Ramanujan's Arithmetic-Geometric Mean

Continued Fractions and Dynamics

Jonathan M. Borwein, FRSC, FAAAS, FAA

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Laureate Professor & Director

CARMA

University of Newcastle, Callaghan 2308 NSW

www.carma.newcastle.edu.au/∼jb616/#TALKS

Joint work with Richard Crandall — also D. Borwein, Fee, Luke, Mayer

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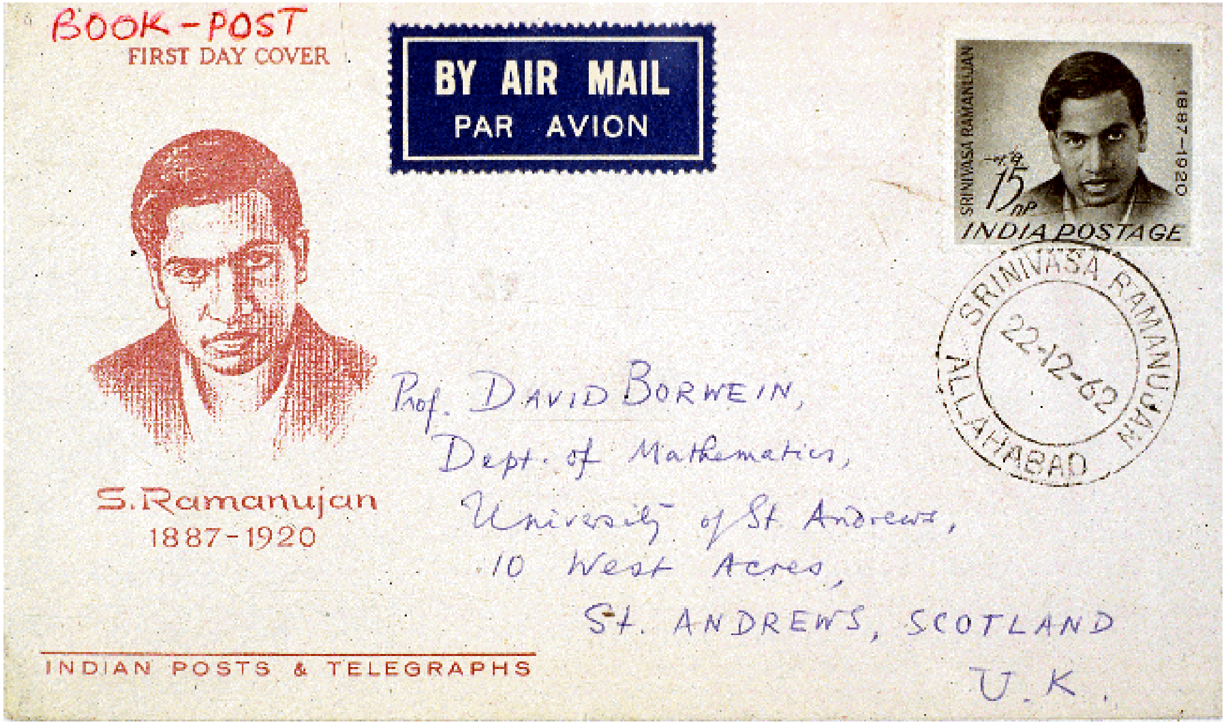
About Verbal Presentations

*I feel so strongly about the wrongness of reading a lecture that my language may seem immoderate. ··· The spoken word and the written word are quite different arts. ··· I feel that to collect an audience and then read one's material is like inviting a friend to go for a walk and asking him not to mind if you go alongside him in your car.** — Sir Lawrence Bragg

- What would he say about reading overheads?

*From page 76 of *Science*, July 5, 1996.

Srinivasa Ramanujan (1887–1920)



- G. N. Watson (1886–1965), on reading Ramanujan's work, describes:

a thrill which is indistinguishable from the thrill I feel when I enter the Sagrestia Nuovo of the Capella Medici and see before me the austere beauty of the four statues representing 'Day,' 'Night,' 'Evening,' and 'Dawn' which Michelangelo has set over the tomb of Guiliano de'Medici and Lorenzo de'Medici.

1. Abstract

The *Ramanujan AGM continued fraction*

$$\mathcal{R}_\eta(a,b) = \cfrac{a}{\eta + \cfrac{b^2}{\eta + \cfrac{4a^2}{\eta + \cfrac{9b^2}{\eta + \ddots}}}}$$

enjoys attractive algebraic properties such as a striking **arithmetic-geometric mean** relation & elegant links with **elliptic-function theory**.

- The fraction **presented a serious computational challenge**, which we could not resist.

♣ Resolving this challenge lead to four quite subtle published papers:

  – two published in *Experimental Mathematics* **13** (2004), 275–286, 287–296 and;

  – two in *The Ramanujan Journal* **13**, (2007), 63–101 and **16** (2008), 285–304.

In **Part I** (colloquium): we show how to rapidly evaluate $\mathcal{R}$ for any positive reals a, b, η. The problematic case being $a \approx b$—then subtle transformations allow rapid evaluation.

- On route we find, e.g., that for rational $a = b$, $\mathcal{R}_\eta$ is an L-series with a 'closed-form.'

- We ultimately exhibit an **algorithm yielding D digits of $\mathcal{R}$ in $O(D)$ iterations**.*

In **Part II** (seminar): we address the harder theoretical and computational dilemmas arising when **(i)** parameters are allowed to be complex, or **(ii)** more general fractions are used.

*The big-O constant is independent of the positive-real triple a, b, η.

2. Preliminaries

PART I. Entry 12 of Chapter 18 of *Ramanujan's Second Notebook* [BeIII] gives the beautiful:

$$\mathcal{R}_\eta(a,b) = \cfrac{a}{\eta + \cfrac{b^2}{\eta + \cfrac{4a^2}{\eta + \cfrac{9b^2}{\eta + \ddots}}}} \tag{1.1}$$

which we interpret—in most of the present treatment—for **real, positive** $a, b, \eta > 0$.

Remarkably, for such parameters, $\mathcal{R}$ satisfies an AGM relation:

$$\mathcal{R}_\eta\left(\frac{a+b}{2}, \sqrt{ab}\right) = \frac{\mathcal{R}_\eta(a,b) + \mathcal{R}_\eta(b,a)}{2} \tag{1.2}$$

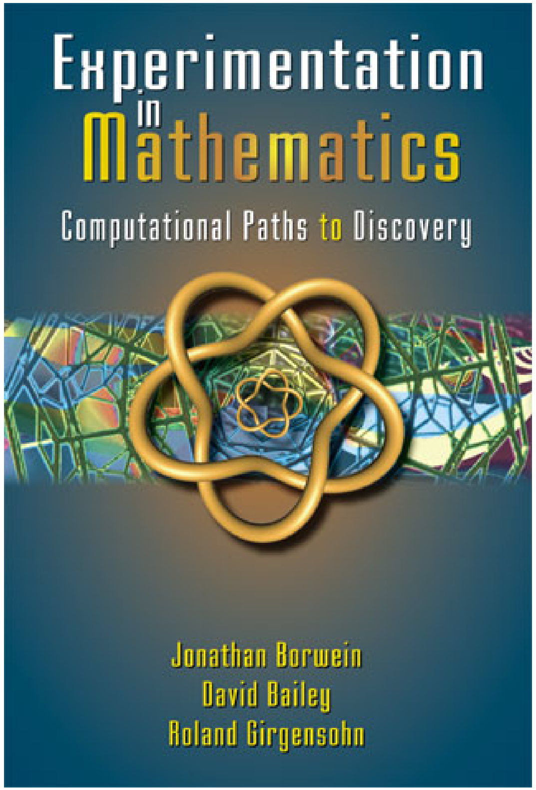
1. (1.2) is one of many relations we develop for computation of $\mathcal{R}_\eta$.

2. **The hard cases occur when b is near to a,** including the case $a = b$.

3. We eventually exhibit an algorithm uniformly of geometric/linear convergence across the positive quadrant $a, b > 0$.

4. Along the way, we find attractive identities, such as that for $\mathcal{R}_\eta(r, r)$, with r rational.

5. Finally, we consider complex a, b—obtaining theorems and conjectures on the domain of validity for the AGM relation (1.2).



Research started in earnest when we noted $\mathcal{R}_1(1,1)$ **'seemed close to'** $\log 2$.

> *Such is the value of experiment: one can be led into deep waters.*

As can be seen by 'cancellation' of the η elements down the fraction:

$$\mathcal{R}_\eta(a,b) = \mathcal{R}_1(a/\eta, b/\eta),$$

valid because the fraction converges.

> Discussed in Ch. 1 of *Experimentation in Mathematics*.

To prove convergence we put $a/\mathcal{R}_1$ in **RCF** (**reduced continued fraction**) form:

$$\mathcal{R}_1(a,b) = \frac{a}{[A_0; A_1, A_2, A_3, \ldots]} \tag{2.1}$$

$$:= \cfrac{a}{A_0 + \cfrac{1}{A_1 + \cfrac{1}{A_2 + \cfrac{1}{A_3 + \ddots}}}}$$

where the A_i are all positive real.

It is here [Ramanujan's work on elliptic and modular functions] that both the profundity and limitations of Ramanujan's knowledge stand out most sharply.
— G.H. Hardy

Inspection of $\mathcal{R}$ yields the RCF elements explicitly and gives the asymptotics of A_n:

For **even** n

$$A_n = \frac{n!^2}{(n/2)!^4} \, 4^{-n} \, \frac{b^n}{a^n} \quad \sim \quad \frac{2}{\pi \, \mathbf{n}} \, \frac{b^n}{a^n}.$$

For **odd** n

$$A_n = \frac{((n-1)/2!)^4}{n!^2} \, 4^{n-1} \, \frac{a^{n-1}}{b^{n+1}} \quad \sim \quad \frac{\pi}{2 \, ab \, \mathbf{n}} \, \frac{a^n}{b^n}.$$

• This representation leads immediately to:

Theorem 2.1: For any positive real pair a, b the fraction $\mathcal{R}_1(a, b)$ converges.

Proof: An RCF converges **iff** $\sum A_i$ diverges. (This is the *Seidel–Stern theorem* [Kh,LW].)

In our case, such divergence is evident for every choice of real $a, b > 0$. ©

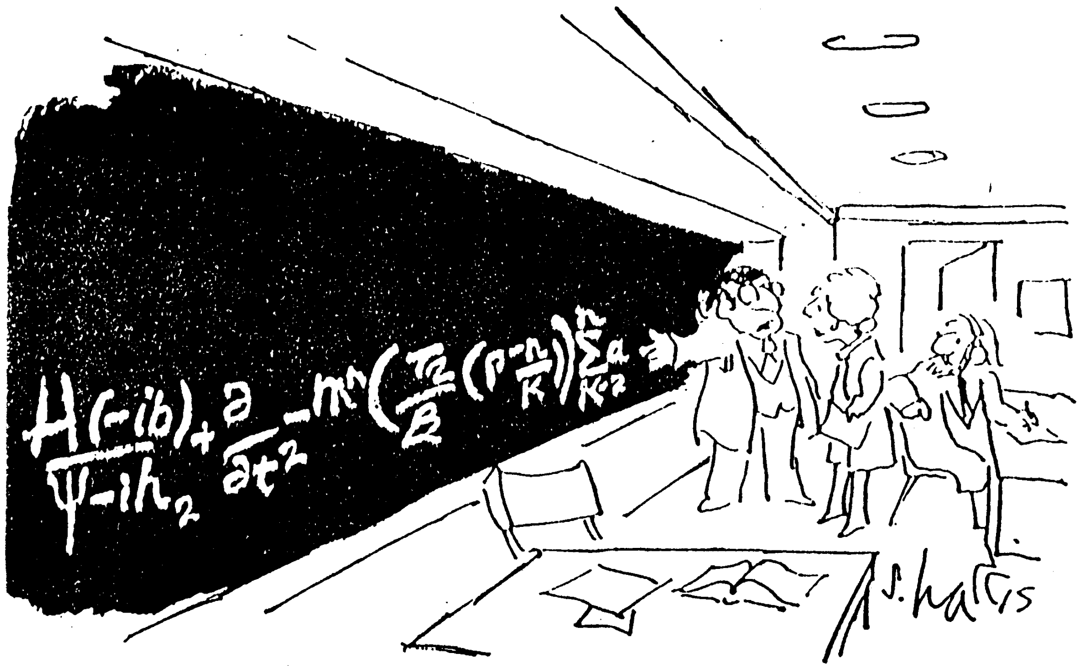
We later show a different fraction for $\mathcal{R}(a)$, and other computationally efficient constructs.

- Note for $a = b$, divergence of $\sum A_i$ is only *logarithmic* — a true indication of slow convergence (we wax more quantitatively later).

- **Our interest started** with asking how, for $a > 0$, to (rapidly) evaluate

$$\mathcal{R}(a) := \mathcal{R}_1(a, a)$$

and thence to prove suspected identities.



"But this *is* the simplified version for the general public."

3. Hyperbolic-elliptic Forms

Links between standard **Jacobi theta functions**

$$\theta_2(q) := \sum q^{(n+1/2)^2}, \qquad \theta_3(q) := \sum q^{n^2}$$

and **elliptic integrals** yield various results. We start with:

Theorem 3.1: For real $y, \eta > 0$ and $q := e^{-\pi y}$

$$\eta \sum_{k \in D} \frac{\operatorname{sech}(k\pi y/2)}{\eta^2 + k^2} = \mathcal{R}_\eta(\theta_2^2(q), \theta_3^2(q)),$$

$$\eta \sum_{k \in E} \frac{\operatorname{sech}(k\pi y/2)}{\eta^2 + k^2} = \mathcal{R}_\eta(\theta_3^2(q), \theta_2^2(q)),$$

where D, E denote respectively the odd, even integers. Consequently, **the Ramanujan AGM identity (1.2) holds for positive triples** η, a, b.

Proof: The sech relations are proved—in equivalent form—in Berndt's treatment (Vol II, Ch. 18) of **Ramanujan's Notebooks** [BeIII].

For the AGM, assume $0 < b < a$. The assignments

$$\theta_2^2(q)/\theta_3^2(q) := b/a, \qquad \eta := \theta_2^2(q)/b$$

are possible (since $b/a \in [0,1)$, see [BB]) and implicitly define q, η, and together with

$$\theta_2^2(q) + \theta_3^2(q) = \theta_3^2(\sqrt{q}),$$

$$2\,\theta_2(q)\theta_3(q) = \theta_2^2(\sqrt{q})$$

and repeated use of the sech sums above yield

$$\mathcal{R}_1\left(\theta_3^2(q)/\eta, \theta_2^2(q)/\eta\right) + \mathcal{R}_1\left(\theta_2^2(q)/\eta, \theta_3^2(q)/\eta\right)$$

$$= 2\mathcal{R}_1\left(\theta_3^2(\sqrt{q})/(2\eta), \theta_2^2(\sqrt{q})/(2\eta)\right).$$

Since

$$\theta_2^2(q) = \eta\, b, \qquad \theta_3^2(q) = \eta\, a$$

the AGM identity (1.2) holds for all pairs with $a > b > 0$. The case $0 < a < b$ is handled by symmetry, or on starting by setting $\theta_2^2(q)/\theta_3^2(q) := a/b$. ©

- The wonderful sech identities above stem from classical work of Rogers, Stieltjes, Preece, and of course Ramanujan [BeIII] in which one finds the earlier work detailed.

- The proof given for the AGM identity has been claimed for various complex a, b sometimes over ambitiously.*

**Mea culpa* indirectly.

- These sech series can be used to establish two numerical series involving the **complete elliptic integral**

$$K(k) := \int_0^{\pi/2} \frac{1}{\sqrt{1-k^2\sin^2(\theta)}}\, d\theta.$$

We write $\mathrm{K} := \mathrm{K}(k)$, $\mathrm{K}' := \mathrm{K}(k')$ with $k' := \sqrt{1-k^2}$.*

Theorem 3.2: For $0 < b < a$ and $k := b/a$ we have

$$\mathcal{R}_1(a,b) = \frac{\pi a\,\mathrm{K}}{2} \sum_{n\in Z} \frac{\operatorname{sech}\left(n\pi\frac{\mathrm{K}'}{\mathrm{K}}\right)}{\mathrm{K}^2+\pi^2a^2n^2}. \tag{3.1}$$

Correspondingly, for $0 < a < b$ and $k := a/b$ we have

$$\mathcal{R}_1(a,b) = 2\pi b\,\mathrm{K} \sum_{n\in D} \frac{\operatorname{sech}\left(n\pi\frac{\mathrm{K}'}{2\mathrm{K}}\right)}{4\mathrm{K}^2+\pi^2b^2n^2}. \tag{3.2}$$

*$K(k)$ is fast computable via the AGM iteration.

Proof: The series follow from the assignments

$$\theta_3^2(q)/\eta := \max(a,b),\ \theta_2^2(q)/\eta := \min(a,b)$$

and Jacobi's *nome* relations

$$e^{-\pi K'/K} = q, \qquad K(k) = \frac{\pi}{2}\theta_3^2(q)$$

inserted appropriately into Theorem 3.1. ©

- The sech-elliptic series (3.1-2) allow fast computation of $\mathcal{R}_1$ for b **not too near** a.

- D digits for $\mathcal{R}_1(a,b)$ requires $O(D\,\mathsf{K}/\mathsf{K}')$ summands.

- So, another motive for the following analysis was **slow convergence of the sech-elliptic forms** for $b \approx a$.

- We also have attractive evaluations such as

$$\mathcal{R}_1\left(1, \frac{1}{\sqrt{2}}\right) = \frac{\pi}{2} \mathsf{K}\left(\frac{1}{\sqrt{2}}\right) \sum_{n \in Z} \frac{\operatorname{sech}(n\pi)}{\mathsf{K}^2(1/\sqrt{2}) + n^2\pi^2}.$$

Here

$$\mathsf{K}\left(\frac{1}{\sqrt{2}}\right) = \frac{\Gamma^2(1/4)}{4\sqrt{\pi}},$$

see [BB].

- There are similar series for $\mathcal{R}_1(1, k_N)$ at the N-*th singular value*, [BBa2].

- A similar relation for $\mathcal{R}_1\left(1/\sqrt{2},1\right)$ obtains via (3.2), and via the AGM relation (1.2) yields the oddity:

$$\mathcal{R}_1\left(\frac{1+\sqrt{2}}{2\sqrt{2}},\frac{1}{2^{1/4}}\right)=\pi\,\mathsf{K}\left(\frac{1}{\sqrt{2}}\right)\sum_{n\in Z}\frac{\mathrm{sech}(n\pi/2)}{4\mathsf{K}^2(1/\sqrt{2})+n^2\pi^2}.$$

- But we have no closed forms for $a\neq b$.

4. Six Forms for $\mathcal{R}(a)$

- Recalling that $\mathcal{R}(a) := \mathcal{R}_1(a, a)$, we next derive relations for the hard case $b = a$.

Interpreting (3.1) as a Riemann-integral in the limit as $b \to a^-$ (for $a > 0$), gives a slew of relations involving the **digamma function** [St,AS] $\psi := \frac{\Gamma'}{\Gamma}$ and the **Gaussian hypergeometric function**

$$F = {}_2F_1(a, b; c; \cdot).$$

- The following identities are presented in an order that can be serially derived:

Evaluating $\mathcal{R}(a)$

Proposition: For all $a > 0$:

$$
\begin{aligned}
\mathcal{R}(a) &= \int_0^\infty \frac{\operatorname{sech}\left(\frac{\pi\, x}{2\, a}\right)}{1+x^2}\, dx \\
&= 2\, a \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{1+(2\, k-1)\, a} \\
&= \frac{1}{2}\left(\psi\left(\frac{3}{4}+\frac{1}{4a}\right)-\psi\left(\frac{1}{4}+\frac{1}{4a}\right)\right) \\
&= \frac{2a}{1+a}\, F\left(\frac{1}{2a}+\frac{1}{2}, 1; \frac{1}{2a}+\frac{3}{2}; -1\right) \\
&= 2\int_0^1 \frac{t^{1/a}}{1+t^2}\, dt \\
&= \int_0^\infty e^{-x/a} \operatorname{sech}(x)\, dx.
\end{aligned}
$$

Exploiting the Various Forms

The first series or t-integral yield a recurrence

$$\mathcal{R}(a) = \frac{2a}{1+a} - R\left(\frac{a}{1+2a}\right),$$

while known relations for digamma [AS,St] lead to

$$\mathcal{R}(a) = C(a) + \frac{\pi}{2}\sec\left(\frac{\pi}{2a}\right) - \tag{4.1}$$

$$\frac{2a^2(1+8a-106a^2+280a^3+9a^4)}{1-12a+25a^2+120a^3-341a^4-108a^5+315a^6}$$

for a "rational-zeta" series [BBC]:

$$C(a) = \frac{1}{2}\sum_{n\geq 1}\{\zeta(2n+1)-1\}\,\frac{(3a-1)^{2n}-(a-1)^{2n}}{(4a)^{2n}}$$

- Note that (4.1), while rapidly convergent for some a, has sec poles, some being cancelled by the rational function.

  - We require $a > 1/9$ for convergence of the rational-zeta sum.

  - However, the recurrence relation above can be used to force convergence of such a rational-zeta series.

- The hypergeometric form for $\mathcal{R}(a)$ is of special interest because [BBa1] of:

The Gauss continued fraction

$$F(\gamma, 1; 1+\gamma; -1) = [\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots] \tag{4.2}$$

$$= \cfrac{1}{\alpha_1 + \cfrac{1}{\alpha_2 + \cfrac{1}{\alpha_3 + \cfrac{1}{\alpha_4 + \ddots}}}}$$

Here, we have explicitly $\alpha_1 = 1$ and

$$\alpha_n = \gamma\,((n-1)/2)!)^{-2}\,(n-1+\gamma)\prod_{j=1}^{(n-3)/2}(j+\gamma)^2$$

$$n = 3, 5, 7, \ldots$$

$$\alpha_n = \gamma^{-1}\,(n/2-1)!^2\,(n-1+\gamma)\prod_{j=1}^{n/2-1}(j+\gamma)^{-2}$$

$$n = 2, 4, 6, \ldots$$

Asymptotic Expansions need not Converge

An interesting aspect of *formal* analysis is based upon the first sech-integral for $\mathcal{R}(a)$. Expanding and using a representation of the even **Euler numbers**

$$E_{2n} := (-1)^n \int_0^\infty \operatorname{sech}(\pi x/2) x^{2n}\, dx$$

one obtains

$$\mathcal{R}(a) \sim \sum_{n \geq 0} E_{2n}\, a^{2n+1},$$

yielding an **asymptotic series of** *zero* **radius** of convergence. Here the E_{2n} commence

$$1, -1, 5, -61, 1385, -50521, 2702765 \ldots$$

Moreover, for the asymptotic error, we have [BBa2,BCP]:

$$\left|\mathcal{R}(a) - \sum_{n=1}^{N-1} E_{2n}\, a^{2n+1}\right| \leq |E_{2N}|\; a^{2N+1},$$

- It is a classic theorem of Borel [St,BBa2] that for every real sequence (a_n) there is a C^∞ function f on R with $f^{(n)}(0) = a_n$.

- Who knew they could be so explicit?

- The oft-stated success of **Padé approximation** is well exemplified in our case.

  If one takes the unique $(3,3)$ Padé form* we obtain

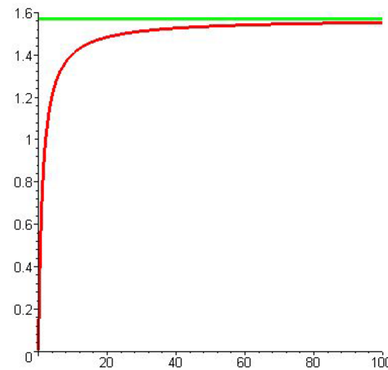
$$\mathcal{R}(a) \approx a\,\frac{1 + 90\,a^2 + 1433\,a^4 + 2304\,a^6}{1 + 91\,a^2 + 1519\,a^4 + 3429\,a^6}.$$

- This is remarkably good for small a; e.g., yielding $\mathcal{R}(1/10) \approx 0.09904494$ correct to the implied precision. For $\mathcal{R}(1/2)$ and the $(30,30)$ Padé approximant† one obtains 4 good digits.

*Thus, top and bottom of $\mathcal{R}(a)/a$ have degree 3 in the variable a^2.

†Numerator and denominator have degree 30 in a^2.

- Though the convergence rate is slower for larger a, the method allows, say, *graphing* $\mathcal{R}$ to reasonable precision.



A plot of $\mathcal{R}$

Having noted a **formal expansion** at $a = 0$, we naturally asked for: **An asymptotic form valid for *large* a**?

- Via a typical asymptotic development, we find *more*, namely **a convergent expansion** for all $a > 1$:

Starting with our second sech integral

$$\mathcal{R}(a) = \int_0^\infty e^{-x/a}\,\mathrm{sech}\,x\,dx$$

we again use the Euler numbers and known **Hurwitz-zeta** evaluations of sech-power integrals for *odd* powers. We obtain a convergent series valid at least for real $a > 1$:

$$\begin{aligned}\mathcal{R}(a) &= \frac{\pi}{2}\sec\left(\frac{\pi}{2a}\right) - 2\sum_{m\in D^+}\frac{\beta(m+1)}{a^m}\\ &= 2\sum_{k\geq 0}\beta(k+1)\left(-\frac{1}{a}\right)^k\end{aligned}$$

Here D^+ denotes positive odd integers and

$$\beta(s) := 1/1^s - 1/3^s + 1/5^s - \cdots$$

is the Catalan primitive L-series mod 4.

- Remarkably, we find the leading terms for large a involve **Catalan's constant** $G := \beta(2)$ via

$$\mathcal{R}(a) = \frac{\pi}{2} - \frac{2G}{a} + \frac{\pi^3}{16a^2} - \cdots,$$

a development difficult to infer from casual inspection of Ramanujan's fraction.

- Even the asymptote $\mathcal{R}(\infty) = \pi/2$ is hard to so infer, though it follows from various of the previous representations for $\mathcal{R}(a)$. Using recurrence relations and various expansions we also obtain results pertaining to the derivatives of $\mathcal{R}$, notably

$$\mathcal{R}'(1) = 8(1 - G), \qquad \mathcal{R}'\left(\frac{1}{2}\right) = \frac{\pi^2}{24}.$$

- A peculiar property of ψ leads to an exact evaluation of the imaginary part of the digamma representation of $\mathcal{R}(a)$ when a lies on the circle

$$C_{1/2} := \{z : |z - \frac{1}{2}| = \frac{1}{2}\}$$

in the complex plane.

Imaginary parts of the needed digamma values have a *closed form* [AS,St], and we obtain

$$\mathrm{Im}(\mathcal{R}(a)) = \frac{\pi}{2} \operatorname{cosech}\left(\frac{\pi y}{2}\right) - \frac{1}{y}$$

for

$$y := i\left(1 - \frac{1}{a}\right) \quad \text{and} \quad a \in C_{1/2}.$$

- Note that y is always real, and we have an elementary form for $\mathrm{Im}(\mathcal{R})$ on the given continuum set. Admittedly we have not yet discussed complex parameters; we do that later.

- The Ramanujan fraction converges at least for $a = b$, $\mathrm{Re}(a) \neq 0$, and it is instructive to compare numerical evaluations of imaginary parts via the above cosech identity.

Firmament — made for Coxeter at ninety

5. The $\mathcal{R}$ Function at Rational Arguments

For positive integers p, q, we have from the above

$$\mathcal{R}\left(\frac{p}{q}\right) = 2p\left(\frac{1}{q+p} - \frac{1}{q+3p} + \frac{1}{q+5p} - \dots\right),$$

which is in the form of a **Dirichlet L-function**. One way to evaluate L-functions is via Fourier-transforms to pick out terms from a general logarithmic series. An equivalent, elementary form for the digamma at rational arguments is a celebrated result of Gauss. In our case

$$\mathcal{R}\left(\frac{p}{q}\right) = \sum_{\text{odd } k>0}^{4p} e^{-2\pi i\, k(q+p)/(4p)} \times$$

$$\left\{-\log\left(1 - e^{2\pi i\, k/(4p)}\right) - \frac{1}{n}\sum_{n=1}^{q+p-1} e^{2\pi i\, kn/(4p)}\right\}.$$

After various simplifications, forcing everything to be real-valued, we arrive at a finite closed form. Namely:

$$\begin{aligned}
\mathcal{R}\left(\frac{p}{q}\right) &= -2p\sum_{n=1}^{p+q-1}\frac{1}{n}\left(\delta_{n\equiv p+q \bmod 4p}-\delta_{n\equiv 3p+q \bmod 4p}\right)\\
&- 2\sum_{0\ <\text{odd}\ k<2p}\cos\left(\frac{(p+q)k\pi}{2p}\right)\log\left(2\sin\left(\frac{\pi k}{4p}\right)\right)\\
&+ 2\pi\sum_{0\ <\text{odd}\ k<2p}\left(\frac{1}{2}-\frac{k}{4p}\right)\sin\left(\frac{(p+q)k\pi}{2p}\right) \qquad (5.1)
\end{aligned}$$

- When $q=1$, so that we seek $\mathcal{R}(p)$ for integer p, the first, rational sum vanishes. The finite series (5.1) with $O(p+q)$ total terms leads quickly to exact evaluations such as:

Exact Evaluations

$$\mathcal{R}(1/4) = \frac{\pi}{2} - \frac{4}{3}, \qquad \mathcal{R}(1/3) = 1 - \log 2,$$

$$\mathcal{R}(\mathbf{1}) = \log \mathbf{2}, \qquad \mathcal{R}(1/2) = 2 - \pi/2,$$

$$\mathcal{R}(2/3) = 4 - \frac{\pi}{\sqrt{2}} - \sqrt{2}\log(1+\sqrt{2}),$$

$$\mathcal{R}(3/2) = \pi + \sqrt{3}\log\left(2-\sqrt{3}\right),$$

$$\mathcal{R}(2) = \sqrt{2}\left\{\frac{\pi}{2} - \log(1+\sqrt{2})\right\},$$

$$\mathcal{R}(3) = \frac{\pi}{\sqrt{3}} - \log 2,$$

- And, of course, many other attractive forms.

- From (5.1) for positive integer q one has

$$\mathcal{R}(1/q) = \text{rational} + (-1)^{(q-1)/2} \log 2 \qquad (q \text{ odd})$$

$$\mathcal{R}(1/q) = \text{rational} + (-1)^{q/2}\,\pi/2 \qquad (q \text{ even})$$

as also follow from $\mathcal{R}(1) = \log 2$, $\mathcal{R}(1/2) = 2 - \pi/2$ and

$$\mathcal{R}\left(\frac{1}{q}\right) = \frac{2}{q-1} - \mathcal{R}\left(\frac{1}{q-2}\right).$$

- An alluring evaluation involves the *golden mean*:

$$\mathcal{R}(5) = \frac{\pi}{\sqrt{\tau\sqrt{5}}} + \log 2 - \sqrt{5}\,\log\tau, \qquad (\tau := (1+\sqrt{5})/2).$$

- Such evaluations—based on (5.1)—can involve quite delicate symbolic work.

- We have not analyzed evaluating $\mathcal{R}(a)$ for *irrational* a by approximating a first via high-resolution rationals, and then using (5.1).

  Such a development would be of both computational and theoretical interest.

- Armed with exact knowledge of $\mathcal{R}(p/q)$ we find some interesting Gauss-fraction results, in the form of rational multiples of

$$F(\gamma, 1; 1+\gamma; -1) = [\alpha_1, \alpha_2, \ldots].$$

For example, (4.2) yields

$$\mathcal{R}(1) = \log 2 = \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{1 + \ddots}}}},$$

But alas the beginnings of this fraction are misleading; subsequent elements a_n run

$$\log 2 = [1, 2, 3, 1, 5, \frac{2}{3}, 7, \frac{1}{2}, 9, \frac{2}{5}, \ldots],$$

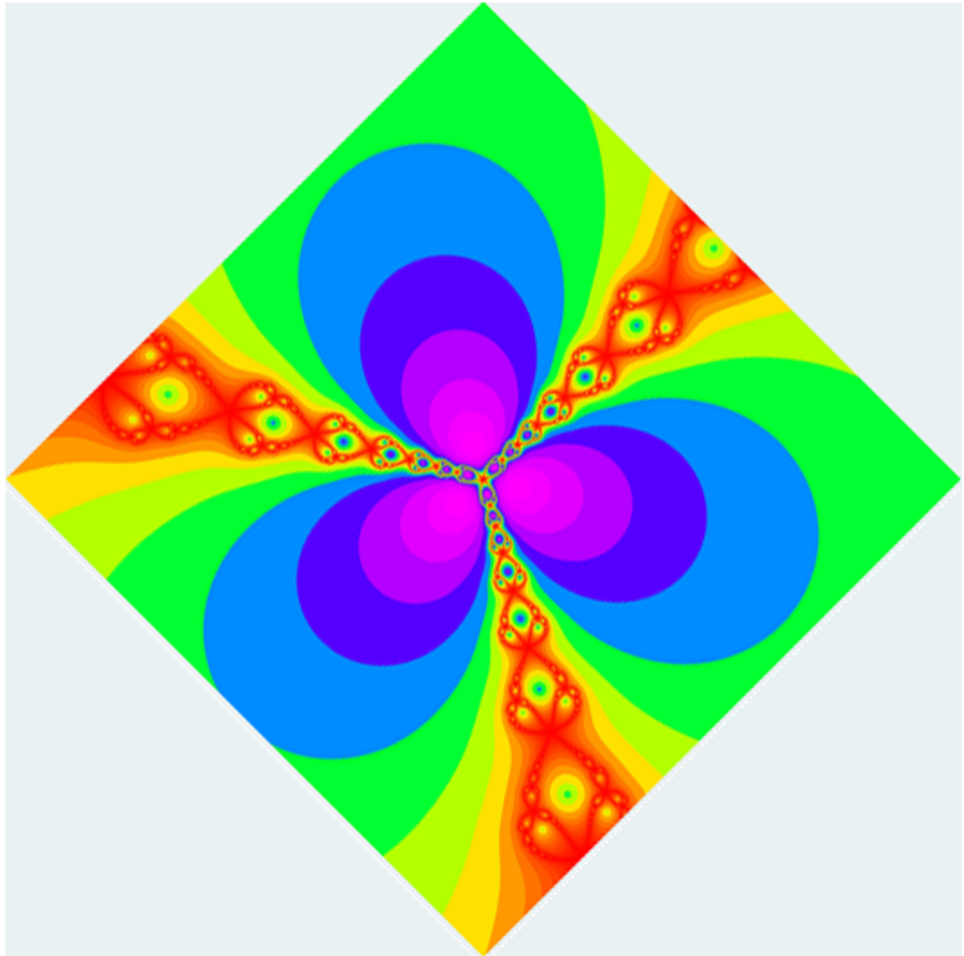
being as $\alpha_n = n, 4/n$ resp. for n odd, even.

Similarly, one can derive

$$2 - 2\log 2 = [1^3, r_2, 2^3, r_4, 3^3, r_6, 4^3, \ldots],$$

where the even-indexed fraction elements r_{2n} are computable rationals.

- Though these RCFs do not have integer elements, the growths of the α_n provide a clue to the convergence rate, which we study in a subsequent section.



6. Transformation of $\mathcal{R}_1(a,b)$

The big step. We noted that the sech-elliptic series (3.1) (also (3.2)) will converge slowly when $b \approx a$, yet in Sections 4, 5 we successfully addressed the case $b = a$.

We now establish a series representation when $b < a$ but b is very near to a.

We employ the wonderful fact that sech is its own Fourier transform, in that

$$\int_{-\infty}^{\infty} e^{i\,\gamma x}\,\mathrm{sech}(\lambda x)\,dx = \frac{\pi}{\lambda}\mathrm{sech}\left(\frac{\pi\gamma}{2\lambda}\right).$$

Using this relation, one can perform a *Poisson transform* of the sech-elliptic series (3.1).

- The success of the transform depends on analyzing

$$I(\lambda, \gamma) := \int_{-\infty}^{\infty} \frac{\mathrm{sech}\lambda x}{1 + x^2} e^{i\,\gamma x}\, dx.$$

One may obtain the differential equation:

$$-\frac{\partial^2 I}{\partial \gamma^2} + I = \frac{\pi}{\lambda} \mathrm{sech}\left(\frac{\pi\gamma}{2\lambda}\right)$$

and solve it — after some *machinations*.

We obtained

$$I(\lambda, \gamma) \;=\; \frac{\pi}{\cos\lambda} e^{-\gamma} + \frac{2\pi}{\lambda} \sum_{d \in D^+} \frac{(-1)^{(d-1)/2} e^{-\pi d \gamma/(2\lambda)}}{1 - \pi^2 d^2/(4\lambda^2)}.$$

where D^+ denotes the positive odd integers.

- When $\lambda = \pi D/2$ for some odd D, the $1/\cos$ pole conveniently cancels a corresponding pole in the summation, and the result can be inferred either by avoiding $d = D$ in the sum and inserting a precise residual term

$$\Delta I = \pi(-1)^{(D-1)/2} e^{-\gamma}(\gamma + 1/2)/\lambda,$$

or more simply by taking a numerical limit as $\lambda \to \pi D/2$.

- When $\gamma \to 0$ we can recover the ψ-function form of the integral of $\mathrm{sech}(\lambda x)/(1+x^2)$.

Via **Poisson transformation** of (3.1) we obtain, for $0 < b < a$,

$$\mathcal{R}_1(a,b) = \mathcal{R}\left(\frac{\pi a}{2\mathsf{K}'}\right) + \frac{\pi}{\cos\frac{\mathsf{K}'}{a}} \frac{1}{e^{2\mathsf{K}/a}-1} \qquad (6.1)$$

$$+ \; 8\pi a\, \mathsf{K}' \sum_{d\in D^+} \frac{(-1)^{(d-1)/2}}{4\mathsf{K}'^2 - \pi^2 d^2 a^2} \frac{1}{e^{\pi d\, \mathsf{K}/\mathsf{K}'}-1}$$

where $k := b/a$, $\mathsf{K} := \mathsf{K}(k)$, $\mathsf{K}' := \mathsf{K}(k')$, and D^+ again denotes the positive odd integers.

A similar Poisson transform obtains from (3.2) in the case $b > a > 0$. Such transforms appear recondite, but we have what we desired: *convergence is rapid for $b \approx a$*: because $K/K' \sim \infty$.

7. Convergence Results

For an RCF $x = [a_0, a_1, \ldots]$ (i.e., each a_i is nonnegative but need not be integer) one has the usual recurrence relations*

$$p_n = a_n \cdot p_{n-1} + 1 \cdot p_{n-2},$$

$$q_n = a_n \cdot q_{n-1} + 1 \cdot q_{n-2},$$

with

$$(p_0, p_{-1}, q_0, q_{-1}) := (a_0, 1, 1, 0).$$

*The corresponding matrix scheme with b_n inserted for '1' applies generally to CF's.

We also have the approximation rule for the convergents

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}},$$

so that convergence rates can be bounded by virtue of the growth of the q_n.

- One may iterate the recurrence in various ways, obtaining for example

$$q_n = \left(1 + a_n a_{n-1} + \frac{a_n}{a_{n-2}}\right) q_{n-2} - \frac{a_n}{a_{n-2}} q_{n-4}.$$

An immediate application is

Theorem 7.1: For the **RCF form of the Gauss fraction**, $F(\gamma, 1; 1+\gamma; -1) = [\alpha_1, \alpha_2, \ldots]$, and for $\gamma > 1/2$ we have

$$\left| F - \frac{p_n}{q_n} \right| < \frac{c}{8^{n/2}},$$

where c is an absolute constant.

Remark: One can obtain sharper γ-dependent bounds. We intend here just to show geometric convergence; i.e. that the number of good digits grows at least linearly in the number of iterates.

Also note that for the $\mathcal{R}(a)$ evaluation of current interest, $\gamma = 1/2 + 1/(2a)$ so that the condition on γ is natural.

Proof: From the element assignments in (4.2) we have

$$\alpha_n \alpha_{n-1} = \frac{4}{(n-1)^2}(n-1+\gamma)(n-2+\gamma);$$

$$1 < n \text{ odd },$$

$$\alpha_n \alpha_{n-1} = \frac{1}{(n/2-1+\gamma)^2}(n-1+\gamma)(n-2+\gamma);$$

$$n \text{ even }.$$

We also have $q_1 = 1$, $q_2 = 1 + 1/\gamma > 2$ so that for sufficiently large n we have $\alpha_n \alpha_{n-1} + 1 > 4$,2 respectively as n is odd, even.

From the estimate $q_n > (\alpha_n \alpha_{n-1} + 1) q_{n-2}$ the desired bound follows. ©

- A clever computational acceleration for Gauss fractions is described in [BBa1,AAR,LW]. Consider the previously displayed fraction $\log 2 = [1, 2, 3, 1, 5, 2/3, \ldots]$.

Generally a "tail" t_N of this construct, meaning a subfraction starting from the N-th element, runs like so:

$$t_N := \cfrac{1}{\frac{4}{N} + \cfrac{1}{N+1+\cfrac{1}{\frac{4}{N+2} + \cfrac{1}{N+3+\ddots}}}}.$$

We *hope* this tail t_N is near the *periodic* fraction

$$[4/N, N, 4/N, N, \ldots] = N(\sqrt{2} - 1)/2.$$

- This suggests that if we evaluate the Gauss fraction and stop at element $4/N$, this one element should be replaced by $2(1+\sqrt{2})/N$.

  - in our own numerical experiments, this trick always adds a few digits precision.

- As suggested in [LW], there are higher-order takes of this idea; e.g., the use of longer periods for the tail sub-fraction.

  - as the reference shows, experimentally, the acceleration can be significant.

Convergence Rates

We now attack convergence of **the Ramanujan RCF**, viz

$$\frac{a}{\mathcal{R}_1(a,b)} = [A_0; A_1, A_2, A_3, \ldots].$$

with the A_i defined subsequent to (2.1).

- The q_n convergents are linear combinations of $a^i \cdot b^j$'s for i, j even integers, and can be explicitly determined.

This leads to

$$q_n \geq 1 + \frac{b^{n-2}}{a^n} \prod_{m \text{ even}}^{n} \left(1 - \frac{1}{m}\right)^2$$

$$> 1 + \frac{1}{2n} \frac{b^{n-2}}{a^n}$$

for n even,

$$q_n \geq \frac{1}{b^2} + \frac{a^{n-1}}{b^{n+1}} \prod_{m \text{ even}}^{n-1} \left(\frac{m}{m+1}\right)^2$$

$$> \frac{1}{b^2} + \frac{1}{n} \frac{a^{n-1}}{b^{n+1}}$$

for n odd.

- We are ready for a convergence result —which can be sharpened — for the *original Ramanujan construct:*

Theorem 7.2: For the **Ramanujan RCF**

$$\frac{a}{\mathcal{R}_1(a,b)} = [A_0; A_1, A_2, A_3, \ldots]$$

we have for $b > a > 0$

$$\left|\frac{a}{\mathcal{R}_1(a,b)} - \frac{p_n}{q_n}\right| < \frac{2nb^4}{(b/a)^n},$$

while for $a > b > 0$ we have

$$\left|\frac{a}{\mathcal{R}_1(a,b)} - \frac{p_n}{q_n}\right| < \frac{nb/a}{(a/b)^n}.$$

Proof: The given bounds follow directly upon inspection of the products $q_n\, q_{n+1}$. ©

- As previously intimated, convergence for $a = b$ is slow. What we can prove is:

Theorem 7.3: For real $a > 0$, we have

$$\left|\frac{a}{\mathcal{R}(a)} - \frac{p_n}{q_n}\right| < \frac{c(a)}{n^{h(a)}},$$

where $c(a), h(a)$ are n-independent constants.

The exponent $h(a)$ can be taken to be

$$h(a) = c_0 \min(1, 4\pi^2/a^2)$$

where the constant c_0 is absolute (can be sharpened—and made more explicit.)

Remark: The bound is computationally poor, but convergence does occur. Indeed, for $a = b$ or even $a \approx b$ we now have many other, rapidly convergent options.

Proof: Inductively, assume for (n-independent) $d(a), g(a)$ and $n \in [1, N-1]$ that $q_n < dn^g$. The asymptotics following (2.1) mean $A_n > f(a)/n$ for an n-independent f.

Then we have a bound for the next q_N:

$$q_N > \frac{f}{N} d(N-1)^g + d\,(N-2)^g\,.$$

For $g < 1$, $0 < x \leq 1/2$ we have

$$(1-x)^g > 1 - gx - gx^2,$$

and the constants d, g can be arranged so that $q_N > dN^g$; hence the induction goes through. ©

We reprise the import of these three theorems:

- (**Theorem 7.1**) The *Gauss fraction* for $\mathcal{R}(a)$ exhibits (*at least*) geometric/linear convergence.

- (**Theorem 7.2)** So does the *original Ramanujan form* $\mathcal{R}_1(a, b)$ when a/b or b/a is (significantly) greater than unity .

- (**Theorem 7.3**) When $a = b$ we still have convergence in the original form.

  As suggested by Theorem 7.3 convergence is far below geometric/linear.

8. A Uniformly Convergent Algorithm

- We now give a complete algorithm to evaluate the original fraction $\mathcal{R}_\eta(a,b)$ for positive real parameters.

- Convergence is uniform — for any positive real triple η, a, b we obtain D good digits in no more than $c\,D$ computational iterations, where c is independent of the size of η, a, b.*

*'Iterations' mean continued-fraction recurrence steps, or series-summand additions.

Algorithm for $\mathcal{R}_\eta(a,b)$ with real $\eta, a, b > 0$:

1. Observe that $\mathcal{R}_\eta(a,b) = \mathcal{R}_1(a/\eta, b/\eta)$ so with impunity we may assume $\eta = 1$ and evaluate only $\mathcal{R}_1$.

2. If ($a/b > 2$ or $b/a > 2$) return original (1.1), or equivalently (2.1);

3. If ($a = b$) $\{$
   if ($a = p/q$ rational) return finite form (5.1); else return the Gauss RCF (4.2) or rational-zeta form (4.1) or (4.3) or some other scheme such as rapid ψ computations; $\}$

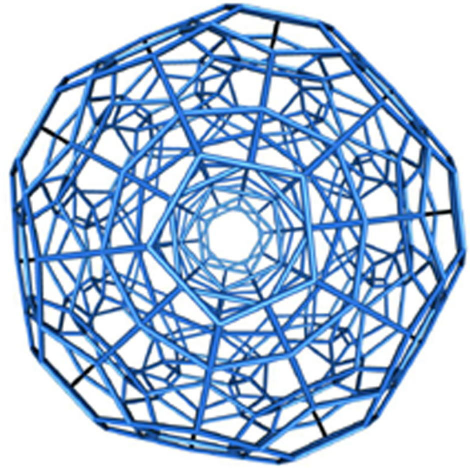
4. If ($b < a$) $\{$
   if (b is *not too close* to a)*, return sech-elliptic result (3.1); else return Poisson-transform result (6.1); $\}$

5. (We have $b > a$) Return, as in (1.2),

$$2\mathcal{R}_1\left((a+b)/2, \sqrt{ab}\right) - \mathcal{R}_1(b,a).$$

©

*Say, $|1 - b/a| > \varepsilon > 0$ for any fixed $\varepsilon > 0$.



- It is an implicit tribute to Ramanujan's ingenuity that the final step (4) of the algorithm allows the entire procedure to go through for *all* positive real parameters.

- One may avoid step (4) by invoking a Poisson transformation of (3.2), but Ramanujan's AGM identity is finer!

PART II: 9. About Complex Parameters

PART II. *Complex* parameters a, b, η are **complex**, as we found via extensive experimentation.

We attack this by assuming $\eta = 1$ and defining

$$\mathcal{D} := \{(a,b) \in \mathbb{C}^2 : \mathcal{R}_1(a,b) \text{ converges}\},$$

i.e., the convergents for (1.1) have a well-defined limit.

- There are literature claims [BeIII] that

  $$\{(a,b) \in \mathbb{C}^2 : \mathrm{Re}(a), \mathrm{Re}(b) > 0\} \subseteq \mathcal{D},$$

  i.e., that convergence occurs whenever both parameters have positive real part.

- **This is false** — as we shall show. The *exact identification* of $\mathcal{D}$ is very delicate.

- $\mathcal{R}_1(a,b)$ *typically diverges* for $|a| = |b|$: we observed numerically* that

$$\mathcal{R}_1\left(\frac{1}{2}+\frac{\sqrt{-3}}{2}, \frac{1}{2}-\frac{\sqrt{-3}}{2}\right)$$

and $\mathcal{R}_1(1,i)$ have '**period two**' — as is generic — while $\mathcal{R}_1(t\,i, t\,i)$ is '**chaotic**' for $t > 0$.

*After a *caution* on checking only even terms!

- We have implicitly used, for positive reals $a \neq b$ and perforce for the Jacobian parameter

$$q := \frac{\min(a,b)}{\max(a,b)} \in [0,1),$$

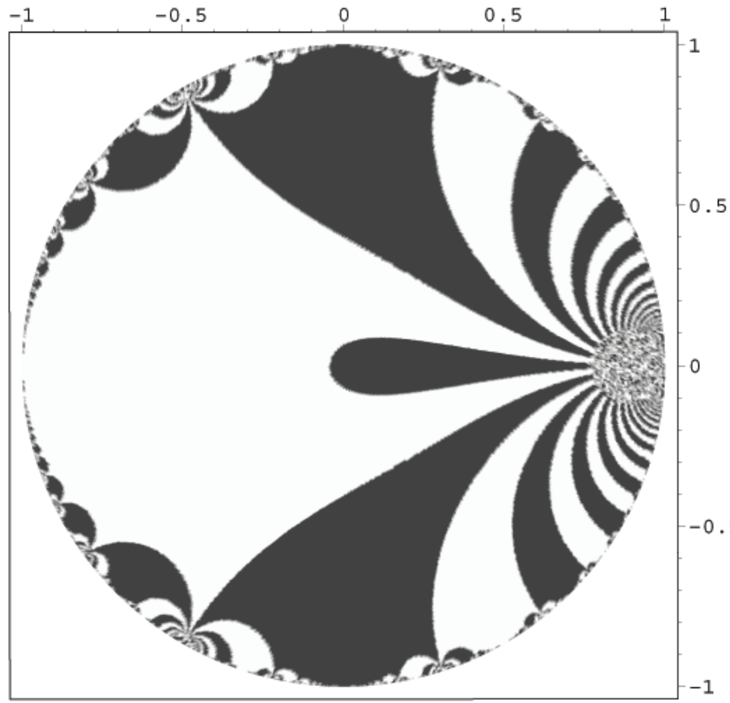
the fact that

$$0 \leq \frac{\theta_2(q)}{\theta_3(q)} < 1.$$

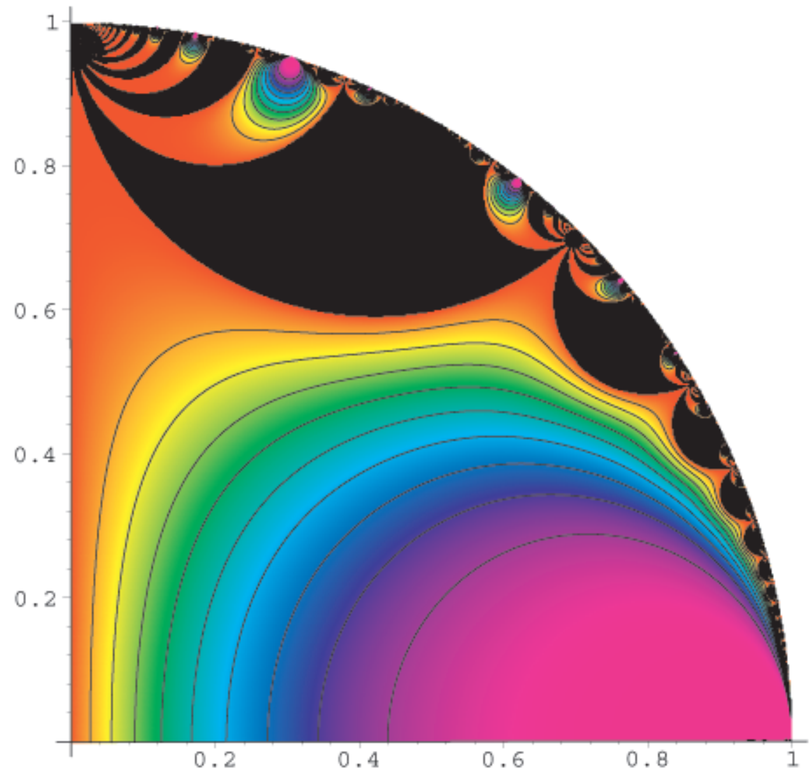
- If, however, one plots *complex* q with this ratio of **absolute value less than one**, a complicated **fractal structure emerges**, as shown in the Figures below — this leads to the *theory of modular forms* [BB].

- Thence the sech relations of Theorem 2.1 are suspect for complex q.

- Numerically, the identities appear to fail when $|\theta_2(q)/\theta_3(q)|$ exceeds unity as graphed in white for $|q| < 1$:



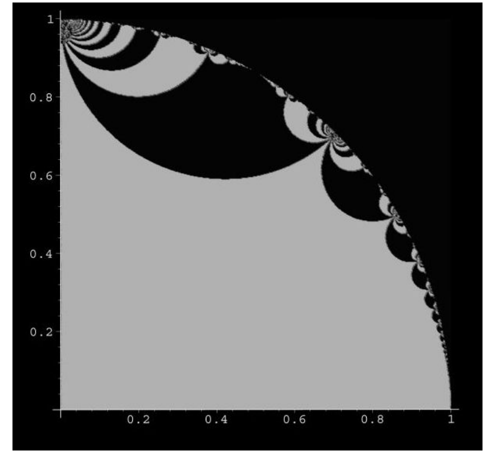
- Such *fractal behaviour* is ubiquitous.



- Where $|\theta_4(q)/\theta_3(q)| > 1$ in first quadrant.*

*Colours show gradations between zero and one.

Though the fraction $\mathcal{R}_1(a,b)$ converges widely, the AGM relation (1.2) does *not* hold across $\mathcal{D}$.



Using $a := 1, b := -3/2 + i/4$ the computationalist will find that the AGM relation fails:

$$\mathcal{R}_1\left(-\frac{1}{4}+\frac{i}{8}, \sqrt{-\frac{3}{2}+\frac{i}{4}}\right) \neq \frac{\mathcal{R}_1\left(1,b\right)+\mathcal{R}_1\left(b,1\right)}{2}.$$

- The *key to determining the domain for the AGM relation* seems to be the ordering of the *moduli* of the relevant parameters.

- We take the elliptic integral $\mathsf{K}(k)$ for complex k to be defined by

$$\mathsf{K}(k) = \frac{\pi}{2} \mathsf{F}\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right).$$

- The hypergeometric function F converges absolutely for k in the disk $(|k| < 1)$ and is continued analytically.

- For any numbers $z = re^{i\phi}$ under discussion, $r \geq 0$, $\arg(z) \in (-\pi, +\pi]$ and so

$$\sqrt{z} := \sqrt{r} e^{i\phi/2}.$$

- We start with some numerically based *Conjectures* now **proven**:

Theorem 9.0 (Analytic continuation): Consider complex pairs (a, b). Then

If $|a| > |b|$ the original fraction $\mathcal{R}_1(a, b)$ exists and agrees with the sech series **(3.1)**.

If $|a| < |b|$ the original fraction $\mathcal{R}_1(a, b)$ exists and agrees with the sech series **(3.2)**.

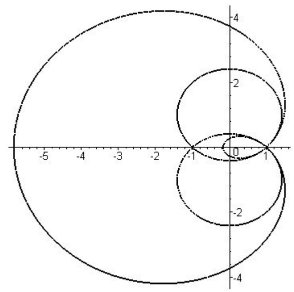
Theorem 9.1: $\mathcal{R}(a) := \mathcal{R}_1(a, a)$ converges **iff** $a \notin \mathcal{I}$. That is, the fraction diverges if and only if a is pure imaginary. Moreover, for $a \in \mathcal{C} \backslash \mathcal{I}$ the fraction converges to a holomorphic function of a in the appropriate open half-plane.

Theorem 9.2: $\mathcal{R}_1(a, b)$ converges for all real pairs; that is whenever Im(a) = Im(b) = 0.

Theorem 9.3: (i) The even/odd parts of $\mathcal{R}_1(1, i)$ (e.g.) converge to **distinct limits**.
(ii) There are $\mathrm{Re}(a), \mathrm{Re}(b) > 0$ such that $\mathcal{R}_1(a, b)$ diverges.

★ Define

- $\mathcal{H} := \{z \in \mathcal{C} : \left|\frac{2\sqrt{z}}{1+z}\right| < 1\}$,
- $\mathcal{K} := \{z \in \mathcal{C} : \left|\frac{2z}{1+z^2}\right| < 1\}$.



Theorem 9.4: If $a/b \in \mathcal{K}$ then $\mathcal{R}_1(a, b)$ and $\mathcal{R}_1(b, a)$ both converge.

Theorem 9.5: $\mathcal{H} \subset \mathcal{K}$ (properly).

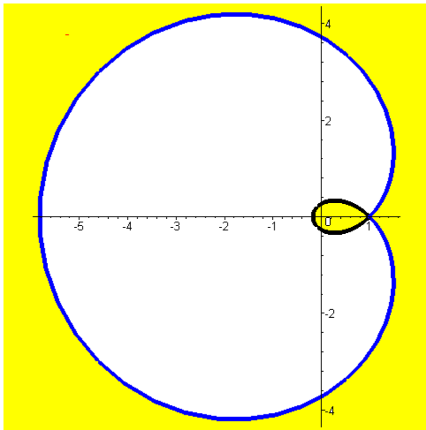
These results combine to give a region of validity for the AGM relation:

Theorem 9.6: If $a/b \in \mathcal{H}$ then $\mathcal{R}_1(a,b)$ & $\mathcal{R}_1(b,a)$ converge, and the arithmetic mean $(a+b)/2$ dominates the geometric mean $\sqrt{ab}$ in modulus.

- As to the problematic issues regarding the AGM relation (1.2) $\cdots$. We performed "**scatter diagram**" analysis (very robust) to find computationally where the AGM relation held in the parameter space.

★ The results (shown to the right in yellow) were quite spectacular!

- And led to the Theorems above.



- With $\mathcal{C}' := \{z \in \mathcal{C} : |z| = 1, z^2 \neq 1\}$, we were led to:

Theorem 9.11: The precise domain of convergence for $\mathcal{R}_1(a,b)$ is

$$\mathcal{D}_0 = \{(a,b) \in \mathcal{C} \times \mathcal{C} : (a/b \notin \mathcal{C}') \text{ or } (a^2 = b^2,\ b \notin \mathcal{I})\}.$$

Hence, for $a/b \in \mathcal{C}'$ we have divergence. Also, $\mathcal{R}_1(a,b)$ converges to an analytic function of a or b on the domain

$$\mathcal{D}_2 := \{(a,b) \in \mathcal{C} \times \mathcal{C} :\ |a/b| \neq 1\} \subset \mathcal{D}_0.$$

- Note, we are not harming Theorems 9.4–9.6 because neither $\mathcal{H}$ nor $\mathcal{K}$ intersects $\mathcal{C}'$.

- The "**bifurcation**" of Theorem 9.11 is very subtle.

Theorem 9.12: Restricting $a/b \in \mathcal{H}$ implies the truth of the AGM relation (1.2) with all three fractions converging.

Proof: For $a/b \in \mathcal{H}$, the ratio $(a+b)/(2\sqrt{ab}) \notin \mathcal{C}'$ and via 9.11 we have sufficient analyticity to apply Berndt's technique of Part I. ©

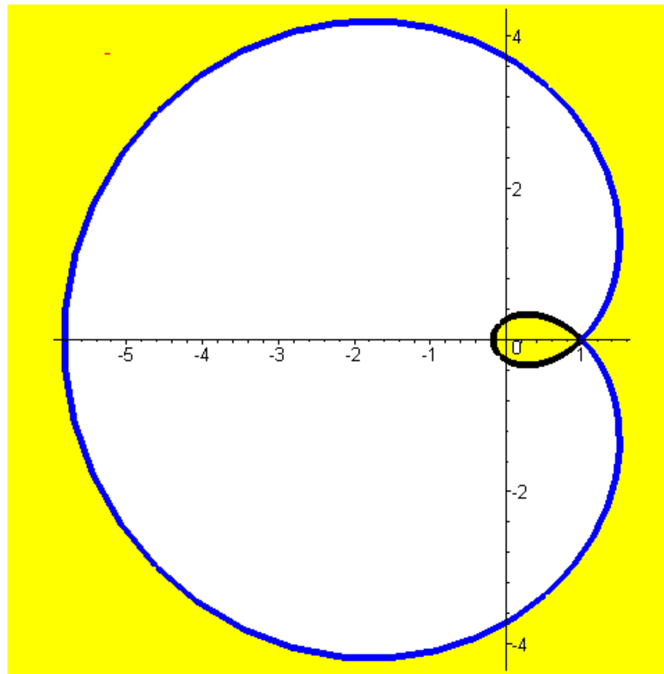
- A picturesque take on Theorems 9.4–9.6 and 9.12 is:

  Equivalently, a/b belongs to the closed exterior of $\partial\mathcal{H}$, which in polar-coordinates is given by the **cardioid-knot**

  $$r^2 + (2\cos\phi - 4)r + 1 = 0$$

  drawn in the complex plane ($r := |a/b|$).

- $a/b \in \mathcal{H}$: the *arithmetic mean dominating the geometric mean in modulus*.



A cardioid-knot, on the (yellow) exterior of which the Ramanujan AGM relation (1.2), (9.1) holds.

Proof: A pair from Theorem 9.11 meets

$$1 \leq \left|\frac{a+b}{2\sqrt{ab}}\right|^2 = \frac{1}{4}\left|\sqrt{z} + \frac{1}{\sqrt{z}}\right|^2,$$

with $z := a/b$. Thus, for $z := re^{i\phi}$ we have $4 \leq r + 2\cos\phi + 1/r$, which defines the exterior* of the cardioid-knot curve. ©

- To clarify, consider the two lobes of $\partial\mathcal{H}$: We fuse the orbits of the $\pm$ instances, yielding:

$$r = 2 - \cos\theta \pm \sqrt{(1-\cos\theta)(3-\cos\theta)}.$$

- Thus, $\mathcal{H}$ has a small loop around the origin, with left-intercept $\sqrt{8} - 3 + 0i$, and a wider contour whose left-intercept is $-3 - \sqrt{8} + 0i$.

*As determined by Jordan crossings.

- The condition $a/b \in \mathcal{H}(\mathcal{K})$ is symmetric: if a/b is in $\mathcal{H}(\mathcal{K})$ then so is b/a, since $r \to 1/r$ leaves the polar formula invariant.

- In particular, the AGM relation holds whenever $(a, b) \in \mathcal{D}$ and a/b lies on the exterior rays:

$$\frac{a}{b} \text{ or } \frac{b}{a} \in [\sqrt{8} - 3, \infty) \cup (\infty, -3 - \sqrt{8}],$$

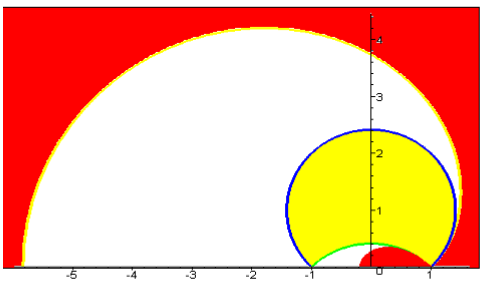
  thus including all positive real pairs (a, b) as well as a somewhat wider class.

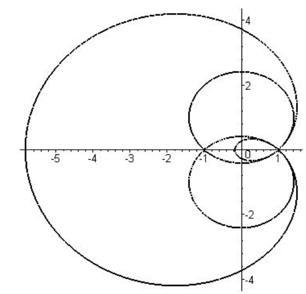
- Similarly, the AGM relation holds for pairs $(a, b) = (1, i\beta)$ with

$$\pm\beta \in [0, 2 - \sqrt{3}] \cup [2 + \sqrt{3}, \infty).$$

Remark. We performed extensive numerical experiments *without faulting* Theorems 9.11 and 9.12.

- Even with $a = b$ we needed $(a, a) \in \mathcal{D}$; recall (i, i) (also $(1, i)$) **provably** is not in $\mathcal{D}$.
  The unit circle only intersects $\mathcal{H}$ at $z = 1$.





Where $\mathcal{R}$ exists (not yellow) and where the AGM holds (red).

$$\left|2\frac{\sqrt{z}}{1+z}\right| \Rightarrow \left|2\frac{z}{1+z^2}\right| < 1.$$

$\diamond$ A key component of our proofs, actually valid in any B^* algebra, is:

Theorem 9.13. Let (A_n), (B_n) be sequences of $k \times k$ complex matrices.

Suppose that $\prod_{j=1}^{n} A_j$ converges as $n \to \infty$ to an **invertible limit** while $\sum_{j=1}^{\infty} \|B_j\| < \infty$. Then

$$\prod_{j=1}^{n} (A_j + B_j)$$

also converges to a finite complex matrix.

- Theorem 9.13 appears new even in $\mathbf{C^1}$!

★ It allows one to linearize *nonlinear matrix recursions* — ignoring $O\left(1/n^2\right)$ terms for convergence purposes.

- This is how the issue we now turn to, of the dynamics of (t_n), arose when applied to the matrix form of the partial fraction for $\mathcal{R}_1$:

$$A_n \quad = \quad \overbrace{I + \frac{1}{2an}\begin{bmatrix} 0 & \left(\frac{a}{b}\right)^{2n} \\ \left(\frac{b}{a}\right)^{2n} & 0 \end{bmatrix}}^{A_n+B_n \text{ is linearization}} - \overbrace{O\left(\frac{1}{n^2}\right)}^{B_n}$$

10. Visual Dynamics from a 'Black Box'

• Six months after the discoveries we had a beautiful proof using genuinely new dynamical results.

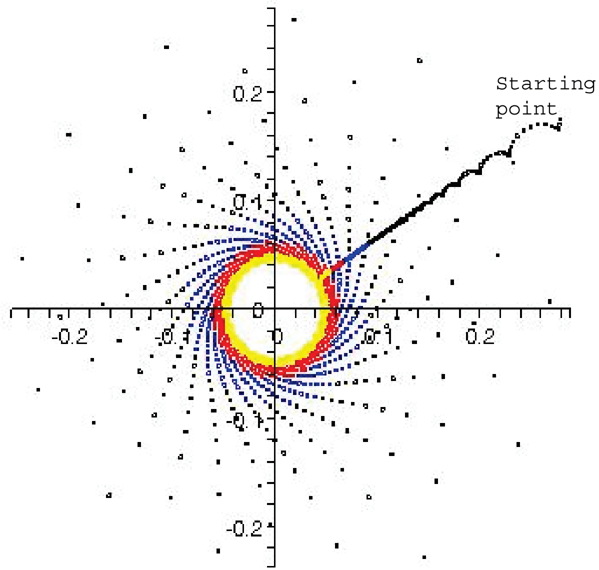
Starting from the *linear dynamical system* $t_0 := t_1 := 1$:

$$t_n \hookleftarrow \frac{1}{n} t_{n-1} + \omega_{n-1} \left(1 - \frac{1}{n}\right) t_{n-2},$$

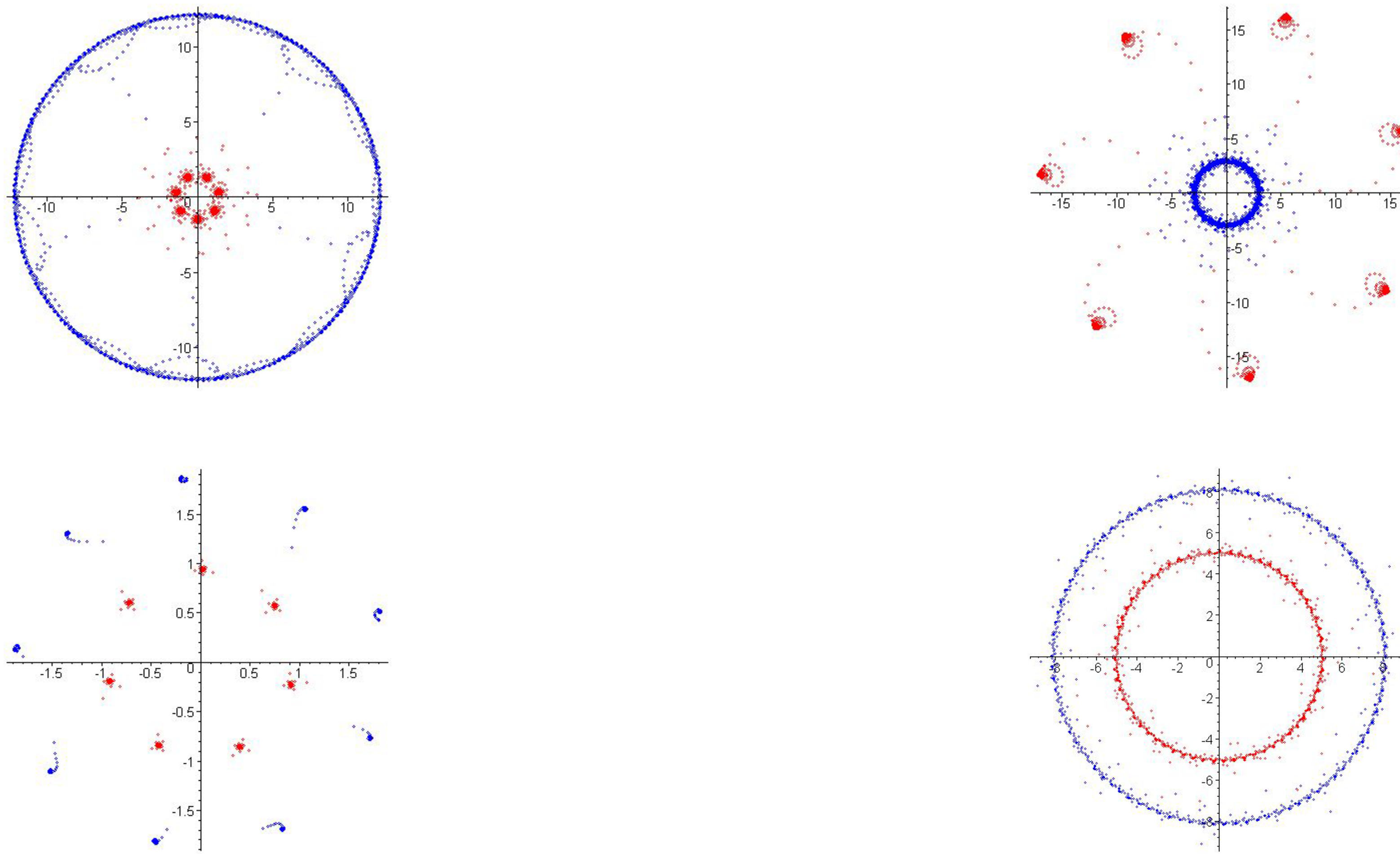
where $\omega_n = a^2, b^2$ for n even, odd respectively — or is much more general.

- Indeed $\sqrt{n}\, t_n$ is bounded $\Leftrightarrow$ $\mathcal{R}_1(a,b)$ *diverges* [actually $t_n = q_{n-1}/n!$]

- Numerically all one learns is that it is “tending to zero slowly”. Pictorially we see significantly more:



- Scaling by $\sqrt{n}$, and coloring odd and even iterates, fine structure appears.



The **attractors** **for various $|a| = |b| = 1$.**

★ This is *now* fully explained, especially the original rate of convergence, which follows by a fine *singular-value* argument. The radii are also determined.

- We used our matrix stability theorem to show the hard case: $|a| = |b|, a \neq b, a \neq 0$ implies divergence of the fraction.

- A *Cinderella* generated applet at `http://www.carma.newcastle.edu.au/~jb616/rama.html` neatly illustrates the behaviour of t_n.

♠ L. Lorentzen (2008) has now provided a fine more conventional proof of Theorem 9.11.

11. The 'Chaotic' Case

Jacobsen-Masson theory used in Theorem 9.1 shows, unlike $\mathcal{R}_1(1,i)$, even/odd fractions for $\mathcal{R}_1(i,i)$ behave "chaotically," neither converge.

When $a = b = i$, (t_n) exhibit a fourfold quasi-oscillation, as n runs through values mod 4.

★ Plotted versus n, the (real) sequence $t_n(\mathbf{1,1})$ exhibits the "serpentine oscillation" of four separate "necklaces."

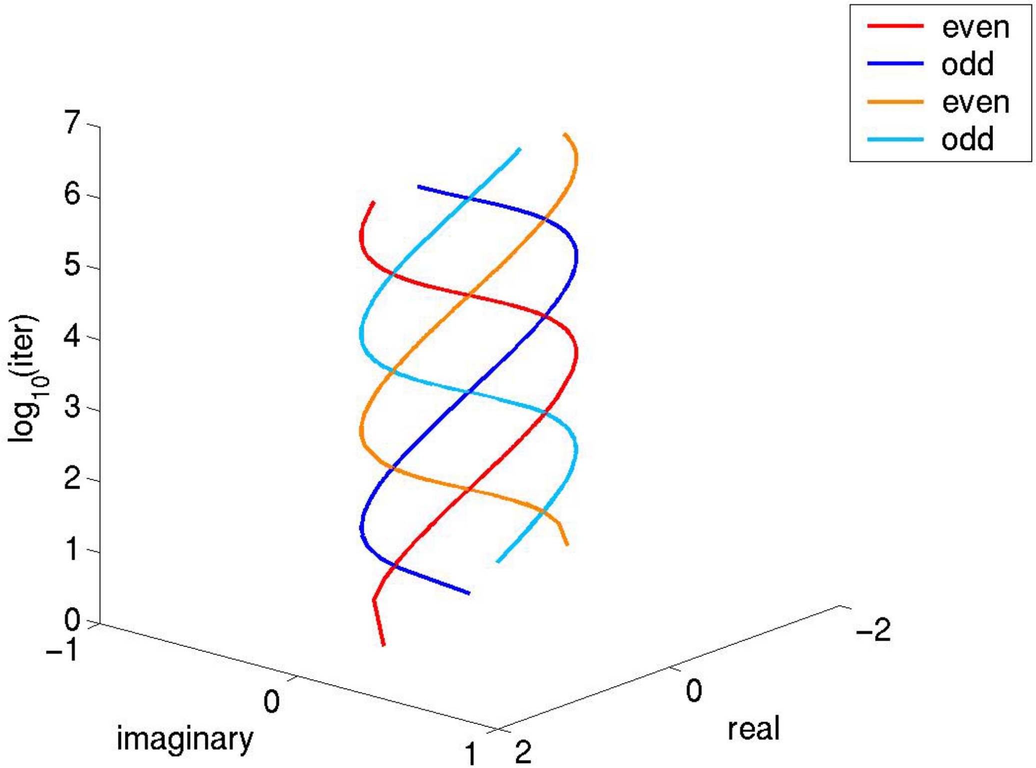
For $a = i$, the detailed asymptotic is

$$t_n(1,1) = \sqrt{\frac{2}{\pi}\cosh\frac{\pi}{2}}\,\frac{1}{\sqrt{n}}\left(1 + O\left(\frac{1}{n}\right)\right) \times$$

$$\begin{cases} (-1)^{n/2}\cos(\theta - \log(2n)/2) & n \text{ is even} \\ (-1)^{(n+1)/2}\sin(\theta - \log(2n)/2) & n \text{ is odd} \end{cases}$$

where

$$\theta := \arg\Gamma((1+i)/2).$$



The subtle four fold serpent.

♠ This behaviour seems *very* difficult to infer directly from the recurrence.

♡ Analysis is based on a striking hypergeometric parametrization which was both **experimentally discovered and computer proved**!

It is

$$t_n(1,1) = \frac{1}{2}F_n(a) + \frac{1}{2}F_n(-a),$$

where

$$F_n(a) := -\frac{a^n 2^{1-\omega}}{\omega\,\beta(n+\omega,-\omega)}\,{}_2\mathrm{F}_1\left(\omega,\omega;n+1+\omega;\frac{1}{2}\right),$$

where

$$\beta(n+1+\omega,-\omega) = \frac{\Gamma\,(n+1)}{\Gamma\,(n+1+\omega)\,\Gamma\,(-\omega)},$$

and $\omega = \frac{1-1/a}{2}$.

12. More General Fractions

Study of $\mathcal{R}$ devolved to *hard but compelling* conjectures on complex dynamics, with many interesting *proven* and *unproven* generalizations (e.g., Borwein-Luke, 2008).

For *any* sequence $a \equiv (a_n)_{n=1}^{\infty}$, we considered fractions like

$$\mathcal{S}_1(a) = \cfrac{1^2 a_1^2}{1 + \cfrac{2^2 a_2^2}{1 + \cfrac{3^2 a_3^2}{1 + \ddots}}}$$

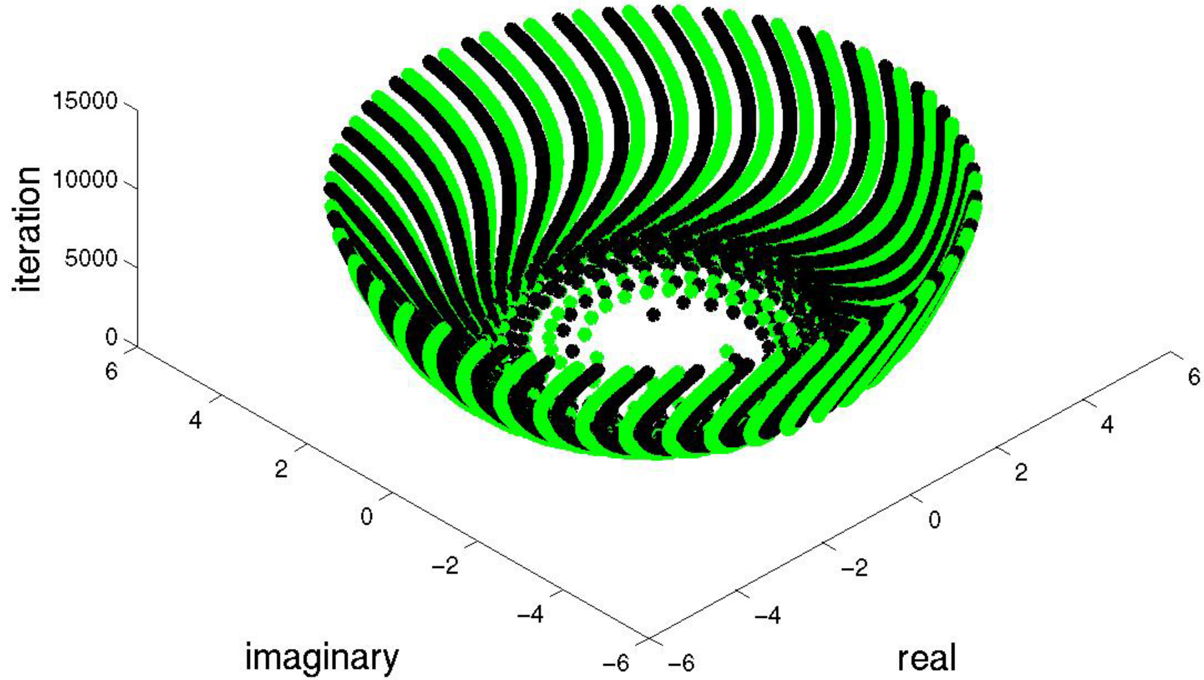
- We studied convergence properties for deterministic and random sequences (a_n).

- For the deterministic case the best results are for periodic sequences, satisfying

$$a_j = a_{j+c}$$

for all j and some finite c.

- The cases (i) $a_n = Const \in \mathbb{C}$, (ii) $a_n = -a_{n+1} \in \mathbb{C}$, (iii) $|a_{2n}| = 1,\ a_{2n+1} = i$, and (iv) $a_{2n} = a_{2m},\ a_{2n+1} = a_{2m+1}$ with $|a_n| = |a_m|\ \forall\ m, n \in \mathbb{N}$, were already covered.



A period three dynamical system

(odd and even iterates)

13. Final Open Problems

- On the basis of numerical experiments, we note that some "deeper" AGM identity might hold.

  - there are pairs $\{a, b\}$ so

$$\frac{\mathcal{R}_1(a,b)+\mathcal{R}_1(b,a)}{2} \neq \mathcal{R}_1\left(\frac{a+b}{2}, \sqrt{ab}\right)$$

    but *the LHS agrees* numerically with some variant, $\mathcal{S}_1((a+b)/2, \sqrt{ab})$, naively chosen from (3.1) or (3.2).

  - such coincidences are remarkable and difficult to predict.

1. What precisely is the domain of pairs for which $\mathcal{R}_1(a, b)$ converges, *and* some AGM holds?

2. Relatedly, when does the fraction depart from its various analytic representations?

3. While $\mathcal{R}(i) := \mathcal{R}_1(i, i)$ does not converge, the ψ-function representation of Section 4 has a definite value at $a = i$. Does some limit such as $\lim_{\epsilon \to 0} \mathcal{R}_1(i + \epsilon, i)$ exist and coincide?

4. Despite a host of closed forms for $\mathcal{R}(a) := \mathcal{R}_1(a, a)$, we know no *nontrivial* closed form for $\mathcal{R}_1(a, b)$ with $a \neq b$.

All physicists and a good many quite respectable mathematicians are contemptuous about proof. Beauty is the first test. There is no permanent place in the world for ugly mathematics.

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