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Some Permutation Representations of Weyl Groups Associated with the Cohomology of Toric Varieties

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Contents

INTRODUCTION

Let R be a crystallographic root system of rank n in a real Euclidean space V. The hyperplanes orthogonal to the roots of R define a simplicial fan Φ_R in V. The simplicial decomposition of the unit sphere induced by this fan is the Coxeter complex Δ_R. The Weyl group W of R acts on Φ_R, and hence on the associated toric variety X_R, and hence also on the rational cohomology $H^*(X_R, \mathbf{Q})$. The main result of this paper is the fact that the representation of W carried by $H^*(X_R, \mathbf{Q})$ is isomorphic to (the literarization of) a certain permutation representation π_R of W.

We prove a number of interesting properties of π_R even though we lack an explicit construction of it. For example, the degree of π_R is $|W|$, the number of orbits is 2^n, and the isotropy groups of π_R (i.e., the stabilizers of points) are generated by reflections, but not necessarily simple reflections. If R is irreducible, then even more can be said. For example, we prove that the isotropy groups of π_R for irreducible R are generated by

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reflections corresponding to subsets of the extended diagram of R. These subgroups are the stabilizers of nonempty faces of the fundamental alcove for the affine Weyl group attached to R.

Since we lack an explicit construction of π_R, we are forced to study it indirectly through the linear (rather than permutation) action of W on $H^*(X_R, \mathbf{Q})$. By a fundamental theorem of Danilov, one knows that the cohomology ring of the toric variety of a simplicial fan Φ is isomorphic to a quotient of the face ring $\mathbf{Q}[\Delta]$ (where Δ is the simplicial complex of the fan) by a certain natural system of parameters Θ. Hence, for the purpose of studying the action of W, one can replace $H^*(X_R, \mathbf{Q})$ with $\mathbf{Q}[\Delta_R]/\Theta$. This is the point of view we adopt in this paper.

It should be noted that the Weyl group representations under consideration here bear a superficial resemblance to another family of representations that have previously been considered by Björner [Bj, §6], Garsia and Stanton [GS], Solomon [S], and Stanley [St2, §§4, 6]. This other family is also obtained as a quotient of $\mathbf{Q}[\Delta_R]$ by an appropriate system of parameters, but the distinction (and it leads to results of a completely different nature) stems from the fact that the systems of parameters in the two cases are not equivalent as W-modules. In our case, the action of W on the parameters is isomorphic to the reflection representation of W, whereas, in the other case, the parameters are fixed pointwise by W. Furthermore, in this latter case, the structure of the W-representation carried by the quotient ring is simply a multi-graded refinement of the regular representation.

Precedents

At the 1985 Durham Symposium on the Symmetric Group, Procesi proved a character formula for the W-representation carried by $H^*(X_R, \mathbf{Q})$ and derived a more explicit formula in the special case $R = A_n$. The details appear in [P].

From the point of view of combinatorics, the toric varieties X_R are of interest because their Betti numbers (essentially Weyl group analogues of the Eulerian numbers) provide examples that fit into Stanley's scheme for proving unimodality of sequences via the Hard Lefschetz Theorem. In this particular case, one can also use the presence of a Weyl group action to give an isotypic refinement of the unimodality of the Betti numbers. That is, one can show [St5, p. 528] that for each irreducible representation ρ of W, the multiplicity of ρ in each (even) degree of $H^*(X_R, \mathbf{Q})$ forms a unimodal sequence. In 1988, Stanley translated Procesi's result for A_n into the language of symmetric functions, and included the "isotypic" Betti numbers implicit in these symmetric functions as examples in his survey of unimodal sequences [St5, p. 529]. Brenti later gave an elementary proof of their unimodality in [Br].

In Stanley's presentation of the Procesi formula, it is self-evident that in the A_n case the character of the cohomology of the toric variety is a permutation character. This observation was first made explicit in [Ste3], where this author posed the question of finding a geometric explanation of this fact.

Recently, Dolgachev and Lunts gave a recursive, geometric technique for generating $H^*(X_R, \mathbf{Q})$ in the case of $R = A_n$ and $R = C_n$ [DL], and from this it is not difficult to deduce that the action of the Weyl group is indeed isomorphic to the linearization of a permutation representation in these cases. They also gave an independent proof of a character formula for $\mathbf{Q}[\Delta_R]/\Theta$ for arbitrary R that is closely related to the one of Procesi. Their results suggested to this author the possibility of finding permutation representations in the general case.

Organization

In Section 1, we consider group actions on complete simplicial fans. If the group action satisfies a certain axiom (see (G1)), then we are able to prove a simple character formula (Theorem 1.4) for the representation carried by the cohomology of the associated toric variety (or equivalently, the representation carried by the face ring modulo Θ). This is a slight generalization of the formulas previously given by Procesi [P] and Dolgachev and Lunts [DL].

In Section 2, we provide a collection of formulas for the h-polynomials of the Coxeter complexes Δ_R, as well as for the subcomplexes invariant under given elements of the Weyl group. We also give formulas for the number of maximal faces in these complexes. These data are needed so that we can apply the character formula of Section 1. We remark that these invariant subcomplexes of Coxter complexes (and their underlying hyperplane arrangements) have been previously considered by Orlik and Solomon [OS].

In Section 3, we provide an explicit basis for the W-invariants of $\mathbf{Q}[\Delta_R]/\Theta$. A corollary of this result is the fact that the number of orbits of π_R is 2^n.

In Section 4, we give the precise statements of the main results of this paper. After reading this Introduction, it should be feasible for the impatient reader to skip directly to this section.

In Section 5, we note that a corollary of one of the results in Section 4 is the fact that if $w \in W$ belongs to no proper reflection subgroup of W, then $\det(1-w)$ is the index of the root lattice in the weight lattice. In the special case for which w is a Coxeter element, this fact can be found as an exercise in [B], but we are unaware of a similar proof of the more general statement.

The proofs of the main results are distributed throughout Sections 6–10 on a case-by-case basis. At first it may seem strange that we have separate sections for the root systems of type is B_n and C_n—they have the same Weyl group, the same Coxeter complex, and the same linear representations on $\mathbf{Q}[\Delta_R]/\Theta$. However, the implications of our main results in these two cases are *not* identical, since it turns out that π_{B_n} and π_{C_n} are not isomorphic as permutation representations. For example, it is possible to assign a grading to π_R that is compatible with $K[\Delta_R]/\Theta$ if $R = C_n$, but not if $R = B_n$.

Our proofs for the exceptional root systems are computer-assisted. The details of how the calculations were carried out are discussed in Section 10.

There are two appendices. In the first, we list the extended diagrams of the irreducible root systems. Although these diagrams can be found elsewhere (e.g., [B, H]), we have included them here for the convenience of the reader. In the second appendix, we list the multiplicities and types of the transitive permutation characters that occur in π_R for each of the exceptional root systems.

1. Group Actions on Simplicial Fans

We first need to review some of the basic algebraic properties of simplicial complexes. Further details can be found in [St3].

Let Δ be an abstract simplicial complex with vertex set I. Thus Δ is a collection of subsets (faces) of I such that for all $F \in \Delta$, $F' \subset F$ implies $F' \in \Delta$. Recall that $\dim F = |F| - 1$, and that $\dim \Delta$ is the maximum dimension of any face of Δ. Henceforth we assume that Δ is finite and $(n-1)$-dimensional.

Let $f(\Delta) = (f_{-1}, f_0, ..., f_{n-1})$ denote the f-vector of Δ, so that f_i is the number of i-dimensional faces of Δ. (In particular, $f_{-1} = 1$.) One may define the h-vector $h(\Delta) = (h_0, ..., h_n)$ and h-polynomial $P_\Delta(q)$ in terms of the f-vector via

$$P_\Delta(q) = \sum_{i=0}^{n} h_i q^i = \sum_{i=0}^{n} f_{i-1} q^i (1-q)^{n-i}.$$

Let K be a field of characteristic zero, and let $\{x_i : i \in I\}$ be a set of independent indeterminates. The *support* of a monomial $\prod_i x_i^{\alpha_i}$ is defined to be $\{i \in I : \alpha_i > 0\}$. The *face ring* $K[\Delta]$ (or Stanley–Reisner ring) is defined to be $K[x_i : i \in I]/\Psi$, where Ψ denotes the ideal generated by monomials whose supports are not faces of Δ. The monomials with support in Δ form a K-basis for $K[\Delta]$, and therefore

$$\frac{P_\Delta(q)}{(1-q)^n} = \sum_{F \in \Delta} \frac{q^{|F|}}{(1-q)^{|F|}} = \sum_{i=0}^{n} f_{i-1} \frac{q^i}{(1-q)^i} \tag{1.1}$$

is the Hilbert series for $K[\Delta]$.

A (homogeneous) system of parameters for $K[\Delta]$ consists of a sequence $\Theta = (\theta_1, \ldots, \theta_n)$, with each $\theta_i \in K[\Delta]$ homogeneous of positive degree, such that $K[\Delta]/\Theta$ is finite-dimensional as a K-vector space. In particular, this requires that $\theta_1, \ldots, \theta_n$ be algebraically independent over K. By abuse of notation, we identify Θ with the ideal it generates in $K[\Delta]$. One says that $K[\Delta]$ (or simply Δ) is Cohen–Macaulay if for some (equivalently every) system of parameters Θ, $K[\Delta]$ is a free module over the polynomial subring $K[\Theta]$. If each θ_i has degree 1 (and Δ is indeed Cohen–Macaulay), it is easy to show that $P_\Delta(q)$ is the Hilbert series for $K[\Delta]/\Theta$ (cf. (1.1)). In particular, the h-vectors for such complexes are nonnegative.

Simplical Fans

Let V be an n-dimensional real Euclidean space, and let Φ be a polyhedral decomposition of V. By this we mean that Φ is a collection of pointed convex polyhedral cones in V with apex at the origin, such that

(F0) $V = \bigcup \Phi$.

(F1) Every face of every cone $C \in \Phi$ is also in Φ.

(F2) If $C_1, C_2 \in \Phi$, then $C_1 \cap C_2$ is a face of both C_1 and C_2.

This paper is concerned only with decompositions Φ that are *simplicial*; i.e., each i-dimensional cone is generated by i vectors. In such cases, the intersection of Φ with the unit sphere in V yields an $(n-1)$-dimensional simplicial complex Δ in which the i-dimensional, cones of Φ correspond to $(i-1)$-dimensional faces of Δ.

If, in addition, the cones of Φ are generated by vectors belonging to some lattice L in V, then Φ defines a (complete) simplicial fan in L [O]. We also describe this condition by saying that Φ is *integral* with respect to L.

Group actions

Let G be a finite subgroup of the orthogonal group $O(V)$, and suppose that G acts as a set of automorphisms of some simplicial decomposition Φ of V. Note that G also acts on the associated complex Δ. We say that Φ carries a *proper* action of G if

(G1) For every $C \in \Phi$ and $w \in G$, $w(C) = C$ implies that w fixes every face of C.

In other words, if w fixes a cone C setwise, then it must fix C pointwise. Under these circumstances, it follows that the vertices (one-dimensional cones) of Φ fixed by w must span $V_w := \{v \in V : wv = v\}$, and therefore the restriction of Φ to V_w yields a simplicial decomposition Φ^w of V_w. In particular, the simplicial complex associated with Φ^w is the Δ-subcomplex

$$\Delta^w := \{F \in \Delta : w(F) = F\}.$$

The action of G on Δ induces a graded KG-module structure on $K[\Delta]$. In order to simplify the description of decompositions of graded modules, let us adopt the convention that if U is a KG-module, then $q^j U$ denotes the graded KG-module obtained by assigning degree j to every element of U. Formal sums of such objects should be interpreted as (external) direct sums. Also, if G acts by permutations on some set S, let us denote by $K\langle S\rangle$ the KG-module obtained by extending the action of G linearly to the vector space freely generated by S.

LEMMA 1.1. *Let Φ be a simplicial decomposition of V with associated complex Δ. If Φ carries a proper G-action, then as graded KG-modules we have*

$$K[\Delta] \cong \sum_{F \in \Delta/G} \frac{q^{|F|}}{(1-q)^{|F|}} \cdot K\langle wF : w \in G\rangle,$$

where F ranges over a set of orbit representatives for G on Δ.

Proof. Let $\mathcal{M}$ denote the basis of $K[\Delta]$ consisting of monomials with support in Δ. Note that $K[\Delta] \cong K\langle \mathcal{M}\rangle$ as graded KG-modules, since G permutes the vertices of Φ, and hence also $\mathcal{M}$. For $F \in \Delta$, let $\mathcal{M}_F$ denote the set of monomials with support F. Since the action of G is proper, each G-orbit on $\mathcal{M}$ contains at most one member of $\mathcal{M}_F$. Furthermore, the action of G on the orbit of any monomial in $\mathcal{M}_F$ is isomorphic to the action of G on the orbit of F, since $w\mathcal{M}_F = \mathcal{M}_{wF}$. Therefore, the graded KG-module generated by $\mathcal{M}_F$ is isomorphic to $P_F(q) \cdot K\langle wF : w \in G\rangle$, where $P_F(q) = q^{|F|}(1-q)^{-|F|}$ denotes the generating function for monomials in $\mathcal{M}_F$. ∎

If $U = \bigoplus_j U_j$ is any graded KG-module, let $\chi[U, q]$ denote the *graded* character of U. In other words, for $w \in G$, let us define

$$\chi[U, q](w) = \sum_j q^j \operatorname{tr}_{U_j}(w).$$

LEMMA 1.2. *If Φ, Δ, G are as above, then for $w \in G$,*

$$\chi[K[\Delta], q](w) = \frac{P_{\Delta^w}(q)}{(1-q)^{\delta(w)}},$$

where $\delta(w) := 1 + \dim \Delta^w = \dim V_w$.

Proof. Continuing the notation of the previous proof, note that $\chi[K[\Delta], q](w)$ is the generating function for monomials in $\mathcal{M}$ fixed by w. Since the action of G is proper, a monomial with support F is fixed by w if and only if F is fixed by w. Hence, $\chi[K[\Delta], q](w)$ is the Hilbert series for $K[\Delta^w]$. ▮

For any $F \in \Delta$, let $G_F = \{w \in G : w(F) = F\}$ denote the isotropy group of F.

COROLLARY 1.3. *If Φ, Δ, G are as above, then for $w \in G$,*

$$P_{\Delta^w}(q) = \sum_{F \in \Delta/G} q^{|F|}(1-q)^{\delta(w)-|F|} \cdot 1_{G_F}^G(w).$$

Proof. The G-character of $K\langle wF : w \in G\rangle$ is $1_{G_F}^G$, the induction of the trivial character from G_F to G. Now compare the formula for $\chi[K[\Delta], q]$ in Lemma 1.2 with the one implicit in Lemma 1.1. ▮

Let $\langle \cdot, \cdot \rangle$ denote the inner product on V, and let $\varepsilon_1, ..., \varepsilon_n$ be a basis for V. Regarding the vertex set I of Δ as a subset of V, let us define (cf. [D, §10.7])

$$\theta_i = \sum_{v \in I} \langle v, \varepsilon_i \rangle \, x_v \in K[\Delta]. \tag{1.2}$$

In general, the scalar products $\langle v, \varepsilon_i \rangle$ could be arbitrary real numbers, and so we have to assume that K is the real field in such cases. However, if Φ is integral with respect to some lattice L, then we may assume $I \subset L$ and choose $\varepsilon_1, ..., \varepsilon_n$ to be a basis for the dual lattice L^*. In that case, all of the coefficients $\langle v, \varepsilon_i \rangle$ would be integral, and thus (1.2) would be a valid definition over any field.

It is easy to show (e.g., using [St1, p. 150]) that $\Theta = (\theta_1, ..., \theta_n)$ is a homogeneous system of parameters for $K[\Delta]$ of degree 1. Note that for $w \in G$, we have

$$w\theta_i = \sum_{v \in I} \langle v, \varepsilon_i \rangle x_{wv} = \sum_{v \in I} \langle w^{-1}v, \varepsilon_i \rangle \, x_v = \sum_{v \in I} \langle v, w\varepsilon_i \rangle \, x_v. \tag{1.3}$$

Hence, the K-span of $\theta_1, ..., \theta_n$ is G-stable. In particular, the ideal Θ is also G-stable, so $K[\Delta]/\Theta$ carries a graded KG-module structure.

THEOREM 1.4. *If Φ, Δ, G are as above, then for $w \in G$,*

$$\chi[K[\Delta]/\Theta, q](w) = P_{\Delta^w}(q) \frac{\det(1-qw)}{(1-q)^{\delta(w)}}.$$

Proof. Since Δ is a triangulation of a sphere (essentially by definition), it follows by a theorem of Reisner [St3, §II.4] that $K[\Delta]$ is Cohen–Macaulay. Hence $K[\Delta]$ is free as a module for $K[\Theta]$. We claim that this implies

$$K[\Delta] \cong K[\Delta]/\Theta \otimes K[\Theta] \tag{1.4}$$

as graded KG-modules. To see this, note that since G is finite, any surjective homomorphism of KG-modules splits. In particular, there is a graded KG-module injection $\sigma: K[\Delta]/\Theta \to K[\Delta]$ that inverts the natural projection $K[\Delta] \to K[\Delta]/\Theta$. Since $K[\Delta]$ is free, it follows that σ maps K-bases for $K[\Delta]/\Theta$ to (free) $K[\Theta]$-bases for $K[\Delta]$. Hence, there is a graded KG-module isomorphism from $K[\Delta]/\Theta \otimes K[\Theta]$ to $K[\Delta]$ in which $u \otimes a \mapsto a\sigma(u)$ for $u \in K[\Delta]/\Theta$, $a \in K[\Theta]$.

Now by (1.3), we see that the action of G on the K-span of $\theta_1, ..., \theta_n$ is isomorphic to the action of G on V. Hence the KG-module structure carried by $K[\Theta]$ is isomorphic to the symmetric algebra of V. In particular, $\chi[K[\Theta], q](w) = \det(1 - qw)^{-1}$. Using Lemma 1.2 to compare the graded characters of the two expressions in (1.4), we thus obtain

$$\frac{P_{\Delta^w}(q)}{(1-q)^{\delta(w)}} = \chi[K[\Delta]/\Theta, q](w) \cdot \frac{1}{\det(1-qw)}. \quad \blacksquare$$

Let $|\Delta| = f_{n-1}(\Delta) = P_\Delta(1)$ denote the number of maximal faces of the complex Δ, and let χ^Δ denote the *ungraded* G-character of $K[\Delta]/\Theta$; i.e., $\chi^\Delta := \chi[K[\Delta]/\Theta, 1]$. Note that $(1-q)^{-\delta(w)} \det(1 - qw)$ is essentially the characteristic polynomial of w on $V_w^\perp$, since $\delta(w) = \dim V_w$. Therefore, in the limit $q \to 1$, we obtain

COROLLARY 1.5. *For $w \in G$, we have $\chi^\Delta(w) = |\Delta^w| \cdot \det_{V_w^\perp}(1 - w)$.*

Now let us suppose that Φ is integral with respect to some lattice L in V. In that case, there is a toric variety X_Φ associated with Φ (e.g., see [O]), and a theorem of Danilov [D, §§10.8, 10.9] asserts that the cohomology ring $H^*(X_\Phi) = H^*(X_\Phi, K)$ is a quotient of the face ring. To be more precise (taking into account the fact that the cohomology is nonzero only in even degrees), we have a graded ring and KG-module isomorphism

$$H^*(X_\Phi) \cong K[\Delta]/\Theta, \tag{1.5}$$

provided that we assign degree 2 to the generators of $K[\Delta]$. As an immediate corollary, we thus obtain the following slight generalization of a result of Dolgachev and Lunts [DL] and Procesi [P, §2].

COROLLARY 1.6. *Let Φ be a complete simplicial fan with associated complex Δ. If Φ carries a proper G-action, then for $w \in G$,*

$$\chi[H^*(X_\Phi), q](w) = \chi[K[\Delta]/\Theta, q^2](w) = P_{\Delta^w}(q^2)\,\frac{\det(1-q^2w)}{(1-q^2)^{\delta(w)}}.$$

As will become clear in what follows, a natural question to ask is whether the ungraded KG-module $H^*(X_\Phi, K)$ is in fact a permutation module. That is, does $H^*(X_\Phi, K)$ (or equivalently, $K[\Delta]/\Theta$) possess a K-basis that is permuted by G? The following result shows that the character is integral and nonnegative—an obvious necessary condition.

PROPOSITION 1.7. *If Φ is integral, then $\chi^\Delta(w) \in \mathbf{N}$ for all $w \in G$.*

Proof. Let $v_1, ..., v_n \in V$ be a basis for L. Any chamber (maximal cone) of Φ is generated by n independent vectors in L, and hence these vectors have coordinates that are integral with respect to $v_1, ..., v_n$. In particular, since any $w \in G$ maps chambers to chambers, it follows that the representing matrix for w on V is rational with respect to this basis. Therefore, $\det_{V_w^\perp}(1-w)$ is rational. However, character values are algebraic integers, so $\chi^\Delta(w) = |\Delta^w| \cdot \det_{V_w^\perp}(1-w)$ must be integral. To prove the nonnegativity, note that since w is of finite order, the eigenvalues of w on V are roots of unity. Therefore, aside from ± 1, they can be arranged in conjugate pairs $e^{\pm i\alpha}$ for $\alpha \in \mathbf{R}$. Since $(1-e^{i\alpha})(1-e^{-i\alpha}) = 2 - 2\cos\alpha \geqslant 0$, it follows that $\det_{V_w^\perp}(1-w) \geqslant 0$. ∎

2. THE COXETER COMPLEX

We now specialize to the setting of root systems. For definitions, see [B] or [H].

Let V continue to denote an n-dimensional real Euclidean space, and let R be a reduced (not necessarily crystallographic) root system of rank n in V. For nonzero $\alpha \in V$, let $s_\alpha \in O(V)$ denote reflection across the hyperplane $\alpha^\perp$; i.e., $s_\alpha(v) = v - [\alpha, v]\alpha$, where $[\alpha, v] := 2\langle\alpha, v\rangle/\langle\alpha, \alpha\rangle$. Let $S = \{\alpha_1, ..., \alpha_n\} \subset R$ be a set of simple roots, and let W be the finite group generated by the reflections $\{s_\alpha : \alpha \in R\}$. We write $s_1, ..., s_n$ for the reflections corresponding to $\alpha_1, ..., \alpha_n$.

Associated with the root system R is a hyperplane arrangement $\{\alpha^\perp : \alpha \in R\}$. These hyperplanes define a simplicial decomposition Φ_R of V; the associated complex Δ_R is known as the Coxeter complex [Bj, H].

Let $C_0 = \{v \in V : \langle\alpha_i, v\rangle \geqslant 0,\ i = 1, ..., n\}$ denote the fundamental chamber of Φ_R relative to the choice of S. It is well-known that the action of W

on the chambers of Φ_R is simply transitive. Furthermore, C_0 is a fundamental domain for W on V; i.e., the W-orbit of each $v \in V$ contains exactly one point of C_0 (e.g., [H, §1.12]). An immediate consequence of this is the fact that the action of W on Φ_R is proper.

Let $\omega_1, ..., \omega_n$ denote the basis of V that is "dual" to S via the Cartan form $[\cdot, \cdot]$; i.e., the basis defined by the conditions $[\alpha_i, \omega_j] = \delta_{ij}$. Note that C_0 is the cone spanned by $\omega_1, ..., \omega_n$. By transitivity, it follows that the vertices of Φ_R are the W-orbits of $\mathbf{R}^+\omega_1, ..., \mathbf{R}^+\omega_n$. More generally, the W-orbits of faces are indexed by subsets of S; indeed, one may take

$$C_J := \{v \in C_0 : \langle \alpha, v \rangle = 0 \text{ for } \alpha \in J\}$$

as a representative of the face-orbit indexed by a given $J \subset S$. Note that this labeling scheme has the property that J indexes a cone of codimension $|J|$.

If R is crystallographic; i.e., $[\alpha, \beta] \in \mathbf{Z}$ for all $\alpha, \beta \in R$, then $\omega_1, ..., \omega_n$ generate a W-invariant lattice Λ_R in V (viz., the weight lattice). Thus in such cases, Φ_R is integral with respect to Λ_R.

By a theorem of Björner [Bj], it is known that Coxeter complexes are shellable, and from the shelling one may easily obtain a simple combinatorial description of the h-polynomials $P_R := P_{\Delta_R}$ as follows. For $w \in W$, let $l(w)$ denote the length of w with respect to S; i.e., the minimum length l of any factorization $w = s_{i_1} s_{i_2} \cdots s_{i_l}$. The *descent set* of w is defined to be $D(w) = \{i : l(ws_i) < l(w)\}$; this generalizes the usual notion of the descent set of a permutation.

THEOREM 2.1 (essentially [Bj, Theorem 2.1]). $P_R(q) = \sum_{w \in W} q^{|D(w)|}$.

In order to explicitly determine the W-character of $K[\Delta_R]/\Theta$ via Theorem 1.4, we need to know the h-polynomials of more than just the Coxeter complexes—we also need the h-polynomials of the w-fixed subcomplexes Δ_R^w for all $w \in W$. It is often the case that these complexes are isometric to Coxeter complexes of smaller rank, but not always. A more reliable way to obtain their h-polynomials is by means of Corollary 1.3. In the following, we take this as the starting point, and subsequently derive a number of simplifications in special cases.

For any $J \subset R$, let W_J denote the subgroup of W generated by $\{s_\alpha : \alpha \in J\}$. In the case $J \subset S$, W_J is known as a parabolic subgroup. In these latter cases, it is sometimes more convenient to regard $J \subset \{1, ..., n\}$, with the obvious identification $i \leftrightarrow \alpha_i$.

PROPOSITION 2.2. *Let R be a root system for the reflection group W.*

(a) *For $w \in W$, $P_{\Delta_R^w}(q) = \sum_{J \subset S} q^{n-|J|}(1-q)^{\delta(w)-n+|J|} \cdot 1_{W_J}^W(w)$.*

(b) $P_R(q) = \sum_{J \subset S} q^{n-|J|}(1-q)^{|J|} [W : W_J]$.

Proof. (a) Apply Corollary 1.3, using the fact that the orbits of faces are indexed by subsets J of S, and the fact that the isotropy group of C_J is W_J [H, p. 22].

(b) Specialize to the case of the identity element. ∎

For $i \leqslant j$, let $W_{[i,j]}$ denote the parabolic subgroup generated by $s_i, s_{i+1}, ..., s_j$, with the convention that $W_{[i+1,i]}$ is the one-element group. In case the Coxeter diagram of R has no forks, the latter of the above two formulas can be expressed as a determinant of order $n+1$.

PROPOSITION 2.3. *If S can be linearly ordered so that $\langle \alpha_i, \alpha_j \rangle = 0$ for $|i-j| > 1$, then*

$$qP_R(q) = |W| \cdot \det[a_{ij}]_{0 \leqslant i,j \leqslant n},$$

where $a_{ij} = 0$ for $i-j > 1$, $a_{ij} = q-1$ for $i-j=1$, and $a_{ij} = q/|W_{[i+1,j]}|$ for $i \leqslant j$.

Proof. The expansion of $\det[a_{ij}]$ is indexed in an obvious way by permutations π of $\{0, 1, ..., n\}$. Since $a_{ij} = 0$ for $i-j > 1$, it is easy to chack that $a_{0,\pi(0)} \cdots a_{n,\pi(n)}$ will be zero unless π is composed of disjoint cycles of the form $(j, j-1, ..., i)$ for $i \leqslant j$. There is a one-to-one correspondence $\pi \leftrightarrow J$ between such permutations and subsets J of $\{1, ..., n\}$ in which J^c consists of the minimal nonzero elements of the cycles.

Suppose that π is one of the above permutations and that $0 = i_0 < i_1 < \cdots < i_l$ are the minimal elements of the cycles of π (and hence, $J^c = \{i_1, ..., i_l\}$). Since $a_{ij} = q-1$ for $i-j=1$, we have

$$\operatorname{sgn}(\pi)\, a_{0,\pi(0)} \cdots a_{n,\pi(n)} = (-1)^{|J|} (q-1)^{|J|} a_{i_0,i_1-1} a_{i_1,i_2-1} \cdots a_{i_l,n}. \quad (2.1)$$

However, since $\langle \alpha_i, \alpha_j \rangle = 0$ for $|i-j| > 1$, it follows that

$$W_J \cong W_{[i_0+1, i_1-1]} \times W_{[i_1+1, i_2-1]} \times \cdots \times W_{[i_l+1, n]}.$$

Since $a_{ij} = q/|W_{[i+1,j]}|$ for $i \leqslant j$, we therefore have

$$q^{n-|J|+1} |W_J|^{-1} = a_{i_0,i_1-1} a_{i_1,i_2-1} \cdots a_{i_l,n}.$$

Thus by (2.1) we obtain

$$\det[a_{ij}] = q \sum_{J \subset S} q^{n-|J|} (1-q)^{|J|} |W_J|^{-1}.$$

Using Proposition 2.2(b), this can be readily identified as $|W|^{-1} qP_R(q)$. ∎

For example, in the case $R = H_3$ (so that W is the symmetry group of an icosahedron), this result implies

$$P_{H_3}(q) = 120q^{-1} \cdot \det \begin{bmatrix} q & q/2 & q/10 & q/120 \\ q-1 & q & q/2 & q/6 \\ 0 & q-1 & q & q/2 \\ 0 & 0 & q-1 & q \end{bmatrix} = 1 + 59q + 59q^2 + q^3.$$

Remark 2.4. If we modify the definition of the matrix $[a_{ij}]$ so that $a_{ij} = t_i - 1$ for $i - j = 1$, and $a_{ij} = t_i / |W_{[i+1,j]}|$ for $i \leqslant j$, then essentially the same argument shows that the polynomial $P_R(t_1, ..., t_n)$ defined by

$$t_0 P_R(t_1, ..., t_n) = |W| \cdot \det[a_{ij}]_{0 \leqslant i,j \leqslant n}$$

is the fine h-polynomial of Δ_R (for the definition, see [St1, p. 146]). By Theorem 2.1 of [Bj], it follows that the number of $w \in W$ with descent set $\{i_1, ..., i_l\}$ is the coefficient of $t_{i_1} \cdots t_{i_l}$ in $P_R(t_1, ..., t_n)$. In the special case $R = A_n$, this fact is essentially equivalent to Example 2.2.4 of [St4].

Now consider the equivalence relation on W in which $w \approx w'$ if there exists some $x \in W$ such that $V_w = xV_{w'}$. This relation is coarser than conjugacy in W. Since Φ_R^w is the restriction of Φ_R to V_w, it follows that if $w \approx w'$, then $\Delta_R^w \cong \Delta_R^{w'}$. Thus, for the purpose of computing the h-polynomials of the w-fixed subcomplexes, it suffices to restrict our attention to a set of equivalence-class representatives for $\approx$.

For $J \subset S$, let $V_J = \{v \in V : \langle \alpha, v \rangle = 0 \text{ for } \alpha \in J\}$ denote the vector space spanned by C_J, and let w_J denote a Coxeter element for W_J, i.e., the product of the reflections s_α for $\alpha \in J$, taken in any order.

LEMMA 2.5. *Let* $w \in W$.

(a) $V_w = xV_J$ *for some* $x \in W$ *and* $J \subset S$ *such that* $w \in xW_Jx^{-1}$.

(b) *If* $w = w_J$ *for some* $J \subset S$, *then* $V_w = V_J$.

Proof. (a) Let C be a chamber of Φ_R^w. Some cone in the W-orbit of C must belong to C_0; i.e., $C = xC_J$ for some $x \in W$ and $J \subset S$. Therefore $V_w = xV_J$, since C spans V_w and C_J spans V_J. Furthermore, since w belongs to the isotropy group of C, it follows that $x^{-1}wx$ belongs to W_J, the isotropy group of C_J.

(b) Clearly $V_J \subset V_w$. For the converse, proceed by induction on $|J|$, the cases $|J| \leqslant 1$ being trivial. Without loss of generality, suppose that $J = \{\alpha_1, ..., \alpha_j\}$ and $w = s_1 \cdots s_j$. If $v \in V_w$, then $s_1(v) = s_2 \cdots s_j(v)$. However, $s_2 \cdots s_j(v) = v + c_2\alpha_2 + \cdots + c_j\alpha_j$ and $s_1(v) = v + c_1\alpha_1$ for suitable scalars $c_i \in \mathbf{R}$, and therefore $c_2\alpha_2 + \cdots + c_j\alpha_j = c_1\alpha_1$. However, the α_i's are linearly independent, so $c_i = 0$ and $s_2 \cdots s_j(v) = s_1(v) = v$. Applying the induction hypothesis to $w' = s_2 \cdots s_j$, we conclude that each s_i fixes v, so $v \in V_J$. ∎

LEMMA 2.6. *If $I, J \subset S$ and $xw_I x^{-1} \in W_J$ for some $x \in W$, then $xW_I x^{-1} \subset W_J$.*

Proof. If $xw_I x^{-1} \in W_J$, then $xw_I x^{-1}$ fixes V_J, so w_I fixes $x^{-1}V_J$. Therefore $V_J \subset xV_I$, by Lemma 2.5(b). Since each reflection $s_\alpha : \alpha \in I$ fixes V_I (and hence $x^{-1}V_J$), it follows that each of $xs_\alpha x^{-1}$ fixes V_J. Thus $xW_I x^{-1}$ is contained in W_J, the subgroup fixing V_J. ∎

Let $\sim$ denote conjugacy of elements or subgroups of W, as appropriate.

COROLLARY 2.7 [OS, p. 276]. *If $I, J \subset S$, then $w_I \sim w_J$ if and only if $W_I \sim W_J$.*

As a consequence of Lemma 2.5, we see that for every $w \in W$ there exists a subset J of S such that $w \approx w_J$. However, there may (and usually will) be further equivalences among the w_J's. In any case, we need only to consider the w-fixed subcomplexes for $w = w_J$. Let us write $P_R^J(q)$ and Δ_R^J as abbreviations for $P_{\Delta_R^w}(q)$ and Δ_R^w in such cases.

THEOREM 2.8. *For any $J \subset S$, we have*

$$|\Delta_R^J| = P_R^J(1) = [N(W_J) : W_J] \cdot |\{I \subset S : W_I \sim W_J\}|.$$

Proof. Let $w = w_J$. By Proposition 2.2(a), we have

$$P_R^J(q) = \sum_{I \subset S} q^{n-|I|}(1-q)^{|I|-|J|} \cdot 1_{W_I}^W(w), \tag{2.2}$$

since $\delta(w) = \dim V_w = n - |J|$ by Lemma 2.5(b). The permutation character $1_{W_I}^W(w)$ is the number of cosets xW_I fixed by w, so by Lemma 2.6,

$$\begin{aligned} 1_{W_I}^W(w) &= |W_I|^{-1} \cdot |\{x \in W : xw_J x^{-1} \in W_I\}| \\ &= |W_I|^{-1} \cdot |\{x \in W : xW_J x^{-1} \subset W_I\}|. \end{aligned} \tag{2.3}$$

In particular, $1_{W_I}^W(w)$ will be nonzero only if there exists $x \in W$ such that $xW_J x^{-1} \subset W_I$. In that case, by comparing the spaces of invariants of the two groups, it follows that $xV_J \supset V_I$. This in turn forces $|I| \geqslant |J|$.

Now set $q = 1$ in (2.2). The only surviving terms will have $|I| = |J|$. For each $x \in W$ satisfying $xW_J x^{-1} \subset W_I$, we must have $xV_J = V_I$ since the two spaces now have the same dimension. Hence, the two subgroups fixing V_J and V_I are conjugate; i.e., $W_J \sim W_I$. For such I, (2.3) simplifies to

$$1_{W_I}^W(w) = |W_J|^{-1} \cdot |\{x \in W : xW_J x^{-1} \subset W_J\}| = [N(W_J) : W_J].$$

To complete the proof, note that there are now a total of $|\{I \subset S : W_I \sim W_J\}|$ nonzero summands in (2.2), each contributing $1_{W_J}^W(w)$. ∎

In case w is a reflection, the above formula simplifies.

COROLLARY 2.9. *If* $w = s_\alpha$ *and* $I = R \cap \alpha^\perp$, *then* $|\Delta_R^w| = |W_I| \cdot |\{\alpha_i \in S : s_\alpha \sim s_i\}|$.

Proof. Without loss of generality, assume $\alpha = \alpha_1$, set $J = \{\alpha_1\}$, and apply Theorem 2.8. In this case, the normalizer of W_J is the centralizer of w, so we have $[N(W_J) : W_J] = |W|/2\,|C(w)|$, where $C(w)$ denotes the set of conjugates of w. Since $xs_\alpha x^{-1} = s_{x\alpha}$, it follows that $\beta \mapsto s_\beta$ defines a two-to-one map from the W-orbit of α onto $C(w)$. (Remember that $s_\alpha(\alpha) = -\alpha$, so $W\alpha = -W\alpha$.) Therefore, $[N(W_J) : W_J] = |W|/|W\alpha|$ is the order of the isotropy group of α. According to [H, p. 22], this subgroup is generated by the reflections $s_\beta \in W$ such that $\langle \alpha, \beta \rangle = 0$. ∎

We remark that by a result of Orlik and Solomon [OS], it is known that the lattice of intersections of the hyperplane arrangement corresponding to Δ_R^J has a characteristic polynomial with integer roots; these roots provide a factorization for $|\Delta_R^J|$.

3. THE W-INVARIANTS OF $K[\Delta_R]/\Theta$

As in the previous section, R continues to denote a root system of rank n in the Euclidean space V, with associated reflection group W, simple roots S, Coxeter complex Δ_R, and system of parameters Θ as defined in (1.2). It is convenient henceforth to use $\chi[R, q]$ as an abbreviation for the graded character $\chi[K[\Delta_R]/\Theta, q]$.

Recall that for every $J \subset S$ there is a corresponding face C_J of Δ_R in the fundamental chamber C_0, and that every W-orbit of faces contains exactly one of the faces C_J. Let $x_J \in K[\Delta_R]$ denote the unique square-free monomial whose support consists of the vertices of C_J, and define

$$y_J := \sum_{w \in W} w(x_J) \in K[\Delta_R]^W.$$

THEOREM 3.1. *The elements* $y_J \pmod{\Theta}$ *for* $J \subset S$ *are a basis for* $(K[\Delta_R]/\Theta)^W$.

Given a shelling order for Δ_R, it is possible to write down an explicit basis for $K[\Delta_R]/\Theta$ consisting of certain square-free monomials (e.g., see Theorem 1.7 of [Bj], or Theorem 4.2 of [G]). However, in order to prove Theorem 3.1, we only need the following weaker result.

LEMMA 3.2 (Danilov [D, §10.7.1]). *$K[\Delta_R]/\Theta$ is spanned by square-free monomials.*

Proof of Theorem 3.1. Any square-free monomial in $K[\Delta_R]$ belongs to the W-orbit of some monomial x_J for $J \subset S$. Since the operator $\sum_{w \in W} w$ acts as a projection onto the space of W-invariants, Lemma 3.2 implies that the elements y_J must span the W-invariants of $K[\Delta_R]/\Theta$. Thus, we need only prove that the elements y_J are linearly independent modulo Θ.

To prove the independence, it suffices to work over the complex field and verify that the multiplicity of the trivial representation in $\mathbf{C}[\Delta_R]/\Theta$ is 2^n, or more specifically, has Hilbert series $(1+q)^n$. For this, apply Theorem 1.4 and Proposition 2.2(a) to obtain

$$\begin{aligned}\chi[R,q](w) &= \sum_{J \subset S} q^{n-|J|}(1-q)^{-n+|J|} \cdot 1_{W_J}^W(w) \cdot \det(1-qw) \\ &= \sum_{J \subset S} q^{n-|J|}(1-q)^{-n+|J|} \cdot 1_{W_J}^W(w) \cdot \chi[\Lambda(V), -q](w), \quad (3.1)\end{aligned}$$

where $\Lambda(V) = \bigoplus_{k \geq 0} \Lambda^k(V)$ denotes the exterior algebra of V as a graded $\mathbf{C}W$-module.

Now let 1_W be the trivial character of W, and let $\langle \chi, \varphi \rangle_W = |W|^{-1} \sum_{w \in W} \chi(w) \bar{\varphi}(w)$ denote the standard Hermitian inner product on complex W-characters (with $\bar{a}$ denoting complex conjugation). For any W-characters χ and φ, one has $\langle \chi \cdot \varphi, 1_W \rangle_W = \langle \chi, \bar{\varphi} \rangle_W$, so (3.1) implies

$$\langle \chi[R,q], 1_W \rangle_W = \sum_{J \subset S} q^{n-|J|}(1-q)^{-n+|J|} \sum_{k \geq 0} (-q)^k \langle \Lambda^k, 1_{W_J}^W \rangle_W, \quad (3.2)$$

where Λ^k denotes the W-character of $\Lambda^k(V)$.

We claim that $\langle \Lambda^k, 1_{W_J}^W \rangle_W = \binom{n-|J|}{k}$. To prove this, first note that, by Frobenius reciprocity, we have

$$\langle \Lambda^k, 1_{W_J}^W \rangle_W = \langle \Lambda^k, 1_{W_J} \rangle_{W_J} = \dim \Lambda^k(V)^{W_J}.$$

The reflection group W_J fixes a subspace V_J, and the roots of W_J span $V_J^\perp$. By a result of Steinberg (see Chapter V, Exercise 2.3 of [B]), it follows that the only W_J-invariants in $\Lambda(V_J^\perp)$ occur in degree zero, and hence

$$\Lambda(V)^{W_J} \cong [\Lambda(V_J) \otimes \Lambda(V_J^\perp)]^{W_J} \cong \Lambda(V_J).$$

Since $\dim V_J = n - |J|$, the claim follows.

Using this information in (3.2), we obtain

$$\langle \chi[R,q], 1_W \rangle_W = \sum_{J \subset S} q^{n-|J|} = (1+q)^n. \quad ∎$$

Since each orbit of a permutation representation includes one copy of the trivial representation, we may conclude the following.

COROLLARY 3.3. *If $H^*(X_R)$ carries a permutation representation of W, then the number of orbits is 2^n, and if it carries a graded permutation representation, then the number of orbits in degree $2k$ is $\binom{n}{k}$.*

4. THE MAIN RESULTS

For the remainder of this paper, we assume that the root system R is crystallographic. The simplicial decomposition Φ_R is therefore a fan in the weight lattice Λ_R, and so there is a toric variety X_R associated with Φ_R. Note also that the reflection group W is a Weyl group in this case. Our main result is that the representation of W carried by the cohomology ring of X_R, *with the grading ignored*, is isomorphic to a permutation representation π_R of W.

Before giving a more detailed statement of the result, we first need to discuss some peculiarities of irreducible root systems (e.g., see [B, H]). If R is irreducible then there is a unique highest root $\alpha_0 \in R$; it can also be described as the unique long root in the fundamental chamber C_0. The affine Weyl group $\tilde{W}$ is generated by W, together with an affine reflection across the hyperplane $H_0 = \{v \in V : \langle \alpha_0, v \rangle = 1\}$. A fundamental alcove for $\tilde{W}$ is the simplex obtained by intersecting C_0 with the negative half-space defined by H_0. By examining the group generated by the (affine) reflections fixing any given vertex of this simplex, one sees that any proper subset of $S \cup \{-\alpha_0\}$ forms a set of simple roots for some root subsystem of R. The isomorphism classes of these subsystems can be readily determined by deleting vertices from the extended diagram of R. For the convenience of the reader, the extended diagrams are provided in Appendix 1.

We state the main results in two parts. In the first part (Theorems 4.1–4.3), we describe conditions sufficient to uniquely determine the permutation representation π_R; this may be taken as the "definition" of π_R. In the second part (Theorem 4.5, Corollary 4.6), we describe further properties of π_R that are consequences of this definition.

THEOREM 4.1. *Let R be a crystallographic root system with Weyl group W. There is a permutation representation π_R of W, unique up to isomorphism, such that*

(a) *the KW-module generated by π_R is isomorphic to $H^*(X_R, K)$ (ungraded).*

(b) *If R is reducible, say $R = R_1 \oplus R_2$, then $\pi_R \cong \pi_{R_1} \otimes \pi_{R_2}$. (Here $\otimes$ denotes the outer tensor product of permutation representations.)*

(c) *If R is irreducible with highest root α_0, then the isotropy groups of π_R are conjugate to Weyl subgroups W_J for $J \subset S \cup \{-\alpha_0\}$.*

Note that the subgroups W_J appearing in (c) need not be parabolic. Following a suggestion of Haiman, we say that a subgroup of W is *quasi-parabolic* if it is generated by a set of reflections by roots that belong to $S \cup \{-\alpha_0\}$.

To clarify the meaning of this result, we need to discuss some properties of permutation representations. Suppose we have a group G acting by permutations on a finite set Y. Let $Y = Y_1 \cup \cdots \cup Y_l$ be the decomposition of Y into G-orbits, and for each orbit Y_i, let G_i be the isotropy group of some $x_i \in Y_i$. Each G_i is well-defined up to conjugacy, and the multiset $\{G_1, ..., G_l\}$ uniquely determines the permutation representation carried by Y up to isomorphism.

The assertion that the linear representation of G carried by $K\langle Y\rangle$ is isomorphic to some other KG-module does not in general determine the isomorphism class of the permutation representation. Indeed, for "most" finite groups, there are numerous linear dependencies among the characters of the transitive permutation representations. Any such dependence relation gives rise to a pair of nonisomorphic permutation representations whose linearizations are isomorphic over $\mathbf{C}$.

It is easy to see that if we are in the reducible case, say $R = R_1 \oplus R_2$, then the associated Weyl groups are related by $W \cong W_1 \times W_2$, and we have $H^*(X_R) \cong H^*(X_{R_1}) \otimes H^*(X_{R_2})$ as KW-modules. Thus to establish the existence of π_R, it suffices to treat the irreducible case. For this, we claim that it suffices to prove the following.

THEOREM 4.2. *Let R be irreducible and crystallographic with highest root α_0, and set $S_0 := S \cup \{-\alpha_0\}$. The character χ^R of $H^*(X_R)$ satisfies a relation of the form*

$$\chi^R = \sum_{J \subset S_0} m_R(J) \cdot 1_{W_J}^W \tag{4.1}$$

for suitable nonnegative integers $m_R(J)$.

To see that this does imply the existence of π_R, recall that the irreducible representations of Weyl groups can be realized over $\mathbf{Q}$ [H, §8.10]. Bearing in mind that (1.5) is valid for rational cohomology [D, §10.9], it follows that the choice of field K is immaterial (provided that the characteristic is 0). In particular, the KW-module generated by a permutation representation (or indeed, any K-linear representation) will be isomorphic to $H^*(X_R, K)$ if and only if it has character χ^R.

The uniqueness of π_R is a corollary of the following.

THEOREM 4.3. *Let R and S_0 be as above.*

(a) *For $I, J \subset S_0$, we have $1_{W_I}^W = 1_{W_J}^W$ if and only if $W_I \sim W_J$.*

(b) *For $J \subset S_0$, the distinct characters $1_{W_J}^W$ are linearly independent.*

Consequently, if we restrict the sum in (4.1) to a collection of conjugacy clas representatives for the quasi-parabolic subgroups W_J, the multiplicities $m_R(J)$ are uniquely determined.

Remark 4.4. (a) The restriction that R be crystallographic is essential. Although we no longer have a toric variety without this assumption, we may still consider the action of W on $K[\Delta_R]/\Theta$. However, in noncrystallographic cases, the character $\chi[R, 1]$ need not be integer-valued, and hence need not be a permutation character. For example, if $R = H_3$ and w is a Coxeter element for $W(H_3)$, then an easy calculation (using Corollary 1.5) shows that $\chi[R, 1](w) = 3 - \sqrt{5}$.

(b) The fact that the grading of $H^*(X_R)$ has been ignored is also essential. We will show that for the root systems of types A and C it is possible to assign a grading to the orbits of π_R so that the graded KW-module generated by π_R is isomorphic to $H^*(X_R)$. However, for the exceptional root systems, the grading of $H^*(X_R)$ is not compatible with any permutation representation of W (see Remark 10.2). In the cases of type D, we prove that there does exist a graded permutation representation of $W(D_n)$ whose graded KW-module is isomorphic to $H^*(X_R)$, but it is not isomorphic to π_R.

The last of our results is concerned with the multiplicities $m_R(J)$.

THEOREM 4.5. *Let R and S_0 be as above, and assume R is of rank n. For each $r \geqslant 0$ and each $(2r+1)$-subset J of S_0, one may choose an $(r+1)$-set $J' \subset J$ so that (up to conjugacy) the multiset of isotropy groups of π_R is $\{W_{S_0 - J'} : J \subset S_0,\ |J| \text{ odd}\}$.*

COROLLARY 4.6. *Assume R is irreducible and of rank n.*

(a) *The number of orbits of π_R is 2^n.*

(b) *Every isotropy group of π_R is a Weyl subgroup having rank at least $n/2$.*

(c) *The number of orbits of π_R whose isotropy groups have rank $n - r$ is $\binom{n+1}{2r+1}$.*

Remark 4.7. (a) Corollary 4.6(a) is also a consequence of Corollary 3.3.

(b) To completely specify π_R, it remains only to describe a rule for choosing the $(r+1)$-subsets J' from each $(2r+1)$-subset J of S_0. Unfortunately, the only such rules we have found are *ad hoc*—we have one for each infinite family A–D, and special ones for the exceptional root systems.

(c) There may be several possible rules for choosing J' from J so that Theorem 4.5 is satisfied. However, in the special case $|J| = 1$, there is no ambiguity—if J is a singleton, then $J' = J$. In other words, the rank n isotropy groups that occur in π_R are obtained by deleting one root from S_0 in each of the $n+1$ possible ways.

(d) In general, the representation π_R depends on R itself, not just W. Even though $W(B_n) = W(C_n)$, $\Delta_{B_n} = \Delta_{C_n}$, and the corresponding representations carried by their toric varieties are isomorphic, we claim that $\pi_{B_n} \not\cong \pi_{C_n}$. Indeed, by the previous remark and inspection of the extended diagrams for B_n and C_n (see Appendix 1), one sees that $W(D_n)$ occurs as an isotropy group in the case $R = B_n$, but not in the case $R = C_n$.

The proofs of Theorems 4.2, 4.3, and 4.5 are given case-by-case in Sections 6–10.

5. A Note on the Index of Connection

In a crystallographic root system R, the index of connection f_R is defined to be the index of the root lattice in the weight lattice, or equivalently, the determinant of the Cartan matrix [B, p. 224; H, p. 40]. In the following digression, we derive another interpretation of f_R by means of Theorem 4.5.

Let W continue to denote the Weyl group of R. We say that an element $w \in W$ is *essential* if it belongs to no proper reflection subgroup of W.

Proposition 5.1. *Coxeter elements are essential.*

Proof. Let $w = s_1 \cdots s_n$ be a Coxeter element for W, and let W' be a reflection subgroup containing w. Let R' denote the root system of W', and let N (resp., N') denote the set of positive roots of R (resp., R') mapped to negative roots by w. Using a well-known characterization of the length function [H, §1.6], we have $|N| = n$. In fact, an easy inductive argument shows that

$$N = \{\alpha_n, s_n(\alpha_{n-1}), s_n s_{n-1}(\alpha_{n-2}), ..., s_n \cdots s_2(\alpha_1)\}. \tag{5.1}$$

On the other hand, we cannot have $|N'| < n$, since otherwise w would be a product of fewer than n reflections, and hence V_w would be of positive dimension, in contradiction with Lemma 2.5(b). Since $N' \subset N$, it follows that $N' = N$. However, by (5.1) it is clear that the reflections corresponding to N generate all of W; hence, $W' = W$. ∎

THEOREM 5.2. *If w is an essential element of W, then* $\det(1-w)=f_R$.

Remark 5.3. The special case of this result in which w is a Coxeter element can be found in Chapter VI, Exercise 1.22 of [B].

Proof. Note that if R is reducible; say, $R=R_1 \oplus R_2$, then we may write $w=w_1 w_2$, with w_i essential in $W(R_i)$. Thus it suffices to restrict out attention to the case in which R is irreducible. Under this assumption, let n be the rank of R, and for $0 \leqslant i \leqslant n$, let W_i denote the reflection subgroup generated by $\{s_0, ..., s_{i-1}, s_{i+1}, ..., s_n\}$, where s_0 denotes reflection by the highest root.

LEMMA 5.4. *If w is essential, then* $\det(1-w)=|\{i : W_i=W\}|$.

Proof. Let W' be a reflection subgroup of W. We have $1_{W'}^{W}(w)=0$ unless some conjugate of W' contains w. However by assumption, no proper reflection subgroup of W contains w, so $1_{W'}^{W}(w)=0$ unless $W'=W$. On the other hand, by Theorem 4.5 (and in particular, Remark 4.7(c)), we know that the character χ^R of $H^*(X_R)$ is of the form

$$\chi^R = \sum_{0 \leqslant i \leqslant n} 1_{W_i}^{W} + \varphi^R,$$

where φ^R is a sum of transitive permutation characters induced by reflection subgroups of rank $<n$. By the previous observation, we therefore have $\chi^R(w)=|\{i : W_i=W\}|$. To complete the proof, note that $V_w=0$, so $\chi^R(w)=\det(1-w)$ by Corollary 1.5. ∎

We remark that the quantity $|\{i : W_i=W\}|$ has a geometric interpretation; namely, as the number of vertices of the fundamental alcove for $\tilde{W}$ whose isotropy group is isomorphic to W. Equivalently, this is the number of nodes in the extended diagram of R whose removal leaves a graph isomorphic to the diagram of R.

We may complete the proof of Theorem 5.2 at this point by observing that the right side of Lemma 5.4 is independent of w, and then appeal to the proof for Coxeter elements cited in Remark 5.3. Alternatively, one can prove directly that if $\sum_{i=1}^{n} c_i \alpha_i$ is the expansion of the highest root in terms of the simple roots, then for $1 \leqslant i \leqslant n$ one has $c_i=1$ if and only if $W_i=W$. This too completes the proof, since $f_R - 1$ is known to be $|\{i \geqslant 1 : c_i=1\}|$, by Exercise 2.2 in Chapter VI of [B]. ∎

6. THE A-SERIES

As noted in the Introduction, the characters of $H^*(X_R)$ for the root systems of type A have already received considerable attention in previous papers, especially [DL, P, and Ste3]. Indeed, aside from proving the existence of a rule satisfying Theorem 4.5, the material in this section is essentially a restatement of what is already known from these papers. We have included some of the details, since this case will serve as a model for what follows.

Let $\varepsilon_1, ..., \varepsilon_{n+1}$ be an orthonormal basis of $\mathbf{R}^{n+1}$, and let $\varepsilon = \varepsilon_1 + \cdots + \varepsilon_{n+1}$. We use the standard realization of A_n in $V = \{v \in \mathbf{R}^{+1} : \langle v, \varepsilon \rangle = 0\}$; namely,

$$A_n = \{\varepsilon_i - \varepsilon_j : 1 \leqslant i \neq j \leqslant n+1\}.$$

It is convenient to regard A_0 as an empty root system in a 0-dimensional space. For the base $S = \{\alpha_1, ..., \alpha_n\}$, we take $\alpha_i = \varepsilon_{i+1} - \varepsilon_i$. The fundamental chamber C_0 thus consists of the vectors in V with increasing coordinates. The Weyl group $W(A_n)$ is the symmetric group S_{n+1}, and the simple reflections s_i are the adjacent transpositions $(i, i+1)$. The highest root is $\varepsilon_{n+1} - \varepsilon_1$.

The chambers of Φ_{A_n} are the simplicial cones

$$C_w = \left\{ \sum c_i \varepsilon_i \in V : c_{w(1)} \leqslant \cdots \leqslant c_{w(n+1)} \right\},$$

where w varies over the permutations of $\{1, ..., n+1\}$. The k-dimensional faces of a given chamber are obtained by replacing $n-k$ of the inequalities among the c_i's with equalities. Thus $f_{k-1}(\Delta_{A_n})$ is the number of ordered partitions of an $(n+1)$-set into $k+1$ nonempty blocks; i.e.,

$$f_{k-1}(\Delta_{A_n}) = (k+1)!\, S(n+1, k+1), \tag{6.1}$$

where $S(n, k)$ denotes a Stirling number of the second kind. By (1.1) and the known generating functions for Stirling numbers (see (7.2)), it follows that

$$\frac{P_{A_n(q)}}{(1-q)^{n+2}} = \sum_{i \geqslant 0} (i+1)^{n+1} q^i, \tag{6.2}$$

so $qP_{A_n}(q)$ is the classical Eulerian polynomial [C, p. 245]. Alternatively, one could derive this by applying Theorem 2.1, which shows that $h_k(\Delta_{A_n})$ is an Eulerian number, viz., the number of permutations in S_{n+1} with k descents.

Now consider the h-polynomials of the subcomplexes $\Delta^w_{A_n}$. Recall from Section 2 that, up to isometry, these complexes depend only on the conjugacy class of w. Rather than create excessive notation, it is best to consider a specific, example, such as $n=6$ and $w = (1, 2, 3)(4, 5)(6)(7)$ (in cycle notation). In this case,

$$V_w = \left\{ \sum c_i \varepsilon_i \in V : c_1 = c_2 = c_3,\ c_4 = c_5 \right\},$$

and V_w is three-dimensional. The faces of $\Delta^w_{A_6}$ can be identified with ordered partitions of $\{1, ..., 7\}$ in which the elements 1–2–3, and respectively, 4–5, always occur in the same block. Treating these elements as single entities, one can see that there is an inclusion-preserving bijection between $\Delta^w_{A_6}$ and Δ_{A_3}. In the general case, we have

$$\Delta^w_{A_n} \cong \Delta_{A_{c(w)-1}}, \tag{6.3}$$

where $c(w)$ denotes the number of cycles of w.

Note also that if $w \in W(A_n)$ is of cycle-type λ, where $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_l)$ is some partition of $n+1$, then

$$\det(1 - qw) = (1-q)^{-1} \prod_{i=1}^{l} (1 - q^{\lambda_i}). \tag{6.4}$$

Using (6.3) and (6.4) to simplify the character formula in Theorem 1.4, we obtain the following result (essentially Proposition 3.3 of [Ste3]).

COROLLARY 6.1. *If $w \in W(A_n)$ is of cycle-type λ, then*

$$\chi[A_n, q](w) = P_{A_{c(w)-1}}(q) \cdot \prod_i \frac{1-q^{\lambda_i}}{1-q}.$$

To prove that χ^{A_n}, the character of $H^*(X_{A_n})$, is indeed a permutation character of the sort described by Theorem 4.2, it is convenient to introduce a graded character ring $\mathscr{R}_A = \bigoplus_{n \geqslant 0} \mathscr{R}^n_A$, in which $\mathscr{R}^0_A = K$, and (for $n > 0$) $\mathscr{R}^n_A$ is the K-vector space spanned by the characters of $W(A_{n-1}) = S_n$. The multiplicative structure is defined by the induction of outer tensor products; i.e.,

$$f \cdot g := \operatorname{ind}^{S_{m+n}}_{S_m \times S_n} (f \otimes g)$$

for all $f \in \mathscr{R}^m_A$, $g \in \mathscr{R}^n_A$. The element $1 \in \mathscr{R}^0_A$ acts as a multiplicative identity in $\mathscr{R}_A$.

It is well-known that $\mathscr{R}_A$ is isomorphic to the ring of symmetric functions via the characteristic map of Frobenius [M1, §7]. Thus we may easily apply the theory of symmetric functions to $\mathscr{R}_A$. In particular, let $h_n \in \mathscr{R}^n_A$ denote the trivial character of $W(A_{n-1})$, and for any partition λ of n, define $h_\lambda := h_{\lambda_1} \cdots h_{\lambda_l}$ so that h_λ is the permutation character of $W(A_{n-1})$ induced by a reflection subgroup isomorphic to

$$S_\lambda := S_{\lambda_1} \times \cdots \times S_{\lambda_l}.$$

The elements h_n correspond to the complete homogeneous symmetric functions; consequently, the h_n's freely generate $\mathscr{R}_A$ as a commutative K-algebra [M1].

Proof of Theorem 4.3. It is well-known and easy to show that all reflection subgroups of $W(A_{n-1})$ are isomorphic to one of the subgroups S_λ for some partition λ of n. Moreover, all of the reflection subgroups corresponding to a given choice of λ are easily seen to be conjugate in $W(A_{n-1})$. Since the h_n's freely generate $\mathscr{R}_A$ over K, it follows that the characters h_λ induced by the subgroups S_λ form a K-basis for $\mathscr{R}^n_A$. In particular, they are distinct (thus proving (a)) and linearly independent (thus proving (b)). ∎

We also need to make use of the virtual characters that correspond to the power-sum symmetric functions. Thus define $p_n \in \mathscr{R}^n_A$ to be the image of the nth power-sum symmetric function, and set $p_\lambda := p_{\lambda_1} \cdots p_{\lambda_l}$ for all partitions λ. From basic properties of the ring of symmetric functions (e.g., [M1]), one knows that the p_n's are also algebraically independent generators of $\mathscr{R}_A$ over K.

By virtue of the Frobenius characteristic map [M1, §7], the p_λ's are essentially the indicator functions for the conjugacy classes of S_n. More precisely, if $\chi \in \mathscr{R}^n_A$ is any character, and $\chi(\lambda)$ is the value of χ at any $w \in S_n$ of cycle-type λ, then we have

$$\chi = \sum_{|\lambda| = n} \frac{1}{z_\lambda} \chi(\lambda)\, p_\lambda, \tag{6.5}$$

where z_λ denotes the size of the S_n-centralizer of a permutation of cycle-type λ. (Here the notation $|\lambda|$ refers to the sum of the parts of λ.) In particular, for the trivial character we have the expansion $h_n = \sum_\lambda z_\lambda^{-1} p_\lambda$. This relationship can also be expressed as a generating function identity

$$H(t) = \sum_\lambda \frac{1}{z_\lambda} p_\lambda t^{|\lambda|} = \exp\left(\sum_{n \geqslant 1} p_n t^n / n \right), \tag{6.6}$$

where $H(t) := 1 + \sum_{n \geqslant 1} h_n t^n$ [M1, p. 17].

A result equivalent to the following was first proved by Procesi [P, p. 160], and then related in essentially this form by Stanley [St5, p. 529] (cf. also [Ste3, §4]). Independently, another proof was recently given by Dolgachev and Lunts [DL].

THEOREM 6.2. *We have*

$$\begin{aligned} 1 + \sum_{n \geqslant 0} \chi[A_n, q]\, t^{n+1} &= \frac{(1-q)\, H(t)}{H(qt) - qH(t)} \\ &= \frac{1 + \sum_{m \geqslant 1} h_m t^m}{1 - \sum_{m \geqslant 2} (q + \cdots + q^{m-1})\, h_m t^m}. \end{aligned}$$

Proof. Using (6.5) to rewrite Corollary 6.1 in terms of the p_λ's, we obtain

$$\begin{aligned} 1+\sum_{n\geqslant 0}\chi[A_n,q]\,t^{n+1} &= \sum_\lambda \frac{1}{z_\lambda}\cdot\frac{P_{A_{l(\lambda)-1}}(q)}{(1-q)^{l(\lambda)}}\cdot\prod_{i=1}^{l(\lambda)}(1-q^{\lambda_i})\,p_{\lambda_i}t^{\lambda_i} \\ &= (1-q)\sum_\lambda\frac{1}{z_\lambda}\cdot\prod_i(1-q^{\lambda_i})\,p_{\lambda_i}t^{\lambda_i}\sum_{k\geqslant 0}(k+1)^{l(\lambda)}q^k \\ &= (1-q)\sum_{k\geqslant 0}q^k\sum_\lambda\frac{1}{z_\lambda}(k+1)^{l(\lambda)}\prod_i(1-q^{\lambda_i})\,p_{\lambda_i}t^{\lambda_i}, \end{aligned}$$

the second equality being a consequence of (6.2). For a fixed choice of k, (6.6) shows that the inner sum is identical to the expression

$$\exp\left[(k+1)\sum_{n\geqslant 1}(1-q^n)\,p_n t^n/n\right]=[H(t)/H(qt)]^{k+1},$$

so we obtain

$$\begin{aligned} 1+\sum_{n\geqslant 0}\chi[A_n,q]\,t^{n+1} &= (1-q)\sum_{k\geqslant 0}q^k[H(t)/H(qt)]^{k+1} \\ &= \frac{(1-q)\,H(t)}{H(qt)-qH(t)}. \quad \blacksquare \end{aligned}$$

Note that Theorem 6.2 immediately implies Theorem 4.2 (for the A-series), as well as the following stronger result.

COROLLARY 6.3. *$K[\Delta_{A_n}]/\Theta$ (and hence also $H^*(X_{A_n})$) carries a graded permutation representation whose isotropy groups are parabolic subgroups of $W(A_n)$.*

Finally, it remains to construct a rule for describing the isotropy groups that is compatible with Theorem 4.5. For this, we begin by noting that a simple rearrangement of terms provides the following equivalent formulation of Theorem 6.2:

$$\sum_{n\geqslant 0}\chi[A_n,q]\,t^n=\frac{\sum_{m\geqslant 0}(1+q+\cdots+q^m)\,h_{m+1}t^m}{1-\sum_{m\geqslant 2}(q+\cdots+q^{m-1})\,h_m t^m}. \tag{6.7}$$

For convenience, we set $[A_n]:=h_{n+1}$. Substituting $q=1$ yields

$$\sum_{n\geqslant 0}\chi^{A_n}t^n=\frac{\sum_{m\geqslant 0}(m+1)[A_m]\,t^m}{1-\sum_{m\geqslant 1}m[A_m]\,t^{m+1}}.$$

Extracting the coefficient of t^n, we obtain

$$\chi^{A_n} = \sum_{r \geq 0} \sum_{m_0 + \cdots + m_r = n - r} (m_0 + 1) m_1 \cdots m_r [A_{m_0}][A_{m_1}] \cdots [A_{m_r}], \tag{6.8}$$

where the inner sum ranges over integers such that $m_0 \geq 0$ and $m_1, ..., m_r \geq 1$.

Proof of Theorem 4.5. The case $n = 1$ is trivial, so assume $n \geq 2$. Let $I = \{1, ..., n+1\}$, and for $J \subset I$, let W_J denote the subgroup of $W(A_n)$ generated by $\{s_j : j \in J\}$, where s_{n+1} denotes reflection by the highest root. For each odd subset $J = \{i_1 < i_2 < \cdots < i_{2r+1}\}$ of I, let us define

$$J' := \{i_1, i_3, i_5, ..., i_{2r+1}\}.$$

By inspection of the extended diagram of A_n (see Appendix 1), it is easy to see that the permutation character induced by $W_{I-J'}$ is $[A_{k+k'}][A_{m_1}] \cdots [A_{m_r}]$, where $k = i_1 - 1$, $k' = n + 1 - i_{2r+1}$, and $m_j = i_{2j+1} - i_{2j-1} - 1$ for $1 \leq j \leq r$. If we fix the parameters k, k' and $m_1, ..., m_r$, then the possible choices for J are obtained by varying the parameters i_{2j} in $m_1 \cdots m_r$ ways. Thus we have

$$\sum_{J \subset I : |J| \text{ odd}} 1_{W_{I-J'}}^{W}$$
$$= \sum_{r \geq 0} \sum_{k + k' + m_1 + \cdots + m_r = n - r} m_1 \cdots m_r [A_{k+k'}][A_{m_1}] \cdots [A_{m_r}].$$

This is clearly equivalent to the expansion for χ^{A_n} provided in (6.8). ∎

It is not difficult to refine the above analysis and use (6.7) to show that if we assign the degree $-1 + i_1 - i_2 + i_3 - \cdots + i_{2r+1}$ to the orbit indexed by $J = \{i_1 < \cdots < i_{2r+1}\}$, then we obtain a grading for π_{A_n} that is isomorphic to the grading of $K[\Delta_{A_n}]/\Theta$.

An explicit description of the decomposition of $\chi[A_n, q]$ into irreducible characters can be found in [Ste3, §4].

7. The C-Series

Using the standard embedding in $V = \mathbf{R}^n$, we have

$$C_n = \{2\varepsilon_i : 1 \leq i \leq n\} \cup \{\pm\varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq n\}.$$

It is convenient to allow $n \geq 0$. For the base $S = \{\alpha_1, ..., \alpha_n\}$, we take $\alpha_1 = 2\varepsilon_1$ and $\alpha_i = \varepsilon_{i+1} - \varepsilon_i$ for $i > 1$. The fundamental chamber consists of the vectors with nonnegative, increasing coordinates. The Weyl group $W(C_n)$ acts as the group of signed permutations of $\varepsilon_1, ..., \varepsilon_n$. The highest root is $2\varepsilon_n$.

As in Section 6, $S(n, k)$ denotes a Stirling number of the second kind.

PROPOSITION 7.1. *We have*

(a) $f_{k-1}(\Delta_{C_n}) = \sum_{j \geq 0} 2^{n-j} \binom{n}{j} k!\, S(n-j, k)$.

(b) $P_{C_n}(q)/(1-q)^{n+1} = \sum_{i \geq 0} (2i+1)^n q^i$.

Proof. (a) The chambers of Φ_{C_n} are the simplicial cones

$$C_{\gamma, w} = \left\{ \sum c_i \varepsilon_i \in V : 0 \leqslant \gamma_1 c_{w(1)} \leqslant \cdots \leqslant \gamma_n c_{w(n)} \right\}, \tag{7.1}$$

where w varies over the permutations of $\{1, ..., n\}$ and $\gamma_i = \pm 1$. The k-dimensional faces of $C_{\gamma, w}$ are obtained by replacing $n-k$ of the inequalities involving the c_i's with equalities. For example,

$$0 = c_3 = c_6 \leqslant -c_1 = c_5 \leqslant c_4 = -c_2$$

describes a typical two-dimensional cone of Φ_{C_6}. In general, the k-dimensional cones of Φ_{C_n} are in one-to-one correspondence with "signed," ordered partitions of $\{0, 1, ..., n\}$ into $k+1$ nonempty blocks such that (1) the first block contains 0 and has no signs attached to it, and (2) each element of the remaining blocks has a sign attached. If there are to be $j+1$ elements in the block containing 0, then there will be $\binom{n}{j}$ choices for this block, 2^{n-j} choices for the signs, and $k!\, S(n-j, k)$ possible ways to complete the partition. Thus there are $2^{n-j} \binom{n}{j} k!\, S(n-j, k)$ such cones.

(b) By (1.1) and part (a), we have

$$\frac{P_{C_n}(q)}{(1-q)^{n+1}} = \sum_{k, j \geq 0} 2^{n-j} \binom{n}{j} k!\, S(n-j, k) \frac{q^k}{(1-q)^{k+1}}$$
$$= \sum_{k, j, i \geq 0} 2^{n-j} \binom{n}{j} \binom{i}{k} k!\, S(n-j, k) q^i.$$

Using the well-known generating function (e.g., [C, p. 207])

$$x^n = \sum_k \binom{x}{k} k!\, S(n, k), \tag{7.2}$$

we thus obtain

$$\frac{P_{C_n}(q)}{(1-q)^{n+1}} = \sum_{j, i \geq 0} (2i)^{n-j} \binom{n}{j} q^i = \sum_{i \geq 0} (2i+1)^n q^i. \quad ∎$$

The elements of $W(C_n)$ can be compactly described as products of signed cycles; for example, $(-1, 2, -3)(-4, 5)(6)(-7)$ represents the element of

$W(C_7)$ that maps $\varepsilon_1 \mapsto \varepsilon_2$, $\varepsilon_2 \mapsto -\varepsilon_3$, $\varepsilon_3 \mapsto -\varepsilon_1$, and so on. A signed cycle is said to be *positive* or *negative* according to the product of its signs. The conjugacy classes are indexed by pairs of partitions (μ, ν) such that $|\mu| + |\nu| = n$; the pair indexing the class of a given $w \in W(C_n)$ may be defined so that the parts of μ (resp., ν) are the lengths of the positive (resp., negative) cycles of w (cf. [Ca, §7]). It is easy to check that if w is of type (μ, ν), then

$$\det(1 - qw) = \prod_i (1 - q^{\mu_i}) \prod_j (1 + q^{\nu_j}). \tag{7.3}$$

In particular, the dimension of V_w is $l(\mu)$, the number of parts of μ.

Now consider the h-polynomials of the subcomplexes $\Delta^w_{C_n}$. For example, if we take $w = (-1, 2, -3)(-4, 5)(6)(-7) \in W(C_7)$, then we have

$$V_w = \left\{ \sum c_i \varepsilon_i \in V : c_1 = c_2 = -c_3, c_4 = c_5 = c_7 = 0 \right\}.$$

Therefore, the faces of $\Delta^w_{C_n}$ can be identified with signed, ordered partitions of $\{0, 1, 2, 3, 6\}$ satisfying the properties described in the proof of Proposition 7.1(a), along with the extra condition that the elements 1–2–3 always occur in the same block and with the same relative signs (if any). By treating 1–2–3 as a single entity, one can see that there is an inclusion-preserving bijection between $\Delta^w_{C_n}$ and Δ_{C_2}. In the general case, if w is of type (μ, ν), then there will be $|\nu|$ coordinates in V_w that are identically 0, and one w-invariant vector for each of the $l(\mu)$ positive cycles. As in Lemma 4.9 of [DL], one obtains

$$\Delta^w_{C_n} \cong \Delta_{C_{l(\mu)}}. \tag{7.4}$$

Using (7.3) and (7.4) to simplify the character formula of Theorem 1.4 yields the following result (originally obtained by Dolgachev and Lunts [DL]).

COROLLARY 7.2. *If $w \in W(C_n)$ is of type (μ, ν), then*

$$\chi[C_n, q](w) = \frac{P_{C_{l(\mu)}}(q)}{(1-q)^{l(\mu)}} \prod_i (1 - q^{\mu_i}) \prod_j (1 + q^{\nu_j}).$$

To analyze the permutation characters of reflection subgroups of $W(C_n)$, we need to use an analogue of the character ring $\mathscr{R}_A$ of Section 6. This construction is well-known and has appeared, e.g., in [M2, §9; Ste2, §5; and Z]. First choose any orthogonal decomposition $\mathbf{R}^{m+n} = V_1 \oplus V_2$ in which V_1 (resp., V_2) is spanned by m (resp., n) of the coordinate vectors ε_i. Upon restriction to V_1 and V_2, we obtain root systems isomorphic to C_m and C_n, and hence an embedding $W(C_m) \times W(C_n) \subsetneq W(C_{m+n})$. It is easy to see that all such embeddings are conjugate. Thus by analogy with the A-series, we may define a graded K-algebra $\mathscr{R}_C = \bigoplus_{n \geq 0} \mathscr{R}_C^n$, in which $\mathscr{R}_C^n$ is the K-vector space spanned by the characters of $W(C_n)$, and the multiplication is obtained by the induction of outer tensor products from $W(C_m) \times W(C_n)$ to $W(C_{m+n})$.

Although there is a risk of possible confusion with the notation of Section 6, we define $h_n \in \mathscr{R}_C^n$ to be the trivial character of $W(C_n)$, and set $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_l}$ for partitions λ. Since the trivial characters of the symmetric groups freely generate $\mathscr{R}_A$, it follows that the h_n's freely generate the subalgebra of $\mathscr{R}_C$ spanned by characters of representations that factor through the "sign-forgetting" homomorphism σ: $W(C_n) \to W(A_{n-1})$.

Let δ_n denote the one-dimensional representation of $W(C_n)$ whose kernel is $W(D_n)$, and let $g_n \in \mathscr{R}_C^n$ denote the character of δ_n. The operation $\rho \mapsto \delta_n \otimes \rho$ on $W(C_n)$-representations induces an automorphism $\delta: \mathscr{R}_C \to \mathscr{R}_C$ of order 2 with the property that $\delta(h_n) = g_n$. By a theorem of Young (e.g., see [M2; Ste2, §5; or Z, §7]), it is known that the distinct irreducible characters of $W(C_n)$ are of the form $\chi \cdot \delta(\varphi)$, where $\chi \in \mathscr{R}_C^k$ and $\varphi \in \mathscr{R}_C^{n-k}$ vary over irreducible characters that factor through σ, and k runs from 0 to n. It follows that $\mathscr{R}_C$ is freely generated by the h_n's and g_n's, and therefore

$$\mathscr{R}_C \cong \mathscr{R}_A \otimes \mathscr{R}_A$$

as graded K-algebras.

We also now ready to analyze the conjugacy classes of reflection subgroups of $W(C_n)$. Let R be an arbitrary root subsystem of C_n, and let W be the reflection subgroup it generates. We define a multigraph $\Gamma(W)$ on n vertices by providing an edge between i and j if $\varepsilon_i - \varepsilon_j \in R$, and also if $\varepsilon_i + \varepsilon_j \in R$. In particular, there is a double edge between i and j ($i \neq j$) if and only if $\varepsilon \pm \varepsilon_j$ are both in R, and there is a loop at vertex i if and only if $2\varepsilon_i \in R$.

The connected components of the graphs $\Gamma(W)$ can be classified as follows: (1) complete graphs without loops or multiple edges, (2) doubly complete graphs without loops (i.e., double edges between distinct points) on two or more points, and (3) doubly complete graphs with loops at every vertex. We say that such a graph on n points is of type A_{n-1}, D_n, or C_n, according to whether it belongs to the first, second, or third of these groups. These labels have the property that W is isomorphic to the Weyl group of the corresponding label, but isomorphism of the groups does not imply isomorphism of the graphs. Also, one should be careful to distinguish an isolated loop (type C_1) from an isolated point with no loop (type A_0).

LEMMA 7.3. *Two reflection subgroups of $W(C_n)$ are conjugate if and only if their graphs are isomorphic.*

Proof. The effect of conjugation on $\Gamma(W)$ is merely permutation of the vertices, and thus does not change the isomorphism class of $\Gamma(W)$. For the converse, recall that the various embeddings of $C_m \oplus C_n$ in C_{m+n} obtained by partitioning the coordinate vectors are all conjugate in $W(C_{m+n})$. It therefore suffices to consider the case of connected graphs. For the graphs of type C_n and D_n, the set of reflections contained in W is completely determined by the graph, so there is nothing further to prove in these cases. If $\Gamma(W)$ is of type A_{n-1}, then W contains the reflections $(1, 2)$ or $(-1, -2)$, $(2, 3)$ or $(-2, -3)$, and so on. By successively conjugating these reflections by (-2), (-3), ... (where necessary), one can show that W is conjugate to the reflection subgroup generated by $(1, 2), ..., (n-1, n)$. ∎

Since there is only one conjugacy class of reflection groups corresponding to each type of graph, we may unambiguously write $[A_{n-1}]$, $[D_n]$ $(n \geqslant 2)$, and $[C_n]$ for the permutation characters of $W(C_n)$ induced by reflection subgroups of these respective types.

In part (c) of the following, h_0 and g_0 both denote the unit element of $\mathscr{R}_C$.

PROPOSITION 7.4. *We have*

(a) $[C_n] = h_n$ $(n \geqslant 1)$.

(b) $[D_n] = h_n + g_n$ $(n \geqslant 2)$.

(c) $[A_{n-1}] = \sum_{k=0}^{n} h_k g_{n-k}$ $(n \geqslant 1)$.

Proof. Of these three assertions, only part (c) deserves elaboration. There is a natural permutation representation of $W(C_n)$ on $2n$ points, say $x_1, y_1, ..., x_n, y_n$. In this representation, the reflections (i, j) act via the interchanges $x_i \leftrightarrow x_j$ and $y_i \leftrightarrow y_j$; the reflections $(-i)$ act via $x_i \leftrightarrow y_i$. Passing to the symmetric algebra generated by this representation, the orbit of the monomial $x_1 \cdots x_n$ generates a transitive permutation representation of $W(C_n)$ whose isotropy group is of type A_{n-1}. A K-linear basis for this representation is provided by the 2^n products of the form $(x_1 \pm y_1) \cdots (x_n \pm y_n)$; moreover, the subspaces V_k spanned by the products with k "+" signs and $n-k$ "−" signs are $W(C_n)$-invariant. Using the fact that the action of $W(C_n)$ on V_n is trivial and on V_0 is isomorphic to δ_n, one can easily verify that the character of V_k is $h_k g_{n-k}$. ∎

Proof of Theorem 4.3. (a) The permutation character induced by an arbitrary reflection subgroup of $W(C_n)$ is a monomial in the variables $[A_{m-1}]$, $[D_m]$ $(m \geqslant 2)$, and $[C_m]$. However, $\mathscr{R}_C$ is a polynomial ring and hence, a unique factorization domain. Since Proposition 7.4 shows that these characters are distinct primes of $\mathscr{R}_C$, it follows that distinct monomials (i.e., distinct classes of reflection subgroups) yield distinct characters.

(b) By inspection of the extended diagram of C_n (Appendix 1), one sees that all of the quasi-parabolic subgroups of $W(C_n)$ have graphs whose components are of types A and C. Since the h_n's and g_n's freely generate $\mathscr{R}_C$, a simple induction argument based on Proposition 7.4 shows that the $[C_n]$'s and $[A_{n-1}]$'s must also freely generate $\mathscr{R}_C$. In particular, there are no dependence relations among the permutation characters induced by reflection subgroups with graphs having all components of type A and C. ∎

We remark that part (b) does not extend to arbitrary reflection subgroups of $W(C_n)$. Using Proposition 7.4, one finds that a dependence relation occurs at rank 2:

$$[A_1] + [C_1] \cdot [C_1] = [D_2] + [A_0] \cdot [C_1].$$

There is an analogue of the Frobenius map for the characters of $W(C_n)$. An explicit description of it can be found e.g. in [Ste2, §5]. Some constructions of more general Frobenius-type maps can be found in [M2] and [Z]. The only aspect of this theory we need here is that for $n \geqslant 1$, there exist certain special virtual characters $p_n^+, p_n^- \in \mathscr{R}_C^n$ (analogous to the power-sums) that (1) freely generate $\mathscr{R}_C$, and (2) act as indicator functions for the positive and negative cycle-types. To explain this second property more precisely, let us first define $p_\mu^+ := p_{\mu_1}^+ \cdots p_{\mu_l}^+$ for partitions μ (and similarly define p_μ^-). Now if $\chi \in \mathscr{R}_C^n$ is any character of $W(C_n)$, then we have

$$\chi = \sum_{|\mu| + |\nu| = n} \frac{1}{z_\mu z_\nu} \cdot \chi(\mu, \nu)\, p_\mu^+\, p_\nu^-, \tag{7.5}$$

where $\chi(\mu, \nu)$ denotes the value of χ at the conjugacy class indexed by (μ, ν) [Ste2, §5].

By analogy with (6.6), let us define

$$H^\pm(t) := \sum_\lambda \frac{1}{z_\lambda} p_\lambda^\pm t^{|\lambda|} = \exp\left(\sum_{n \geqslant 1} p_n^\pm t^n/n \right), \tag{7.6a}$$

$$H_C(t) := 1 + \sum_{n \geqslant 1} h_n t^n = \sum_{n \geqslant 0} [C_n] t^n, \tag{7.6b}$$

$$H_A(t) := 1 + \sum_{n \geqslant 1} [A_{n-1}]\, t^n. \tag{7.6c}$$

PROPOSITION 7.5. *We have*

(a) $H_C(t) = H^+(t)\,H^-(t)$.

(b) $H_A(t) = H^+(t)^2$.

Proof. By (7.5) and (7.6a), we have

$$H^+(t)\,H^-(t) = \sum_{\mu,\nu} \frac{1}{z_\mu z_\nu} p_\mu^+ p_\nu^- t^{|\mu|+|\nu|} = 1 + \sum_{n\geq 1} h_n t^n,$$

and thus (a) follows. For (b), apply the substitution $p_n^- \mapsto -p_n^-$. Again by (7.5) and (7.6a), we obtain

$$H^+(t)\,H^-(t)^{-1} = \sum_{\mu,\nu} \frac{1}{z_\mu z_\nu} (-1)^{l(\nu)} p_\mu^+ p_\nu^- t^{|\mu|+|\nu|} = 1 + \sum_{n\geq 1} g_n t^n.$$

Proposition 7.4(c) shows that $H_A(t)$ is the product of the above two series. ∎

The following result has also been proved by Dolgachev and Lunts [DL].

THEOREM 7.6. *We have*

$$\sum_{n\geq 0} \chi[C_n, q]\,t^n = \frac{(1-q)\,H_C(t)\,H_C(qt)}{H_A(qt) - qH_A(t)}$$
$$= \frac{\sum_{k,k'\geq 0} [C_k][C_{k'}]\,q^k t^{k+k'}}{1 - \sum_{m\geq 1} (q + \cdots + q^m)[A_m]\,t^{m+1}}.$$

Proof. By (7.5) and Corollary 7.2, we have

$$\sum_{n\geq 0} \chi[C_n, q]\,t^n$$
$$= \sum_{\mu,\nu} \frac{t^{|\mu|+|\nu|}}{z_\mu z_\nu} \frac{P_{C_{l(\mu)}}(q)}{(1-q)^{l(\mu)}} \prod_i (1-q^{\mu_i})\,p_{\mu_i}^+ \prod_j (1+q^{\nu_j})\,p_{\nu_j}^-$$
$$= (1-q) \sum_{\mu,\nu} \frac{t^{|\mu|+|\nu|}}{z_\mu z_\nu} \prod_i (1-q^{\mu_i})\,p_{\mu_i}^+$$
$$\times \prod_j (1+q^{\nu_j})\,p_{\nu_j}^- \sum_{k\geq 0} (2k+1)^{l(\mu)} q^k$$
$$= (1-q) \sum_{k\geq 0} q^k \sum_{\mu,\nu} \frac{t^{|\mu|+|\nu|}}{z_\mu z_\nu}$$
$$\times \prod_i (2k+1)(1-q^{\mu_i})\,p_{\mu_i}^+ \prod_j (1+q^{\nu_j})\,p_{\nu_j}^-,$$

the second equality being a consequence of Proposition 7.1(b). Now by (7.6a), we have

$$\sum_{\mu} \frac{t^{|\mu|}}{z_\mu} \prod_i (2k+1)(1-q^{\mu_i})\, p^+_{\mu_i} = \frac{H^+(t)^{2k+1}}{H^+(qt)^{2k+1}},$$

$$\sum_{\nu} \frac{t^{|\nu|}}{z_\nu} \prod_j (1+q^{\nu_j})\, p^-_{\nu_j} = H^-(t)\, H^-(qt),$$

so the above sum simplifies to

$$\begin{aligned}\sum_{n \geqslant 0} \chi[C_n, q]\, t^n &= (1-q) \sum_{k \geqslant 0} q^k [H^+(t)/H^+(qt)]^{2k+1} H^-(t)\, H^-(qt) \\ &= \frac{(1-q)\, H^+(t)\, H^+(qt)\, H^-(t)\, H^-(qt)}{H^+(qt)^2 - qH^+(t)^2}\end{aligned}$$

To complete the proof, note that Proposition 7.5 shows that this is identical to the first of the two claimed series for $\chi[C_n, q]$. The second series follows directly from the first. ∎

By inspection of the extended diagram of C_n, one can see that the graphs of the quasi-parabolic subgroups of $W(C_n)$ are characterized by the fact that they have at most two components of type C, with the remainder having type A. Thus an immediate consequence of Theorem 7.6 is Theorem 4.2, as well as the following graded refinement.

COROLLARY 7.7. *$K[\Delta_{C_n}]/\Theta$ (and hence also $H^*(X_{C_n})$) carries a graded permutation representation whose isotropy groups are quasi-parabolic subgroups of $W(C_n)$.*

To construct a rule that satisfies Theorem 4.5, first substitute $q = 1$ in Theorem 7.6 and extract the coefficient of t^n, obtaining

$$\chi^{C_n} = \sum_{r \geqslant 0} \sum_{k+k'+m_1+\cdots+m_r=n-r} m_1 \cdots m_r [C_k][C_{k'}][A_{m_1}] \cdots [A_{m_r}],$$

where the inner sum ranges over integers such that $k, k' \geqslant 0$ and $m_1, \ldots, m_r \geqslant 1$.

Proof of Theorem 4.5. Assume $n \geqslant 2$. Let $I = \{1, \ldots, n+1\}$, and for $J \subset I$, write W_J for the subgroup of $W(C_n)$ generated by $\{s_j : j \in J\}$, where s_{n+1} denotes reflection by the highest root. Again for each odd subset $J = \{i_1 < i_2 < \cdots < i_{2r+1}\}$ of I, we define

$$J' := \{i_1, i_3, i_5, \ldots, i_{2r+1}\}.$$

By inspection of the extended diagram (Appendix 1), one sees that the permutation character induced by $W_{I-J'}$ is $[C_k][C_{k'}][A_{m_1}] \cdots [A_{m_r}]$, where

$k = i_1 - 1$, $k' = n + 1 - i_{2r+1}$, and $m_j = i_{2j+1} - i_{2j-1} - 1$ for $1 \leqslant j \leqslant r$. If we fix the parameters k, k', and m_i, then there are $m_1 \cdots m_r$ choices for J. The remainder of the proof now proceeds in the same manner as for the A-series. ∎

One may define a grading for π_{C_n} that is isomorphic to the grading of $K[\Delta_{C_n}]/\Theta$ by assigning degree $-1 + i_1 - i_2 + \cdots + i_{2r+1}$ to the orbit indexed by $J = \{i_1 < \cdots < i_{2r+1}\}$. It is also possible to give an explicit decomposition of χ^{C_n} into irreducible characters analogous to [Ste3, §4], although we do not pursue the details here.

8. The B-Series

Using the standard embedding in $V = \mathbf{R}^n$, we have

$$B_n = \{\varepsilon_i : 1 \leqslant i \leqslant n\} \cup \{\pm \varepsilon_i \pm \varepsilon_j : 1 \leqslant i < j \leqslant n\}.$$

As in the previous section, it is convenient to allow the rank n to be an arbitrary nonnegative integer. For the base $S = \{\alpha_1, ..., \alpha_n\}$, we take $\alpha_1 = \varepsilon_1$ and $\alpha_i = \varepsilon_{i+1} - \varepsilon_i$ for $i > 1$. For $n \geqslant 2$, the highest root is $\varepsilon_{n-1} + \varepsilon_n$.

Since the only difference between B_n and C_n is the length of certain roots, we have $W(B_n) = W(C_n)$, $\Phi_{B_n} = \Phi_{C_n}$, $\Delta_{B_n} = \Delta_{C_n}$, and $K[\Delta_{B_n}]/\Theta \cong K[\Delta_{C_n}]/\Theta$. Thus several of the results of Section 7 are valid for B_n without modification; namely, Propositions 7.1, 7.4, and 7.5, Corollary 7.2, Lemma 7.3, and Theorem 7.6. However, the proofs of Theorems 4.2, 4.3, and 4.5 we gave for C_n are not valid for B_n (provided $n > 2$), since the quasi-parabolic subgroups are not the same. (Compare the extended diagrams of B_n and C_n in Appendix 1.)

We continue to use the character ring $\mathscr{R}_C$ and the notion of the graph of a reflection subgroup of $W(B_n)$, as defined in Section 7. However, for aesthetic reasons, in this section we write B_n for the graph of type C_n, and $[B_n] = [C_n] = h_n$ for the trivial character of $W(B_n)$.

Proof of Theorem 4.3. The proof of part (a) given in Section 7 is equally valid for this case, since it applies to all reflection subgroups of $W(B_n)$. It remains to prove (b).

By inspection of the extended diagram of B_n, one can see that the graphs of the quasi-parabolic subgroups for B_n are characterized by the fact that they have at most one component of type B and at most one component of type D; the remaining components are of type A. Recall from Proposition 7.4 that $[B_n] = h_n$, $[D_n] = h_n + g_n$, and $[A_{n-1}] = a_n$, where $a_n := \sum_{k=0}^{n} h_k g_{n-k}$. Thus we seek to prove that the characters

$$a_\lambda, \quad h_k a_\lambda (k \geqslant 1), \quad (h_l + g_l) a_\lambda \ (l \geqslant 2), \quad h_k (h_l + g_l) a_\lambda \ (k \geqslant 1, l \geqslant 2),$$

are linearly independent, where $a_\lambda = a_{\lambda_1} a_{\lambda_2} \cdots$ varies over all partitions λ. Since the h_n's and g_n's are algebraically independent over K, this is now a purely formal problem in the domain of polynomial rings.

As we remarked in Section 7, $\mathscr{R}_C$ is also freely generated by the h_n's and the a_n's, so we may analyze the above expressions by regarding g_n as a polynomial function of $h_1, a_1, h_2, a_2, \ldots$. From this point of view, the problem is to prove that

$$1, \quad h_k, \quad h_l + g_l, \quad h_k(h_l + g_l) \qquad (k \geqslant 1, l \geqslant 2) \tag{8.1}$$

are linearly independent over $K[a_1, a_2, \ldots]$. To simplify this task further, we prove that even after applying a certain specialization for the h_n's to (8.1), we still obtain a linearly independent set.

For this, let x and y be indeterminates and define $h_n \mapsto x^n - yx^{n-1}$, or equivalently, $\sum h_n t^n \mapsto (1-ty)/(1-tx)$. If we explicitly solve for g_n, we obtain

$$\sum_{n \geqslant 0} g_n t^n = \left(1 + \sum_{n \geqslant 1} a_n t^n\right)\left(\sum_{n \geqslant 0} h_n t^n\right)^{-1} \mapsto \left(1 + \sum_{n \geqslant 1} a_n t^n\right) \frac{1-tx}{1-ty}.$$

Hence for $n \geqslant 1$, we have

$$g_n \mapsto y^n - xy^{n-1} + \text{lower order terms},$$

where "lower" refers to the total degree with respect to x and y.

If there were a dependence relation among the elements of (8.1), then there would also be a dependence relation among their leading terms (with respect to total degree in x and y). But these leading terms are independent of $a_1, a_2, \ldots$, so it suffices merely to prove the linear independence of (8.1) over K in the special case $h_n = x^n - yx^{n-1}$, $g_n = y^n - xy^{n-1}$.

In this case, we claim more specifically that (8.1) forms a K-basis for $K + (x-y) K[x, y]$. To prove this, note that $h_k = (x-y)x^{k-1}$ and $h_l + g_l = (x-y)(x^{l-1} - y^{l-1})$, so it suffices to show that

$$x^{k-1}, \quad x^{l-1} - y^{l-1}, \quad (x-y)(x^{l-1} - y^{l-1})x^{k-1} \qquad (k \geqslant 1, l \geqslant 2)$$

is a K-basis for $K[x, y]$. However, $\{x^{k-1} : k \geqslant 1\}$ is a basis for the subspace $K[x]$; we claim that the remaining elements form a basis for the complementary subspace $(x-y) K[x, y]$. Indeed, each of the remaining terms is divisible by $x - y$, and their quotients

$$x^{l-2} + x^{l-3}y + \cdots + y^{l-2}, \quad x^{k-1}(x^{l-1} - y^{l-1}) \qquad (k \geqslant 1, l \geqslant 2)$$

are clearly a basis for $K[x, y]$. ∎

Next we consider the problem of writing χ^{B_n} as a linear combination of permutation characters induced by quasi-parabolic subgroups.

PROPOSITION 8.1. *For $n \geqslant 1$, we have*

$$\sum_{k=0}^{n} [B_k][B_{n-k}] = 2[B_n] + [B_{n-1}][A_0] - [A_{n-1}] + \sum_{k=2}^{n} [D_k][B_{n-k}].$$

Proof. This follows directly from Proposition 7.4. ∎

Proof of Theorem 4.2. If we substitute $q=1$ in Theorem 7.6 and apply Proposition 8.1, we obtain

$$\sum_{n \geqslant 0} \chi^{B_n} y^n = \frac{\begin{pmatrix} 1 + \sum_{m \geqslant 1} (2[B_m] + [B_{m-1}][A_0] \\ - [A_{m-1}])t^m + \sum_{m \geqslant k \geqslant 2} [D_k][B_{m-k}] t^m \end{pmatrix}}{1 - \sum_{m \geqslant 1} m[A_m] t^{m+1}},$$

or equivalently, $\sum_{n \geqslant 1} \chi^{B_n} t^n = B(t) \cdot (1 - \sum_{m \geqslant 1} m[A_m]\, t^{m+1})^{-1}$, where

$$B(t) = (2 + [A_0] t) \sum_{m \geqslant 1} [B_m] t^m + \sum_{m \geqslant 1} (m-1)[A_m]\, t^{m+1}$$
$$+ \sum_{m \geqslant k \geqslant 2} [D_k][B_{m-k}]\, t^m.$$

In this form, it is evident that χ^{B_n} is an integral sum of permutation characters induced by reflection subgroups of $W(B_n)$ whose graphs have at most one component each of types B and D. These are precisely the quasi-parabolic subgroups of $W(B_n)$. ∎

Remark 8.2. The analogue of Corollary 7.7 is false for the B-series. Even for $n=3$, one can check that the grading of $K[\Delta_{B_n}]/\Theta$ is not compatible with any permutation representation of $W(B_n)$ whose isotropy groups are quasi-parabolic.

If we expand the above series for χ^{B_n} in detail, we obtain

$$\chi^{B_n} = 2\chi^B_{\mathrm{I}} + \chi^B_{\mathrm{II}} + \chi^B_{\mathrm{III}} + \chi^B_{\mathrm{IV}},$$

where

$$\chi^B_{\mathrm{I}} = \sum_{r \geqslant 0} \sum_{m_0 + \cdots + m_r = n-r} m_1 \cdots m_r [B_{m_0}][A_{m_1}] \cdots [A_{m_r}] \tag{8.2a}$$

$$\chi^B_{\mathrm{II}} = \sum_{r \geqslant 1} \sum_{m_0 + \cdots + m_{r-1} = n-r} m_1 \cdots m_{r-1} [B_{m_0}][A_0][A_{m_1}] \cdots [A_{m_{r-1}}] \tag{8.2b}$$

$$\chi^B_{\mathrm{III}} = \sum_{r \geqslant 1} \sum_{m_1 + \cdots + m_r = n-r} (m_1 - 1)\, m_2 \cdots m_r [A_{m_1}] \cdots [A_{m_r}] \tag{8.2c}$$

$$\chi^B_{\mathrm{IV}} = \sum_{r \geqslant 0} \sum_{k_1 + k_2 + m_1 + \cdots + m_r = n-r} m_1 \cdots m_r [B_{k_1}][D_{k_2}][A_{m_1}] \cdots [A_{m_r}], \tag{8.2d}$$

where $k_1 \geqslant 0$, $k_2 \geqslant 2$, and $m_i \geqslant 1$.

Proof of Theorem 4.5. Assume $n \geqslant 3$. Let $I = \{1, ..., n+1\}$, and for $J \subset I$, write W_J for the subgroup of $W(B_n)$ generated by $\{s_j : j \in J\}$, where s_{n+1} denotes reflection by the highest root. For each odd subset $J = \{i_1 < i_2 < \cdots < i_{2r+1}\}$ of I, we define

$$J' := \begin{cases} \{i_2, i_4, ..., i_{2r}, i_{2r+1}\} & \text{if } \{1, n\} \subset J, \\ \{i_1, i_3, ..., i_{2r-1}, i_{2r+1}\} & \text{otherwise.} \end{cases}$$

We claim that this rule satisfies Theorem 4.5; i.e., we claim

$$\chi^{B_n} = \sum_{J \subset I : |J| \text{ odd}} 1_{W_{I-J'}}^{W}. \tag{8.3}$$

To prove this claim, we need to break this sum into several pieces according to the presence or absence of the elements $1, n$, and $n+1$ in J. The reader is advised to refer to the extended diagram in Appendix 1 for what follows.

Case I. $n, n+1 \notin J$. We have $J' = \{i_1, i_3, ..., i_{2r+1}\}$, and

$$1_{W_{I-J'}}^{W} = [B_{k_1}][D_{k_2}][A_{m_1}] \cdots [A_{m_r}],$$

where $k_1 = i_1 - 1 \geqslant 0$, $k_2 = n + 1 - i_{2r+1} \geqslant 2$, and $m_j = i_{2j+1} - i_{2j-1} - 1$. If we fix the parameters k_j and m_j, then there are $m_1 \cdots m_r$ possible choices for J (corresponding to the possible choices for $i_2, i_4, ...$). Hence by (8.2d), we see that the total contribution of this case to (8.3) equals χ_{IV}^{B}.

Case II. $n+1 \in J$, $1 \notin J$. In this case, we still have $J' = \{i_1, i_3, ..., i_{2r+1}\}$, but

$$1_{W_{I-J'}}^{W} = [B_{m_0}][A_{m_1}] \cdots [A_{m_r}],$$

where $m_0 = i_1 - 1$, and $m_j = i_{2j+1} - i_{2j-1} - 1$ for $j > 0$. If we fix the parameters m_j, then there are $m_1 \cdots m_r$ possible choices for J. Hence by (8.2a), this case contributes to (8.3) an amount equal to χ_{I}^{B}.

Case II′. $n \in J, n+1 \notin J$. In this case, we either have $J' = \{i_2, i_4, ..., i_{2r+1}\}$ or $J' = \{i_1, i_3, ..., i_{2r+1}\}$, depending on whether $1 \in J$ or $1 \notin J$. Moreover,

$$1_{W_{I-J'}}^{W} = [B_{m_0}][A_{m_1}] \cdots [A_{m_r}],$$

where either (1) $m_0 = i_2 - 1$, $m_j = i_{2j+2} - i_{2j} - 1$ $(0 < j < r)$ and $m_r = n - i_{2r}$ (if $1 \in J$), or (2) $m_0 = i_1 - 1$, $m_j = i_{2j+1} - i_{2j-1} - 1$ (if $1 \notin J$). If we fix m_j, then there are $m_1 \cdots m_{r-1}$ possible choices for J with $1 \in J$ and $m_1 \cdots m_{r-1}(m_r - 1)$ with $1 \notin J$, for a total of $m_1 \cdots m_r$ choices. Hence this case also yields an amount equal to χ_{I}^{B}.

Case III. $1, n+1 \in J$, $n \notin J$. We have $J' = \{i_1, i_3, ..., i_{2r+1}\}$, and

$$1^W_{W_{I-J'}} = [A_{m_1}] \cdots [A_{m_r}],$$

where $m_j = i_{2j+1} - i_{2j-1} - 1$. If we fix m_j, then there are $m_1 \cdots m_{r-1}(m_r - 1)$ choices for J, so by (8.2c), the total contribution of this case to (8.3) equals χ^B_{III}.

Case IV. $1, n, n+1 \in J$. We have $J' = \{i_2, i_4, ..., i_{2r+1}\}$. Since removal of n and $n+1$ leaves a graph of type A_0, we obtain

$$1^W_{W_{I-J'}} = [B_{m_0}][A_{m_1}] \cdots [A_{m_{r-1}}][A_0],$$

where $m_0 = i_2 - 1$, and $m_j = i_{2j+2} - i_{2j} - 1$ $(j > 0)$. If we fix m_j, then there are $m_1 \cdots m_{r-1}$ choices for J, so by (8.2b), this case yields an amount equal to χ^B_{II}.

In summary, this analysis shows that the terms of (8.3) are in one-to-one correspondence with the terms in the expansion of χ^{B_n} in (8.2). ∎

9. The D-Series

Now consider the root system $D_n = \{\pm\varepsilon_i \pm \varepsilon_j : 1 \leqslant i < j \leqslant n\}$ in $V = \mathbf{R}^n$. We sometimes include D_2 and D_3 in the analysis, even though D_2 is reducible and $D_3 \cong A_3$. The Weyl group $W(D_n)$ acts as the group of signed permutations of $\varepsilon_1, ..., \varepsilon_n$ with even numbers of sign changes. For the base $S = \{\alpha_1, ..., \alpha_n\}$, we take $\alpha_1 = \varepsilon_1 + \varepsilon_2$ and $\alpha_i = \varepsilon_{i+1} - \varepsilon_i$ for $i > 1$. For $n \geqslant 3$, the highest root is $\varepsilon_{n-1} + \varepsilon_n$.

Unlike the other classical root systems, D_n has the property that the subcomplexes $\Delta^w_{D_n}$ need not be isomorphic to Coxeter complexes of lower rank. The more general structures that arise in this way form a two-parameter family of complexes that were first considered by Zaslavsky [Za]. To describe these complexes, let l be an integer in the range $0 \leqslant l \leqslant n$, and consider the simplicial hyperplane arrangement

$$\{\varepsilon_i^\perp : 1 \leqslant i \leqslant l\} \cup \{\alpha^\perp : \alpha \in D_n\}.$$

These hyperplanes define a complete simplicial fan $\Phi_{D^l_n}$ in V; we use $\Delta_{D^l_n}$ to denote the associated simplicial complex. Note that for $l = 0$ and $l = n$ we obtain the Coxeter complexes for D_n and C_n, respectively.

Lemma 9.1. *For $n \geqslant 2$, we have*

$$P_{D^l_n}(q) = P_{C_n}(q) - 2^{n-1}(n-l)\, qP_{A_{n-2}}(q).$$

Proof. Recall that the chambers $C_{\gamma,w}$ of Φ_{C_n} are indexed by the permutations w of $\{1, ..., n\}$ and n-tuples $\gamma = (\gamma_1, ..., \gamma_n)$ such that $\gamma_i = \pm 1$ (cf. (7.1)). The chambers of $\Phi_{D_n^l}$ are of two types: (a) the C_n-chambers $C_{\gamma,w}$ with $w(1) \leqslant l$, and (b) for each $(n-1)$-tuple $\gamma = (\gamma_2, ..., \gamma_n)$ and choice of w such that $w(1) > l$, the chambers

$$C'_{\gamma,w} := C_{\gamma^+,w} \cup C_{\gamma^-,w} = \left\{ \sum c_i \varepsilon_i \in V : |c_{w(1)}| \leqslant \gamma_2 c_{w(2)} \leqslant \cdots \leqslant \gamma_n c_{w(n)} \right\},$$

where $\gamma^{\pm} = (\pm 1, \gamma_2, ..., \gamma_n)$. Note that the bounding hyperplanes for $C'_{\gamma,w}$ consist of $(\varepsilon_{w(2)} \pm \varepsilon_{w(1)})^{\perp}$ and $(\gamma_i \varepsilon_{w(i)} - \gamma_{i-1} \varepsilon_{w(i-1)})^{\perp}$ for $i > 2$.

Consider now the problem of enumerating the number of k-dimensional cones C of $\Phi_{D_n^l}$ that are confined to exactly j of the coordinate hyperplanes $\varepsilon_i^{\perp}$. For $j \geqslant 2$, the faces of $C'_{\gamma,w}$ with this property are also faces of Φ_{C_n}. From the proof of Proposition 7.1, we already know that there are $2^{n-j}\binom{n}{j}\, k!\, S(n-j, k)$ such cones.

If $j = 1$, then C cannot be a face of $C'_{\gamma,w}$. Returning to the combinatorial indexing of the cones of Φ_{C_n} (cf. the proof of Proposition 7.1), we see that the cones of this type are in one-to-one correspondence with signed, ordered partitions of $\{1, ..., n\}$ into $k+1$ nonempty blocks in which the first block is a singleton $\{i\}$ such that $1 \leqslant i \leqslant l$, and the elements of each remaining block have a sign attached to them. Hence there are $2^{n-1} l \cdot k!\, S(n-1, k)$ cones of this type.

Now consider the case $j = 0$. There are a total of $2^n k!\, S(n, k)$ cones of this type in Φ_{C_n}, but some of these may not be cones of $\Phi_{D_n^l}$. The problem arises from the faces of $C'_{\gamma,w}$ that are not orthogonal to either of the roots $\varepsilon_{w(2)} \pm \varepsilon_{w(1)}$. These faces are unions of pairs of cones of Φ_{C_n}; they correspond to the signed, ordered partitions of $\{1, ..., n\}$ whose first block is a singleton $\{i\}$ with $i > l$ and such that there are signs attached to each of the n elements. The pairs of cones that are joined in $\Phi_{D_n^l}$ are the ones whose corresponding partitions are related by interchanging the sign attached to the singleton i. Since there are $2^{n-1}(n-l)(k-1)!\, S(n-1, k-1)$ such pairs of partitions, it follows that there are a total of $2^n k!\, S(n, k) - 2^{n-1}(n-l)(k-1)!\, S(n-1, k-1)$ cones of this type.

Combining the number of cones in these three cases and comparing the result with (6.1) and Proposition 7.1(a) yields

$$\begin{aligned} f_{k-1}(\Delta_{C_n}) - f_{k-1}(\Delta_{D_n^l}) &= 2^{n-1}(n-l)\, k!\, S(n-1, k) + 2^{n-1}(n-l)(k-1)!\, S(n-1, k-1) \\ &= 2^{n-1}(n-l)[f_{k-2}(\Delta_{A_{n-2}}) + f_{k-3}(\Delta_{A_{n-2}})]. \end{aligned}$$

Applying (1.1), we obtain

$$P_{C_n}(q) - P_{D_n^l}(q) = 2^{n-1}(n-l)[q(1-q)\, P_{A_{n-2}}(q) + q^2 P_{A_{n-2}}(q)]. \quad \blacksquare$$

We remark that the conjugacy classes of $W(C_n)$ that belong to $W(D_n)$ are the ones indexed by the partition-pairs (μ, ν) with $l(\nu)$ even. For even n, these classes are coarser than the conjugacy classes of $W(D_n)$ itself (cf. [Ca, §7]).

THEOREM 9.2. *If* $w \in W(D_n)$ *is of type* (μ, ν) *and* $k = |\{i : \mu_i \geqslant 2\}|$, *then*

$$\chi[D_n, q](w) = \begin{cases} P_{C_{l(\mu)}}(q) \prod_i \frac{(1-q^{\mu_i})}{(1-q)} \prod_j (1+q^{\nu_j}) & \text{for} \quad \nu \neq \varnothing, \\ P_{D^k_{l(\mu)}}(q) \prod_i \frac{(1-q^{\mu_i})}{(1-q)} & \text{for} \quad \nu = \varnothing. \end{cases}$$

Proof. First consider the case $\nu \neq \varnothing$. Since the only invariants of negative cycles in V are 0, it follows that in this case, V_w lies on one or more of the coordinate hyperplanes $\varepsilon_i^\perp$. However, it is easy to see that the restrictions of both Φ_{D_n} and Φ_{C_n} to $\varepsilon_i^\perp$ are isometric. Thus by (7.4), we have

$$\Delta^w_{D_n} \cong \Delta^w_{C_n} \cong \Delta_{C_{l(\mu)}}.$$

The first of the two claimed formulas is now a consequence of (7.3) and Theorem 1.4.

Now consider the case $\nu = \varnothing$. For this it is better to first consider an example; say, $w = (1, -2, -3, 4)(5, 6)(7)$. We have $(\mu, \nu) = (421, \varnothing)$, and a basis for V_w consists of the vectors $\varepsilon'_1 = \varepsilon_1 - \varepsilon_2 + \varepsilon_3 + \varepsilon_4$, $\varepsilon'_2 = \varepsilon_5 + \varepsilon_6$, and $\varepsilon'_3 = \varepsilon_7$. Note that the restrictions of the hyperplanes $(\varepsilon_1 \pm \varepsilon_5)^\perp$ and $(\varepsilon_1 - \varepsilon_2)^\perp$ to V_w are (with complements taken relative to V_w) $(\varepsilon'_1 \pm \varepsilon'_2)^\perp$ and $(\varepsilon'_1)^\perp$, respectively. More generally, it is not hard to see that the set of all hyperplanes in V_w obtainable by restriction this way are of the form $(\varepsilon'_i \pm \varepsilon'_j)^\perp$ $(i<j)$, together with $(\varepsilon'_1)^\perp$ and $(\varepsilon'_2)^\perp$. In other words, $\Delta^w_{D_7} \cong \Delta_{D^2_3}$.

In the general case, there is a natural basis of w-invariant vectors $\varepsilon'_1, ..., \varepsilon'_l$, where $l = l(\mu)$ is the number of positive cycles of w. There is a root $\alpha \in D_n$ such that the restriction of $\alpha^\perp$ to V_w is one the hyperplanes $(\varepsilon'_i)^\perp$ if and only if the cycle of w corresponding to ε'_i has length at least 2. The remaining hyperplanes are of the form $(\varepsilon'_i \pm \varepsilon'_j)^\perp$, so $\Delta^w_{D_n} \cong \Delta_{D^k_l}$, where k is the number of non-fixed points of w. The second of the claimed formulas now follows from (7.3) and Theorem 1.4. ∎

There is an obvious analogue of the graded character rings $\mathscr{R}_A$ and $\mathscr{R}_C$ for the D-series; namely, $\mathscr{R}_D = \bigoplus_{n \geqslant 0} \mathscr{R}^n_D$, where $\mathscr{R}^n_D$ is the K-vector space spanned by the characters of $W(D_n)$, and the multiplication is obtained by the induction of outer tensor products from $W(D_n) \times W(D_m)$ to $W(D_{m+n})$. We should note that for the purposes of this definition, we may regard $W(D_1)$ as the trivial group and treat $\mathscr{R}^0_D$ as a one-dimensional space containing a unit element for $\mathscr{R}_D$. We should also point out that all of the embeddings $W(D_n) \times W(D_m) \subsetneq W(D_{m+n})$ obtained by partitioning the basis vectors ε_i into two sets are easily seen to be conjugate, so the product is well-defined.

For $\chi \in \mathscr{R}_D$, let $\chi \mapsto \hat{\chi}$ denote the (graded) ring homomorphism $\mathscr{R}_D \to \mathscr{R}_C$ provided by induction of characters, and recall that δ denotes the involution on $\mathscr{R}_C$ that corresponds to tensoring by the sign character induced by $W(D_n)$. For $n \geqslant 1$, let $s \in W(C_n)$ denote reflection by the root $2\varepsilon_1$. Since conjugation by s is an automorphism of $W(D_n)$, it follows that if we define $\chi^\sigma(w) := \chi(sws)$ for $\chi \in \mathscr{R}_D^n$ and $w \in W(D_n)$, then $\chi \mapsto \chi^\sigma$ defines a K-linear involution on $\mathscr{R}_D$ (but not a ring homomorphism).

PROPOSITION 9.3. *Assume* $\chi \in \mathscr{R}_D^n$, $n \geqslant 1$.

(a) $\chi = \chi^\sigma$ *if and only if* χ *is the restriction of some* $\varphi \in \mathscr{R}_C^n$.

(b) *If* $\chi = \chi^\sigma$ *and* χ *is the restriction of* $\varphi \in \mathscr{R}_C^n$, *then* $\hat{\chi} = \varphi + \delta\varphi$.

(c) *The map* $\chi \mapsto \hat{\chi}$ *is one-to-one on* $\{\chi \in \mathscr{R}_D^n : \chi^\sigma = \chi\}$.

(d) *As graded rings,* $\mathscr{R}_D \cong \mathscr{R}_C^\delta \oplus \mathscr{R}_A^+$, *where* $\mathscr{R}_A^+$ *denotes the ideal of* $\mathscr{R}_A$ *generated by homogeneous elements of positive degree, regraded by doubling degrees.*

Proof. (a) The effect of σ is to interchange the values of χ on the pairs of conjugacy classes of $W(D_n)$ that are obtained by splitting conjugacy classes of $W(C_n)$. Hence χ is σ-invariant if and only if it is constant on conjugacy classes of $W(C_n)$.

(b) It suffices to assume that φ is an irreducible character. If $\varphi \neq \delta(\varphi)$, then by standard techniques of Clifford theory (e.g., [CR, §11] or [Ste1, §6A]), χ is also irreducible; otherwise, if $\varphi = \delta\varphi$, then $\chi = \chi^+ + \chi^-$, where χ^+ and χ^- are two irreducible, inequivalent $W(D_n)$-characters. Thus by Frobenius reciprocity, if $\varphi \neq \delta(\varphi)$, then $\hat{\chi} = \varphi + \delta(\varphi)$, and if $\varphi = \delta(\varphi)$, $\hat{\chi}^\pm = \varphi$ and $\hat{\chi} = 2\varphi = \varphi + \delta(\varphi)$.

(c) From (b), we have $\hat{\chi}(w) = 2\chi(w)$ for all $w \in W(D_n)$.

(d) Since σ is an involution, we have $\mathscr{R}_D \cong I^+ \oplus I^-$ (as vector spaces), where I^+ and I^- denote the eigenspaces of σ with eigenvalues $+1$ and -1. Depending on how the embedding of $W(D_m) \times W(D_n)$ in $W(D_{m+n})$ is chosen, s will commute with either $W(D_m)$ or $W(D_n)$, and therefore

$$(\chi_1 \cdot \chi_2)^\sigma = \chi_1^\sigma \cdot \chi_2 = \chi_1 \cdot \chi_2^\sigma,$$

provided that χ_1 and χ_2 are homogeneous and of positive degree in $\mathscr{R}_D$. This shows that both eigenspaces of σ are ideals of $\mathscr{R}_D$, so we have $\mathscr{R}_D \cong I^+ \oplus I^-$ as graded rings. From (b) and (c), it is clear that $\chi \mapsto \hat{\chi}$ is an isomorphism from I^+ onto the δ-invariant subring $\mathscr{R}_C^\delta$. The fact that $I^- \cong \mathscr{R}_A^+$ can be deduced from Theorem 7.5 of [Ste1]. ∎

Let R be a root subsystem of D_n, let W be the reflection subgroup it generates, and let $\Gamma(W)$ be the corresponding graph, as defined in Section 7. Note that the connected components of $\Gamma(W)$ must be of types A and D. The *parity* of R (or W) is defined to be 0, 1, or -1 as follows. If $\Gamma(W)$ is connected and of type D_n ($n \geqslant 2$) or A_{2n} ($n \geqslant 0$), then the parity is zero. If $\Gamma(W)$ is of type A_{2n-1} ($n \geqslant 1$) then the parity is defined to be $(-1)^m$, where m is the number of roots of the form $\varepsilon_i + \varepsilon_j$ in R. Finally, if $\Gamma(W)$ is disconnected, the parity is defined to be the product of the parities of the irreducible components of R.

LEMMA 9.4. *Two reflection subgroups of $W(D_n)$ are conjugate if and only if their graphs are isomorphic and they have the same parity.*

Proof. For $1 \leqslant i \leqslant n$, let $t_i \in W(C_n)$ denote reflection by the root $2\varepsilon_i$. If two reflection subgroups of $W(D_n)$ are conjugate, then they are also conjugate in $W(C_n)$, so by Lemma 7.3 they must have isomorphic graphs. Furthermore, if R is any root subsystem of D_n, then conjugation by t_i preserves the parity of all irreducible components of R, except possibly the component that is not orthogonal to ε_i. If this component has a graph of type D_m or A_{2m} then the parity is zero, so in these cases parity is preserved even up to conjugacy in $W(C_n)$. However, if the component is of type A_{2m-1}, then the effect of conjugation by t_i is to interchange the presence or absence of $2m-1$ roots of the form $\varepsilon_i + \varepsilon_j$ in R. Hence this changes the parity of one component, so conjugation by an even number of the reflections t_i (and hence by all of $W(D_n)$) will therefore preserve parity.

For the converse, let W be a reflection subgroup of $W(D_n)$. We seek to prove that all other reflection subgroups of $W(D_n)$ with the same parity and having graphs isomorphic to $\Gamma(W)$ must be conjugate to W. By Lemma 7.3, such groups must certainly be conjugate by some element $x \in W(C_n)$. If $\Gamma(W)$ has any component of type D_m or A_{2m}, then the normalizer of W in $W(C_n)$ includes elements not in $W(D_n)$ (in the former case, reflections of the form t_i; in the latter case, a product of $2m+1$ such reflections). Therefore, even if $x \notin W(C_n)$, there will still exist a suitable $x_0 \in W(D_n)$ such that $xWx^{-1} = x_0 W x_0^{-1}$. Otherwise, if every component of $\Gamma(W)$ is of type A_{2m-1} for some m, then the parity of W is ± 1, and the argument of the previous paragraph shows that conjugation by t_i changes the parity of W. Therefore in this case, xWx^{-1} will be a $W(D_n)$-conjugate of W if and only if $x \in W(D_n)$. ∎

Proof of Theorem 4.3. (a) If two reflection subgroups induce the same permutation character in $W(D_n)$, then they will also induce the same permutation character in $W(C_n)$. Hence by the proof of Theorem 4.3(a) that we gave for the C-series, the subgroups must be conjugate in $W(C_n)$; in particular, they must have isomorphic graphs. Since the property of having parity 0 depends only on the graph, Lemma 9.4 implies that the only possible counterexamples must involve a pair of reflection subgroups with opposite (nonzero) parity and isomorphic graphs, with the connected components being of type $A_{2\mu_1-1}, ..., A_{2\mu_l-1}$ for some partition μ.

The permutation representations of $W(D_n)$ induced by the reflection subgroups with graphs having the above type can be obtained from the action of $W(D_n)$ on the orbit generated by a vector $v=\sum c_i \varepsilon_i \in V$ such that $|\{i : |c_i| = j\}| = 2\mu_j$ for $j = 1, ..., l$. The parity of the isotropy group of v is the product of the signs of the c_i's. If we choose v so that $c_i > 0$ for all i, then there will exist a signless permutation w_μ of cycle-type 2μ that fixes 2^l elements in the orbit of v. On the other hand, if we change v by substituting $c_1 \to -c_1$, then every element in the orbit of v has an odd number of negative coordinates. For such vectors there must exist an integer j such that both j and $-j$ occur as coordinates; however, no such vector can be fixed by w_μ. Hence, subgroups of opposite parity do not induce the same character.

(b) We first claim that the (distinct) permutation characters of $W(C_n)$ induced by reflection subgroups of $W(D_n)$ are linearly independent. To prove this, first observe that these characters are monomials in the variables $[D_m]$ $(m \geqslant 2)$ and $[A_{m-1}]$ $(m \geqslant 1)$. Now recall from Proposition 7.4 that $[A_{m-1}] = a_m$ and $[D_m] = h_m + g_m$, where $a_m = \sum_k h_k g_{n-k}$. Since $h_1, a_1, h_2, a_2, ...$ are algebraically independent generators of $\mathscr{R}_C$, we may regard g_m and $[D_m]$ as polynomial functions of $h_1, a_1, h_2, a_2, ...$ From this point of view, we seek to prove that $[D_2], [D_3], ...$ are algebraically independent over $K[a_1, a_2, ...]$.

For this we define a total ordering on the monomials h_λ that form a basis for $\mathscr{R}_C$ over $K[a_1, a_2, ...]$. We say that h_λ precedes h_μ provided that either (1) $|\lambda| < |\mu|$ or (2) $|\lambda| = |\mu|$, and λ precedes μ in lexicographic order.

For $m \geqslant 2$, the highest order terms of g_m with respect to this order are $-h_m + h_{m-1}h_1$, so the highest monomial in $[D_m]$ is $h_{m-1}h_1$. More generally, it follows that if μ is any partition with parts $\geqslant 2$, then the highest monomial in $[D_{\mu_1}] \cdots [D_{\mu_l}]$ is h_λ, where $\lambda = (\mu_1 - 1, ..., \mu_l - 1, 1, ..., 1)$ (a partition of length $2l$). Since the mapping $\mu \mapsto \lambda$ is injective, we may conclude that $[D_2], [D_3], ...$ are indeed algebraically independent, so the claim follows.

To complete the proof, note that this argument shows that if there were a nontrivial dependence relation in $\mathscr{R}_D$ involving permutation characters induced by reflection subgroups of $W(D_n)$, then there would have to be one involving a set of $W(D_n)$-characters that are preimages of a single $W(C_n)$-

character χ. However, the only nonconjugate reflection subgroups of $W(D_n)$ that induce the same permutation character in $W(C_n)$ are the pairs with isomorphic graphs and opposite (nonzero) parity, thanks to the algebraic independence of $[D_m]$ and $[A_{m-1}]$. Such pairs of characters have the same degree, and by part (a) they are distinct, so there is no dependence relation between any such pair. ∎

As a corollary of Proposition 9.3 and the above proof, we obtain

PROPOSITION 9.5. *If $\chi \in \mathscr{R}_D^n$ and $\chi = \chi^\sigma$, then χ is the character of a permutation representation whose isotropy groups are reflection subgroups of $W(D_n)$ if and only if*

(1) *$\hat{\chi}$ is a polynomial function of $[D_m]$ $(m \geq 2)$ and $[A_{m-1}]$ $(m \geq 1)$ with nonnegative integer coefficients, and*

(2) *for each partition μ, the coefficient of $[A_{2\mu_1-1}] \cdots [A_{2\mu_l-1}]$ in $\hat{\chi}$ is even.*

For $0 \leq k \leq n$, let $[D_{k,n-k}] \in \mathscr{R}_C^n$ denote the permutation character of $W(C_n)$ induced by $W(D_n) \cap (W(C_k) \times W(C_{n-k}))$. Note that these subgroups are not reflection groups. The following result provides a decomposition of the graded $W(C_n)$-character of $K[\Delta_{D_n}]/\Theta$ obtained by induction from $W(D_n)$.

THEOREM 9.6. *We have*

$$\sum_{n \geq 2} \hat{\chi}[D_n, q]\, t^n = \frac{D(q, t)}{1 - \sum_{m \geq 1} (q + \cdots + q^m)[A_m]\, t^{m+1}},$$

where

$$\begin{aligned} D(q, t) = {} & [D_2](1+q)^2 t^2 + \sum_{m \geq 3} \sum_{0 \leq k \leq m} [D_{k,m-k}]\, q^k t^m \\ & + 2 \sum_{m \geq 2} (q + \cdots + q^m)[A_m]\, t^{m+1} \\ & + [A_0]\, t \sum_{m \geq 2} (q^2 + \cdots + q^m)[A_m]\, t^{m+1}. \end{aligned}$$

Proof. By Theorem 9.2, we see that $\chi[D_n, q]$ is σ-invariant (i.e., constant on the conjugacy classes of $W(C_n)$), so by Proposition 9.3(b) we have $\hat{\chi}[D_n, q](w) = 2\chi[D_n, q](w)$ for $w \in W(D_n)$ (and 0 otherwise). Thus if we define $\varphi_n \in \mathscr{R}_C^n$ so that

$$\chi[C_n, q] + \delta\chi[C_n, q] = \hat{\chi}[D_n, q] + \varphi_n, \tag{9.1}$$

then Lemma 9.1, Theorem 9.2, and Corollary 7.2 collectively imply that $\varphi_n(w)=0$ unless w is of type $(\mu, \varnothing)$, and in that case,

$$\begin{aligned}\varphi_n(w) &= 2[P_{C_{l(\mu)}}(q) - P_{D^k_{l(\mu)}}(q)] \prod_i \frac{1-q^{\mu_i}}{1-q} \\ &= 2^{l(\mu)}(l(\mu)-k(\mu))\, qP_{A_{l(\mu)-2}}(q) \prod_i \frac{1-q^{\mu_i}}{1-q},\end{aligned}$$

where $k=k(\mu)=|\{i:\mu_i \geqslant 2\}|$. Note that the above expression is nonzero only if μ has at least one part equal to 1. In that case, we have $(l(\mu)-k(\mu))/z_\mu = 1/z_\lambda$, where λ is the partition obtained by deleting a 1 from μ. Thus by (7.5) and (6.2), we obtain

$$\begin{aligned}\sum_{n\geqslant 1} \varphi_n t^n &= 2q(1-q)\, p_1^+ t \sum_\lambda \frac{2^{l(\lambda)} t^{|\lambda|}}{z_\lambda} \frac{P_{A_{l(\lambda)-1}}(q)}{(1-q)^{l(\lambda)+1}} \prod_i (1-q^{\lambda_i})\, p_{\lambda_i}^+ \\ &= 2q(1-q)\, p_1^+ t \sum_{k\geqslant 0} q^k \sum_\lambda \frac{t^{|\lambda|}}{z_\lambda} \prod_i (2k+2)(1-q^{\lambda_i})\, p_{\lambda_i}^+ .\end{aligned}$$

Hence by (7.6a), Proposition 7.5(b), and the fact that $2p_1^+ = [A_0]$, we obtain

$$\begin{aligned}\sum_{n\geqslant 1} \varphi_n t^n &= q(1-q)[A_0]\, t \sum_{k\geqslant 0} q^k [H_A(t)/H_A(qt)]^{k+1} \\ &= \frac{q(1-q)[A_0]\, tH_A(t)}{H_A(qt)-H_A(t)} \\ &= \frac{[A_0]\, qt(1+\sum_{m\geqslant 0}[A_m]\, t^{m+1})}{1-\sum_{m\geqslant 1}(q+\cdots+q^m)[A_m]\, t^{m+1}}. \qquad (9.2)\end{aligned}$$

Next we observe that $[D_{k,n-k}]$ can be obtained by first restricting $[C_k][C_{n-k}]$ to $W(D_n)$, and then inducing back to $W(C_n)$. Thus by Proposition 9.3(b), we have $[D_{k,n-k}] = h_k h_{n-k} + g_k g_{n-k}$. Now we apply Theorem 7.6 to obtain

$$\begin{aligned}&\sum_{n\geqslant 0} (\chi[C_n, q] + \delta\chi[C_n, q])\, t^n \\ &\quad = \frac{(1-q)\sum_{0\leqslant k\leqslant m}(h_k h_{m-k} + g_k g_{m-k})\, q^k t^m}{H_A(qt)-H_A(t)} \\ &\quad = \frac{\sum_{0\leqslant k\leqslant m}[D_{k,m-k}]\, q^k t^m}{1-\sum_{m\geqslant 1}(q+\cdots+q^m)[A_m]\, t^{m+1}}. \qquad (9.3)\end{aligned}$$

It is a routine execise to use (9.1), (9.2), and (9.3) to derive an expression for the formal series $\sum_{n\geq 2}\hat{\chi}[D_n, q]\,t^n$. Note that (9.2) is of the form $[A_0]qt+O(t^2)$ and (9.3) is of the form $2+[A_0](1+q)t+O(t^2)$. These low-order terms must be removed in order to obtain the claimed expansion, since it begins with terms of order t^2. ∎

Remark 9.7. (a) This result shows that $K[\Delta_{D_n}]/\Theta$ carries the structure of a graded permutation representation of $W(D_n)$. Indeed, by Proposition 9.3(c), it is enough to show that $\hat{\chi}[D_n, q]$ can be obtained by inducing a σ-invariant permutation character of $W(D_n)$; this is easy to see directly from the above expansion of $\hat{\chi}[D_n, q]$.

(b) There is no graded version of Theorem 4.2 for the D-series. If the character of the homogeneous component of degree 1 in $K[\Delta_{D_n}]/\Theta$ were indeed a permutation character whose isotropy subgroups were reflection subgroups, then by Theorem 9.6, $[D_{n-1,1}]$ would have to be a polynomial function of $[D_m]$ and $[A_{m-1}]$. However, using techniques similar to those employed in the proof of Theorem 4.3(b), one can show that for $n\geq 4$, $[D_{n-1,1}]$ cannot be expressed in such a form. (In particular, $\mathscr{R}_D^n$ is not spanned by the permutation characters induced by reflection subgroups.)

In the following, it is convenient to set $[D_1]:=[A_0]=h_1+g_1$.

Proof of Theorem 4.2. If we set $q=1$ in Theorem 9.6 and apply the identity

$$\begin{aligned}\sum_{k=0}^{n}[D_{k,n-k}] &= \sum_{k=0}^{n} h_k h_{n-k} + g_k g_{n-k}\\ &= \sum_{k=0}^{n} (h_k+g_k)(h_{n-k}+g_{n-k}) - 2h_k g_{n-k}\\ &= 4[D_n]-2[A_{n-1}]+\sum_{k=1}^{n-1}[D_k][D_{n-k}],\end{aligned}$$

then we obtain $\sum_{n\geq 2}\hat{\chi}^{D_n}t^n = D(t)(1-\sum_{m\geq 1} m[A_m]\,t^{m+1})^{-1}$, where

$$\begin{aligned}D(t) = 4\sum_{m\geq 2}[D_m]\,t^m &+ \sum_{m\geq 3}\sum_{0<k<m}[D_k][D_{m-k}]\,t^m\\ &+ (2+[A_0]t)\sum_{m\geq 2}(m-1)[A_m]\,t^{m+1}.\end{aligned}$$

In this form, it is evident that χ^{D_n} is the character of a permutation representation whose isotropy groups are reflection subgroups of $W(D_n)$ (cf. Proposition 9.5). Furthermore, these isotropy groups have graphs with at most two components of type D. On the other hand, by inspection of the extended diagram of D_n in Appendix 1 (for $n \geqslant 4$), one sees that the graphs of the quasi-parabolic subgroups of $W(D_n)$ are characterized by this property. ∎

If we expand the above series for $\hat{\chi}^{D_n}$ in detail, we obtain

$$\hat{\chi}^{D_n} = 4\chi_{\mathrm{I}}^{D} + \chi_{\mathrm{II}}^{D} + 2\chi_{\mathrm{III}}^{D} + \chi_{\mathrm{IV}}^{D},$$

where

$$\chi_{\mathrm{I}}^{D} = \sum_{r \geqslant 0} \sum_{m_0 + \cdots + m_r = n-r} m_1 \cdots m_r [D_{m_0}][A_{m_1}] \cdots [A_{m_r}], \tag{9.4a}$$

$$\chi_{\mathrm{II}}^{D} = \sum_{r \geqslant 0} \sum_{k+k'+m_1+\cdots+m_r = n-r} m_1 \cdots m_r [D_k][D_{k'}] \times [A_{m_1}] \cdots [A_{m_r}], \tag{9.4b}$$

$$\chi_{\mathrm{III}}^{D} = \sum_{r \geqslant 1} \sum_{m_1 + \cdots + m_r = n-r} (m_1 - 1)\, m_2 \cdots m_r [A_{m_1}] \cdots [A_{m_r}], \tag{9.4c}$$

$$\chi_{\mathrm{IV}}^{D} = \sum_{r \geqslant 2} \sum_{m_1 + \cdots + m_{r-1} = n-r} (m_1 - 1)\, m_2 \cdots m_{r-1} [A_0] \times [A_{m_1}] \cdots [A_{m_{r-1}}], \tag{9.4d}$$

where $k, k', m_1, ..., m_r \geqslant 1$, $m_0 \geqslant 2$, and $k + k' \geqslant 3$.

Proof of Theorem 4.5. Assume $n \geqslant 4$. Let $I = \{1, ..., n+1\}$, and for $J \subset I$, write W_J for the subgroup of $W = W(D_n)$ generated by $\{s_j : j \in J\}$, where s_{n+1} denotes reflection by the highest root. For each odd subset $J = \{i_1 < i_2 < \cdots < i_{2r+1}\}$ of I, let $J_0 = \{1, 2, n, n+1\} \cap J$, and define J to be of *class 1* if $J_0 = \{1, n\}$, $\{2, n\}$, $\{1, 2, n\}$, $\{1, 2, n+1\}$, or $\{1, 2, n, n+1\}$; otherwise, define J to be of *class 2*. Now define

$$J' := \begin{cases} \{i_1, i_2, i_4, ..., i_{2r}\} & \text{if } J \text{ is of class 1,} \\ \{i_1, i_3, ..., i_{2r-1}, i_{2r+1}\} & \text{if } J \text{ is of class 2.} \end{cases}$$

We claim that

$$\hat{\chi}^{D_n} = \sum_{J \subset I:\, |J| \text{ odd}} 1_{W_{I-J'}}^{\hat{W}}, \tag{9.5}$$

where $\hat{W} = W(C_n)$. To prove this, we break the sum into four parts identical to (9.4a)–(9.4d). The reader is advised to refer to the extended diagram in Appendix 1 for what follows.

Case I. $J_0 = \{1, 2, n, n+1\}$. In this case, J is of class 1. Since removal of the first two indices leaves a graph of type A_0, we obtain

$$1_{W_{I-J'}}^{\hat{W}} = [A_0][A_{m_1}] \cdots [A_{m_{r-1}}],$$

where $m_j = i_{2j+2} - i_{2j} - 1$ for $j < r-1$ and $m_{r-1} = i_{2r} - i_{2r-2}$. If we fix the parameters m_j, then there will be $m_1 \cdots m_{r-2}(m_{r-1}-1)$ choices for J, so by (9.4d) this case yields an amount equal to χ^D_{IV}.

Case II. $J_0 = \varnothing$, $\{1, 2, n\}$, or $\{1, 2, n+1\}$. If $J_0 = \varnothing$, then J is of class 2, and

$$1^{\hat{W}}_{W_{I-J'}} = [D_k][D_{m_0-k}][A_{m_1}] \cdots [A_{m_r}],$$

where $k = i_1 - 1 \geqslant 2$, $m_0 - k = n + 1 - i_{2r+1} \geqslant 2$, and $m_j = i_{2j+1} - i_{2j-1} - 1$ for $j > 0$. If we fix k and m_j, then there are $m_1 \cdots m_r$ choices for J. Otherwise, in the latter two cases, J is of class 1 and one of the components of $W_{I-J'}$ is of type A_0, yielding

$$1^{\hat{W}}_{W_{I-J'}} = [A_0][D_{m_0-1}][A_{m_1}] \cdots [A_{m_{r-1}}],$$

where $m_0 = n + 2 - i_{2r} \geqslant 3$ and $m_j = i_{2j+2} - i_{2j} - 1$ for $j > 0$. If we fix m_j, then there are $m_1 \cdots m_{r-1}$ choices for J and 2 choices for J_0. Since $[A_0] = [D_1]$, it follows by (9.4b) that the total contribution of these subcases to (9.5) equals χ^D_{II}.

Case III. $J_0 = \{1, n+1\}$, $\{2, n+1\}$, $\{1, n, n+1\}$, or $\{2, n, n+1\}$. In these cases, J is of class 2, and

$$1^{\hat{W}}_{W_{I-J'}} = [A_{m_1}] \cdots [A_{m_r}],$$

where $m_1 = i_3 - 2$, $m_r = n - i_{2r-1}$, and $m_j = i_{2j+1} - i_{2j-1} - 1$ $(1 < j < r)$. If we fix m_j, then there are a total of $2(m_1 - 1) m_2 \cdots m_{r-1}(m_r - 1)$ choices for J in the first two cases, and $2(m_1 - 1) m_2 \cdots m_{r-1}$ choices in the latter two cases, for a total of $2(m_1 - 1) m_2 \cdots m_r$ choices. Hence by (9.4c), the total contribution of this case to (9.5) equals $2\chi^D_{\mathrm{III}}$.

Case IV. $|J_0| = 1$ or 2, and $J_0 \neq \{1, n+1\}$, $\{2, n+1\}$. In these cases, J is of class 2, unless $J_0 = \{1, n\}$. Regardless of which alternative occurs, we have

$$1^{\hat{W}}_{W_{I-J'}} = [D_{m_0}][A_{m_1}] \cdots [A_{m_r}],$$

for suitable parameters $m_0 \geqslant 2$ and $m_1, ..., m_r \geqslant 1$ depending on J. If we fix the parameters m_j, and J_0 is one of the four possible singletons, there will be $(m_1 - 1) m_2 \cdots m_r$ choices for J (depending on how the indices m_j are ordered). If J_0 is one of the four possible doubletons, then there will be $m_2 \cdots m_r$ possible choices for J. Thus there are a total of $4 m_1 \cdots m_r$ choices, so by (9.4a) this case yields $4\chi^D_{\mathrm{I}}$.

These four cases complete the proof of (9.5). To prove that this rule for choosing J' from J actually satisfies Theorem 4.5, observe that the reflection subgroups with nonzero parity occur only in Case III. Moreover, the collection of sets of the form $I - J'$ that occur in Case III is stable with respect to interchanging 1 and 2. Therefore $\Sigma_J 1_{W_{I-J'}}^W$ is σ-invariant, so the result follows from (9.5) and Proposition 9.3(c). ∎

10. The Exceptional Cases

Our proofs of the main results for the exceptional root systems are computer-assisted. The programs were written in *Maple* and are freely available through the *Internet* [Ste4]. In this section we describe the algorithms we used, and discuss some details peculiar to the individual cases.

The core of the algorithm is built around the ability to quickly execute the following tasks for arbitrary Weyl groups:

(T1) Determine the conjugacy class that contains a given element w.

(T2) Determine the size of each conjugacy class.

(T3) Produce a representative of each conjugacy class.

For the classical Weyl groups this is a straightforward task, and is perhaps most easily accomplished using permuation representations of degree $n+1$ for $W(A_n)$, and of degree $2n$ for $W(B_n)$ and $W(D_n)$. It is necessary to solve T2 and T3 only once for each exceptional group—the answers can then be stored in arrays. One technique for generating the conjugacy class data is to pass to a permutation representation (e.g., the action of W on the long roots), and then use standard probabilistic algorithms for permutations groups. The results can then be checked against the information in [Ca].

Once the conjugacy class data has been generated, then an effective means for solving T1 can be based largely on characteristic polynomials. Certainly two elements $w_1, w_2 \in W$ can be conjugate only if $\det(1 - qw_1) = \det(1 - qw_2)$. A second necessary condition for conjugacy in Weyl groups with more than one root length is that in any representations of w_1 and w_2 as products of reflections, the number of long roots that occur in both expressions must agree mod 2. Among the exceptional groups, these two criteria suffice to distinguish all conjugacy classes except for two pairs in $W(F_4)$, 6 pairs in $W(E_7)$, and 4 pairs and one triple in $W(E_8)$. In the cases where the test fails, it suffices to pass to a permutation representation of W induced by a parabolic subgroup of maximum possible size—in these representations, the permutations in previously indistinguishable conjugacy classes have different cycle types.

Using T1–T3, it is easy to compute the permutation character of W induced by any reflection subgroup W'. Indeed, for $x \in W$, we have

$$1_{W'}^{W}(x)=\frac{1}{|W'|}|\{w\in W: wxw^{-1}\in W'\}|$$
$$=\frac{|Z(x)|}{|W'|}\cdot|C(x)\cap W'|=\frac{|W|}{|W'|}\cdot\frac{|C(x)\cap W'|}{|C(x)|}, \qquad (10.1)$$

where $Z(x)$ denotes the W-centralizer of x, and $C(x)$ denotes the conjugacy class of x in W. Hence, if $x_1, x_2, \dots$ (resp., $y_1, y_2, \dots$) are a list of conjugacy class representatives for W (resp., W') as produced by T3, then via T1 one can construct the inclusion map $i \mapsto \sigma(i)$ defined by the property that y_i is conjugate in W to $x_{\sigma(i)}$. Once the inclusion map is known, then by (10.1), we have

$$1_{W'}^{W}(x_j)=\frac{|W|}{|W'|}\sum_{i:\sigma(i)=j}\frac{|C'(y_i)|}{|C(x_j)|}, \qquad (10.2)$$

where $C'(y)$ denotes the W'-conjugacy class of y. Given T2, these quantities are easily computable.

For a given Weyl group W with root system R, the algorithm proceeds as follows.

Step 1. Define an equivalence relation on subsets $J \subset S$ by setting $I \sim J$ if and only if the parabolic subgroups W_I and W_J are conjugate. Using T1 and Corollary 2.7, determine the equivalence classes of this relation.

Step 2. For a representative J of each equivalence class of the relation contructed in Step 1, use (10.2) to compute the permutation character of W induced by W_J.

Step 3. Use Proposition 2.2(a), Theorem 1.4, and the permutation characters computed in Step 2 to determine the character χ^R.

In order to verify Theorem 4.3(a), as well as to minimize redundant computations, we need to determine the equivalence relation of conjugacy among the quasi-parabolic subgroups of W. This is more delicate than the relation of Step 1, but as a first approxiimation we can exploit the fact that each proper subset of S_0 is a base for some root subsystem of R.

Step 4. For each n-subset I of S_0, use the methods of Step 1 to determine the equivalence relation on subsets of I corresponding to conjugacy (in W_I) of parabolic subgroups of W_I. Determine the transitive closure of the relation (on subsets of S_0) obtained from the union of these $n+1$ equivalence relations. (Sets belonging to the same equivalence class thus index quasi-parabolic subgroups that are provably conjugate in W.)

Step 5. Use (10.2) to compute the permutation character of W induced by a representative from each equivalence class of the relation constructed in Step 4. List each instance of representatives from distinct classes producing identical characters.

Step 6. Show that the distinct characters obtained in Step 5 are linearly independent, thus verifying Theorem 4.3(b). Use Gaussian elimination to write χ^R as a rational linear combination of these permutation characters. Check that the coefficients are nonnegative integers, thus verifying Theorem 4.2.

To construct a rule satisfying Theorem 4.5, we proceed as follows. Let $\pi_1, ..., \pi_l$ denote the distinct permutation characters computed in Step 5, and let $m_1, ..., m_l$ denote their multiplicities in χ^R, so that $\chi^R = \sum_i m_i \pi_i$. Define $X_i = \{I \subset S_0 : 1_{W_I}^W = \pi_i\}$ for $1 \leqslant i \leqslant l$. Now let $n - r_i$ denote the common value of $|I|$ for $I \in X_i$, and define

$$Y_i = \{J \subset S_0 : |J| = 2r_i + 1, \text{ and } |I \cap J| = r_i \text{ for some } I \in X_i\}.$$

The set Y_i consists of the possible $(2r_i + 1)$-subsets J with the property that there exists an $(r_i + 1)$-subset J' of J such that $1_{W_{S_0 - J'}}^W = \pi_i$. To verify Theorem 4.5, it therefore suffices to exhibit a choice of m_i elements from Y_i ($i = 1, ..., l$) with no repetitions among the choices (i.e., a system of distinct representatives). Since Corollary 3.3 shows that $\sum m_i = 2^n$, this is only possible if every one of the 2^n subsets of odd cardinality is chosen exactly once. Once this is done, a rule for choosing J' from each odd subset J can be obtained by determining which family Y_i is represented by J, and then choosing any $J' \subset J$ such that $2|J'| + 1 = |J|$ and $S_0 - J' \in X_i$. (The existence of such a choice is intrinsic to the definition of Y_i.)

The final step of the algorithm is the following.

Step 7. Use standard augmenting-path techniques (e.g., [LP]) to construct a system of distinct representatives for the family of sets consisting of m_i copies of Y_i for $i = 1, ..., l$, thus verifying Theorem 4.5.

The one remaining issue is the verification of Theorem 4.3(a). In particular, the crucial point is whether the equivalence relation constructed in Step 4 is strong enough to decide which quasi-parabolic subgroups of W induce the same permutation character. In other words, do the members of distinct equivalence classes induce distinct permutation characters of W? For the root systems G_2, F_4, and E_8, the answer turns out to be affirmative, so this verifies Theorem 4.3(a) for these cases.

For the root system E_6, there are 5 triples of equivalence classes that each induce the same permutation character, and for E_7, there are 15 pairs. Thus to verify Theorem 4.3(a) in these cases, we need to prove that these "indistinguishable" classes actually belong to the same conjugacy classes of subgroups of W. To do this, first recall that the isotropy group of any $v \in V$

is the reflection subgroup generated by $\{s_\alpha \in W : \langle \alpha, v \rangle = 0\}$. Moreover, since the W-orbit of v includes (exactly) one point in the fundamental chamber, it follows that these isotropy groups are conjugate to parabolic subgroups. Hence, given the information computed in Step 1, we can easily determine when the isotropy groups of two different vectors are conjugate in W.

To construct a vector v with a prescribed isotropy group is especially simple when the subgroup is generated by a subset of some simple system for W that is prescribed in advance. In such a context, one merely constructs a vector that is orthogonal to the roots of the desired reflections, and not orthogonal to the remaining roots in the given simple system.

Let us say that a simple root $\alpha_i \in S$ is *special* if by removing the corresponding node from the extended diagram of R we obtain a graph isomorphic to the ordinary diagram of R. (This is equivalent to having the corresponding fundamental weight be minuscule.) Note that in such cases $S_0 - \{\alpha_i\}$ forms a simple system for R. For E_6 there are two special roots, and for E_7 there is one. Thus we can supplement Step 4 of the algorithm with the following computation.

Step 4.1. For each special root α_i, construct vectors $v \in V$ whose isotropy groups represent each of the conjugacy classes of parabolic subgroups of W, relative to the simple system $S_0 - \{\alpha_i\}$. (The conjugacy class representatives are available from Step 1.) For each such v, determine the unique vector v_0 in the W-orbit of v that belongs to the fundamental chamber (relative to S), and determine the isotropy group of v_0 (a parabolic subgroup of W relative to S). Add the relation implied by the fact that the isotropy groups of v and v_0 are conjugate in W to the equivalence relation constructed in Step 4.

Once this step is added to the algorithm, we find that the new equivalence relation is able to prove conjugacy for all but one of the indistinguishable triples of equivalence classes for E_6, and all but three of the pairs of E_7.

For the remaining four cases, we argue as follows. Consider the subspaces U of V that are spanned by sets of roots. The Weyl group permutes these spaces, so the isotropy groups of subspaces belonging to the same orbit provide a large class of examples of conjugate subgroups. In particular, the last remaining triple of equivalence classes for E_6 has representatives that can be identified as the isotropy groups for the three lines in V spanned by the two special roots and the highest root. Since the roots form a single orbit, these three classes of groups must indeed be conjugate. For E_7, it is necessary to consider isotropy groups of higher dimensional subspaces.

PROPOSITION 10.1. *Let U be a subspace of V spanned by a subset of R, and let $R' = R \cap U$ be the root subsystem of R in U. If the diagram automorphisms of R' are inner, then the isotropy group of U is the reflection subgroup of W generated by $\{s_\alpha : \alpha \in R \cap U\} \cup \{s_\alpha : \alpha \in R \cap U^\perp\}$.*

Proof. Certainly all of the above reflections belong to the isotropy group of U. Conversely, let $S' = \{\beta_1, ..., \beta_k\}$ be a base for R'. If w belongs to the isotropy group of U, then w acts as an isometry of U that preserves R'. In particular, wS' is also a base for R'. Since $W_{R'}$ acts transitively on bases, there must exist $x \in W_{R'}$ such that $wS' = xS'$, and therefore $\beta_i \mapsto x^{-1}w\beta_i$ induces a diagram automorphism of R'. If the diagram automorphisms are inner, then it must be possible to choose x so that $\beta_i = x^{-1}w\beta_i$ for all i. Thus $x^{-1}w$ must fix U pointwise, so $x^{-1}w$ is a product of reflections by roots that are orthogonal to U [H, p. 22]. ∎

If W_1 and W_2 are two reflection subgroups of W that are known to be conjugate, then the subspaces U_1 and U_2 spanned by the roots of their reflections must belong to the same W-orbit, and hence their isotropy groups must be conjugate. Using this idea we can produce the following second supplement to Step 4—it suffices to prove conjugacy for the remaining classes that induce identical permutation characters.

Step 4.2. (Assume $R = E_7$.) For $R' = A_1$, A_2, and $A_1 \oplus A_3$, determine all equivalence classes in the relation of Step 4.1 that index subgroups isomorphic to $W(R')$. For a representative subgroup W_J of each class, determine whether the subspace U_J spanned by the roots of W_J contains only the roots of R that belong to R'. In cases where this is affirmative, determine the isotropy groups of the subspaces spanned by the roots of every subgroup in the equivalence class, using the criterion of Proposition 10.1. For the quasi-parabolic subgroups in this list, add to Step 4.1 the relations implied by the fact that these subgroups are conjugate in W.

Remark 10.2. There is no graded analogue of Theorem 4.2 for the exceptional root systems. Perhaps the simplest way to prove this is to note that if w is a Coxeter element for W, then $V_w = 0$ (Lemma 2.5(b)), so the restricted complex Δ_R^w is trivial. Thus by Theorem 1.4, we have $\chi[R, q](w) = \det(1 - qw)$. However, the exceptional Weyl groups all have the property that $\det(1 - qw)$ has (some) negative coefficients. (This is easy to verify—either by direct calculation, or from the tables in [Ca].) Hence, in the exceptional cases, there is no grading of any permutation representation of W that is consistent with the grading of $K[\Delta_R]/\Theta$.

11. Open Problems

There are a number of natural questions suggested by the results of this paper for which we have been unable to find satisfactory answers. The first of these is due to R. Stanley.

Question 11.1. *If Φ is a complete simplicial fan that carries a proper action of a finite group G, is the representation carried by $H^*(X_\Phi, K)$ (or equivalently, $K[\Delta]/\Theta$) isomorphic to a permutation representation of G?*

Recall that in the above context, Proposition 1.7 shows that the (ungraded) character of $H^*(X_\Phi, K)$ is integral and nonnegative. If G is a cyclic group of prime order, then integrality and nonnegativity are sufficient to imply that a G-character is a permutation character. However, this is not the case for the cyclic group on four elements; it might suffice to look for counterexamples involving this group.

Problem 11.2. For each (crystallographic) root system R, find a basis for $K[\Delta_R]/\Theta$ that is permuted by W.

The above problem is more natural (but still open) for the cases $R = A_n$ and $R = C_n$, since in these cases the permutation representation π_R is compatible with the grading of $K[\Delta_R]/\Theta$. We should also point out that from any shelling of Δ_R one can obtain a natural graded basis for $K[\Delta_R]/\Theta$ (e.g., see Theorem 1.7 of [Bj], or Theorem 4.2 of [G]). However, at least in the case of A_n, these bases are not permuted by W.

Problem 11.3. Find a natural construction of the permutation representation π_R.

For any particular root system, one can use the description of the isotropy groups of π_R to easily construct an "artificial" permutation representation isomorphic to π_R (e.g., see [Ste3] for the A-series). However, by finding a "natural" construction that works uniformly for all root systems, one might hope to find more elegant proofs of the main results of this paper. In particular, given an explicit construction of π_R, one would expect to be able to prove directly and uniformly that the character of π_R agrees with the formula for χ^R in Corollary 1.5. Also, in a hypothetical "natural" construction of π_R it should be clear that the isotropy groups of π_R are quasi-parabolic.

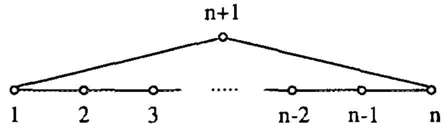
Problem 11.4. Find a rule for describing the isotropy groups of π_R that applies uniformly to all irreducible root systems R.

Presumably, there should be a rule of this type that is compatible with the constraints of Theorem 4.5. One would also expect that a solution of Problem 11.3 would yield insight into this problem as well.

APPENDIX 1

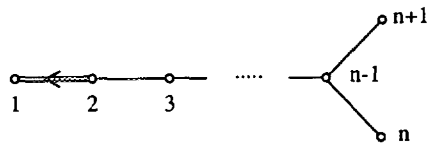
The following are the extended diagrams for the irreducible root systems. The nodes corresponding to simple roots are labeled 1, 2, ..., n, and the highest root is labeled $n+1$.

$A_n, n \geqslant 2$

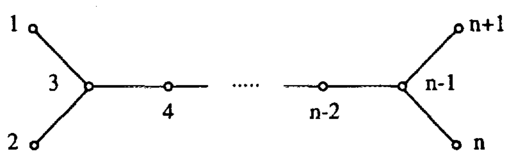


$C_n, n \geqslant 2$

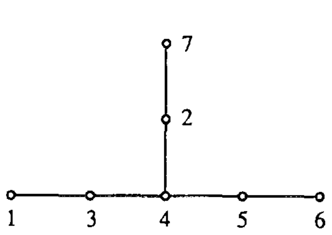
$B_n, n \geqslant 3$

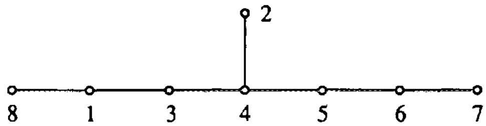


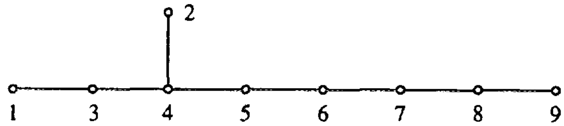
$D_n, n \geqslant 4$

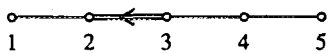


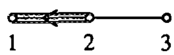
E_6











Appendix 2

The following tables provide the decomposition of π_R into transitive permutation characters for each exceptional root system R. Assuming R is of rank n, the first column lists subsets I of $\{1,2,\ldots,n+1\}$. The indices correspond to roots of the extended diagram, numbered according to the conventions of Appendix 1. (In particular, $n+1$ is the index of the highest root.) The quasi-parabolic subgroups W_I corresponding to the sets I that appear in this column are representatives for the distinct conjugacy classes of subgroups of W that occur as isotropy groups in π_R. In the second column is the isomorphism class of the root system of W_I, and in the third column is the multiplicity of $1_{W_I}^W$ in π_R.

G_2

I	R_I	m_I
1	A_1	1
12	G_2	1
13	A_1^2	1
23	A_2	1

F_4

I	R_I	m_I
13	A_1^2	1
123	C_3	1
124	A_1A_2	4
134	A_1A_2	3
234	B_3	1
235	A_1C_2	1
1234	F_4	1
1235	A_1C_3	1
1245	A_2^2	1
1345	A_1A_3	1
2345	B_4	1

E_6

I	R_I	m_I
125	A_1^3	1
1234	A_4	2
1235	$A_1^2A_2$	11
1245	A_1A_3	7
2345	D_4	1
12345	D_5	5
12346	A_1A_4	14
12356	$A_1A_2^2$	12
13456	A_5	3
12467	$A_1^2A_3$	1
123456	E_6	3
123467	A_1A_5	3
123567	A_2^3	1

E_7

I	R_I	m_I
1235	$A_1^2A_2$	4
1245	A_1A_3	2
1257	A_1^4	2
12345	D_5	1
12346	A_1A_4	12
12356	$A_1A_2^2$	14
13456	A_5	3
12357	$A_1^3A_2$	8
12457	$A_1^2A_3$	10
23457	A_1D_4	2
13467	A_2A_3	6
123456	E_6	2
123457	A_1D_5	6

I	R_I	m_I
123467	A_2A_4	11
123567	$A_1A_2A_3$	18
124567	A_1A_5	8
134567	A_6	5
234567	D_6	2
123578	$A_1^3A_3$	2
134578	A_1A_5	1
234578	$A_1^2D_4$	1
1234567	E_7	2
1234578	A_1D_6	2
1234678	A_2A_5	2
1235678	$A_1A_3^2$	1
1345678	A_7	1

E_8

I	R_I	m_I
1257	A_1^4	1
12346	A_1A_4	2
12356	$A_1A_2^2$	4
12357	$A_1^3A_2$	15
12457	$A_1^2A_3$	10
23457	A_1D_4	1
13467	A_2A_3	4
123457	A_1D_5	5
123467	A_2A_4	12
123567	$A_1A_2A_3$	29
124567	A_1A_5	11
134567	A_6	4
234567	D_6	1

I	R_I	m_I
123468	$A_1^2A_4$	22
123568	$A_1^2A_2^2$	28
234578	A_2D_4	4
134678	A_3^2	9
124579	$A_1^3A_3$	1
1234567	E_7	1
1234568	A_1E_6	4
1234578	A_2D_5	9
1234678	A_3A_4	15
1235678	$A_1A_2A_4$	24
1245678	A_1A_6	11
1345678	A_7	6
2345678	D_7	3
1234579	$A_1^2D_5$	1

I	R_I	m_I
1235679	$A_1^2A_2A_3$	4
1245679	$A_1^2A_5$	3
2345679	A_1D_6	1
1235689	$A_1A_2^3$	1
2345789	A_3D_4	1
12345678	E_8	1
12345679	A_1E_7	1
12345689	A_2E_6	1
12345789	A_3D_5	1
12346789	A_4^2	1
12356789	$A_1A_2A_5$	1
12456789	A_1A_7	1
13456789	A_8	1
23456789	D_8	1

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