96.06 A rational number of the form a^a with a irrational

Here is a very well known elementary proof that there are irrational numbers a and b so that a^b is rational. We know $\sqrt{2}$ is irrational. If $\sqrt{2}^{\sqrt{2}}$ is rational, then we are done, while if it is not rational, then

$$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = 2$$

and we are done. This elementary but non-constructive argument motivates, but does not quite answer, the question of whether there is a single irrational number a so that a^a is rational. The answer to that question is a resounding affirmative and is also elementary.

Theorem 1: Let $I = \left((1/e)^{1/e}, \infty\right)$. Every rational number in I is either of the form a^a for an irrational a or is in the very thin set $\{1, 4, 27, 256, \dots, n^n, \dots\}$.

Proof: Think of I as an interval on the y-axis and note that the function x^x maps the interval $(1/e, \infty)$ in a continuous one-to-one way onto I since $(x^x)' = x^x(1 + \ln x)$ is positive on $(1/e, \infty)$. Given a rational number $r \in I$, let a be the corresponding number on the x-axis so that $a^a = r$. To prove the theorem we must show that if a is rational, then it must be a positive integer. Here is a reformulation. Let $a = \frac{n}{m}$ and $r = \frac{b}{c}$ be reduced fractions so that

$$\left(\frac{n}{m}\right)^{n/m} = \frac{b}{c};$$

then $m = 1$. We have $n^n c^m = b^m m^n$. Since both fractions are reduced, $(n, m) = (b, c) = 1$, so that a prime divides the factor c^m on the left-hand side if, and only if, it divides m^n on the right-hand side. Thus

$$c^m = m^n. \tag{1}$$

Assume $m > 1$. Because of equation (1), there are a prime p, positive integer exponents i and j, and integers k and l both relatively prime to p such that $c = p^i k$ and $m = p^j l$. Furthermore $im = jn$, so that $i(p^j l) = jn$. Thus p^j divides jn. But p is a factor of m and $(m, n) = 1$, so p^j divides j, whence $p^j \leqslant j$. Since $2 \leqslant p$,

$$2^j \leqslant j.$$

However, the contrary inequality, $j < 2^j$ for every positive integer j, is easily proved, either by mathematical induction or as follows:

$$(1 + 1)^j = \sum_{i=0}^{j} \binom{j}{i} \geqslant \sum_{i=0}^{j} 1 = j + 1 > j.$$

This contradiction proves that $m = 1$.

The real numbers can be partitioned into three classes: the rational numbers, the irrational algebraic numbers, and the transcendental numbers. An irrational algebraic is an irrational number that is a root of a polynomial having integer coefficients, for example $\sqrt{2}$ (a root of $x^2 - 2$). A transcendental is an irrational number that is not the root of any polynomial with integer coefficients. This paragraph's content is explained less briefly in many places; see [1] for example.

The Gelfond-Schneider Theorem asserts that if a ($a \neq 0$, $a \neq 1$) is either a rational or an irrational algebraic, and if b is an irrational algebraic, then a^b is transcendental. [1, 2, 3, 4, 5] The proof of this theorem is not elementary, but the theorem itself is very easy to understand and has many useful applications. We will give three applications.

First, there is at least one transcendental number (and actually there are many of them). Secondly, we can remove the non-constructive nature of the proof that formed the third sentence of this paper. We now know that the number $\sqrt{2}^{\sqrt{2}}$ is transcendental and hence irrational. (The irrationality of $\sqrt{2}^{\sqrt{2}}$ was first proved by Kuzmin [6].)

The third application of the Gelfond-Schneider Theorem is to help us get an affirmative answer to the question of whether there is a single irrational number a so that a^{a^a} is rational. This follows immediately from the following theorem.

Theorem 2: Every positive rational number is either of the form a^{a^a} for an irrational a or is in the very thin set

$$\left\{1,\ 16,\ 7\,625\,597\,484\,987,\ \dots\ n^{n^n},\ \dots\right\}.$$

Proof: We will show that, for every positive rational number r not of the form n^{n^n} for some integer n, there is a unique irrational number a so that $a^{a^a} = r$. To prove this, we first observe that the third hyperpower, ${}^3x := x^{x^x} = x^{(x^x)}$ is a strictly increasing continuous function: $(0, \infty) \to (0, \infty)$. At $x = 0$,

$$\lim_{x \to 0^+} x^{x^x} = \left(\lim_{x \to 0^+} x\right)^{\lim_{x \to 0^+} x^x} = 0^1 = 0.$$

Also its derivative is strictly positive on $(0, \infty)$ since $({}^3x)' = x^{x^x}x^{x-1}f(x)$, where $f(x) = 1 + x \ln x(1 + \ln x)$ is clearly positive on $(0, 1/e] \cup [1, \infty)$, while on $(1/e, 1)$, $f(x) > 1 + \ln x(1 + \ln x) = (\ln x + \frac{1}{2})^2 + \frac{3}{4}$. Thus the function ${}^3x$ provides a one-to-one correspondence between the positive x-axis and the positive y-axis. Furthermore, its inverse maps *every* positive rational number r that is not of the form n^{n^n} for some integer n into an irrational number a so that ${}^3a = r$. To prove this, let r be rational and $a = \frac{n}{m}$ be a reduced fraction such that ${}^3a = r$. Let $\alpha = \frac{n}{m}$, $\beta = (\frac{n}{m})^{\frac{n}{m}}$. Assume that $m \geqslant 2$. Then β is a root of the integer polynomial $m^n x^m - n^n$. Also it was shown in Theorem 1 that β is irrational. In short, α is rational but not an integer, and β is an irrational algebraic number. By the Gelfond-Schneider Theorem, ${}^3a = \alpha^\beta$ is transcendental and thus irrational. This means that $m = 1$ and that ${}^3a = n^{n^n}$ for some positive integer n.

The obvious question that now arises is whether there is a $k \geqslant 4$ and an irrational number a so that ${}^ka$ is rational? We cannot answer this question, even when $k = 4$. We conjecture that the answer is just as strongly positive as it was with ${}^2a$ and ${}^3a$. In fact, we will hazard that whenever a is a positive non-integer rational number, and $k \geqslant 4$, ${}^ka$ is irrational.

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