{27}$$

When only nonnegative weights are used, we are not guaranteed a Pareto optimal solution unless the solution to (27) is unique (Theorem 3.1.3, Miettinen, 1999). Of course, when the feasible region is not convex, we may have so-called "duality gap points" which are Pareto optimal solutions that cannot be obtained via this method (other approaches can be used in this case); for an example of these duality gaps, see ReVelle and McGarity (1997), p. 560. Since we are not concerned with enumeration all the Pareto optimal solutions, these gap points do not pose a problem in this setting. Our resulting single objective problem using the weighting method is thus given as follows where w_1, w_2, w_3, w_4 are positive weights. [3]

[3] Note that the fourth objective appears with a negative sign since it involved a maximization.

$$\begin{cases} \min w_1 z_1 + w_2 z_2 + w_3 z_3 - w_4 z_4 \\ \text{s.t.} \\ (19), (21a)\text{-}(21e), (22a)\text{-}(22e), (23a)\text{-}(23e), (24a)\text{-}(24e) \\ d_i \in \{0,1\}, \forall i \in S \\ RLD_i, RMD_i, RHD_i, COM_i, IND_i \in \{0,1\}, \forall i \in S_{99} \\ y_{RLD}, y_{RMD}, y_{RHD}, y_{COM}, y_{IND}, w_{RLD}, w_{RMD}, w_{RHD}, w_{COM}, w_{IND} \in \{0,1\} \end{cases} \tag{28}$$

We note that the feasible region to both (25) and (28) is the same and we denote it by the set F.

4. Feasibility, convexity, and existence results for the Smart Growth multiobjective model

In this section, we present some theoretical results concerning the existence of the multiobjective optimization model for Smart Growth (25) as well as for the weighting problem (28). The first area concerns the feasible region for (25) and (28).

4.1 Feasibility

As stated, the feasible region F to the multiobjective model (25) or (28) is not guaranteed to be nonempty. To see this consider for example the case when the constants a_3 and a_4 are chosen so that

$$\sum_{i \in S_{11}} units_i + \sum_{i \in S_{99}} units_i < a_3 < a_4 ,$$

clearly, in this case (21a) can never be satisfied since

$$\sum_{i \in S_{11}} units_i d_i + \sum_{i \in S_{99}} units_i RLD_i \le \sum_{i \in S_{11}} units_i + \sum_{i \in S_{99}} units_i < a_3 < a_4 .$$

A similar conclusion is true for constraints (21b)-(21e) when the other constants $a_5, \ldots, a_{12}$ are considered. In spite of this example, one can still analyze the cases when the feasible region is nonempty to come up with checkable conditions as we show later in this section.

First note that the constraints (24a)-(24d) can always be satisfied by taking the following values for the variables, $r_1^q, r_0^q, c_1^q, c_0^q$ where S_q is the set of parcel indices for quadrant q:

$$r_0^q = \min_{i \in S_q} \{row^S(i)\},\ r_1^q = \max_{i \in S_q} \{row^N(i)\},\ c_0^q = \min_{i \in S_q} \{col^W(i)\}, c_1^q = \max_{i \in S_q} \{col^E(i)\}$$

Constraint (24e) is also satisfied by assumption that the row-column coordinate system for the parcels is set up in such a way that all the parcels are in R_+^2. This is not a restrictive assumption since it just means a translation of the axes if not the case. Also, we need the following reasonable assumption to hold.

Assumption 2
For each parcel i,
a. $row^N(i) > row^S(i)$, i.e., the parcel has a positive “height”,
b. $col^E(i) > col^W(i)$, i.e., the parcel has a positive “width”.

The main analysis of feasibility for this problem concerns the various cases for constraints (21a)-(21e) and the relationship with the other constraints

$(19), (22a) - (22e), (23a) - (23e), d_i \in \{0,1\}, \forall i \in S, RLD_i, RMD_i, RHD_i, COM_i, IND_i \in \{0,1\}, \forall i \in S_{99}$, $y_{RLD}, y_{RMD}, y_{RHD}, y_{COM}, y_{IND}, w_{RLD}, w_{RMD}, w_{RHD}, w_{COM}, w_{IND} \in \{0,1\}$. In particular, it is important to note that except for (19), all these constraints are separable by zonal type. Thus, one can analyze to some extent, the feasibility separately and analogously for residential low density, residential medium density, etc. We start by analyzing (19), (21a), (22a), and (23a) in terms of the residential low density zone. There are three cases to consider for the constraints (19), (21a), (22a), and (23a), depending on the relative values of $\sum_{i \in S_{11}} units_i$, a_3, and a_4. Since by Assumption 1, $0 < a_3 < a_4$, these three checkable cases are as follows:

Case 1: $a_3 \leq \sum_{i \in S_{11}} units_i \leq a_4$, Case 2: $a_3 < a_4 < \sum_{i \in S_{11}} units_i$, Case 3: $\sum_{i \in S_{11}} units_i < a_3 < a_4$.

4.1.1 Case 1

In this case, the residential low density zone has a sufficient number of units from the pool of assigned parcels S_{11} and does not need any from the unassigned set S_{99}. Hence, the following are feasible values for constraints (21a), (22a), and (23a). Take $d_i = 1, \forall i \in S_{11}$, $RLD_i = 0, \forall i \in S_{99}, y_{RLD} = 0, w_{RLD} = 0$. The value of $y_{RLD} = 0$ corresponds to not needing any parcels from S_{99} from (22a) in the residential low density zone and $w_{RLD} = 0$ confirms via (23a), that all parcels in S_{11} will be developed.

4.1.2 Case 2

In this case, there are more units in S_{11} than is actually needed, i.e., $a_4 < \sum_{i \in S_{11}} units_i$, so that some parcels should not be developed. We also don't need to develop any from the unassigned set S_{99}. The main question is how to identify which parcels will remain undeveloped. Due to the binary nature of the development variables d_i ,the function $\sum_{i \in S_{11}} units_i d_i$ is "lumpy". This feature makes the following infeasibility possible. Suppose that the lower and upper bounds on the number of units are $a_3 = 380, a_4 = 400$ but that $\sum_{i \in S_{11}} units_i = 415$ showing that case 2 is appropriate. To satisfy (21a), we would want to find a set of parcels $D \subseteq S_{11}$ such that $\sum_{j \in D} units_j \in [15,35]$ and then set $d_j = 0, j \in D, d_i = 1, \forall i \in S_{11} - D$. This would not be possible if the smallest number of units for any parcel in S_{11} were larger than 35. If we designate the parcel in S_{11} with the smallest number of units as j, this result is clear since for any D such that $\{j\} \subseteq D \subseteq S_{11}$, $\sum_{i \in S_{11} - D} units_i \leq \sum_{i \in S_{11} - \{j\}} units_i < a_3 < a_4$. A natural question is whether (21a) can be satisfied with some units from the S_{99} pool of unassigned parcels in this case. The answer is "no" since by (22a), because $\sum_{i \in S_{11}} units_i > a_4 > a_3$, which forces $y_{RLD} = 0$ (otherwise a contradiction) which in turn forces $RLD_i = 0, \forall i \in S_{99}$ via the other part of (22a), namely, $\sum_{i \in S_{99}} RLD_i units_i \leq My_{RLD}$. Thus, in this case there is also no feasible solution to (25) or (28) due to the "lumpiness" of the data. However, we can make the following assumption that is checkable in practice to obtain feasible values.

Assumption 3
If $a_3 < a_4 < \sum_{i \in S_{11}} units_i$ *, then there exists a set* $D \subseteq S_{11}$ *such that* $\sum_{i \in S_{11}-D} units_i \in [a_3, a_4]$.

Under Assumption 3, the following are feasible values for (21a), (22a), (23a):
$d_i = 1, \forall i \in S_{11} - D, d_i = 0, \forall i \in D, RLD_i = 0, \forall i \in S_{99}, y_{RLD} = 0, w_{RLD} = 1$. The value of $y_{RLD} = 0$ corresponds to not needing any parcels from S_{99} via (22a) in the residential low density zone. Note that $w_{RLD} = 0$ would not be feasible since it would imply via (23a) that all parcels in S_{11} would be developed; by contrast, $w_{RLD} = 1$ is feasible in this case.

There are many ways to test Assumption 3, one way is to solve the following relatively small mixed integer linear program.

$$
\begin{aligned}
&\min g^+\Delta^+ + g^-\Delta^- \\
&s.t. \\
&\sum_{i \in S_{11}} units_i - \sum_{i \in S_{11}} units_i d_i - \Delta^+ \le a_4 \\
&\sum_{i \in S_{11}} units_i - \sum_{i \in S_{11}} units_i d_i + \Delta^- \ge a_3 \\
&d_i \in \{0,1\}, \forall i \in S_{11} \\
&\Delta^+, \Delta^- \ge 0
\end{aligned}
$$

where g^+, g^- are positive penalty parameters and Δ^+, Δ^- are respectively, positive and negative deviations from the upper and lower goals. First note that unbounded solutions to this mixed integer linear program are not possible since we are trying to minimize the sum of the nonnegative penalties and given the form of the other constraints. A solution to this optimization problem in which the objective function is greater than zero indicates the existence of the desired set D; namely, take $D = \{i \in S_{11} \mid d_i^* = 1\}$ where d_i^* is the optimal value of these binary variables. Conversely, it is clear that if the optimal objective function value is equal to zero, then Case 1 holds.

4.1.3 Case 3

The third case to consider is when $\sum_{i \in S_{11}} units_i < a_3 < a_4$. In this situation there is an insufficient number of units just in the S_{11} group of residential low density parcels and necessarily some from the unassigned pool of S_{99} need to be considered. This is a complicating factor since unlike the first two cases, no "sharing" of these unassigned parcels between zones was needed. By constrast, in this case, depending on the cases for the other four zonal types, via constraint (19) we need to balance how we allocate these unassigned parcels to be developed in the five zonal types.

First we need to make an assumption similar to Assumption 3 to avoid the "lumpiness" problem identified above; this condition can also be checked in practice.

Assumption 4
If $\sum_{i \in S_{11}} units_i < a_3 < a_4$*, then there exists a set* $E_{RLD} \subseteq S_{99}$ *such that*
$\sum_{i \in S_{11}} units_i + \sum_{i \in E_{RLD}} units_i \in [a_3, a_4]$.

Under Assumption 3, the following are feasible values for (21a), (22a), (23a):
$d_i = 1, \forall i \in S_{11}, RLD_i = 1, \forall i \in E_{RLD}, RLD_i = 0, \forall i \in S_{99} - E_{RLD}, y_{RLD} = 1, w_{RLD} = 0$. The value of $y_{RLD} = 1$ corresponds to needing some parcels from S_{99} from (22a) in the residential low density zone and $w_{RLD} = 0$ confirms via (23a) that all parcels in S_{11} will be developed.

Unlike the first two cases, case 3 also needs to take into account that the number of S_{99} parcels used does not exceed the total available. Thus, if case 3 were valid only for the residential low density zone parcels, we would have to additionally check that $|E_{RLD}| \leq |S_{99}|$. More generally, assuming that Assumption 4 is applied to the other four zones with "acres" replacing "units" for the commercial and industrial zones and that the corresponding sets are designated as:
$E_{RMD} \subseteq S_{99}$ (residential medium density),
$E_{RHD} \subseteq S_{99}$ (residential high density),
$E_{COM} \subseteq S_{99}$ (commercial),
$E_{IND} \subseteq S_{99}$ (industrial),
we would need to make the following checkable assumption.

Assumption 5
$|E_{RLD}| + |E_{RMD}| + |E_{RHD}| + |E_{COM}| + |E_{IND}| \leq |S_{99}|$

Thus, with the Assumptions 3,4, and 5 in place where needed, we have described feasible values for the residential low density related variables. A similar analysis for the other four zones leads to the same sort of results with the values for the other zonal variables computed analogously. We note that in Section 5, when we solve (25) for land parcels in Montgomery County, Maryland, that the number of units and or area available within each group is less than the lower bound except for the acres available for industrial use which had 253.83 acres available and the bounds were 178.8 and 268.2 acres. Thus, the first four zones corresponded to case 3 and the industrial zone related to case 1.

4.1.4 Other approaches for feasibility

Another approach, assuming that there was some flexibility in the values of the constants $a_3, \ldots, a_{12}$, would be to choose the pairs $(a_3, a_4), \ldots, (a_{11}, a_{12})$ so that the lower and upper bounds were separated enough so that case 1 was in effect. Consequently, the analysis for case 1 shows explicit values for the variables that would represent a feasible solution. This is not an unreasonable course of action given that the modeler generally has some flexibility in determining aspects of the constraints.

Another possibility is to enlarge the feasible region by relaxing the constraints (21a)-(21e) along the lines of goal programming (Winston, 1994) consistent with (4) giving the following constraints:

$$\begin{cases} under_{RLD} \geq a_3 - \sum_{i \in S_{11}} units_i d_i - \sum_{i \in S_{99}} units_i RLD_i, under_{RLD} \geq 0 \\ over_{RLD} \geq -a_4 + \sum_{i \in S_{11}} units_i d_i + \sum_{i \in S_{99}} units_i RLD_i, over_{RLD} \geq 0 \end{cases} \quad (21a')$$

$$\begin{cases} under_{RMD} \geq a_5 - \sum_{i \in S_{12}} units_i d_i - \sum_{i \in S_{99}} units_i RMD_i, under_{RMD} \geq 0 \\ over_{RMD} \geq -a_6 + \sum_{i \in S_{12}} units_i d_i + \sum_{i \in S_{99}} units_i RMD_i, over_{RMD} \geq 0 \end{cases} \quad (21b')$$

$$\begin{cases} under_{RHD} \geq a_7 - \sum_{i \in S_{13}} units_i d_i - \sum_{i \in S_{99}} units_i RHD_i, under_{RHD} \geq 0 \\ over_{RHD} \geq -a_8 + \sum_{i \in S_{13}} units_i d_i + \sum_{i \in S_{99}} units_i RHD_i, over_{RHD} \geq 0 \end{cases} \quad (21c')$$

$$\begin{cases} under_{COM} \geq a_9 - \sum_{i \in S_{14}} acres_i d_i - \sum_{i \in S_{99}} acres_i COM_i, under_{COM} \geq 0 \\ over_{COM} \geq -a_{10} + \sum_{i \in S_{14}} acres_i d_i + \sum_{i \in S_{99}} acres_i COM_i, over_{COM} \geq 0 \end{cases} \quad (21d')$$

$$\begin{cases} under_{IND} \geq a_{11} - \sum_{i \in S_{15}} acres_i d_i - \sum_{i \in S_{99}} acres_i IND_i, under_{IND} \geq 0 \\ over_{IND} \geq -a_{12} + \sum_{i \in S_{15}} acres_i d_i + \sum_{i \in S_{99}} acres_i IND_i, over_{IND} \geq 0 \end{cases} \quad (21e')$$

Also, we define the new penalty objective function as:

$$z_5 = g^-_{RLD}\ \mathrm{under}_{\mathrm{RLD}} + g^+_{RLD}\ \mathrm{over}_{\mathrm{RLD}} + g^-_{RMD}\ \mathrm{under}_{\mathrm{RMD}} + g^+_{RMD}\ \mathrm{over}_{\mathrm{RMD}} + g^-_{RHD}\ \mathrm{under}_{\mathrm{RHD}} + g^+_{RHD}\ \mathrm{over}_{\mathrm{RHD}} + g^-_{COM}\ \mathrm{under}_{\mathrm{COM}} + g^+_{COM}\ \mathrm{over}_{\mathrm{COM}} + g^-_{IND}\ \mathrm{under}_{\mathrm{IND}} + g^+_{IND}\ \mathrm{over}_{\mathrm{IND}}$$

where the coefficients $g^-_{RLD}, g^+_{RLD}, g^-_{RMD}, g^+_{RMD}, g^-_{RHD}, g^+_{RHD}, g^-_{COM}, g^+_{COM}, g^-_{IND}, g^+_{IND} > 0$.
We define the relaxed versions of (25) and (28) with $w_5 > 0$ as follows:

$$
\left\{
\begin{array}{l}
\min z_1 = \sum_{q=1}^{Q} \left(r_1^q - r_0^q\right)^2 + \left(c_1^q - c_0^q\right)^2,\ (planner) \\
\min z_2 = \sum_{i \in S} \left(\Delta\_imperv_i\right)\left(area_i\right) d_i,\ (environmentalist) \\
\min z_3 = \sum_{i \in \bar{S}} area_i\, d_i, (conservationist) \\
\max \mathrm{z}_4 = \\
\sum_{\mathrm{i} \in \mathrm{S}_{11}} \mathrm{value_i d_i} + \sum_{\mathrm{i} \in \mathrm{S}_{12}} \mathrm{value_i d_i} + \sum_{\mathrm{i} \in \mathrm{S}_{13}} \mathrm{value_i d_i} + \sum_{\mathrm{i} \in \mathrm{S}_{14}} \mathrm{value_i d_i} + \sum_{\mathrm{i} \in \mathrm{S}_{15}} \mathrm{value_i d_i} \\
+ \sum_{\mathrm{i} \in \mathrm{S}_{99}} \left(\mathrm{value_i RLD_i + value_i RMD_i + value_i RHD_i + value_i COM_i + value_i IND_i}\right) \\
(planner) \\
\min \mathrm{z}_5\ (growth\ rate\ bounds\ penalties) \\
s.t. \\
(19), (21a') - (21e'), (22a) - (22e), (23a) - (23e), (24a) - (24e) \\
d_i \in \{0,1\}, \forall i \in S \\
RLD_i, RMD_i, RHD_i, COM_i, IND_i \in \{0,1\}, \forall i \in S_{99} \\
y_{RLD}, y_{RMD}, y_{RHD}, y_{COM}, y_{IND}, w_{RLD}, w_{RMD}, w_{RHD}, w_{COM}, w_{IND} \in \{0,1\}
\end{array}
\right.
\tag{25'}
$$

and

$$
\left\{
\begin{array}{l}
\min w_1 z_1 + w_2 z_2 + w_3 z_3 - w_4 z_4 + w_5 z_5 \\
\mathrm{s.t.} \\
(19), (21a') - (21e'), (22a) - (22e), (23a) - (23e), (24a) - (24e) \\
d_i \in \{0,1\}, \forall i \in S \\
RLD_i, RMD_i, RHD_i, COM_i, IND_i \in \{0,1\}, \forall i \in S_{99} \\
y_{RLD}, y_{RMD}, y_{RHD}, y_{COM}, y_{IND}, w_{RLD}, w_{RMD}, w_{RHD}, w_{COM}, w_{IND} \in \{0,1\}
\end{array}
\right.
\tag{28'}
$$

As shown below, the feasible region to these relaxed problems is always nonempty.

4.2 Convexity and existence results

In this section we present convexity and existence results for the Smart Growth multiobjective problem defined above. First, we note that for computational reasons, it is important to have both the objective function as well as constraint set of our multiobjective problem convex. This result will ensure that all local solutions are in fact global ones; see Bazaraa *et al.* (1979) for details. As we'll see, all the pieces of the weighted objective function (i.e., taking positively weighted combinations of the individual objectives) will be linear except for the compactness measure, which will be shown to be convex quadratic, resulting in a convex quadratic objective function overall. The constraints are linear except for the binary restrictions on selected variables. But relaxing these constraints to having these variables in $[0,1]$ as is done for computing subproblems, we see that the constraint set will be linear. Thus, we will have a mixed integer convex quadratic problem to be solved overall. The purpose of the next few results is to show that the overall weighted objective function of our problem will be convex in the variables.

Lemma 1

The function $f\left(r_1^q, r_0^q, c_1^q, c_0^q\right) = \left(r_1^q - r_0^q\right)^2 + \left(c_1^q - c_0^q\right)^2$ *is convex in the variables* $r_1^q, r_0^q, c_1^q, c_0^q$.

Proof.

We have

$$\nabla^2 f\left(r_1^q, r_0^q, c_1^q, c_0^q\right) = \begin{pmatrix} 2 & -2 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & -2 & 2 \end{pmatrix}$$ which is of the form $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$. But the eigenvalues of A_1 and A_2 are $\{0,4\}$. Since the eigenvalues of a block diagonal matrix are simply the union of the eigenvalues of each block matrix,

$\Rightarrow$ eigenvalues of $\nabla^2 f\left(r_1^q, r_0^q, c_1^q, c_0^q\right)$ are $\{0,0,4,4\}$

$\Rightarrow \nabla^2 f\left(r_1^q, r_0^q, c_1^q, c_0^q\right)$ is (symmetric) positive semi-definite

$\Rightarrow f$ is convex. QED

This leads to the next result concerning convexity of the weighting problem.

Theorem 1

The weighted objective $w_1 z_1 + w_2 z_2 + w_3 z_3 - w_4 z_4$ *is convex in its variables as long as the weights* $w_1, w_2, w_3, w_4 \geq 0$.

Proof.

First note that the developer's objective function z_4 is linear in its variables so that $-w_4 z_4$ is also linear. Since linear functions are convex and since nonnegative sums of convex functions are convex, the desired result follows from Lemma 1 noting that the Hessian is of the form

$$\begin{pmatrix} H_1 & 0 & \dots & 0 \\ 0 & \ddots & 0 & \vdots \\ \vdots & 0 & H_Q & \\ 0 & \dots & & 0 \end{pmatrix}$$ where $H_q = \nabla^2 f\left(r_1^q, r_0^q, c_1^q, c_0^q\right), q = 1, \dots, Q$. QED

We note that when minimizing the area of the rectangle enclosing the developed parcels, we do not have a convex function that is to be minimized. This is clear in light of Lemma 1 and Theorem 1 if we look at the Hessian matrix of $\left(r_1 - r_0\right)\left(c_1 - c_0\right)$ in terms of the variables

r_1, c_1, r_0, c_0. The Hessian matrix is $\begin{pmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 \end{pmatrix}$ which has eigenvalues $\{-\sqrt{2}, -\sqrt{2}, \sqrt{2}, \sqrt{2}\}$ so that this Hessian is not positive semi-definite, hence this function is not convex.

We still need to ensure that (24a)-(24d) accurately define the borders of the rectangle around the developed parcels. This result as well as a guarantee of a solution to (28) or (28') is shown in the next theorem.

Theorem 2

i) Problem (28) always has an optimal solution if Assumptions 3,4, and 5 hold and applied to each zone type, depending on cases 1, 2, or 3,
ii) Problem (28') always has a solution,
iii) At an optimal solution to either problem (28) or (28') if Assumption 2 holds, then the constraints (24a)-(24d) insure that $r_1^q, r_0^q, c_1^q, c_0^q$ correspond respectively to the northernmost, southernmost, easternmost, and westernmost borders of all the developed parcels in quadrant q for $q = 1, \ldots, Q$.

Proof.

The problem (28) has a continuous objective function so that by the Weierstrass Theorem, it suffices to show that the feasible region is nonempty and compact. Since Assumptions 3, 4, and 5 hold, we know that the feasible region to (28) is nonempty. The boundedness of the variables $d_i \forall i \in S, RLD_i, RMD_i, RHD_i, COM_i, IND_i, \forall i \in S_{99}$ and $y_{RLD}, y_{RMD}, y_{RHD}, y_{COM}, y_{IND}, w_{RLD}, w_{RMD}, w_{RHD}, w_{COM}, w_{IND}$ is immediate since they are all binary. We will show that the other four sets of variables $r_1^q, r_0^q, c_1^q, c_0^q$ are also bounded. First note that all of these variables are bounded below by zero via (24e). From (24a) and (24c) we see that for each quadrant q,

$$r_0^q \le \min_i \{row^S(i)\} + (1 - d_i)M \le M_1$$
$$c_0^q \le \min_i \{col^W(i)\} + (1 - d_i)M \le M_2$$

for suitable positive constants M_1, M_2. Thus, it suffices to consider upper bounds on just r_1^q and c_1^q. By the symmetry in both the constraints (24) and the quadratic part of the objective function, it suffices to just analyze r_1^q since similar reasoning will apply to c_1^q. Assume that for some quadrant q we have a sequence $\{(r_1^q)^k\} \to \infty$ as $k \to \infty$. Such a sequence of values is clearly feasible via (24b). But the function $f\left((r_1^q)^k, r_0^q, c_1^q, c_0^q\right) = \left((r_1^q)^k - r_0^q\right)^2 + (c_1^q - c_0^q)^2 \to \infty$ as $\{(r_1^q)^k\} \to \infty$. We can see this since for suitably large values of the index $w > 0$, and for the fixed index k, $(r_1^q)^{k+w} = (r_1^q)^k + \Delta^w$ with $\Delta^w > 0$ and $\lim_{w \to \infty} \Delta^w = \infty$ and $f\left((r_1^q)^{k+w}, r_0^q, c_1^q, c_0^q\right) - f\left((r_1^q)^k, r_0^q, c_1^q, c_0^q\right) = \Delta^2 + 2\Delta\left((r_1^q)^k - r_0^q\right) > 0$ as long as $\Delta > 2\left|(r_1^q)^k - r_0^q\right|$.

Since (28) involves a minimization, no such sequence $\{(r_1^q)^k\} \to \infty$ could be optimal. A similar argument holds for c_1^q and all other quadrants. Thus, there exist constants $\beta_1, \beta_2 > 0$ such that the solution set to (28) is the same as that of (*), the problem that is the same as (28) but with the additional restrictions that $r_1^q \leq \beta_1$ and $c_1^q \leq \beta_2$ for each quadrant q. But (*) has all variables bounded so does (28). The closedness of the feasible region is also guaranteed since all the constraints are linear or binary. The desired result to i.) then follows.

As for part ii.), we need only consider the nonemptyness of the feasible region and the boundedness of the over- and under-achievement variables given the above discussion for (28). But the penalties $g_{RLD}^-, g_{RLD}^+, g_{RMD}^-, g_{RMD}^+, g_{RHD}^-, g_{RHD}^+, g_{COM}^-, g_{COM}^+, g_{IND}^-, g_{IND}^+ > 0$ and because we are minimizing the sum of these penalized deviations in (21a')-(21e') with no other constraints on the under- and over-achievement variables (except nonnegativity), by an optimality argument similar to part i), the solution set to (28') is the same as that of a problem (**) which adds upper bounds on these under- and over-achievement variables. Hence, all variables are bounded.

As for the nonemptyness of the feasible region to (28'), due to the relaxation of constraints (21a)-(21e), we see that the following is a feasible solution:

$d_i = 1, \forall i \in S_{11} \cup S_{12} \cup S_{13} \cup S_{14} \cup S_{15}$, $d_i = 0, \forall i \in S_{99}$, for each $q = 1, \ldots, Q$,

$r_0^q = \min_{i \in S_q} \{row^S(i)\}$, $r_1^q = \max_{i \in S_q} \{row^N(i)\}$, $c_0^q = \min_{i \in S_q} \{col^W(i)\}$, $c_1^q = \max_{i \in S_q} \{col^E(i)\}$

$$\text{if } \begin{cases} \sum_{i \in S_{11}} units_i > a_3 \Rightarrow y_{RLD} = 0, RLD_i = 0, \forall i \in S_{99}, w_{RLD} = 1 \\ \sum_{i \in S_{11}} units_i \leq a_3 \Rightarrow y_{RLD} = 1, RLD_i = 0, \forall i \in S_{99}, w_{RLD} = 0 \end{cases}$$

Lastly, we take the under- and over-achievement variables as follows:

$$under_{RLD} = \max\left\{a_3 - \sum_{i \in S_{11}} units_i d_i - \sum_{i \in S_{99}} units_i RLD_i, 0\right\}, \text{ and}$$

$$over_{RLD} = \max\left\{-a_4 + \sum_{i \in S_{11}} units_i d_i + \sum_{i \in S_{99}} units_i RLD_i, 0\right\}$$

Similarly for the other four zones with "RLD" replaced by "RMD", "RHD", "COM", and "IND".

As for part iii.), let $x^* = \{d_i^* \forall i \in S, RLD_i^*, RMD_i^*, RHD_i^*, COM_i^*, IND_i^*, \forall i \in S_{99},$
$y_{RLD}^*, y_{RMD}^*, y_{RHD}^*, y_{COM}^*, y_{IND}^*, w_{RLD}^*, w_{RMD}^*, w_{RHD}^*, w_{COM}^*, w_{IND}^*,$
$(r_1^q)^*, (r_0^q)^*, (c_1^q)^*, (c_0^q), q = 1, \ldots, Q^*\}$ be an optimal solution to (28). There are two cases to consider.

For case 1, assume that for a particular quadrant q there is at least one developed parcel in a solution. By (24a), (24b), and Assumption 2 we see that

$$(r_0^q)^* \leq row^S(i) < row^N(i) \leq (r_1^q)^* \tag{29}$$

for a developed parcel i so that $(r_1^q)^* - (r_0^q)^* > 0$. Suppose for sake of contradiction that for all indices i, (24b) holds as a strict inequality. Consider the feasible value for r_1^q of $\hat{r}_1^q = (r_1^q)^* - \Delta$, where Δ is sufficiently small and satisfies $0 < \Delta < (r_1^q)^* - (r_0^q)^*$ and all other values are the same as in x^*. Then we have the following.

$$\left[\left(\hat{r}_1^q\right)^* - \left(r_0^q\right)^*\right]^2 + \left[\left(c_1^q\right)^* - \left(c_0^q\right)^*\right]^2$$
$$= \left[\left(\left(r_1^q\right)^* - \Delta\right) - \left(r_0^q\right)^*\right]^2 + \left[\left(c_1^q\right)^* - \left(c_0^q\right)^*\right]^2$$
$$= \left[\left(r_1^q\right)^* - \left(r_0^q\right)^*\right]^2 + \left[\left(c_1^q\right)^* - \left(c_0^q\right)^*\right]^2 - 2\left[\left(r_1^q\right)^* - \left(r_0^q\right)^*\right]\Delta + \Delta^2$$
$$< \left[\left(r_1^q\right)^* - \left(r_0^q\right)^*\right]^2 + \left[\left(c_1^q\right)^* - \left(c_0^q\right)^*\right]^2$$

as long as the function $\theta(\Delta) = -2\left[\left(r_1^q\right)^* - \left(r_0^q\right)^*\right]\Delta + \Delta^2 < 0$. This is guaranteed since $\theta(\Delta)$ has roots at $\left\{0, 2\left(\left(r_1^q\right)^* - \left(r_0^q\right)^*\right)\right\}$ and is negative in between these roots. Thus, we have shown a contradiction to the optimality of x^* showing that there must be an index i for this quadrant such that (24b) holds as an equality. Similar reasoning applies to (24a) (24c), and (24d) so that the desired result follows.

For case 2, assume that for the quadrant q no parcels are developed in an optimal solution x^*. In this case, the northernmost, southernmost, easternmost, and westernmost borders are somewhat arbitrary since the set of developed parcels is vacuous. However, we know that to make sense, we must have $\left(r_1^q\right)^* \geq \left(r_0^q\right)^*, \left(c_1^q\right)^* \geq \left(c_0^q\right)$. But (24a)-24d) show that

$$r_0^q \leq row^S(i) + M \text{ and } row^N(i) - M \leq r_1^q \tag{30}$$
$$c_0^q \leq col^W(i) + M \text{ and } col^E(i) - M \leq c_1^q \tag{31}$$

which, in conjunction with the other constraints, allows for any ordering between the pairs of variables $\left[\left(r_1^q\right)^*, \left(r_0^q\right)^*\right]$, and $\left[\left(c_1^q\right)^*, \left(c_0^q\right)\right]$ given that M is a sufficiently large positive value. Hence, by an optimality argument, it must be the case that $\left(r_1^q\right)^* = \left(r_0^q\right)^*, \left(c_1^q\right)^* = \left(c_0^q\right)$ to minimize the objective function term for this quadrant. Such values clearly make sense in light of the vacuous set of developed parcels for the quadrant.

For (28'), a similar argument holds and can be used to show that either the over- or under-achievement penalty holds as an equality at optimality. QED

Theorem 3
i) The Smart Growth problem (25) always has an optimal solution if Assumptions 3, 4, and 5 hold and applied to each zone type, depending on cases 1, 2, or 3, or
ii) The relaxed Smart Growth problem (25') always has a solution.
Proof.
From Theorem 2, we know that the weighted problem (28) always has a solution for nonnegative weights w_1, w_2, w_3, w_4. When these weights are strictly positive, by Theorem 3.1.2. in Miettinen (1999), the solution of the weighting problem corresponds to a Pareto optimal point of (25). Thus, any choice of positive weights insures a Pareto optimal solution to (25). A similar line of reasoning holds for problem (25') with the penalty objective z_5 taking on a weight of w_5. QED

5. Numerical results for Montgomery County, Maryland

In this section we present numerical results based on land parcels in Montgomery County, Maryland for solving the multiobjective optimization problem (25). As described above, Pareto optimal solutions to (25) can be obtained as solutions to the weighted version of the problem (28), which are instances of quadratic mixed integer programs (QMIPs) with about 3,500 variables (most of which are binary) and over 23,000 constraints.

5.1 Database of land parcels for Montgomery County, Maryland

Montgomery County, Maryland is located north of Washington D.C. and borders the state of Virginia as shown in Figure 3. Covering some 1,300 square kilometers (500 square miles) of Maryland's territory and occupied by over 873,000 inhabitants[4], this county is the most populated one in Maryland. Using a database of Montgomery County land parcel information in geographic information system (GIS) format, we were able to analyze both current and potential development of the area. Figure 4 shows the northwestern section of the county used in this study, comprising our database of some 913 undeveloped and 4,837 previously developed parcels.

For the purposes of examining the compactness objectives, we have divided the county into four quadrants (Q1, Q2, Q3, Q4) as presented in Figure 5a. Since the borders of the parcels were not perfectly aligned with the quadrant divisions, the centroid of each parcel was used to determine into which quadrant the parcel should be assigned. If the centroid was within the bounds of the quadrant, then the whole parcel was assigned to that quadrant. We note that quadrant 3 (Q3), all things being equal, had the greatest chance for significant compact land development given its relatively small number of previously developed parcels. After partitioning the parcels based on this centroid rule, the resulting quadrants and their associated parcels appear in Figure 5b. Once the parcels were assigned to the quadrants, the parcel coordinates were normalized to reduce the numerical value of the coordinates, useful for more balanced results in the weighted optimizations. Specifically, the minimum northing (row) and easting (column) values among all parcels, was deducted from the northing and easting coordinates for each parcel. Thus, the westernmost point of the westernmost parcel of the set had a horizontal coordinate of zero; similarly, the southernmost point of the southernmost parcel had a vertical coordinate of zero.

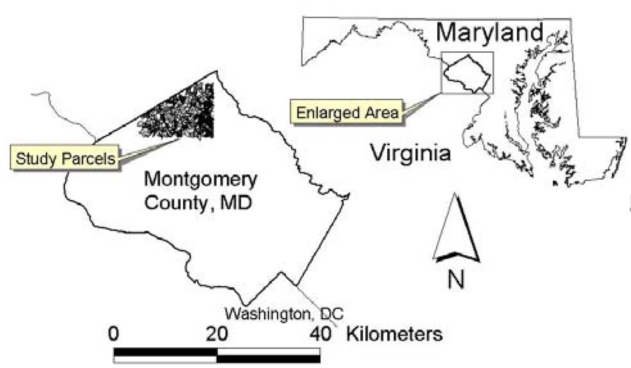

Fig. 3 Montgomery County, Maryland.

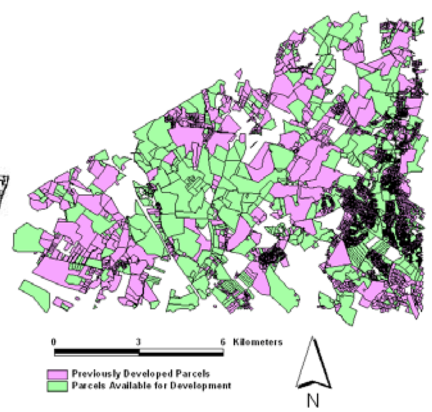

Fig. 4. Montgomery County Database Segmented By Previously Developed Parcels (Purple) and Those Available For Development (Green).

[4] According to 2000 Census survey. Source: http://www.co.mo.md.us/cntymap.htm

Fig. 5a. Division of Montgomery County Into Four Quadrants.

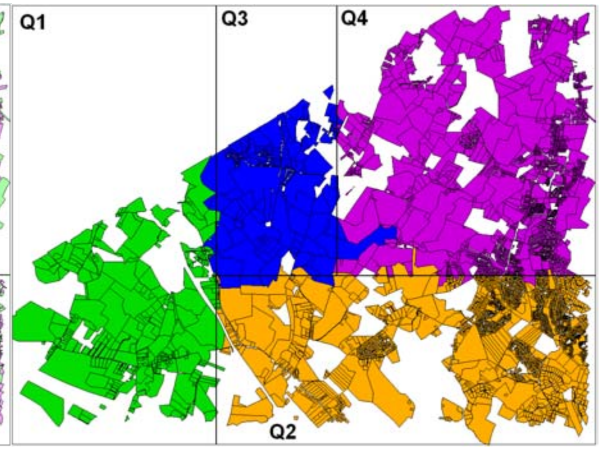

Fig. 5b. Parcels Assigned to Each Quadrant Using the Centroid Rule.

To illustrate the effect of the environmentally sensitive parcels involved in the conservationist's objective function, we selected 53 parcels from our database. Their locations are shown in Figure 6 along with the relative number in each of the quadrants as indicated in Table 4.

Table 4
Number of Environmentally Sensitive Parcels Distribution By Quadrant

Quadrant	**Number of Parcels**
2	21
3	30
4	2

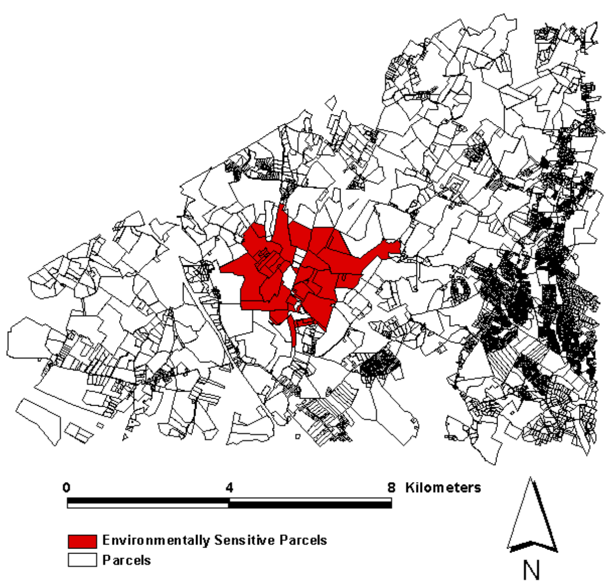

Fig. 6. Set of Environmentally Sensitive Parcels.

5.2 Nine cases considered

In this and subsequent sections we describe some findings associated with Pareto optimal land development solutions to (25) using the weighting method, i.e., solving (28). The resulting QMIPs were generated making use of the MPL modeling language and solved using the XPRESS-MP solver. In all, nine sets of weights for the four stakeholders were applied; these weights appear in Table 5 shown below and are displayed in 4-tuples of weights of the form (w_1, w_2, w_3, w_4). These weights, w_1, w_2, w_3, w_4 correspond respectively to, the planner, the environmentalist, the conservationist, and the developer. For example, case 9 represents a weight of "1" for each of the stakeholders.

These nine cases corresponding to the assignment of different weights in (28) can be broken down into two main groups. Group one is composed of solving (28) just from the single perspective of one of the stakeholders. For example, case 1 has weights of (1,0,0,0) which corresponds to considering only the planner's perspective. Consequently there are four of these single objective cases in all: case 1 ("Planner Alone"), case 3 ("Environmentalist Alone"), case 5 ("Conservationist Alone"), and case 7 ("Developer Alone"). Land development plans determined as solutions to these four optimizations do not necessarily represent Pareto optimal solutions (unless they are unique). These results are meant more for purposes of comparison with the Pareto optimal solutions, which we describe next.

The second group of cases considers strictly positive weights for each of the stakeholder perspectives resulting in Pareto optimal solutions (hence "Pareto" in the title of these cases). Case 9 involves an equal weight of 1 for each of the stakeholder's objectives. This is contrasted with the other four cases (cases 2,4, 6, and 8) in which one of the stakeholders is "highlighted" with the largest weight of one assigned to it; the weights for the other three stakeholders is set to 0.001. For example, case 2 assigns a weight of one to the planner and 0.001 to the other three stakeholders. Table 6 presents the values of the different objectives evaluated for each of the nine cases under consideration. In addition, the last column of this table represents the relative gap value, i.e., $|best\,solution - best\,bound| / best\,bound$, used with the solver, a value of zero generally not leading to reasonable solution times.

Analysis of these nine cases will be concentrated on two areas described below. First, what are the tradeoffs between the various stakeholders when Pareto optimal as opposed to single objective solutions are used? Secondly, we concentrate on the planner's compactness objective and highlight some key findings.

Table 5
Weights Assigned to Each Stakeholder's Objective

Case		Planner (Compactness)	Environmentalist (Imperviousness Change)	Conservationist (Env. Sensitive Area)	Developer (Profit)	Relative Gap
1	Planner Alone	1	0	0	0	5e-005
2	Planner Pareto	1	0.001	0.001	0.001	5e-005
3	Environmentalist Alone	0	1	0	0	5e-005
4	Environmentalist Pareto	0.001	1	0.001	0.001	5e-005
5	Conservationist	0	0	1	0	5e-005

	Alone					
6	Conservationist Pareto	0.001	0.001	1	0.001	5e-005
7	Developer Alone	0	0	0	1	5e-005
8	Developer Pareto	0.001	0.001	0.001	1	5e-004
9	All Perspectives	1	1	1	1	5e-005

5.3 Analysis of tradeoffs involving single and multiple objective solutions

From Table 6, we observe how the individual objectives reach their optimal values when they are evaluated alone (shown in bold) and when other stakeholder interests are taken into account. Thus, this table provides valuable information on the explicit tradeoffs that are made in considering all the stakeholder perspectives and is therefore important in the Smart Growth planning process. Note that the conservationist achieves an objective of 0 (i.e., no environmentally sensitive parcels are developed) when considering its single objective optimization.

Consider first the perspective of the planner who is trying to maximize the compactness of the developed land in all four of the quadrants taken separately. If we consider just the planner's single objective by itself (case 1), we see that the optimal level of compactness of the developed land [5] is 286.06 square kilometers (115.45 square miles). Normalizing so that this value is 100%, we see that the planner does worse when the other three stakeholders's objectives are optimized one at a time. In particular, the compactness measures worsens by 11.4%, 13.9%, and 16.7, respectively, when optimizing just for the environmentalist, the conservationist, and the developer, reflecting that other concerns are more important for these other stakeholders. However, it is interesting to note that when the five "Pareto" cases are considered ("Planner Pareto", "Environmentalist Pareto", "Conservationist Pareto", "Developer Pareto", "All Perspectives"), the optimal compactness matches that of when just the planner is considered. This suggests the importance of this objective to all stakeholders when given a positive weight, also it is related to some extent to the magnitude of the compactness objective.

The environmentalist obtains a solution with minimum change in imperviousness when his perspective is considered by itself, resulting in an optimal value of 0.01593 square kilometers. Once the other stakeholders are considered either separately or in a Pareto fashion, the environmentalist does worse. The environmentalist's objective appears to be more sensitive than the planner's in that the former worsens by about 50% under the "Planner Pareto" perspective but the Planner does no worse than its single objective under "Environmentalist Pareto". Of course, part of the explanation is possible due to the weights considered and the scales involved. The environmentalist's objective does particularly badly, as one might suspect, when one views the "Developer Pareto" perspective. In this case, Table 6 indicates a 49.1% worsening in the change in imperviousness due to accommodating the developer's objective with a higher weight. Also, the environmentalist's objective suffers about the same amount when considering the "Conservationist Pareto" case. Consequently, the environmentalist appears to be the most sensitive to the objectives of the other stakeholders in that it has the largest percentage deviations from optimality when considers the other stakeholders.

The conservationist is able to steer development out of the environmentally sensitive areas when this is the only perspective. However, only a slight change occurs in this objective function when the other perspectives receive a small positive weight (the "Conservationist Pareto" case). Lastly, Table 6 indicates some significant worsening in the developer's optimal objective function

[5] As measured by minimizing the square of the length of the diagonal of the compactness rectangle.

when the other stakeholders are involved. For example, the developer's profit drops by over 33% when considering the "Environmentalist Pareto" case.

Table 6
Value of the Objective Functions By Case

Case	Description	Maximum Distance Squared km^2 (mi^2)	Percentage of Optimal	Imperviousness Change km^2 (mi^2)	Percentage of Optimal
1	Planner Alone	**286.06 (110.45)**	100.0%	0.01840 (0.0071)	115.5%
2	Planner Pareto	286.06 (110.45)	100.0%	0.02391 (0.00923)	150.1%
3	Environmentalist Alone	318.72 (123.06)	111.4%	**0.01593 (0.00615)**	100.0%
4	Environmentalist Pareto	286.06 (110.45)	100.0%	0.01594 (0.00616)	100.1%
5	Conservationist Alone	325.96 (125.86)	113.9%	0.01755 (0.00678)	110.2%
6	Conservationist Pareto	286.06 (110.45)	100.0%	0.02377 (0.00918)	149.2%
7	Developer Alone	333.69 (128.84)	116.7%	0.02384 (0.0092)	149.6%
8	Developer Pareto	286.06 (110.45)	100.0%	0.02374 (0.00917)	149.1%
9	All Perspectives	286.06 (110.45)	100.0%	0.02337 (0.00902)	146.7%
	Numbers are **better** if:	**Smaller**	**Smaller**	**Smaller**	**Smaller**

Case	Description	Env. Sensitive Area km^2 (mi^2)	Percentage of Optimal	Profit Millions of $ U.S.	Percentage of Optimal
1	Planner Alone	0.87 (553.80)	infinite	$1,317.56	69.2%
2	Planner Pareto	2.68 (1,712.49)	infinite	$1,686.95	88.7%
3	Environmentalist Alone	1.86 (1,192.03)	infinite	$1,148.84	60.4%
4	Environmentalist Pareto	1.30 (833.64)	infinite	$1,273.13	66.9%
5	Conservationist Alone	**0.00 (0.00)**	0/0 0/0	$1,266.36	66.5%
6	Conservationist Pareto	0.02 (11.67)	infinite	$1,891.61	99.4%
7	Developer Alone	4.09 (2,616.62)	infinite	**$1,902.89**	100.0%
8	Developer Pareto	1.77 (1,132.39)	infinite	$1,899.82	99.8%

9	All Perspectives	2.56 (1,640.65)	infinite	$1,672.72	87.9%
	Numbers are **better** if:	**Smaller**		**Larger**	**Larger**

Another set of observations regarding the tradeoffs between the different stakeholders concerns the choice in the number of units or area. From Table 7 we see that the developer scenarios involved selecting nearly or equal to the maximum possible number of parcels without exceeding the upper bound. This makes sense since all things being equal, the profit increases with more parcels being developed. The limiting factors are the upper bounds and other constraints or perspectives that need to be considered. We also note that the environmentalist chose to develop nearly or equal to the minimum amounts required by the bounds. The other two stakeholders (the conservationist and the planner) appear to be more constrained in their land development choices, possibly by their respective objectives. This leads to numbers of units and areas, which are relatively less close to the upper and lower bounds of allowable development.

Table 7
Number of Units or Acres Developed By Each Perspective In Each Zone, Lower and Upper Bounds

Case	**Description**	**Units RLD**	**Units RMD**	**Units RHD**	**Area Commercial km^2 (mi^2)**	**Area Industrial km^2 (mi^2)**
1	Planner Alone	1,745	8,190	4,887	1.641 (0.634)	0.985 (0.380)
2	Planner Pareto	2,331	12,285	6,384	1.641 (0.634)	1.027 (0.397)
3	Environmentalist Alone	1,554	8,190	4,256	1.094 (0.423)	0.724 (0.279)
4	Environmentalist Pareto	1,554	8,190	4,256	1.096 (0.423)	0.724 (0.279)
5	Conservationist Alone	1,554	8,296	4,687	1.481 (0.572)	1.027 (0.397)
6	Conservationist Pareto	2,329	12,282	6,372	1.638 (0.632)	1.027 (0.397)
7	Developer Alone	2,331	12,285	6,384	1.641 (0.634)	1.027 (0.397)
8	Developer Pareto	2,331	12,285	6,381	1.641 (0.634)	1.027 (0.397)
9	All Perspectives	2,331	12,285	6,384	1.641 (0.634)	0.740 (0.286)
	Lower Bound	1,554	8,190	4,256	1.094 (0.423)	0.724 (0.279)
	Upper Bound	2,331	12,285	6,384	1.641 (0.634)	1.085 (0.419)
	Available	971	3,359	1,926	0.690 (0.266)	1.027 (0.397)
	Available from 99	7,572	30,998	62,242	31.612 (12.205)	31.612 (12.205)

	Total Available[6]	8,543	34,357	64,168	32.301 (12.472)	32.639 (12.602)

As noted above, the largest change due to tradeoffs is noted for the environmentalist since the objective of this perspective increases from 100% to 150.1% under the "Planner Pareto" perspective. To understand the different aspects related to this tradeoff we divided the parcels involved in these two cases into four groups as depicted in Figure 7.

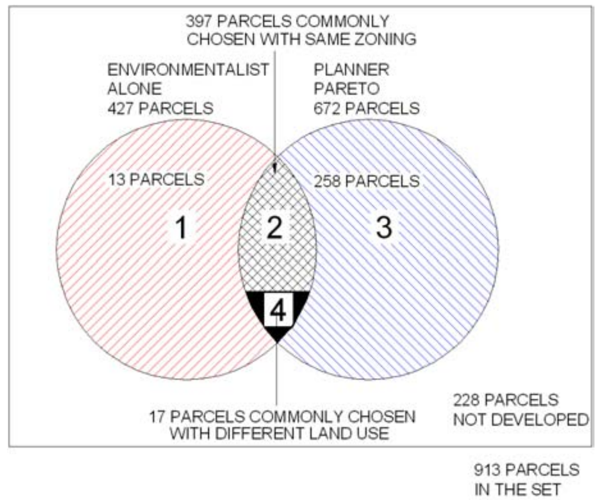

Fig. 7. Comparison of the Environmentalist and Planner Perspectives.

The four groups are formed as follows:

- Group 1: Parcels chosen for development only by the environmentalist,
- Group 2: Parcels chosen by both perspectives in which the same zoning designations were assigned,
- Group 3: Parcels chosen for development only by the "Planner Pareto" perspective,
- Group 4: Parcels chosen by both perspectives but with different zoning assignment.

There is a fifth group formed by the parcels that were already developed. This last group has a tremendous impact on the compactness measurement but doesn't provide any value to any of the other three objectives. A large part of this difference in the environmentalist's objective under the "Planner Pareto" case caused by the "Group 3" parcels. This group contributed to almost half of the increase in the environmentalist's objective under the "Planner Pareto" case.

5.4 Analysis of the compactness objective

Compactness of the developed area is a key component to Smart Growth. All things being equal, the more compact the area, the less the amount of infrastructure is needed and the more the efficient the development plan. Further, open and undisturbed areas are then afforded more space and less fragmentation. In this section we describe some results concerning the compactness of the area of the developed parcels. Since the model in (25) consider compactness of the quadrant separately, it is convenient to consider the compactness of each of the quadrants individually. Figure 8 presents a detail of each quadrant with two key rectangles drawn one inside the other. The

[6] Assuming that all the available parcels from 99 go to each category indicated.

inner rectangle is drawn around all the parcels that were previously developed and the outer rectangle encloses all the parcels in that quadrant, or the "quadrant rectangle" for short. Note that parcels that don't belong to the quadrant in question have been removed for clarity of presentation. By inspection one can notice how each quadrant has a different potential for compactness. For example there is only one parcel in quadrant 2 that would, if developed, change the measure of the compactness (as defined in this paper) since there is only one parcel that has a portion of its area between the inner and outer rectangles. Conversely, quadrant 3 has more potential for compactness given its configuration of parcels that are already developed or available for development. From this observation a key ratio related to the "efficiency" of the compactness can be defined for each quadrant. Specifically, we can take the ratio of the diagonal of the inner rectangle to the diagonal of the quadrant rectangle. Clearly, a lower value means a greater "potential" for more compactness. Table 8 provides these "potential compactness" ratios for each quadrant based on the ratio of the diagonals. Also in this table we see the ratio based on areas. While not optimizing on the area *per se*, it is still instructive to compare these area ratios by quadrant.

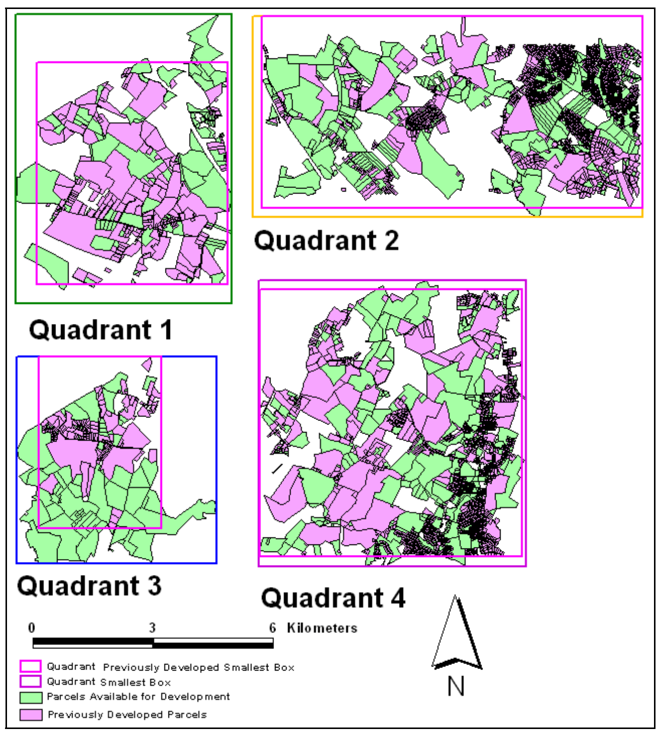

Fig. 8. Inner and Quadrant Rectangles for Quadrant 1(top left), Quadrant 2 (top right), Quadrant 3 (bottom left) and Quadrant 4 (bottom right).

Table 8
Compactness Potential Ratio Calculation for Each Quadrant

Quadrant	**Square of the Diagonal for Already Developed Parcels km^2**	**Square of the Diagonal for the Quadrant km^2**	**Ratio**	**Square of the Area for Already Developed Parcels km^2**	**Square of the Area for the Quadrant km^2**	**Ratio**

1	53.45	76.05	0.7028	26.41	36.90	0.7159
2	113.12	115.38	0.9804	45.49	47.57	0.9562
3	27.62	51.40	0.5374	13.07	25.69	0.5088
4	87.11	95.28	0.9142	43.55	47.52	0.9164

Table 8 confirms what by visual inspection can be seen in Figure 8; quadrant 3 with the smallest ratio has the greatest value of compactness which could be dramatically increased by developing parcels outside the box containing those already developed. From Table 6 we see that the "Developer Alone" perspective has the greatest increase in compactness. Based on this observation we compare the cases that involve the developer in the two most potentially affected quadrants (1 and 3). In Table 9, we show the ratio of the diagonal squared of the rectangle for all developed parcels—previously existing plus those selected by the model—to the diagonal squared for the rectangle of just the previously existing ones. As anticipated, since quadrants 2 and 4 had substantial pre-existing development (demonstrated by the higher ratios from Table 8), the ratio in Table 9 is 1.00 or nearly this value for all the nine cases. By contrast, quadrants 1 and 3 had the lowest diagonal ratios from Table 8 which allowed for larger diagonals in the total developed rectangle. This result is manifested in Table 9 by ratios that are significantly greater than 1.

Table 9
Compactness Ratios for Each Quadrant Based on the Square of the Diagonals

Compactness Ratios

Case	Description	Q1 Ratio	Q2 Ratio	Q3 Ratio	Q4 Ratio
1	Planner Alone	1.00	1.00	1.00	1.05
2	Planner Pareto	1.00	1.00	1.00	1.05
3	Environmentalist Alone	1.39	1.00	1.43	1.05
4	Environmentalist Pareto	1.00	1.00	1.00	1.05
5	Conservationist Alone	1.42	1.02	1.43	1.09
6	Conservationist Pareto	1.00	1.00	1.00	1.05
7	Developer Alone	1.38	1.00	1.86	1.09
8	Developer Pareto	1.00	1.00	1.00	1.05
9	All Perspectives	1.00	1.00	1.00	1.05

Quadrant 1 is analyzed in Figure 9 and shows how when the developer's perspective is considered by itself, the development takes place outside of the area where the previous development exists. This result can be understood by comparing Figures 8 and 9 and noting for example in the "Developer Alone" portion of Figure 9, the red parcel in the northeast corner. This parcel was not within the previously developed rectangle of Figure 8. This result is understandable since the developer's objective is to maximize profits by developing as many parcels as possible. Since the "Developer Alone" case does not consider the compactness objective at all, the parcels chosen may very well be outside of the rectangle of the previously developed parcels. We note however that by giving even a small weight to the compactness objective (using the "Developer Pareto" perspective), the solution involved parcels within a box smaller than the one around the "Developer Alone" case. This means that the developer was able to choose at least one solution that maximized the profit and at the same time maintained the development within a smaller rectangle than the "Developer Alone" case.

For quadrant 3, Figure 10 indicates that the "Developer Alone" perspective again resulted in a larger area for development as compared to the "Developer Pareto" and "All Perspectives" cases. In fact, the "Developer Alone" perspective appears to have used the entire quadrant in terms of the bounding rectangle.

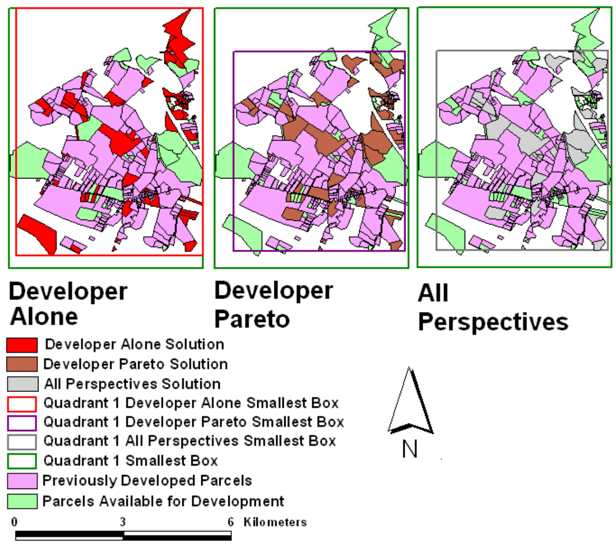

Fig. 9. Solution for the Developer Alone (left), Developer Pareto (center) and All Perspectives (right) Quadrant #1.

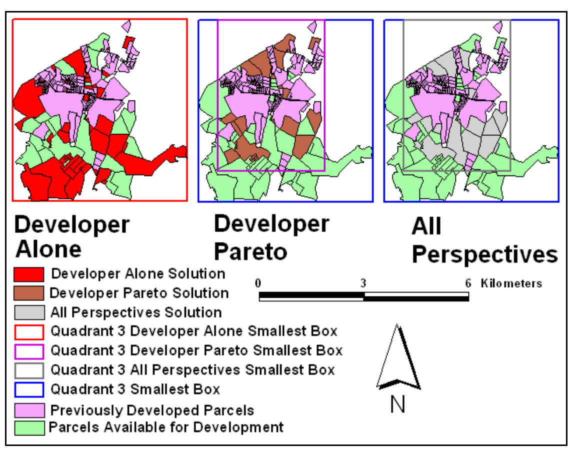

Fig. 10. Solution for the Developer Alone (left), Developer Pareto (center) and All Perspectives (right) Quadrant #3.

These results highlight the importance of considering compactness of the developed area in conjunction with the other stakeholder interests.

6. Conclusions

In this paper we have presented a multiobjective optimization formulation for Smart Growth in land development based on recognizing the objectives of four different types of stakeholders: the government planner, the environmentalist, the conservationist, and the land developer. This paper presented potential objective functions that might be posed by these various stakeholders. For some stakeholders, multiple alternative objective functions were presented. Ultimately, one objective function from each stakeholder was selected and the resulting model was applied in the context of an illustrative example for a GIS-based data set in Montgomery County, Maryland.

This model had both linear and quadratic objective functions subject to linear and binary constraints. Using the weighting method (Cohen, 1978) for determining Pareto optimal points resulted in quadratic mixed integer programs (QMIPs) to be solved for each choice of positive weights applied to the stakeholder objective functions. The quadratic objective resulted from considering compactness of the developed area and represented the government planner's perspective. While other researchers have considered alternative formulations for compactness, our choice is advantageous since it represents a computationally attractive approach to model efficient infrastructure development. Indeed, as we‘ve shown, the weighted problems are convex QMIPs so that their relaxed versions, solved as part of the integer programming solution methodology, insure that local solutions are global ones. Combined with a state-of-the-art solver for QMIPs we have been able to solve rather large instances of these problems with some 3500 variables (mostly binary) and over 23,000 constraints. To illustrate the tradeoffs between stakeholders' individual objectives, we have considered nine different sets of weights and provided an analysis of the results.

This paper demonstrates the value of applying concepts of multiobjective optimization to the complex problem of Smart Growth and land use planning. The specific stakeholders identified and their proposed objective functions, while reasonable, are more illustrative in nature of how these concepts can be applied to this problem. The framework shown here can easily be modified to include other stakeholders' views or different objective functions from the stakeholders already identified. The value of this work is that it necessitates all those involved in the decision making process to formulate explicit and quantifiable descriptions of their goals and constraints. Having such formulations could serve to streamline discussions between a group of different parties with a stake in the future development of a county, state, or region.

Also demonstrated in this paper is the value of GIS technology in addressing decision making that involves a geographic component. The GIS was used at the front-end of this analysis to derive and store the quantities that were the focus of each of the stakeholders' objectives as well as many of the constraints. Further, after optimizations were completed, the GIS served to provide a visual presentation of the alternative outcomes associated with the nine illustrative scenarios that were considered.

We also presented several mathematical results concerning both the existence of a solution to this multiobjective optimization problem as well as the convexity of the QMIP weighting problems that are solved. The existence result based on reasonable assumptions, insures that that there will always be a Pareto optimal point, which is a useful result for future modeling efforts along these lines.

Acknowledgements

The authors gratefully acknowledge support received to perform this study from the National Center for Smart Growth Research and Education at the University of Maryland, College Park, MD. The authors also thank the Maryland Department of Planning for the use of their MdProperty View and Land Use Land Cover data.

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