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A NOTE ON ABEL POLYNOMIALS AND ROOTED LABELED FORESTS

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A special case of the Abel polynomials counts rooted labeled forests. This interpretation is used to obtain a combinatorial proof of the formula expressing x^n as a sum of these polynomials.

Dedicated to Frank Harary and his exceptional intuition

Various polynomials can be associated with combinatorial structures. For example, one instance of the Abel polynomials is the generating function for forests of labeled rooted trees. Specifically, if $A_n(a,x):=x(x-an)^{n-1}$ is the nth Abel polynomial, then

$$A_n(x):=A_n(-1,x)=\sum_{k=0}^{n} t_{nk}x^k \tag{1}$$

where t_{nk} is the number of forests on n labeled vertices consisting of k rooted trees. This is equivalent to the statement that $t_{nk}=\binom{n-1}{k-1}n^{n-k}$ which has been proved by various people, e.g., [3, 5]. Mullin and Rota [6] asked if (1) could be demonstrated combinatorially and this was done by Françon [1]. However such a proof for the inverse formula:

$$x^n=\sum_{k=0}^{n}\binom{n}{k}(-k)^{n-k}A_k(x) \tag{2}$$

was still lacking.

In [4] we showed that identities like (1) and (2) can be proved in a combinatorial manner by associating with the given polynomials a partially ordered set (poset). One identity follows by summing over the poset and the other by Möbius inversion. The purpose of this note is to describe such a poset for the Abel polynomials and hence provide a combinatorial proof of (2).

Let $\mathcal{F}_n$ be the set of all forests on n vertices consisting of labeled rooted trees. To describe a partial order on $\mathcal{F}_n$ we need only specify which forests cover a given $F\in\mathcal{F}_n$ (in a poset, x covers y if $x>y$ and there is no z with $x>z>y$). Let $E(F)$

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be the set of edges of F and $R(F)$ be the set of roots of F. For every pair $v_1, v_2 \in R(F)$ there are two forests, F_1 and F_2, covering F. This pair of forests is defined by $E(F_i) = E(F) \cup \{v_1 v_2\}$ and $R(F_i) = R(F) - \{v_i\}$; $i = 1, 2$. The Hasse diagram for the poset $\mathscr{F}_3$ is displayed in Fig. 1.

Define two functions f, g: $\mathscr{F}_n \to Q[x]$ by $f(F) = A_{k(F)}(x)$ and $g(F) = x^{k(F)}$ where $k(F)$ is the number of components of F. If $\hat{0} \in \mathscr{F}_n$ is the unique forest with no edges, then

$$A_n(x) = \sum_{k=0} t_{nk} x^k = \sum_k \sum_{\substack{F \in \mathscr{F}_n \\ k(F)=k}} x^k = \sum_{F \in \mathscr{F}_n} x^{k(F)}$$

or

$$f(\hat{0}) = \sum_{F \in \mathscr{F}_n} g(F). \tag{3}$$

Since the ideal $I_F = \{F_1 \in \mathscr{F}_n \mid F_1 \geq F\}$ is isomorphic to $\mathscr{F}_{k(F)}$, (3) implies that for all $F \in \mathscr{F}_n$,

$$f(F) = \sum_{\substack{F_1 \geq F \\ F_1 \in \mathscr{F}_n}} g(F_1).$$

Hence by Möbius inversion,

$$x^n = g(\hat{0}) = \sum_{F \in \mathscr{F}_n} \mu(\hat{0}, F) f(F) = \sum_{F \in \mathscr{F}_n} \mu(\hat{0}, F) A_{k(F)}(x), \tag{4}$$

where $\mu(\hat{0}, F)$ is defined inductively by $\mu(\hat{0}, \hat{0}) = 1$, $\mu(\hat{0}, F) = -\sum_{F' < F} \mu(\hat{0}, F')$ (see Rota [7] for details about Möbius functions). By way of example, the value of $\mu(\hat{0}, F)$ is indicated next to F itself in Fig. 1.

To simplify (4), we must evaluate the Möbius functions for the poset $\mathscr{F}_n$. If $F \in \mathscr{F}_n$ is composed of rooted trees $T_1, T_2, \ldots, T_k$, then the interval $[\hat{0}, F]$ is isomorphic to the direct product $[\hat{0}, T_1] \times [\hat{0}, T_2] \times \cdots \times [\hat{0}, T_k]$ in the natural way and $\mu(\hat{0}, F) = \mu(\hat{0}, T_1)\, \mu(\hat{0}, T_2) \cdots \mu(\hat{0}, T_k)$. Hence it suffices to calculate $\mu(\hat{0}, T)$ where T is a single rooted tree. First we must describe the elements of $[\hat{0}, T]$.

Given a tree T and vertices v, w in T, we let v–w denote the unique path from v to w in T. Let T have root r. The *depth of a vertex v*, depth v, is the length of r–v (depth $r = 0$). We will always measure depth with respect to the maximal tree T of $[\hat{0}, T]$. If u is on v–w we write v–u–w. The *subtree corresponding to v in T*, $T(v)$, is the subtree induced by all vertices w in T such that r–v–w.

Lemma 1. *Given $F_1 \in [\hat{0}, T]$, consider any tree $T_1 \subseteq F_1$ with root r_1, and any $v \neq r_1$ in T_1, then*

(a) depth $r_1 <$ depth v,

(b) $T_1(v) = T(v)$.

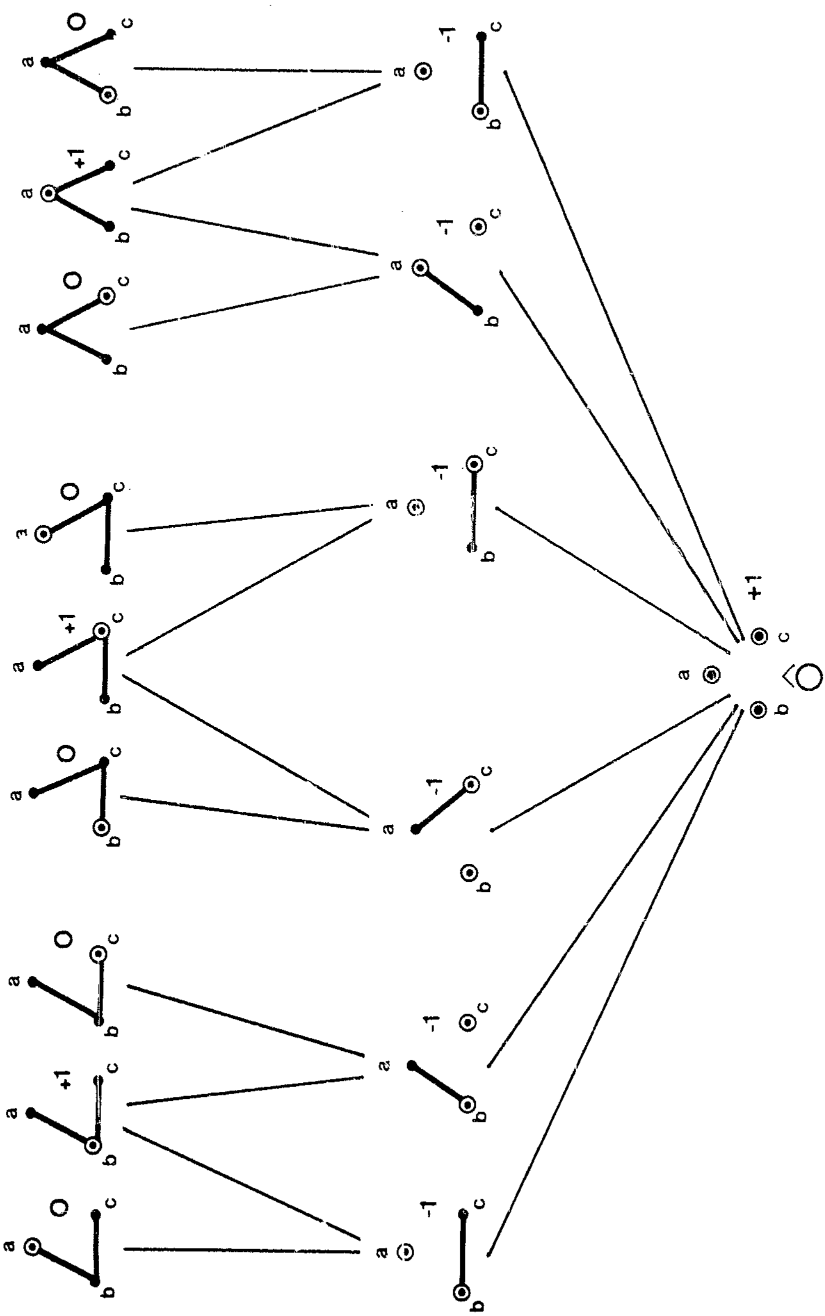

Fig. 1. $\mathcal{F}_3$ and its Möbius function. Roots are circled and $\mu(F)$ is directly to the right of F.

Proof. (a) Assume that depth $v \leq$ depth r_1. Without loss of generality we may assume that depth v is minimal among all $v \in T_1$. Hence r–v–r_1 since otherwise v–r_1 contains other vertices of T_1 of smaller depth.

If $v = r$, then r is not a root in F_1. But $R(F_1) \supseteq R(T)$ so that r is not a root in T, a contradiction. If $v \neq r$, then the minimality of depth v guarantees that r–v contains an edge $uv \in E(T) - E(T_1)$. However we can only add an edge to F_1 if it connects two roots and v is not a root. Hence we will never be able to add uv to F_1 in order to create T, another contradiction.

(b) Since $T_1 \subseteq T$ we have $T_1(v) \subseteq T(v)$. As both $T_1(v)$ and $T(v)$ are connected, to prove $T_1(v) = T(v)$ we need only show that both trees have the same vertex set. So suppose that $w \in T(v) - T_1(v)$ and consider v–w. Following this path from v to w, let xy be the first edge in $T(v)$ that is not in $T_1(v)$. Hence $x \in T_1$ and $y \notin T_1$. But x is not the root of T_1 so, as before, we will never be able to add the edge xy to F_1. □

Note that condition (a) implies that $R(F_1)$ is completely determined by $E(F_1)$ since each tree $T_1 \subseteq F_1$ is rooted at the vertex of minimal depth in T. Hence to specify a forest in $[\hat{0}, T]$ we need only specify its edge set.

Corollary 2. *Given* $F_1, F_2 \in [\hat{0}, T]$, *then* $F_1 \leq F_2$ *if and only if* $E(F_1) \subseteq E(F_2)$.

Proof. The 'only if' part of the corollary follows immediately from the definition of the covering relation in $\mathscr{F}_n$. For the other implication we need only show that we can connect pairs of roots in F_1 to obtain the rest of the edges in F_2, i.e. for every $uv \in E(F_2) - E(F_1)$ we must show that $u, v \in R(F_1)$.

Without loss of generality, let depth $u =$ depth $v - 1$ so that r–u–v. If $u \notin R(F_1)$, then $T(u) \subseteq F_1$ by Lemma 1(b). This implies that $uv \in E(F_1)$, contrary to our assumption. However, if $v \notin R(F_1)$, then the tree of F_1 containing v has root r_1 with depth $r_1 <$ depth v by Lemma 1(a). Hence uv lies on $r_1 - v$ and is thus in $E(F_1)$, another contradiction. □

Corollary 3. *The interval* $[\hat{0}, T]$ *is a lattice with, for all* $F_1, F_2 \in [\hat{0}, T]$,

$F_1 \vee F_2 =$ *the forest in* $[\hat{0}, T]$ *with edge set* $E(F_1) \cup E(F_2)$,

$F_1 \wedge F_2 =$ *the forest in* $[\hat{0}, T]$ *with edge set* $E(F_1) \cap E(F_2)$.

Proof. This result follows from Corollary 2 and the fact that $\cup$ and $\cap$ are the meet and join for subsets of a set. The details are similar to what we have proved in full above and are omitted. □

We are now in a position to calculate $\mu(\hat{0}, T)$ for any tree $T \in \mathscr{F}_n$. In what follows an *endpoint* is a vertex of degree one, an *endline* is an edge incident with an endpoint, and a *bush* is a tree all of whose edges are endlines containing the root.

Proposition 4. *For any* $T \in \mathscr{F}_n$,

$$\mu(\hat{0}, T) = \begin{cases} (-1)^{n-1} & \text{if } T \text{ is a bush,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Consider first the case where T is a bush with root r. Given any subset $S \subseteq E(T)$ there is a forest $F_1 \in [\hat{0}, T]$ with $E(F_1) = S$. Merely root the tree determined by S at r. Since all the isolated points must be roots, we can add edges to F_1 until we obtain T.

In fact these are the only elements of $[\hat{0}, T]$ since given $F_1 \in [\hat{0}, T]$ we have $E(F_1) \subseteq E(T)$ and, by Lemma 1(a), r must be a root since depth $v = 1$ for $v \neq r$. Hence there is a bijection between $[\hat{0}, T]$ and the boolean algebra on $|E(T)|$ elements. By Corollaries 2 and 3 this bijection is an isomorphism of lattices, and so in this case

$$\mu(\hat{0}, T) = (-1)^{|E(T)|} = (-1)^{n-1}.$$

If T is not a bush, consider the atoms (elements covering $\hat{0}$) of $[\hat{0}, T]$. Each atom, F_1, consists of $n-1$ isolated roots and a single edge which we claim must be an endline $r_1 v$ with endpoint v. Clearly any such forest can be completed to T by adding edges. Conversely, if $F_1 \in [\hat{0}, T]$ with unique edge $r_1 v$, then depth $r_1 <$ depth v. Thus $T_1(v) = v = T(v)$ so v must be an endpoint.

Now if T is not a bush, then some edge of T is not an endline. It follows that this edge is not in any atom and, by Corollary 3, that T is not the join of the atoms of $[\hat{0}, T]$. Invoking Hall's theorem [2: 7 p. 349] we see that $\mu(\hat{0}, F) = 0$. □

Corollary 5. *For all* $F \in \mathscr{F}_n$,

$$\mu(\hat{0}, F) = \begin{cases} (-1)^{n-k(F)} & \text{if } F \text{ is a forest of bushes,} \\ 0 & \text{otherwise.} \end{cases}$$

Applying this last result to (4), we see that x^n is expressible as:

$$x^n = \sum_{k=0}^{n} \sum_{\substack{k(F)=k \\ F \text{ a forest of bushes}}} (-1)^{n-k} A_k(x).$$

But the number of forests on n vertices consisting of k bushes is easily seen to be $\binom{n}{k} k^{n-k}$. The number of choices for the roots is $\binom{n}{k}$ and k^{n-k} counts the number of ways to connect the remaining $n-k$ vertices to those roots. Equation (2) follows at once.

Note added in proof

David Reiner [8] has also discovered the poset $\mathscr{F}_n$. The computation of its Möbius function in this note is new.

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