SOME PROPERTIES OF LATIN SQUARES

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INTRODUCTION

- Why study latin squares?
  - Applications
  - Puzzles
  - It’s fun!

LATIN SQUARES

- A **latin square of order *n*** is an $n \times n$ matrix containing n distinct symbols such that each symbol appears in each row and column exactly once. The symbols are usually denoted by 0, 1,…, $n-1$.
- **Example.** Latin square of order 4:

0	1	2	3
1	2	3	0
2	3	0	1
3	0	1	2

❑ **Theorem 1.** There is a latin square of order n for each $n \geq 1$.

This is Theorem 2.2.3. G. Mullen & C. Mummert, 44.

Find the latin square:

0	3	3	1
1	2	3	2
2	2	1	3
2	0	0	1

1	0	2	2
0	2	0	3
1	2	3	0
3	1	1	1

3	1	3	2
0	3	2	0
1	2	0	3
2	3	2	1

2	1	2	3
0	0	3	0
3	1	2	2
1	3	0	1

3	2	2	0
2	1	0	3
2	0	3	1
0	3	1	3

0	1	3	2
1	0	3	2
3	2	1	1
3	0	2	2

1	0	2	2
2	2	3	0
3	1	2	3
0	3	0	1

0	1	3	0
1	3	1	3
3	1	0	2
0	3	2	1

2	3	0	1
0	1	2	3
1	2	3	0
3	0	1	2

0	3	3	1
1	2	3	2
2	2	1	3
2	0	0	1

1	0	2	2
0	2	0	3
1	2	3	0
3	1	1	1

3	1	3	2
0	3	2	0
1	2	0	3
2	3	2	1

2	1	2	3
0	0	3	0
3	1	2	2
1	3	0	1

3	2	2	0
2	1	0	3
2	0	3	1
0	3	1	3

0	1	3	2
1	0	3	2
3	2	1	1
3	0	2	2

1	0	2	2
2	2	3	0
3	1	2	3
0	3	0	1

0	1	3	0
1	3	1	3
3	1	0	2
0	3	2	1

2	3	0	1
0	1	2	3
1	2	3	0
3	0	1	2

Computational Problem

A central problem in the theory of latin squares is to determine how many latin squares of each size exist.

n	***#LS***
1	1
2	2
3	12
4	576
5	161,280
6	812,851,200
7	61,479,419,904,000
8	108,776,032,459,082,956,800
9	5,524,751,496,156,892,842,531,225,600
10	9,982,437,658,213,039,871,725,064,756,920,320,000
11	776,966,836,171,770,144,107,444,346,734,230,682,311,065,600,000

Taken from N. J. A. Sloane, **A002860**, *On-Line Encyclopedia of Integer Sequences* (1996-2008)
http://www.research.att.com/~njas/sequences/A002860

n	*#LS*
1	1
2	2
3	12
4	576
5	[illegible]
6	[illegible]
7	[illegible]
8	108,776,032,459,082,956,800
9	5,524,751,496,156,892,842,531,225,600
10	9,982,437,658,213,039,871,725,064,756,920,320,000
11	776,966,836,171,770,144,107,444,346,734,230,682,311,065,600,000

$n \geq 12$?

Taken from N. J. A. Sloane, **A002860**, *On-Line Encyclopedia of Integer Sequences* (1996-2008)
http://www.research.att.com/~njas/sequences/A002860

TYPES OF LATIN SQUARES

- A latin square of order *n* is said to be **reduced** if its first row and first column are in the standard order 0, 1,…, *n–1*.
- **Example.** This is an example of a *reduced latin square* of order 4:

0	1	2	3
1	2	3	0
2	3	0	1
3	0	1	2

n	*#RLS*
1	1
2	1
3	1
4	4
5	56
6	9,408
7	16,942,080
8	5.35×10^{11}
9	3.78×10^{17}
10	7.58×10^{24}
11	5.36×10^{33}

Taken from N. J. A. Sloane, **A000315**, *On-Line Encyclopedia of Integer Sequences* (1996-2008) http://www.research.att.com/~njas/sequences/A000315.

n	*#RLS*
1	1
2	1
3	1
4	4
9	3.78×10^{17}
10	7.58×10^{24}
11	5.36×10^{33}

$n \geq 12$?

Taken from N. J. A. Sloane, **A000315**, *On-Line Encyclopedia of Integer Sequences* (1996-2008) http://www.research.att.com/~njas/sequences/A000315.

TYPES OF LATIN SQUARES

- Let L_n denote the number of distinct latin squares of order n and let l_n denote the number of distinct reduced latin squares of order n:

Theorem 2. For any $n \geq 2$, $L_n = n!\,(n-1)!\,l_n$

Example:

For $n = 5$, $l_5 = 56$ and you will get

$L_5 = (5!)(4!)(56) = 161{,}280$

This is Theorem 2.2.6. G. Mullen & C. Mummert, 45.

TYPES OF LATIN SQUARES

- A latin square of order *n* is said to be **semi-reduced** if its first row is in the standard order.
- **Example.** This is an example of a *semi-reduced latin square* of order 5:

0	1	2	3	4
2	4	1	0	3
3	2	0	4	1
4	0	3	1	2
1	3	4	2	0

PERMUTATIONS OF LATIN SQUARES

- Permutations of latin squares:
  1. column permutation
  2. row permutation
  3. relabeling

  - A **permutation** of a set is an arrangement of its elements in a certain order.
  - The **number of permutations** of n elements is: $n! = n(n-1)(n-2)(n-3)\ldots(3)(2)(1)$.

PERMUTATIONS OF LATIN SQUARES

- **Column permutation**

0	1	2	3
1	2	3	0
2	3	0	1
3	0	1	2

⇨

0	1	3	2
1	2	0	3
2	3	1	0
3	0	2	1

- **Row permutation**

0	1	3	2
1	2	0	3
2	3	1	0
3	0	2	1

⇨

0	1	3	2
2	3	1	0
1	2	0	3
3	0	2	1

PERMUTATIONS OF LATIN SQUARES

- **Relabeling**:

$$2 \rightarrow 0$$
$$0 \rightarrow 1$$
$$1 \rightarrow 2$$
$$3 \rightarrow 3$$

2	0	1	3
3	1	2	0
0	2	3	1
1	3	0	2

⇒

0	1	2	3
3	2	0	1
1	0	3	2
2	3	1	0

PERMUTATIONS OF LATIN SQUARES

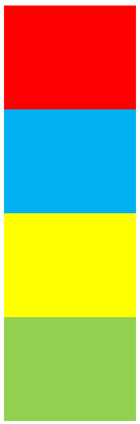
- **Relabeling:**

$0 \rightarrow$

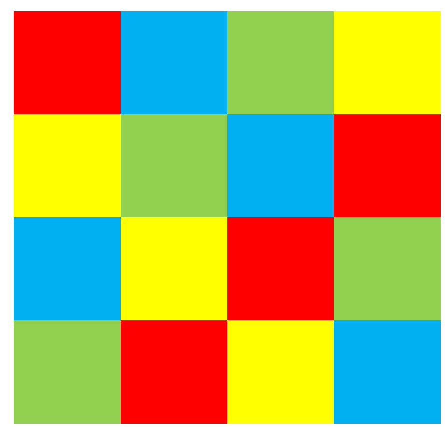
$1 \rightarrow$

$\alpha \rightarrow$

$\alpha^2 \rightarrow$



0	1	α^2	α
α	α^2	1	0
1	α	0	α^2
α^2	0	α	1



PERMUTATIONS OF LATIN SQUARES

Example. All the latin squares of order 3:

Theorem 2. For any $n \geq 2$, $L_n = n!\,(n-1)!\,l_n$

1		
0	1	2
1	2	0
2	0	1

Interchange the n columns

2		
0	2	1
1	0	2
2	1	0

3		
1	2	0
2	0	1
0	1	2

4		
1	0	2
2	1	0
0	2	1

5		
2	0	1
0	1	2
1	2	0

6		
2	1	0
0	2	1
1	0	2

Interchange the last n-1 rows

7		
0	1	2
2	0	1
1	2	0

8		
0	2	1
2	1	0
1	0	2

9		
1	0	2
0	2	1
2	1	0

10		
1	2	0
0	1	2
2	0	1

11		
2	0	1
1	2	0
0	1	2

12		
2	1	0
0	2	1
1	0	2

You get the twelve latin squares of order 3

SETS OF ORTHOGONAL LATIN SQUARES

0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0
2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1
0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0
2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1
0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0
2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1
0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0
2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1
0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0
2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1
0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0
2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1
0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0
2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1
2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1

SUPERIMPOSING LATIN SQUARES

- Given two latin squares of the same size we can **superimpose** them, that is, we can place or lay one latin square over the other to create a square of ordered pairs.
- **Example.**

0	1	2
1	2	0
2	0	1

LS_1

0	1	2
2	0	1
1	2	0

LS_2

(0,0)	(1,1)	(2,2)
(1,2)	(2,0)	(0,1)
(2,1)	(0,2)	(1,0)

$S(LS_1, LS_2)$

r–orthogonality

- ***r*** = $P(LS_1, LS_2)$ is the number of distinct ordered pairs you get when you superimpose LS_1 and LS_2.

- LS_1 and LS_2 are said to be ***r*–orthogonal** if you get *r* distinct ordered pairs when you superimpose them.

r–orthogonality

- **Example.** A pair of 8-orthogonal latin squares of order 4:

0	1	2	3
1	2	3	0
2	3	0	1
3	0	1	2

LS_1

0	1	2	3
3	0	1	2
2	3	0	1
1	2	3	0

LS_2

(0,0)	(1,1)	(2,2)	(3,3)
(1,3)	(2,0)	(3,1)	(0,2)
(2,2)	(3,3)	(0,0)	(1,1)
(3,1)	(0,2)	(1,3)	(2,0)

$S(LS_1,LS_2)$

r–orthogonality

- **Example.** A pair of 8-orthogonal latin squares of order 4:

0	1	2	3
1	2	3	0
2	3	0	1
3	0	1	2

LS_1

0	1	2	3
3	0	1	2
2	3	0	1
1	2	3	0

LS_2

(0,0)	(1,1)	(2,2)	(3,3)
(1,3)	(2,0)	(3,1)	(0,2)
(2,2)	(3,3)	(0,0)	(1,1)
[illegible]	[illegible]	[illegible]	[illegible]

Note that $P(\mathrm{LS}_1,\mathrm{LS}_2) = P(\mathrm{LS}_2,\mathrm{LS}_1)$.

Orthogonal Latin Squares

- Two latin squares of order n are **orthogonal** if $r = n^2$.

- Pair of orthogonal latin squares of order 3:

0	1	2
1	2	0
2	0	1

LS_1

0	1	2
2	0	1
1	2	0

LS_2

⇒

(0,0)	(1,1)	(2,2)
(1,2)	(2,0)	(0,1)
(2,1)	(0,2)	(1,0)

$S(LS_1, LS_2)$

r–orthogonality

- The **spectrum** (for *r*–orthogonality) is the set of all the possible values of *r*.

- The **frequency** (for *r*–orthogonality) is the number of pairs of latin squares of order *n* that are *r*–orthogonal.

r–orthogonality

- **Example**:

  For latin squares of order 4 the spectrum is {4, 6, 8, 9, 12, 16} and the frequency for those values of *r* is →

r	*f*
4	4
5	0
6	12
7	0
8	6
9	24
10	0
11	0
12	48
13	0
14	0
15	0
16	2

- **Example:**

  For latin squares of order 6 the spectrum is {6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 34}

r–orthogonality

- **Theorem 3:** For a positive integer n, a pair of r–orthogonal latin squares of order n, exists if and only if $r \in \{n, n^2\}$ **or** $n + 2 \le r \le n^2 + 2$, except when
  - $n = 2$ and $r = 4$;
  - $n = 3$ and $r \in \{5, 6, 7\}$;
  - $n = 4$ and $r \in \{7, 10, 11, 13, 14\}$;
  - $n = 5$ and $r \in \{8, 9, 20, 22, 23\}$;
  - $n = 6$ and $r \in \{33, 36\}$.

This is Theorem 3.104. C.J. Colbourn and J.H. Dinitz.

r–orthogonality

- **Example**:

  For latin squares of order 4

r	4	5	6	7	8	9	10	11	12	13	14	15	16
f	*	0	*	0	*	*	0	0	*	0	0	0	*

r–orthogonality

- **Proposition**: There exist a pair of latin squares of order *n* that are *r*–orthogonal if and only if there exist a reduced latin square of order *n* and a semi-reduced latin square of order *n* that are *r*-orthogonal.
  - **Example**: The number of pairs of latin squares of order 5 is $L_5 \times L_5 = 26{,}011{,}238{,}400$. The number of pairs of reduced latin squares and semi-reduced latin squares is $l_5 \times sl_5 = 75{,}264$.

Note: $\boldsymbol{sl_n}$ is the number of distinct semi-reduced latin squares of order *n*

r_t–orthogonality

- Let {LS_1,…, LS_t} be a set of $t \geq 2$ latin squares. Then, $\boldsymbol{r_t}$ is the sum of all the $r = P(LS_i, LS_j)$, with $1 \leq i, j \leq t$ and $i \neq j$.

$$r_t = \sum_{i=1}^{t-1} \sum_{j=i}^{t-1} P(LS_i\,, LS_{j+1})$$

- **Example.** Let {LS_1, LS_2 , LS_3} be a set of three latin squares of order *n:*

$$r_3 = P(LS_1, LS_2) + P(LS_1, LS_3) + P(LS_2, LS_3)$$

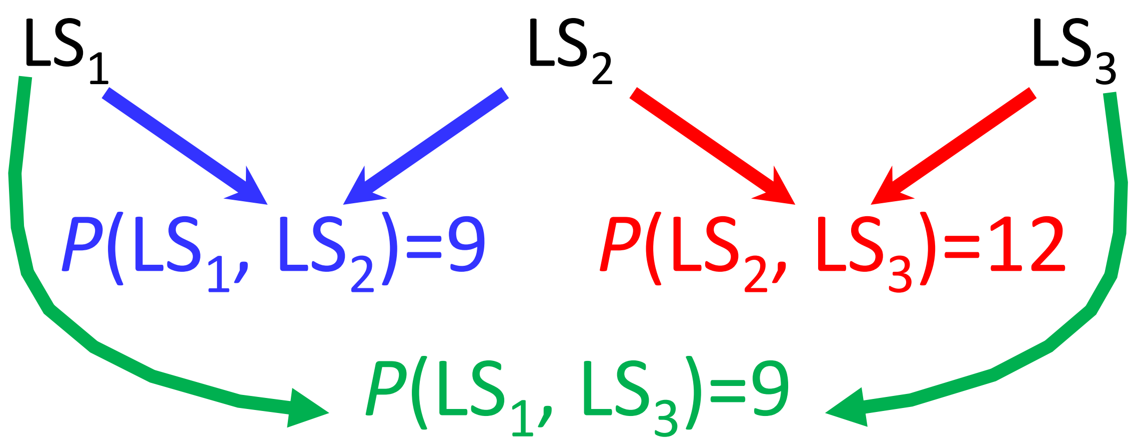
r_t–orthogonality

- **Example:** Here we have a set of three latin squares of order 4 with $r_3 = 9 + 12 + 9 = 30$

0	1	2	3
1	3	0	2
2	0	3	1
3	2	1	0

0	1	2	3
1	2	3	0
2	3	0	1
3	0	1	2

0	1	2	3
2	3	0	1
1	2	3	0
3	0	1	2



r_t–orthogonality

- The **spectrum** (for r_t-orthogonality) is the set of all the possible values of r_t.

- The **frequency** (for r_t-orthogonality) is the number of sets of t latin squares of order n that have an r_t–orthogonality, and it is denoted by h_{r_t}.

Mutually Orthogonal Latin Squares

- A collection $\{LS_1, LS_2, LS_3, \ldots, LS_t\}$ of $t \geq 2$ latin squares of order n is said to be **mutually orthogonal** if every pair of distinct latin squares in the collection is orthogonal.
- **Example.** Let $\{LS_1, LS_2, LS_3\}$ be a set of 3 latin squares of order n.

  This set is orthogonal if $P(LS_1, LS_2)=n^2$, $P(LS_2, LS_3)=n^2$ and $P(LS_1,LS_3)=n^2$.

Mutually Orthogonal Latin Squares

- **Example:** Here we have a set of orthogonal latin squares of order 4:

0	1	2	3
1	0	3	2
2	3	0	1
3	2	1	0

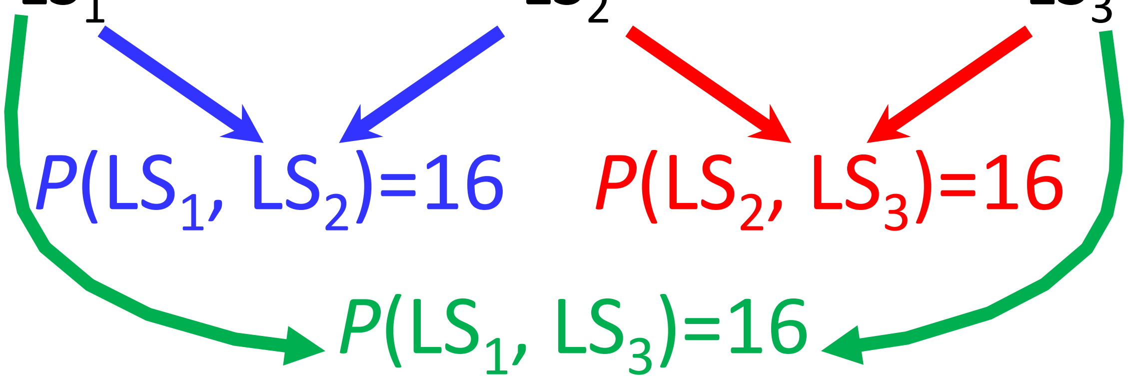
LS_1

0	2	3	1
1	3	2	0
2	0	1	3
3	1	0	2

LS_2

0	3	1	2
1	2	0	3
2	1	3	0
3	0	2	1

LS_3



Mutually Orthogonal Latin Squares

Questions:

Is there a collection of mutually orthogonal latin squares for every order?

If they exist, how big is the largest collection of mutually orthogonal latin squares for each order?

Mutually Orthogonal Latin Squares

- Let $N(n)$ denote the size of the largest collection of *mutually orthogonal latin squares* (MOLS) of order n (that exist).
  - **Theorem 4.** $N(n) \leq n-1$ for any $n \geq 2$.
  - **Theorem 5.** If q is a prime power, then $N(q) = q - 1$.
    - $q = p^r$ where p is prime number and $r \in \mathbb{N}$
  - **Theorem 6.** $N(n) \geq 2$ for all n except 2 and 6.
    - $N(2) = 1$ and $N(6) = 1$
  - **Theorem 7.** Let $n = q_1, \ldots, q_r$, where q_i are distinct prime powers and $q_1 < \ldots < q_r$. Then $N(n) \geq q_1 - 1$.

Theorem 4 is Theorem 2.2.8. G. Mullen & C. Mummert, 46.
Theorem 5 is Theorem 2.2.10. G. Mullen & C. Mummert, 47.
Theorem 6 is Theorem 2.2.19. G. Mullen & C. Mummert, 50.
Theorem 7 is Theorem 2.2.24. G. Mullen & C. Mummert, 52.

Mutually Orthogonal Latin Squares

Question:

Are there mutually orthogonal latin squares of order *n* if *n* is not a prime power?

OUR PROJECTS

Research Questions

- **What is the maximum r_t–orthogonality, $M_n(t)$?**
- **Are there any properties related to $M_n(t)$?**
- **What is the frequency and the spectrum (for r_t–orthogonality) for sets of three or more latin squares of order n?**

RESULTS

t	$M_t(6)$
2	34
3	96
4	188
5	$300 \leq M_5(6) < 340$

Taken from R. Arce & J. Cordova & I. Rubio. (2009)

- We have tables with the spectrum for $n = 4$ and 5 with $2 \leq t \leq n-1$ and for $n = 6$ with $t = 2$.
- We have tables with the frequency for $h_2(4)$, $h_3(4)$, $h_2(5)$, $h_3(5)$ and $h_4(5)$.

Computational Problem

- Reduce number of comparisons and time
- Restrict focus to special sets of latin squares
- Eliminate unnecessary comparisons

Computational Problem

- The plan:
  - Distribute the work:
    - Cores
    - Processors
    - Computers
  - Design a specialized circuit that compares latin squares.

What is Known

- Number of distinct latin squares when $n \leq 11$.
- Number of distinct reduced latin squares when $n \leq 11$.
- The spectrum for r_2–orthogonality and for $n = 4$ and 5 with $2 \leq t \leq n-1$.
- The frequency for $hr_2(4)$, $hr_3(4)$, $hr_2(5)$, $hr_3(5)$, $hr_4(5)$ *and* $hr_2(6)$.
- If n is a prime power and $t \leq n-1$, then

$$\mathrm{M}_t(n) = n^2 \binom{t}{2}$$

- The number of latin squares of order $n = p^r$, where p is a prime number and $r \in \mathrm{N}$, that are mutually orthogonal.

What is Unknown

- The number of distinct latin squares when $n \geq 12$.
- The number of distinct reduced latin squares when $n \geq 12$.
- The spectrum for r_t–orthogonality with $t > 2$ and $n \geq 6$.
- The frequency for the r_t–orthogonality when $t > 2$ and $n \geq 6$.
- The maximum r_t–orthogonality when $t = 5$ and $n = 6$
- The maximum r_t–orthogonality when n is not a prime power and $2 < t \leq n-1$.
- Mutually orthogonal latin squares of order n when n is not a prime power.

FUTURE WORK

- Optimize the computing approach to apply it to the case n = 6 because the time of computing the $M_3(6)$ is 205.52541 years.
- Find properties of the latin squares that produce the $M_t(n)$.
- Find a formula for $M_t(n)$ when n is not a prime power and $t \leq n-1$.
- Estimate the probability that two random latin squares of order n are going to be mutually orthogonal.

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