TQFTs
[early incomplete draft]
version 1h

Kevin Walker

May 11, 2006

Contents

Chapter 0

Introduction

Notes on this early, incomplete draft.

- There are some inconsistencies, since some of the chapters were salvaged from an earlier version.
- I need to go through and fix details re dual fields, reversing orientation of manifolds, etc. (This stuff was deliberately omitted in the first draft to save time.) (At present this has been done in some chapters but not others.)
- need to add references and bibliography
- some parts have not been proof read
- my notational preferences evolved as this was written, so there are some notational inconsistencies

---

Notes for this introduction (to-do list):

- give outline of all the chapters
- acknowledgements (many or few?)
- this is partly expository, partly new stuff
- **key idea:** Computing the path integral by actually integrating against some measure on the space of fields is notoriously difficult, so to develop a mathematically rigorous theory we need to take a detour around this obstacle. Most previous approaches follow Segal and Atiyah *[need refs]* and take a rather wide detour: a TQFT is defined (roughly) to be any choice of vector spaces for n-manifolds and vectors for $n{+}1$-manifolds which satisfy algebraic properties suggested by the path integral. Here we make a tighter detour around the path integral. We retain the notion of fields on manifolds and require

that the vector spaces for n-manifolds be constructed out of these fields by local relations (or dually local projections). These local relations carry the same information as the path integral of the n+1-dimensional ball B^{n+1} with all possible boundary conditions. One advantage of this approach is that the higher algebra / higher codimension / multi-tier structure *[cite Freed (and also xxxx?)]* is automatically present all the way down to dimension zero. Another advantage is that we can construct most (all?) known TQFTs in a uniform framework.

- I've tried to give good citations, but there are bound to be inadvertant omissions (comments welcome) [this is for the future — there are very few citations at the moment]
- hard to decide what order to put things; some readers might want to skip ahead to examples before reading all of the general theory
- I've gone part way toward making things general in a category-theoretic sense, but not all the way. (e.g. I usually assume the target category for the TQFT is complex vector spaces, but the proofs for the most part work more generally.)
- need to comment on references; should I add historical notes at the end of chapters?

Chapter 1

From Path Integrals to Local Relations

In this chapter we give a non-rigorous argument showing that topologically invariant path integrals are more or less equivalent to the systems of fields and local relations described in detail (and rigorously) in the Chapter 4. We'll pretend that the integrals make sense and can be manipulated in the usual ways, and see that this leads directly to local relations. In turns out that fields and local relations make a good foundation for a rigorous treatment of TQFTs, especially the high codimension parts of the theory. In Chapter 6 we will show that in many cases *[be more specific here?]* the full path integral can be recovered combinatorially from the local relations.

1.1 Path Integrals and Projections

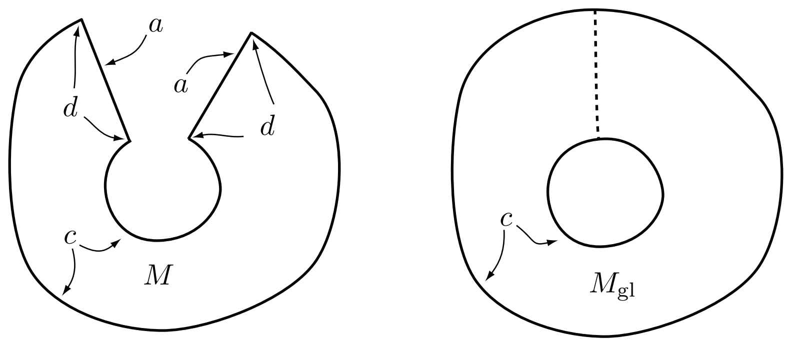
Fix a "top dimension", which we denote $n+1$ for historical reasons.

Let $\mathcal{C}$ be a system of fields for manifolds of dimension less than or equal to $n+1$. Since this is a non-rigorous argument, it's not worth the trouble to define precisely what a system of fields is. Standard examples are $\mathcal{C}(X)$ is the set of all maps from X to some fixed space B, or $\mathcal{C}(X)$ is the set of all equivalence classes of G-bundles with connection over X. In both cases, if X has boundary then we usually fix a boundary condition $c \in \mathcal{C}(\partial X)$ and let $\mathcal{C}(X; c)$ denote the set of all fields which restrict to c on the boundary. We give formal axioms for fields in Chapter 3.

For each $n+1$-manifold M and each $c \in \mathcal{C}(\partial M)$, assume we have a function

$$T : \mathcal{C}(M; c) \to \mathbb{T}.$$

Here $\mathbb{T} \subset \mathbb{C}$ denotes the the circle group. (In more traditional notation, $T = e^{iS}$ for some $S : \mathcal{C}(M; c) \to \mathbb{R}$. S is called the *action* for the theory.) Assume that T is local, in the sense that it is multiplicative (i.e. S is additive) with respect to disjoint unions and it commutes with gluing. In other words, we can cut M into pieces and



1.1.1 Fields on glued manifold

compute T on $\mathcal{C}(M)$ in terms of T on $\mathcal{C}$ of the pieces. We further assume that T is invariant under the action of boundary-fixing homeomorphisms on $\mathcal{C}(M;c)$. In other words, T is topologically invariant.

Now assume that we have defined some sort of integral and class of integrable functions on $\mathcal{C}(M;c)$, and that T is in this class. Define the path integral

$$Z(M;c) = \int_{x\in\mathcal{C}(M;c)} T(x) \in \mathbb{C}.$$

Suppose $\partial M = Y \cup -Y \cup W$ and let M_{gl} denote M glued to itself along $\pm Y$. Let $c \in \mathcal{C}(\partial M_{\text{gl}}) = \mathcal{C}(W_{\text{gl}})$ and let c also denote the corresponding field on W. Let $d \in \mathcal{C}(\partial Y)$ be the restriction of c to $\partial Y \subset \partial W$. (See Figure (1.1.1).) Assume that we can also integrate a class of functions on fields on n-manifolds, and that

1.1.2 $$Z(M_{\text{gl}};c) = \int_{x\in\mathcal{C}(M_{\text{gl}};c)} T(x) = \int_{a\in\mathcal{C}(Y;d)} \int_{y\in\mathcal{C}(M;a,a,c)} T(y).$$

In other words, assume that we can "integrate along fibers".

Let Y be an n-manifold and $d \in \mathcal{C}(\partial Y)$. Consider the n+1-manifold $Y \times I$ (pinched at $\partial Y \times I$ if Y has boundary, so that $\partial(Y \times I) = Y \cup -Y$). For $a, b \in \mathcal{C}(Y;d)$ define

$$K_Y(a,b) = Z(Y \times I; a, b) = \int_{x\in\mathcal{C}(Y\times I;a,b)} T(x).$$

Since $(Y \times I) \cup_Y (Y \times I) \cong Y \times I$, we have, by (1.1.2),

1.1.3 $$\int_{y\in\mathcal{C}(Y;d)} K_Y(a,y)K_Y(y,b) = K_Y(a,b).$$

Let $\mathcal{F}(Y;d)$ denote an appropriate space of functions $\mathcal{C}(Y;d) \to \mathbb{C}$. Define $\pi_Y : \mathcal{F}(Y;d) \to \mathcal{F}(Y;d)$ by

$$\pi_Y(f)(b) = \int_{y\in\mathcal{C}(Y;d)} f(y)K_Y(y,b).$$

Then it follows from (1.1.3) that π_Y is a projection,

$$\pi_Y^2 = \pi_Y.$$

Finally, define $Z(Y;d) \subset \mathcal{F}(Y;d)$ to be the image of this projection,

$$Z(Y;d) = \pi_Y(\mathcal{F}(Y;d)). \qquad \boxed{1.1.4}$$

1.2 Functorial Properties of the Path Integral

For an n+1-manifold M, let $Z(M)$ be the function

$$\begin{aligned} Z(M) : \mathcal{C}(\partial M) &\to \mathbb{C} \\ c &\mapsto Z(M;c). \end{aligned}$$

Then clearly

$$Z(M) \in Z(\partial M). \qquad \boxed{1.2.1}$$

(Proof: Glue a collar of ∂M to M and apply (1.1.2). The result is still M.) It is also easy to see that for any closed n-manifolds Y_1 and Y_2, there is a natural identification

$$Z(Y_1 \sqcup Y_2) = Z(Y_1) \otimes Z(Y_2),$$

and that for any n+1-manifolds M_1 and M_2

$$Z(M_1 \sqcup M_2) = Z(M_1) \otimes Z(M_2) \in Z(\partial M_1) \otimes Z(\partial M_2).$$

Integration gives a pairing

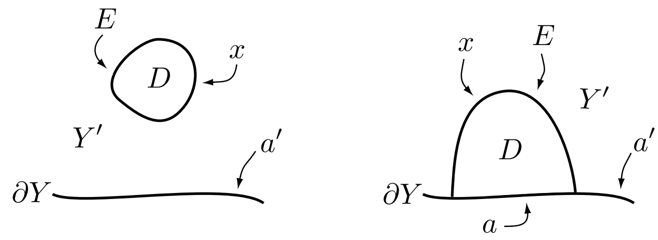
$$\begin{aligned} Z(Y) \otimes Z(-Y) &\to \mathbb{C} \\ f \otimes g &\mapsto \int_{x\in\mathcal{C}(Y)} f(x)g(x). \end{aligned}$$

If $\partial M = Y \sqcup -Y \sqcup W$, this pairing induces a trace map

$$Z(\partial M) = Z(Y) \otimes Z(-Y) \otimes Z(W) \xrightarrow{\mathrm{tr}_Y} Z(W) = Z(\partial M_{\mathrm{gl}}),$$

and

$$Z(M_{\mathrm{gl}}) = \mathrm{tr}_Y(Z(M)).$$



1.3.1 Disk in Y (two cases)

The above constitutes a gluing (without corners) formula for $n+1$-manifolds.

Similarly, if Y is an n-manifold with boundary and $d \in \mathcal{C}(\partial Y)$, there is a pairing

$$\begin{aligned} Z(Y;d) \otimes Z(-Y;d) &\to \mathbb{C} \\ f \otimes g &\mapsto \int_{x \in \mathcal{C}(Y;d)} f(x)g(x). \end{aligned}$$

If $\partial M = Y \cup -Y \cup W$ (setup for gluing with corners *[include figure?]*), then for each $c \in \mathcal{C}(\partial Y)$ there is a trace map

$$Z(Y;d) \otimes Z(-Y;d) \otimes Z(W;d,d) \xrightarrow{\mathrm{tr}_{Y,d}} Z(W;d,d).$$

These maps (for each d) combine to determine a map

$$Z(\partial M) \xrightarrow{\mathrm{tr}_Y} Z(\partial M_{\mathrm{gl}}),$$

(see (6.1.4)) and again

$$Z(M_{\mathrm{gl}}) = \mathrm{tr}_Y(Z(M)).$$

The above constitutes a gluing (with corners) formula for $n+1$-manifolds.

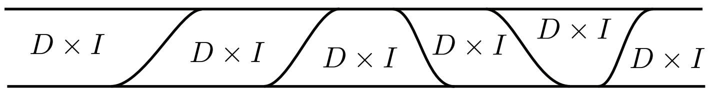
1.3 Locality of $Z(Y)$

We want to show that $Z(Y;d)$ is defined by local relations. Let $D \subset Y$ be in n-ball (possibly intersecting ∂Y), and let Y' be the closure of $Y \setminus D$ (see Figure (1.3.1)). Let $E = \partial D \cap Y'$. Let $a \cup_{\partial E} a' \in \mathcal{C}(\partial Y)$, where $a \in \mathcal{C}(\partial D \cap \partial Y; \partial a)$ and $a' \in \mathcal{C}(\partial Y' \cap \partial Y; \partial a)$. (If D is in the interior of Y then a is the unique field over the empty $n-1$-manifold.) Then

$$\mathcal{C}(Y; a \cup a') = \bigcup_{x \in \mathcal{C}(E;\partial a)} \mathcal{C}(D; a, x) \times \mathcal{C}(Y'; a', x).$$

Restriction then gives an identification

$$\mathcal{F}(Y; a \cup a') = \prod_{x \in \mathcal{C}(E;\partial a)} \mathcal{F}(D; a, x) \otimes \mathcal{F}(Y'; a', x).$$



1.3.2 Decomposing a collar into local collars

(This is a product rather than a direct sum because non-zero components are allowed for infinitely many x.) Define $P_D : \mathcal{F}(Y; a \cup a') \to \mathcal{F}(Y; a \cup a')$ by

$$P_D = \prod_{x \in \mathcal{C}(E;\partial a)} \pi_{(D;x)} \otimes \mathrm{id}\,.$$

(Here we have suppressed a and a' from the notation on the left hand side.) By the same argument as before, $P_D^2 = P_D$. P_D is the local projection corresponding to gluing a collar $D \times I$ to $D \subset Y$.

Now the key point: $Y \times I$ can be constructed by gluing $D_i \times I$ to $Y \times \{0\}$, where D_i runs through an open cover of Y. (See Figure (1.3.2).) It follows that π_Y can be written as composition of the P_{D_i}'s. This implies that

$$\mathrm{im}(\pi_Y) = \bigcap_{D_i} \mathrm{im}(P_{D_i}).$$

(Note that π_Y and the P_{D_i}'s all commute, because the corresponding topological gluings commute.)

Requiring that $f \in \mathrm{im}(P_D)$ is a local condition, in the sense that it only depends on what's going on inside a disk in Y. Thus we have obtained a local description of $\mathrm{im}(\pi_Y)$.

Dually, we can define $y \sim z$ if $P_D(y) = P_D(z)$, where y and z are linear combinations of fields on D. These are the local relations with which we will be concerned for the rest of the book.

In practice, it turns out to be easier to define systems of local relations than it is to define a path integral, so in the Chapter 4 we axiomatize these local relations.

Note that P_D encodes the same information as $Z(D \times I) = Z(B^{n+1})$, so specifying a local relation is tantamount to specifying the path integral of the n+1-ball.

Chapter 2

$\mathbb{Z}_2$ Homology as a TQFT

[kill this chapter?? is it needed?]

In this chapter we use the $\mathbb{Z}_2$ ($= \mathbb{Z}/2\mathbb{Z}$) homology of a surface to introduce the key concepts of local relation, cylinder category, and gluing theorem. All of this is redone in greater generality in Chapter 4, so some readers may want to skip directly to that chapter. This chapter can be thought of as a warm-up for Chapter 4.

The proofs in this chapter are overkill if the goal is merely to investigate $\mathbb{Z}_2$ homology, but they have the virtue of generalizing to many other contexts.

More specific (less general) generalizations of this "$\mathbb{Z}_2$-homology theory" can be found in Section 8.1, where we consider a larger class of local relations on 1-submanifolds, and Section 8.5, where we replace $\mathbb{Z}_2$ with an arbitrary finite group.

2.1 The Basic Construction

Let Y be a closed, oriented 2-manifold without boundary. Consider the set $\mathcal{C}(Y)$ of all unoriented 1-dimensional submanifolds of C of Y (including the empty submanifold), modulo the equivalence relation generated by:

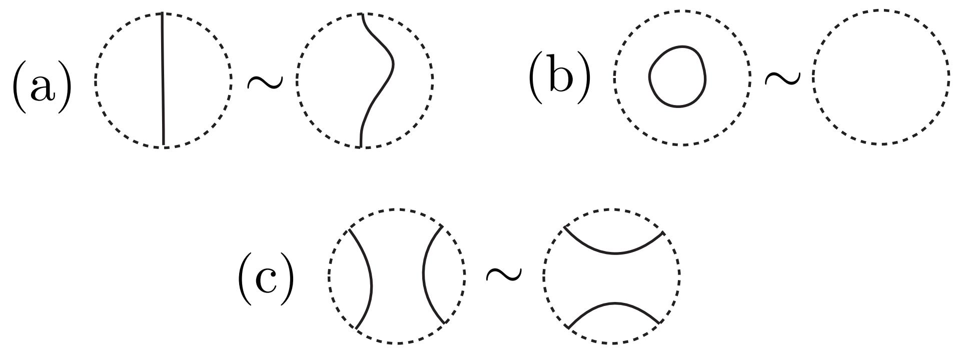
(a) $C \sim C'$ if C and C' are related by an isotopy supported in a disk;

(b) $C \sim C'$ if C' is obtained from C by removing a small trivial circle; and

(c) $C \sim C'$ if C and C' are related by an elementary homology contained in a disk.

This relation is shown schematically in Figure (2.1.1). Here and throughout we adopt the well-known convention that relations illustrated on disks apply to all curves (or fields or whatever) which agree outside of the disk in the figure.

For our purposes, the most important thing about this equivalence relation is that it is *local*, in the sense that it is supported within a disk.



2.1.1 Local relations for $\mathbb{Z}_2$ homology

Of course, $\mathcal{C}(Y)/\sim$ is naturally isomorphic, as a set, to $H_1(Y;\mathbb{Z}_2)$. But we will pretend that we know nothing of homological algebra and instead rely mainly on the fact that $\sim$ is a local relation.

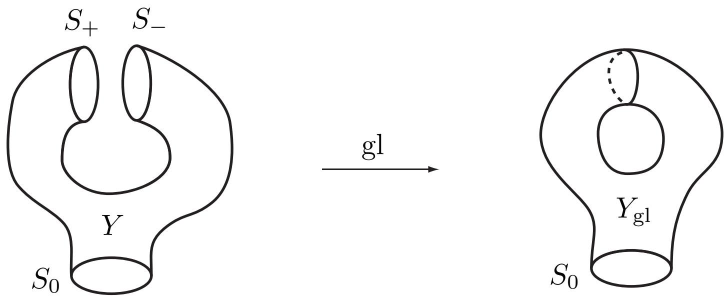
Let $\widetilde{A}(Y)$ be the vector space consisting of all finite formal $\mathbb{C}$-linear combinations of elements of $\mathcal{C}(Y)$, and define $A(Y) = \widetilde{A}(Y)/\sim$. $A(Y)$ is a vector space of dimension $|H_1(Y;\mathbb{Z}_2)|$.

We would like to be able to calculate $A(Y)$ and construct an explicit basis. To do that we will need a gluing theorem, and to state the gluing theorem we will need to extend the above constructions to 2-manifolds with boundary, which we now proceed to do.

If S is a 1-manifold, define $\mathcal{C}(S)$ to be the set of unoriented, properly embedded 0-dimensional submanifolds of S (including the empty submanifold). Let Y be a compact, oriented 2-manifold with boundary, and let $a \in \mathcal{C}(\partial Y)$. Define $\mathcal{C}(Y;a)$ to be the set of all unoriented 1-dimensional submanifolds of C of Y such that $\partial C = a$. Define $\widetilde{A}(Y;a)$, $\sim$, and $A(Y;a)$ as above, *with all isotopies fixing* ∂Y.

2.2 Gluing

Let Y be a 2-manifold with gluing data: ∂Y is decomposed into three disjoint pieces, $\partial Y = S_+ \sqcup S_- \sqcup S_0$, and there is an identification $S_+ = -S_-$ (i.e. S_+ is identified with S_- with reversed orientation). (See Figure (2.2.1).) Let Y_{gl} denote the 2-manifold obtained by gluing S_+ to S_-. Note that $\partial Y_{\text{gl}} = S_0$. Given $(a,b,c) \in \mathcal{C}(S_+)\times\mathcal{C}(S_-)\times\mathcal{C}(S_0) = \mathcal{C}(\partial Y)$, we have the vector space $A(Y;a,b,c)$. We would like to calculate $A(Y_{\text{gl}};c)$ in terms of the vector spaces $A(Y;a,b,c)$ for various a and b. ($c \in \mathcal{C}(S_0) = \mathcal{C}(\partial Y_{\text{gl}})$ will be fixed throughout the following discussion.)



2.2.1 Gluing 2-manifolds

For all $x \in \mathcal{C}(S_+) = \mathcal{C}(S_-)$ gluing yields a map

$$\mathrm{gl}_x : \mathcal{C}(Y; x, x, c) \to \mathcal{C}(Y_{\mathrm{gl}}; c),$$

which induces a linear map

$$\mathrm{gl}_x : \widetilde{A}(Y; x, x, c) \to \widetilde{A}(Y_{\mathrm{gl}}; c).$$

The latter map is compatible with the local relations, and so induces a linear map

$$\mathrm{gl}_x : A(Y; x, x, c) \to A(Y_{\mathrm{gl}}; c).$$

By general position, every $C \in \mathcal{C}(Y_{\mathrm{gl}}; c)$ is isotopic to some $\mathrm{gl}(C')$ where $C' \in \mathcal{C}(Y; x, x, c)$ for some x. Therefore we have a *surjective* linear map

$$\mathrm{gl} : \bigoplus_{x \in \mathcal{C}(S_\pm)} A(Y; x, x, c) \to A(Y_{\mathrm{gl}}, c).$$

If we can describe the kernel of this map, we will have proved the desired gluing theorem.

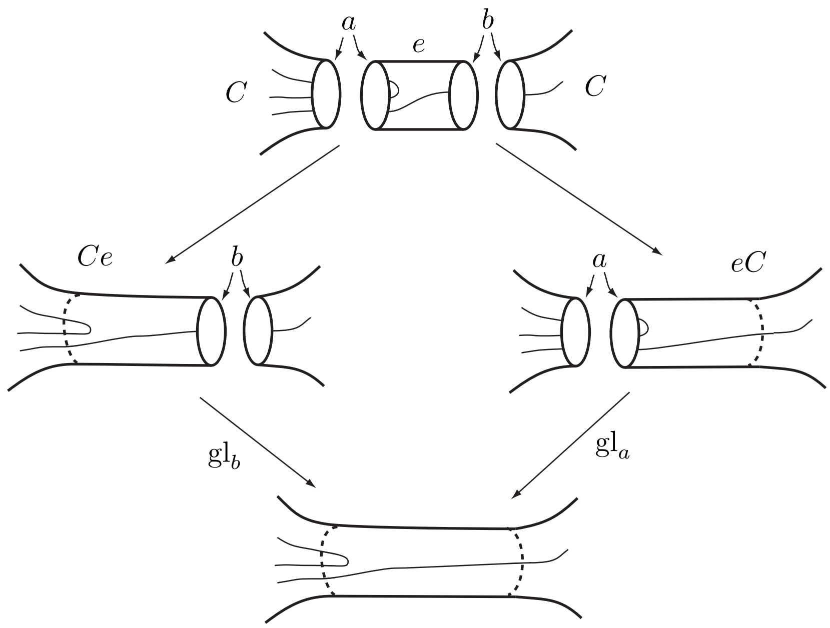
Let $C \in \mathcal{C}(Y; a, b, c)$ for some $a, b \in \mathcal{C}(S_\pm)$, and let $e \in \mathcal{C}(S_\pm \times I; a, b)$. (See Figure (2.2.2).) Let $Ce = C \cup_{S_+} e \in \mathcal{C}(Y; b, b, c)$ and $eC = C \cup_{S_-} e \in \mathcal{C}(Y; a, a, c)$. Then clearly

$$\mathrm{gl}_b(Ce) \sim \mathrm{gl}_a(eC) \in \mathcal{C}(Y_{\mathrm{gl}}; c)$$

(via an isotopy which shifts along the gluing locus in Y_{gl}), and so, moving from $\mathcal{C}(Y_{\mathrm{gl}}; c)$ to $A(Y_{\mathrm{gl}}; c)$,

$$Ce - eC \in \ker(\mathrm{gl}) \subset \bigoplus_{x \in \mathcal{C}(S_\pm)} A(Y; x, x, c).$$

(Here C also denotes its equivalence class in $A(Y; a, b, c)$.)



2.2.2 More than one way to glue a collar

We claim that elements of the form $Ce - eC$ generate all of ker(gl). Consider the following commutative diagram:

$$\begin{array}{ccc} \bigoplus_x \widetilde{A}(Y;x,x,c) & \xrightarrow{\text{gl}} & \widetilde{A}(Y_{\text{gl}};c) \\ \downarrow p & & \downarrow q \\ \bigoplus_x A(Y;x,x,c) & \xrightarrow[\text{gl}]{} & A(Y_{\text{gl}};c) \end{array}$$

(The quotient maps are denoted by p and q.) The top gluing map is injective and the other three maps are surjective. Let K denote the subspace of $\bigoplus_x \widetilde{A}(Y;x,x,c)$ generated by $\ker(p)$ and elements of the form $Ce - eC$. It suffices to show that $\text{gl}(K) \subset \widetilde{A}(Y_{\text{gl}};c)$ contains all elements of the form $A - A'$, where A and A' differ by a generating local relation supported in some disk $D \subset Y_{\text{gl}}$. Arbitrary isotopies on Y_{gl} are generated by (a) isotopies supported away from the gluing locus in Y_{gl} and (b) a shift isotopy supported in a collar of the gluing locus which moves the gluing locus off of itself. If E and $E' \in \mathcal{C}(Y_{\text{gl}};c)$ differ by an isotopy of type (a), then $E - E' = \text{gl}(F)$ where $F \in \ker(p)$. If E and $E' \in \mathcal{C}(Y_{\text{gl}};c)$ differ by an isotopy of type (b), then $E = gl(eC)$ and $E' = \text{gl}(Ce)$ for some C and e as above. It follows that if E and $E' \in \mathcal{C}(Y_{\text{gl}};c)$ differ by a general isotopy, then $E - E' \in \text{gl}(K)$. Hence we can assume that the disk D above is disjoint from the gluing locus, since D can be isotoped off of the gluing locus. Clearly in this case $A - A' = \text{gl}(F)$ for some $F \in \ker(p)$.

We have now proved

Theorem. *With notation as above, let* $L \subset \bigoplus_x A(Y;x,x,c)$ *be the subspace generated by all elements of the form* $Ce - eC$*, where* $C \in \mathcal{C}(Y;a,b,c)$ *and* $e \in \mathcal{C}(S_\pm \times I;a,b)$ *for some* $a,b \in \mathcal{C}(S_\pm)$*. Then there is a natural isomorphism*

$$A(Y_{\mathrm{gl}};c) \cong \bigoplus_x A(Y;x,x,c)/L.$$

□

2.3 Cylinder Categories

The gluing theorem can be usefully restated in term of actions of cylinder categories. Let S be a 1-manifold. Let $A(S)$, the *cylinder category* associated to S, be the category with objects $\mathcal{C}(S)$ and morphisms $\mathrm{mor}(a,b) = A(S \times I;a,b)$. Composition of morphisms is given in the obvious way by gluing cylinders.

Note that $A(-S)$ is naturally isomorphic to $A(S)^{\mathrm{op}}$, and that $A(S \sqcup S')$ is naturally isomorphic to $A(S) \times A(S')$.

If $\partial Y = S \sqcup R$ and $c \in \mathcal{C}(R)$, then Y and c determine a representation of $A(S)$ as follows. To each object a of $A(S)$ (i.e. $a \in \mathcal{C}(S)$), we associate the vector space $A(Y;a,c)$. To each morphism $f : a \to b$ of $A(S)$ (i.e. $f \in \mathcal{C}(S \times I;a,b)$), we associate the linear map

$$\begin{aligned} G_f : A(Y;a,c) &\to A(Y;b,c) \\ x &\mapsto x \cup f. \end{aligned}$$

In other words, G_f is given by gluing a copy of f to Y along S.

Now let Y be a surface with gluing data as above, $\partial Y = S_+ \sqcup S_- \sqcup S_0$. Let $A = A(S_+) = A(S_-)^{\mathrm{op}}$. Fix $c \in \mathcal{C}(S_0)$. For $(a,b) \in \mathcal{C}(S_+ \sqcup S_-)$, define $V_{ab} = A(Y;a,b,c)$. As described above, the collection of vector spaces $\{V_{ab}\}$ affords a representation of $A \times A^{\mathrm{op}} = A(S_+ \sqcup S_-)$. For all a there is a linear map

$$\mathrm{gl}_a : V_{aa} \to A(Y_{\mathrm{gl}};c),$$

and for every morphism $e : x \to y$ of A there is a commutative diagram

$$\begin{array}{ccc} & V_{aa} & \\ \nearrow^{\mathrm{id}_a \times e} & & \searrow^{\mathrm{gl}_a} \\ V_{ab} & & A(Y_{\mathrm{gl}};c) \\ \searrow_{e \times \mathrm{id}_b} & & \nearrow_{\mathrm{gl}_b} \\ & V_{bb} & \end{array}$$

(See Figure (2.2.2).)

Theorem. *$A(Y_{\mathrm{gl}}; c)$ is the universal object with the above two properties. In other words, if there is a vector space W and linear maps $f_a : V_{aa} \to W$ for all x, such that for all $e : x \to y$ the following diagram commutes*

$$\begin{array}{ccccc} & & V_{aa} & & \\ & \overset{\mathrm{id}_a \times e}{\nearrow} & & \overset{f_a}{\searrow} & \\ V_{ab} & & & & W \\ & \underset{e \times \mathrm{id}_b}{\searrow} & & \underset{f_b}{\nearrow} & \\ & & V_{bb} & & \end{array}$$

then there exists a map $g : A(Y_{\mathrm{gl}}; c) \to W$ such that $f_a = g \cdot \mathrm{gl}_a$ for all x. □

This is just a restatement of the first version of the gluing theorem. It has the advantage of generalizing to the case where target category is not a vector space.

For example the theorem remains true if we systematically, in the above discussion, replace $A(Y, a)$ with the set of all properly embedded 1-submanifolds in Y, modulo isotopy, with boundary a. In this case $A \times A^{\mathrm{op}}$ acts on the collection of sets $\{V_{ab}\}$, and $A(Y_{\mathrm{gl}}; c)$ is the universal set having the above properties.

The above universal construction is usually called the *coend* (dual to *end*) of the $A \times A^{\mathrm{op}}$ action. (See for example [Mac98, p. 226].) From our point of view, it would be better to call this the *gluing* of a representation V of $A \times A^{\mathrm{op}}$, but "coend" is already well-established so we stick with that terminology.

2.4 Semisimplicity

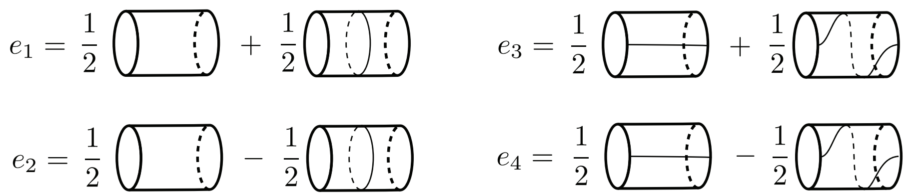
Back to the $\mathbb{Z}_2$-homology vector spaces $A(Y; a)$. The categories $A(S)$ (where S is a closed 1-manifold) have the additional property of being *semisimple*. (See Appendix B for details.) This means, among other things, that there is a fixed list $\mathcal{L}(S)$ of irreducible representations of $A(S)$, and that any representation ρ of $A(S)$ can be written as

$$\rho \cong \bigoplus_{\alpha \in \mathcal{L}(S)} \rho_\alpha \otimes \alpha,$$

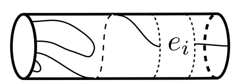
where the isomorphism is natural, and $\rho_\alpha = \hom(\alpha, \rho)$. It is also the case that $\mathcal{L}(A(S \sqcup S')) = \mathcal{L}(A(S)) \times \mathcal{L}(A(S'))$ (i.e. the irreps of $A \times A'$, where A and A' are semi-simple, are isomorphic to the products of irreps of A and A').

More specifically, let e_1, e_2, e_3, e_4 be the morphisms of $A(S^1)$ shown in Figure (2.4.1). The e_i's are a complete set of minimal idempotents for $A(S^1)$. That is, $e_i e_i = e_i$, $e_i e_j = 0$ if $i \neq j$, any morphism f of $A(S^1)$ can be written as $\sum_1^4 g_i e_i h_i$ for some $\{g_i, h_i\}$, and no larger set of idempotents has this property.

Let $\alpha_i = A(S^1) e_i$ as a left $A(S^1)$ representation. Geometrically, α_i is $S^1 \times I$ with e_i fixed on one end, arbitrary curves-mod-relations on the rest of $S^1 \times I$, and $A(S^1)$ acting on the other end (see Figure (2.4.2)).



2.4.1 Idempotents for $A(S^1)$



2.4.2 The representation α_i

If α is an irrep of $A(S(\partial Y))$, define

$$A(Y;\alpha) = A(Y;\cdot)_\alpha = \hom(\alpha, A(Y;\cdot)).$$

More concretely, if $\partial Y = S^1$ then $A(Y;\alpha_i)$ is naturally isomorphic to the space of all curves (mod relations) on Y which coincide with e_i in a collar of ∂Y. In general, we put an idempotent at each closed component of ∂Y.

Reflection in the I direction gives functor $A(S) \to A(S)^{\mathrm{op}}$ for any closed 1-manifold S, and the square of this functor is the identity. This allows us to identify representations of $A(S)$ with representations of $A(S)^{\mathrm{op}}$. For the $\mathbb{Z}_2$ homology theory, $\alpha \cong \alpha^{\mathrm{op}}$ for all irreps α, but this doesn't hold for general theories. (Also, there are no Frobenius-Schur complications (see (13.0.1)) for the $\mathbb{Z}_2$ homology theory.) If $\alpha \otimes \beta$ is an irrep of $A \times A^{\mathrm{op}}$, then the coend of $\alpha \otimes \beta$ is naturally isomorphic to $\mathbb{C}$ if $\alpha = \beta^{\mathrm{op}}$, and zero if $\alpha \ncong \beta^{\mathrm{op}}$. From this follows a third version of the gluing theorem:

Theorem. *Let Y be a surface with boundary $\partial Y = S_+ \sqcup S_- \sqcup S_0$, and let Y_{gl} denote Y with S_+ and S_- identified, as above. Let β be an irrep of $A(S_0)$. Then*

$$A(Y_{\mathrm{gl}};\beta) \cong \bigoplus_{\alpha \in \mathcal{L}(A(S_+))} A(Y;\alpha,\alpha,\beta).$$

□

[talk about 3-dim'l part of theory?]

[need to make boundary orientation and left/right action conventions consistent]

Chapter 3

Topological Fields

The techniques of the previous chapter generalize easily to manifolds of arbitrary dimension and a wider variety of functors $\mathcal{C}$ and local relations. In this chapter we formalize the properties of $\mathcal{C}$ which are needed to make the constructions work. (General requirements for local relations are given in Chapter 4.) Functors $\mathcal{C}$ satisfying the requirements will be called *topological fields*. (The term "field" is borrowed from physics, where typical examples are the set of maps from a manifold into a linear space, or the set of connections on a bundle over a manifold.)

In a nutshell, a topological field is a collection of functors from manifolds of dimensions $0, \ldots, n$ to sets which behave well under cutting and pasting. Our main examples are the set of maps from a manifold into some target space B, and the set of embeddings of cell complexes (perhaps with additional structure and specified local combinatorics) into a manifold. One of the goals of this chapter is to lay the groundwork for more exotic examples.

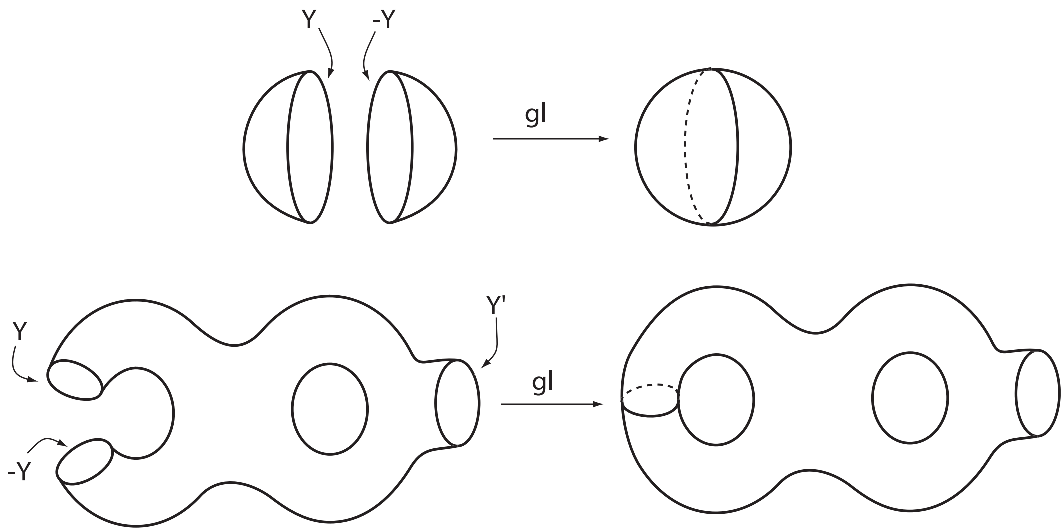
(Impatient readers would do well to skip this chapter and just assume that "topological field" means one of the main examples.)

We start by listing the properties of manifolds that we will use. We then define topological fields as a collection of functors which preserve these properties in the appropriate way.

[I haven't yet carefully checked that the field axioms given below are complete. I might need to add a couple of more axioms.]

3.1 Manifolds

Our manifolds will always be oriented and compact. They might have additional structure (e.g. *[need examples of additional structure]*). They could be PL or smooth. *[for smooth, need to say more about corners; also could be topological?]* Morphisms between manifolds of the same dimension will always be orientation-preserving homeomorphisms (or diffeomorphisms in the case of smooth manifolds) unless specified otherwise.



3.1.1 Gluing manifolds

Let $\mathbf{M}_i$ denote the category whose objects are compact, oriented manifolds of dimension i, and whose morphisms are orientation-preserving homeomorphisms. Included in each $\mathbf{M}_i$ is $\emptyset$, the empty manifold of dimension i.

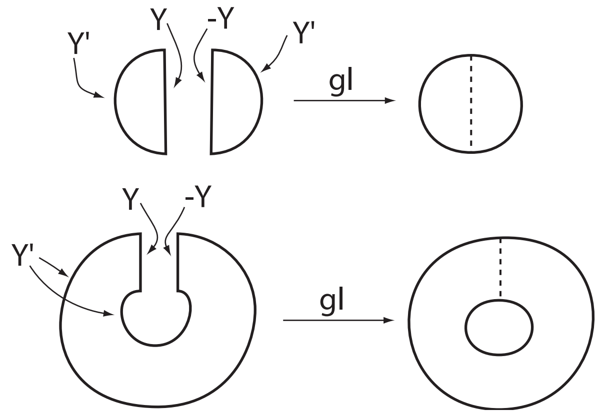
The collection of categories $\{\mathbf{M}_i\}_{i\geq 0}$ has the following familiar additional structure.

Boundary. There is a functor $\partial : \mathbf{M}_i \to \mathbf{M}_{i-1}$. On objects ∂ is the usual boundary of manifolds (with the "outward normal last" orientation convention, say *[is this the convention I want to use?]*). For $f : Y \to W$ a morphism, ∂f is defined to be f restricted to ∂Y, a morphism from ∂Y to ∂W. If $\partial Y = \emptyset$, we say that Y is closed.

Orientation reversal. There is an orientation reversal functor $- : \mathbf{M}_i \to \mathbf{M}_i$. For objects, $-A$ is A with the opposite orientation. For morphisms, $-f$ is identical to f as a map of sets. $-^2$ is the identity functor. $-$ commutes with ∂; i.e., $\partial(-A) = -\partial A$.

Disjoint union. There is a functor $\sqcup : \mathbf{M}_i \times \mathbf{M}_i \to \mathbf{M}_i$ given by disjoint union. ($\mathbf{M}_i \times \mathbf{M}_i$ denotes the abstract product of categories.) Up to canonical isomorphism, this operation is commutative and associative. The empty i-manifold acts as the identity element. $\sqcup$ gives $\mathbf{M}_i$ the structure of a commutative monoid.

Gluing without corners. Given an i-manifold M and an identification $\partial M = Y \sqcup -Y \sqcup Y'$, there is a glued manifold $\mathrm{gl}_Y(M)$ obtained by identifying the two copies of Y in ∂M. (See Figure (3.1.1).) There is a natural isomorphism $\partial(\mathrm{gl}_Y(M)) = Y'$, and $-\mathrm{gl}_Y(M)$ can be identified with $\mathrm{gl}_Y(-M)$. If the gluing region $\pm Y \subset \partial M$ is clear from context, we write simply $\mathrm{gl}(M)$. If $M = M_1 \sqcup M_2$, $Y \subset \partial M_1$, and $-Y \subset \partial M_2$, then we sometimes use the notation $M_1 \cup_Y M_2$ or simply $M_1 \cup M_2$.



3.1.2 Gluing manifolds with corners

Gluing with corners. Let M be an i-manifold and $\partial M = Y \cup -Y \cup Y'$. (That is, ∂M has an identification with the gluing-without-corners $Y \cup -Y \cup Y'$, and $\partial Y' = \partial(Y \sqcup -Y)$.) Then there is a glued manifold $\mathrm{gl}_Y(M)$ obtained by identifying the two copies of Y in ∂M. (See Figure (3.1.2).) There is a natural isomorphism $\partial(\mathrm{gl}_Y(M)) = \mathrm{gl}_{\partial Y}(Y')$, and $-\mathrm{gl}_Y(M)$ can be identified with $\mathrm{gl}_Y(-M)$. Note that gluing without corners is a special case of gluing with corners (corresponding to $\partial Y = \emptyset$). If the gluing region $\pm Y \subset \partial M$ is clear from context, we write simply $\mathrm{gl}(M)$. If $M = M_1 \sqcup M_2$, $Y \subset \partial M_1$, and $-Y \subset \partial M_2$, then we sometimes use the notation $M_1 \cup_Y M_2$ or simply $M_1 \cup M_2$.

Product with I. There is a functor $\mathbf{M}_i \to \mathbf{M}_{i+1}$ given by $M \mapsto M \times I$ and $f \mapsto f \times \mathrm{id}$. (Here I denotes the unit interval $[0,1] \subset \mathbb{R}$.) Unless stated otherwise, our products will be "pinched" along the boundary, so that $\partial(M \times I) = (-M) \cup_{\partial M} M$ (rather than $(-M) \cup (\partial M \times I) \cup M$).

Collar neighborhoods. Let $\partial M = Y \cup Y'$. Then there is a collaring morphism (homeomorphism) $M \to M \cup_Y (Y \times I)$, well-defined up to isotopy rel boundary.

[?? add composition of prod bordisms ??]

3.2 Topological Fields

Before defining topological fields, we give some examples. The reader is encouraged to keep these examples in mind when reading the subsequent abstract axioms for topological fields.

Mapping spaces. Given a topological space B, define

$$\mathcal{MF}_B(X) = \{f : X \to B\},$$

the set of all continuous maps from a manifold X to B. We will be particularly interested in the case where B is the classifying space $B\Gamma$ of a finite group Γ. When the space B is either irrelevant or clear from context, we will omit the subscript B from the notation.

$\mathcal{MF}$ is a functor (either covariant or contravariant, since all morphisms of $\mathcal{M}_i$ are invertible), and it preserves monoidal structure (that is, $\mathcal{MF}(X_1 \sqcup X_2) = \mathcal{MF}(X_1) \times \mathcal{MF}(X_2)$, and similarly for morphisms). If $f \in \mathcal{MF}(X)$, we define ∂f to be f restricted to ∂X. Maps can be glued together if they agree on the gluing region, and all maps on the glued manifold are obtained in this way.

3.2.1 **Designs.** Define

$$\mathcal{S}(X) = \{\text{codimension-1 unoriented PL submanifolds of X}\}.$$

The submanifolds should be properly embedded and transverse to ∂X.

Submanifolds can be glued together if they agree on the gluing region, but note that not all elements of $\mathcal{S}(\mathrm{gl}(X))$ are obtained from $\mathcal{S}(X)$ in this way. A submanifold of $\mathrm{gl}(X)$ is so obtained if and only if it is transverse to the image in $\mathrm{gl}(X)$ of the gluing region. Of course this is, in various senses that can be made precise, almost all of $\mathcal{S}(\mathrm{gl}(X))$, but this "almost all" instead of "all" will complicate our axioms for topological fields.

[give more precise def: embedded complex, specified local behavior, labels, orientations]

The above functors can be generalized to fancier "designs" on manifolds. *[maybe find a better term than "designs".]* (See for example *[forward refs; e.g. G2]*.) Our generic name for such generalizations will be $\mathcal{DF}$. For example, we could define $\mathcal{DF}(Y)$, for a 2-manifold Y, to be the set of all trivalent graphs G in Y, with oriented edges and colors assigned to each edge and each component of $Y \setminus G$. *[need fig]* As before, these should be properly embedded graphs, transverse to ∂Y. Define $\mathcal{DF}(R)$, for a 1-manifold R, to be the set of all properly embedded, oriented 0-submanifolds of R (i.e. a finite collection of signed points in the interior of R), with colors assigned to both the points and the regions between the points. Define $\mathcal{DF}(Z)$, for a 0-manifold Z, to be a coloring of the components of Z. Boundaries of fields are defined in the obvious ways, restricting orientations and colors.

Let $Y = R \cup -R \cup R'$. A field (i.e. colored, oriented graph) on Y glues up to one on $\mathrm{gl}_R(Y)$ if and only if its restrictions to the two copies of R agree with *orientations reversed*. This suggests that we should define, for any 1-manifold S, a bijection

$$\mathcal{DF}(S) \leftrightarrow \mathcal{DF}(-S), \quad a \leftrightarrow \widehat{a},$$

where $\widehat{a}$ has the same colors and points as a, but with reversed orientations on the points.

We now axiomatize topological fields (i.e. list the properties of the above examples which are needed to make later constructions work).

Let $\mathbf{S}$ denote the category of sets. A topological field (of top dimension n) consists of a collection of functors

$$\mathcal{C}_i : \mathbf{M}^i \to \mathbf{S}$$

$(0 \le i \le n)$ which satisfy the following conditions. (We will usually omit the subscript i from the notation.)

Boundary. There are boundary maps

$$\partial : \mathcal{C}(X) \to \mathcal{C}(\partial X).$$

Also, for $c \in \mathcal{C}(\partial X)$ we define

$$\mathcal{C}(X; c) = \partial^{-1}(c) \subset \mathcal{C}(X);$$

$\mathcal{C}(X; c)$ should be thought of as the set of fields on X with boundary conditions c on ∂X.

These boundary maps should be compatible with morphisms and the boundary functor on manifolds: for all i-manifolds X and Y and $f : X \to Y$, the following diagram commutes.

$$\begin{array}{ccc} \mathcal{C}_i(X) & \xrightarrow{\mathcal{C}_i(f)} & \mathcal{C}_i(Y) \\ \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} \\ \mathcal{C}_{i-1}(\partial X) & \xrightarrow{\mathcal{C}_{i-1}(\partial f)} & \mathcal{C}_{i-1}(\partial Y) \end{array}$$

In other words, ∂ is a natural transformation between the two functors $\mathcal{C}_i$ and $\mathcal{C}_{i-1} \circ \partial$ from $\mathbf{M}_i$ to $\mathbf{S}$.

Examples. For $\alpha \in \mathcal{MF}(X)$, $\partial\alpha = \alpha|_{\partial X}$. For $\beta \in \mathcal{S}(X)$ (or $\beta \in \mathcal{DF}(X)$), $\partial\beta$ is the (transverse) intersection of β with ∂X (preserving colors and orientations for $\mathcal{DF}$).

Orientation reversal. If X is closed, there is a bijection

$$\mathcal{C}(X) \leftrightarrow \mathcal{C}(-X), \quad a \leftrightarrow \widehat{a}.$$

For general X, there are bijections

$$\mathcal{C}(X; c) \leftrightarrow \mathcal{C}(-X, \widehat{c}), \quad a \leftrightarrow \widehat{a}.$$

Examples. For $\mathcal{MF}$, these are the obvious bijections resulting from ignoring the orientation of X. For $\mathcal{DF}$, the bijections could involve reversing orientations on parts of the design; see discussion at (3.2.1).

Disjoint union. The fields preserve monoidal structure: there is an identification

$$\mathcal{C}(X_1 \sqcup X_2) = \mathcal{C}(X_1) \times \mathcal{C}(X_2),$$

and these identifications are compatible with morphisms, orientation reversal, and boundaries in the obvious ways. Note that this implies that $\mathcal{C}(\emptyset)$ consists of a single element.

3.2.2 **Gluing without corners.** Given an identification

$$\partial X = Y \sqcup -Y \sqcup W$$

(see Figure (3.1.1)), the disjoint union axiom provides an identification

$$\mathcal{C}(\partial X) = \mathcal{C}(Y) \times \mathcal{C}(-Y) \times \mathcal{C}(W).$$

Using the boundary and orientation reversal maps, we get two maps from $\mathcal{C}(X)$ to $\mathcal{C}(Y)$.

$$\begin{array}{ccccccc} \mathrm{Eq}_Y(\mathcal{C}(X)) & \hookrightarrow & \mathcal{C}(X) & \xrightarrow{\partial} & \mathcal{C}(\partial X) & \xrightarrow{\mathrm{pr}} & \mathcal{C}(Y) \\ &&&&& \searrow^{\mathrm{pr}} & \uparrow{\scriptstyle -} \\ &&&&&& \mathcal{C}(-Y) \end{array}$$

Let $\mathrm{Eq}_Y(\mathcal{C}(X))$ denote the equalizer of these two maps. (That is, the set of all fields in $\mathcal{C}(X)$ on which the two maps agree. In the case where Y and $-Y$ lie in separate components of X, $\mathrm{Eq}_Y(\mathcal{C}(X))$ is a fibered product.) Then there is an injection $\mathrm{gl} : \mathrm{Eq}_Y(\mathcal{C}(X)) \to \mathcal{C}(\mathrm{gl}_Y(X))$ such that the following diagram commutes.

$$\begin{array}{ccccc} \mathrm{Eq}_Y(\mathcal{C}(X)) & \hookrightarrow & \mathcal{C}(X) & \xrightarrow{\partial} & \mathcal{C}(\partial X) \\ {\scriptstyle \mathrm{gl}}\downarrow & & & & \downarrow{\scriptstyle \mathrm{pr}_W} \\ \mathcal{C}(\mathrm{gl}_Y(X)) & & \xrightarrow{\qquad\partial\qquad} & & \mathcal{C}(W) \end{array}$$

We will often identify $\mathrm{Eq}_Y(\mathcal{C}(X))$ is its image $\mathrm{gl}(\mathrm{Eq}_Y(\mathcal{C}(X))) \subset \mathcal{C}(\mathrm{gl}_Y(X))$. If $X = X_1 \sqcup X_2$, $Y \subset \partial X_1$ and $-Y \subset \partial X_2$, then we will often use the notation $\alpha_1 \cup \alpha_2$ instead of $\mathrm{gl}((\alpha_1, \alpha_2))$. (Here $\alpha_i \subset \mathcal{C}(X_i)$.)

Furthermore, we require that for any $x \in \mathcal{C}(\mathrm{gl}_Y(X))$ and any neighborhood N of the image of Y in $\mathrm{gl}_Y(X)$ there exists a homeomorphism f of $\mathrm{gl}_Y(X)$, isotopic to the identity and supported in N, such that $f_*(x) \in \mathrm{Eq}_Y(\mathcal{C}(X))$. In other words, any field on the glued manifold is close to a field obtained by gluing.

[Need to define restriction $c|S$ for codim-0 submanifold S.]

Examples. For $\mathcal{MF}$, $\mathrm{Eq}_Y(\mathcal{MF}(X))$ is all of $\mathcal{MF}(\mathrm{gl}_Y(X))$. For $\mathcal{S}$, $\mathrm{Eq}_Y(\mathcal{S}(X))$ consists of all submanifolds of $\mathrm{gl}_Y(X)$ which are transverse to the image of the gluing region Y in $\mathrm{gl}_Y(X)$.

Gluing with corners. Assume an identification

$$\partial X = Y \cup -Y \cup W,$$

with Y and $-Y$ disjoint in ∂X. (See Figure (3.1.2).) *[correct Y' vs W inconsistency?]* Let $S = \partial W = \partial(Y \sqcup -Y)$, and let $\mathrm{Eq}_S(\mathcal{C}(Y \sqcup -Y \sqcup W)) \subset \mathcal{C}(\partial X)$ be as described in (3.2.2) (gluing without corners on ∂X). Define $\mathcal{C}^{\pitchfork}(X) = \partial^{-1}(\mathrm{Eq}_S(\mathcal{C}(Y \sqcup -Y \sqcup W)))$. $\mathcal{C}^{\pitchfork}(X)$ has two maps (one of which involves orientation reversal) to $\mathcal{C}(Y)$; let $\mathrm{Eq}_Y(\mathcal{C}^{\pitchfork}(X))$ denote the equalizer of these two maps.

$$\begin{array}{ccccccc}
 & & \mathcal{C}(X) & \xrightarrow{\partial} & \mathcal{C}(\partial X) & & \\
 & & \uparrow & & \uparrow{\scriptstyle \mathrm{gl}} & & \\
\mathrm{Eq}_Y(\mathcal{C}^{\pitchfork}(X)) & \hookrightarrow & \mathcal{C}^{\pitchfork}(X) & \xrightarrow{\partial} & \mathrm{Eq}_S(\mathcal{C}(Y \sqcup -Y \sqcup W)) & \xrightarrow{\mathrm{pr}} & \mathcal{C}(Y) \\
 & & & & & \searrow^{\mathrm{pr}} & \uparrow{\scriptstyle -} \\
 & & & & & & \mathcal{C}(-Y)
\end{array}$$

Let $\mathrm{Eq}_{\partial Y}(\mathcal{C}(W)) \subset \mathcal{C}(W)$ be as in (3.2.2), for the gluing $W \to \mathrm{gl}_{\partial Y}(W) = \partial(\mathrm{gl}_Y(X))$. There is a map $\partial : \mathrm{Eq}_Y(\mathcal{C}^{\pitchfork}(X)) \to \mathrm{Eq}_{\partial Y}(\mathcal{C}(W))$ induced by $\partial : \mathcal{C}^{\pitchfork}(X) \to \mathcal{C}(\partial X)$. The gluing with corners axiom requires that there is an injection $\mathrm{gl} : \mathrm{Eq}_Y(\mathcal{C}^{\pitchfork}(X)) \to \mathcal{C}(\mathrm{gl}_Y(X))$ such that the following diagram commutes.

$$\begin{array}{ccc}
\mathrm{Eq}_Y(\mathcal{C}^{\pitchfork}(X)) & \xrightarrow{\partial} & \mathrm{Eq}_{\partial Y}(\mathcal{C}(W)) \\
{\scriptstyle \mathrm{gl}}\downarrow & & \downarrow{\scriptstyle \mathrm{gl}} \\
\mathcal{C}(\mathrm{gl}_Y(X)) & \xrightarrow{\partial} & \mathcal{C}(\mathrm{gl}_{\partial Y}(W))
\end{array}$$

Furthermore, we require that for any $x \in \mathcal{C}^{\pitchfork}(X)$ and any neighborhood N of the image of Y in $\mathrm{gl}_Y(X)$ there exists a homeomorphism f of $\mathrm{gl}_Y(X)$, isotopic (rel boundary) to the identity and supported in N, such that $f_*(x) \in \mathrm{Eq}_Y(\mathcal{C}^{\pitchfork}(X))$. In other words, any field on the glued manifold whose boundary is obtained by gluing is close to a field obtained by gluing.

As before, we will sometimes use the notation $\alpha_1 \cup \alpha_2$ instead of $\mathrm{gl}((\alpha_1, \alpha_2))$ when we are gluing distinct components of X together. The subspace $\mathrm{Eq}_Y(\mathcal{C}^{\pitchfork}(X)) \subset \mathcal{C}(\mathrm{gl}_Y(X))$ is called the *gluing subspace* relative to this decomposition of $\mathrm{gl}_Y(X)$. Note that the gluing without corners axiom (3.2.2) is a special case of this one.

[add products, collars, homeos of products(?)]

3.3 Extended Isotopy

Plan:

- def: extended isotopy generated by isotopy and gluing of product fields to collars; gluing e-isotopic fields yields e-isotopic fields

- note that this is a local relation, since both isotopy and collars can be localized (need further field assumptions here? (for collars))
- work through several examples (ones that will be used later)
- purpose of this section: do some tedious proofs here so as not to interrupt the flow (w?) in later chapters

---

to do:

- ? generalize to not-necessarily-linear case?
- need to be clearer about A maybe/maybe-not being linear (examples will help with this)
- need to define "codim-k" near the start
- ?? need to define "transverse" in field section

Chapter 4

Basic Constructions and Gluing I

In this chapter we define local relations/subspaces, introduce the basic construction for codimension 1 (dimension n) manifolds, and prove the codimension 1 gluing theorem. *[need to be consistent about hyphenating (or not) codimension 1 or codimension-1]*

Local relations (or dually subspaces) are defined as ones which can be specified inside an n-ball. Given fields and local relations, the basic construction associates to each n-manifold Y with field boundary conditions c an object $A(Y;c)$ (either a vector space or set in most examples). Given an $n-1$-manifold S, gluing defines an associative composition on the various $A(S \times I; c_0, c_1)$, and we get a category $A(S)$. The category $A(\partial Y)$ acts on the collection $\{A(Y;\cdot)\}$, and there is a gluing theorem describing the glued spaces $A(Y_{\text{gl}})$ in terms of the various $A(Y;c)$ and the action of $A(\partial Y)$. We give several versions of this gluing theorem.

4.1 Local Relations and Local Subspaces

Local relations (or dually, local subspaces) carry the same information as the path integral of B^{n+1} (see Chapters 1 and 6).

Let $\mathcal{C}$ be a field functor and $B = B^n$ be the standard n-dimensional ball. A *system of local relations* for $\mathcal{C}$ is a collection of equivalence relations $\sim$ in $\mathcal{C}(B;c)$ (for all $c \in \mathcal{C}(\partial B)$) such that:

- If $a, b \in \mathcal{C}(B;c)$ are related by an isotopy, then $a \sim b$. (In other words, $\sim$ is at least as strong as isotopy. This is equivalent to the topological invariance of the action.) 4.1.1
- If $a \in \mathcal{C}(B;c)$ and S is a codimension 0 submanifold of ∂B, then $((c|_S \times I) \cup a) \sim a$. In other words, we can glue a product field onto a subset of 4.1.2

the boundary of a and the result is equivalent to a. (This is in some sense the closure of the isotopy condition.) *[maybe define "extended isotopy" in the fields chapter]*

4.1.3
- Let $E \subset B$ be a subball (possibly intersecting ∂B) of B, equipped with a homeomorphism to the standard ball B. Then $\sim$ on the various $\mathcal{C}(E; c')$ induces relations on the various $\mathcal{C}(B; c)$, and we require that this induced relation be implied by $\sim$ on $\mathcal{C}(B; c)$. (In other words, applying $\sim$ to a subball yields no new relations.)

Let $A(B; c) = \mathcal{C}(B; c)/\sim$.

More generally, we could replace $\mathcal{C}(B; c)$ above with $\widetilde{A}(B; c)$, the space of finite linear combinations of elements of $\mathcal{C}(B; c)$. In this case $\sim$ is equivalent to a quotient map $\widetilde{A}(B; c) \to A(B; c)$.

Dually, let $\widetilde{Z}(B; c)$ be the space of all functions from $\mathcal{C}(B; c)$ to $\mathbb{C}$. A *system of local subspaces* for $\mathcal{C}$ is a collection of subspaces $Z(B; c) \subset \widetilde{Z}(B; c)$ (for all $c \in \mathcal{C}(\partial B)$) such that:

- If $a, b \in \mathcal{C}(B; c)$ are related by an isotopy, $f(a) = f(b)$ for all $f \in Z(B; c)$.
- If $a \in \mathcal{C}(B; c)$ and S is a codimension 0 submanifold of ∂B, then $f((c|_S \times I) \cup a) = f(a)$ for all $f \in Z(B; c)$.
- Let $E \subset B$ be a subball (possibly intersecting ∂B) of B, equipped with a homeomorphism to the standard ball B. Then the various subspaces $Z(E; c')$ induce subspaces $Z'(B; c)$, and we require that $Z'(B; c) \subset Z(B; c)$.

$Z(B; c)$ should be thought of as the image of the local projections in Chapter 1.

4.2 The Basic Construction in Codimension 1

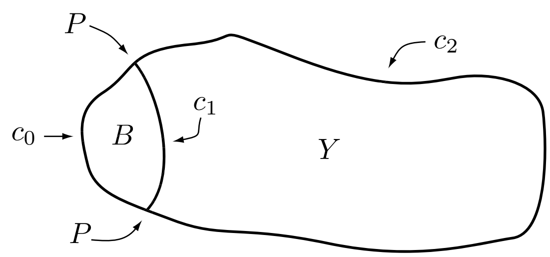
The local relations and subspaces of the previous section give rise to vector spaces for arbitrary n-manifolds. The construction is fairly obvious, but it takes a while to state it precisely. Briefly, $A(Y; c)$ is the quotient of $\widetilde{A}(Y; c)$ by local relations derived from all balls $B \subset Y$. Dually, $Z(Y; c)$ in the intersection of the subspaces of $\widetilde{Z}(Y; c)$ derived from all $B \subset Y$.

Let Y^n be a closed n-manifold. For each ball $B \subset Y$ we have a decomposition $Y = B \cup Y'$ and

$$\mathcal{C}(Y) \supseteq \bigcup_{c \in \mathcal{C}(\partial B)} \mathcal{C}(B; c) \times \mathcal{C}(Y'; \widehat{c}).$$

Hence

$$\widetilde{A}(Y) \supseteq \bigoplus_{c \in \mathcal{C}(\partial B)} \widetilde{A}(B; c) \otimes \widetilde{A}(Y'; \widehat{c}).$$



4.2.1 Subball intersecting boundary

Let $\sim_B$ be the equivalence relation on $\mathcal{C}(Y)$ or $\widetilde{A}(Y)$ induced by the local relations on B and the above decompositions. Let $\sim_Y$ be the equivalence relation generated by $\{\sim_B\}$, where B ranges over all subballs of Y. Define $A(Y) = \widetilde{A}(Y)/\sim_Y$ (or $A(Y) = \mathcal{C}(Y)/\sim_Y$).

Dually, we have, for each $B \subset Y$,

$$\widetilde{Z}(Y) \subseteq \prod_{c \in \mathcal{C}(\partial B)} \widetilde{Z}(B; c) \otimes \widetilde{Z}(Y'; \widehat{c}).$$

Let

$$Z'_B \subseteq \prod_{c \in \mathcal{C}(\partial B)} Z(B; c) \otimes \widetilde{Z}(Y'; \widehat{c}) \subset \widetilde{Z}(Y).$$

Define

$$Z(Y) = \bigcap_B Z'_B,$$

where B ranges over all subballs of Y. *[need to revise notation above; $\otimes$ is not the right thing; fix below too]*

More generally, let Y be an n-manifold with boundary, $c \in \mathcal{C}(\partial Y)$, and $B \subset Y$ be a subball, possibly intersecting ∂Y. Let $S_0 = \partial B \cap \partial Y$, $\partial B = S_0 \cup S_1$, $\partial Y = S_0 \cup S_2$, and $P = \partial S_0 = \partial S_1 = \partial S_2$. (See Figure (4.2.1).) Let $c = c_0 \cup c_2$ with respect to the decomposition $\partial Y = S_0 \cup S_2$, and let $\partial c_0 = \partial c_2 = d \in \mathcal{C}(P)$. Let $Y = Y' \cup B$, so that $\partial Y' = (-S_1) \cup S_2$. Then we have decompositions

$$\mathcal{C}(Y; c) \supseteq \bigcup_{c_1 \in \mathcal{C}(S_1; d)} \mathcal{C}(B; c_0 \cup c_1) \times \mathcal{C}(Y'; \widehat{c_1} \cup c_2)$$

and

$$\widetilde{A}(Y; c) \supseteq \bigoplus_{c_1 \in \mathcal{C}(S_1; d)} \widetilde{A}(B; c_0 \cup c_1) \otimes \widetilde{A}(Y'; \widehat{c_1} \cup c_2).$$

Then as before we can define $\sim_B$ on $\mathcal{C}(Y; c)$ or $\widetilde{A}(Y; c)$, and the various $\sim_B$ generate an equivalence relation $\sim_Y$. We define $A(Y; c) = \widetilde{A}(Y; c)/\sim_Y$ or $A(Y; c) = \mathcal{C}(Y; c)/\sim_Y$.

Dually, we have, for each $B \subset Y$,

$$\widetilde{Z}(Y;c) \subseteq \prod_{c_1 \in \mathcal{C}(S_1;d)} \widetilde{Z}(B;c_0 \cup c_1) \otimes \widetilde{Z}(Y';\widehat{c_1} \cup c_2).$$

Let

$$Z'_B \subseteq \prod_{c_1 \in \mathcal{C}(S_1;d)} Z(B;c_0 \cup c_1) \otimes \widetilde{Z}(Y';\widehat{c_1} \cup c_2) \subset \widetilde{Z}(Y;c).$$

As before, define

$$Z(Y;c) = \bigcap_B Z'_B,$$

where B ranges over all subballs of Y.

Clearly both A and Z are functorial; boundary-fixing homeomorphisms of Y act on $A(Y;c)$ and $Z(Y;c)$. (Since all morphisms are invertible, we needn't distinguish between covariant and contravariant functors.)

At this point we have given two definitions of $A(B^n;c)$ and $Z(B^n;c)$: input for local relations or subspaces and also the above constructions for subballs of B^n. We must show that these two definitions agree. More generally we will show that for any n-manifold Y the above constructions of $A(Y;c)$ (or $Z(Y;c)$) yield the same answer if we restrict the subballs in the construction to come from some open covering of Y.

Let $A^r(Y;c)$ denote the result of the restricted construction. Clearly $A(Y;c) \subseteq A^r(Y;c)$ (more subballs means a stronger equivalence relation), so we must show that $A(Y;c) \supseteq A^r(Y;c)$. An arbitrary boundary-fixing isotopy of Y can be decomposed into isotopies supported in balls of the given open cover of Y. Thus the restricted equivalence relation is at least as strong as isotopy. Let $B \subset Y$ be an arbitrary subball in the interior of Y. There is an isotopy of Y which carries B into the interior of one of the covering subballs. But (4.1.3) in the definition of local relation *[need better way of referring here (and below)]* now insures that that any relation supported in B is also part of the restricted relation. If B intersects ∂Y, we convert B to an interior subball by gluing a collar onto ∂Y. (This relies on (4.1.2) in the definition.) The collar is added in pieces, each of which is supported in the boundary of the given covering of Y.

4.3 Cylinder Categories

Our next goal is to describe how A and Z behave under cutting and gluing. This will entail understanding the effect of shifting a collar of the cutting $n-1$-submanifold across the cutting submanifold (see Section 4.4), which motivates the the definitions in this section.

Let S be a closed $n-1$-manifold. Define $A(S)$, the *cylinder category* associated to S, to be the category consisting of:

- Objects: $A(S)^0 = \mathcal{C}(S)$.
- Morphisms: $A(S)^1_{ab} = A(S\times I; \widehat{a}, b)$. In other words, the morphisms from a to b are fields on the cylinder $S\times I$ with boundary conditions given by $\widehat{a}$ and b, modulo local relations. (In most of our examples this will be a vector space.)
- Composition: Given by gluing two cylinders together to get another cylinder. Associativity of composition follows from the associativity of gluing.

More generally, if S has boundary and $c \in \mathcal{C}(\partial S)$ we can define a category $A(S;c)$ with objects $\mathcal{C}(S;c)$ and morphisms $A(S\times I; \widehat{a}, b)$, where $a, b \in \mathcal{C}(S;c)$. Here, as usual, $S\times I$ denotes a cylinder with the vertical boundary $\partial S\times I$ pinched, so that $\partial(S\times I)$ is naturally identified with $(-S)\cup S$ (the double of S) and $\widehat{a}\cup b \in \mathcal{C}(\partial(S\times I))$.

If S is the empty $n{-}1$-manifold, then $A(S)$ is the trivial category consisting of a single object and (multiples of) its identity morphism.

There is a natural identification

$$A(S;c)^{\mathrm{op}} = A(-S;\widehat{c}).$$

On objects this isomorphism is given by $a \mapsto \widehat{a}$, and on morphisms it is induced by the orientation preserving map $r : (x,t) \mapsto (x, 1-t)$ from $S\times I$ to $(-S)\times I$.

Composing the above functor with orientation reversal on $(-S)\times I$ gives a 4.3.1
contravariant functor from $A(S;c)$ to itself. On objects this is the identity, $a \mapsto a$. On morphisms it is $y \mapsto y^* \stackrel{\mathrm{def}}{=} \widehat{r(y)}$, where $y \in A(S\times I; \widehat{a}, b)$ and r flips the I factor of $S\times I$ as above.

It is easy to see that there is a natural identification

$$A(S_1 \sqcup S_2) = A(S_1)\times A(S_2).$$

Recall that a right representation of a category A is a functor from A^{op} to $\mathcal{V}_1$, the category of vector spaces and linear maps. In other words, for each object x of A we have a vector space W_x, for each morphism $e : x \to y$ of A we have a linear map $\rho_e : W_x \to W_y$, and composition of morphisms is preserved ($\rho_{ef} = \rho_e\rho_f = \rho_f \circ \rho_e$). If A has linear morphisms, then ρ above is required to be a linear map.

Let Y be a n-manifold with boundary (possibly empty). Then the collection of vector spaces $A(Y;\cdot)$ affords a representation of $A(\partial Y)$. A morphism of $A(\partial Y)$ is an equivalence class of fields on $\partial Y\times I$, and gluing this as a collar to Y defines the action of the representation. Associativity of gluing implies that composition is preserved. We denote this representation of $A(\partial Y)$ by $A(Y)$. In other words, $A(Y)$ consists of the collection of vector spaces $A(Y;\cdot)$ together with the above action of $A(\partial Y)$.

Dually, $Z(Y;\cdot)$ affords a left representation of $A(\partial Y)$. The action of $e \in A(\partial Y)^1_{ab}$ on $x \in Z(Y;b)$ is defined by $(e\cdot x)(u) = x(u\cup e)$, where $u \in \mathcal{C}(Y;a)$. As before, we denote this representation (collection of vector spaces plus left action of $A(\partial Y)$) by $Z(Y)$.

For any closed $n-1$-manifold S define $Z(S) = \mathrm{Rep}(A(S))$, the category whose objects are left representations of $A(S)$ and whose morphisms are natural transformations (also called intertwiners in this context). Then we have, for all n-manifolds Y,

4.3.2
$$Z(Y) \in Z(\partial Y).$$

This is the categorified version of (1.2.1) and (6.1.1).

4.3.3 Assume now that $A(\partial Y)$ is semisimple (see Appendix B) and let α be a right irrep of $A(\partial Y)$. Define

$$A(Y;\alpha) = \mathrm{mor}(\alpha, A(Y)),$$

the space of intertwiners from α to $A(Y)$. Let $\mathcal{L}$ be a complete set of irreps for $A(\partial Y)$. Then there is a canonical isomorphism

$$A(Y) \cong \bigoplus_{\alpha \in \mathcal{L}} A(Y;\alpha) \otimes \alpha.$$

In this context, irreps are sometimes referred to as "labels" or "particles".

If $e_\alpha \in A(\partial Y)^1_{bb}$ is a minimal idempotent for α (see Appendix B *[need more specific reference]*), then there is a natural isomorphism

$$A(Y;\alpha) \cong A(Y;b)e_\alpha.$$

In other words, $A(Y;\alpha)$ is naturally isomorphic to the subspace of fields (mod relations) in $A(Y;b)$ which have representatives containing e_α in a collar of ∂Y.

Dually, if β is a left irrep of $A(\partial Y)$, we define

$$Z(Y;\beta) = \mathrm{mor}(\beta, Z(Y)),$$

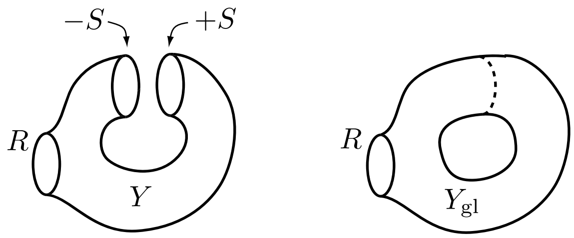
and we have, canonically,

$$Z(Y) \cong \bigoplus_{\beta \in \mathcal{L}} Z(Y;\beta) \otimes \beta$$

(where now $\mathcal{L}$ denotes a complete set of left irreps of $A(\partial Y)$). Note: In the old-fashioned axiomatization of TQFTs (see, e.g., [Walk91]) these are the spaces associated to surfaces with labeled boundary.

4.4 Gluing Codimension-1 Manifolds

Let Y be an n-manifold on the verge of being glued without corners. In other words, we have an identification $\partial Y = (-S) \sqcup S \sqcup R$ (see Figure (4.4.1)), and define Y_{gl} to be Y with S glued to $-S$. Fix (for the remainder of this section) $c \in \mathcal{C}(R) = \mathcal{C}(\partial Y_{\mathrm{gl}})$.



4.4.1 More gluing

Our goal is to describe $A(Y_{\text{gl}}; c)$ in terms of the various $A(Y; \cdot, \cdot, c)$ and the action of $A(-S \sqcup S) = A(S)^{\text{op}} \times A(S)$ on these spaces.

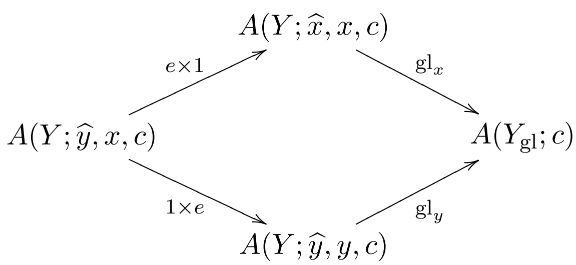
We'll start with the most abstract formulation of the codimension-1 gluing theorem, and then work our way toward more concrete statements.

Theorem. *Let Y, S, Y_{gl}, c be as above.* 4.4.2

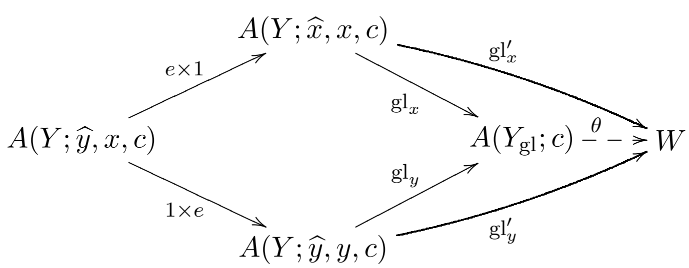
(a) For each object x of $A(S)$ there is a map

$$\text{gl}_x : A(Y; \widehat{x}, x, c) \to A(Y_{\text{gl}}; c).$$

(b) For each morphism $e : x \to y$ of $A(S)$ the following diagram commutes



(c) $A(Y_{\text{gl}}; c)$ is the universal object (vector space, set, or whatever flavor of A we're using) with properties (a) and (b). In other words, given a W and maps $\text{gl}'_x : A(Y; \widehat{x}, x, c) \to W$ (for all x) such that the diagram analogous to the one in (b) above commutes for all e, there is a unique $\theta : A(Y_{\text{gl}}; c) \to W$ such that $\text{gl}'_x = \theta \circ \text{gl}_x$ for all x.



In other words, $A(Y_{\text{gl}}; c)$ is the coend (see Appendix A [need more specific reference]*) of the action of $A(S)^{\text{op}} \times A(S)$ on $A(Y; \cdot, \cdot, c)$.*

Proof. Part (a) is obvious; fields which agree on $\pm S$ can be glued to yield a field on Y_{gl}, and all local relations on Y are also local relations on Y_{gl}.

Part (b) follows from the fact that shifting a collar across the gluing submanifold changes the field by an isotopy. The top arrows in the diagram correspond to gluing e to $-S$, and then gluing $-S$ to S to obtain a field on Y_{gl}. The bottom arrows correspond to gluing e to S first. The two fields thus obtained differ by an isotopy of Y_{gl} which shifts a collar across $\pm S$, so they represent the same element of $A(Y_{\mathrm{gl}}; c)$ and the diagram commutes.

Part (c) requires a little more work. Let $g \in \mathcal{C}(Y_{\mathrm{gl}}; c)$ and $\bar{g}$ be the equivalence class of g in $A(Y_{\mathrm{gl}}; c)$. After an isotopy, we may assume that g is transverse to the gluing submanifold (image of $\pm S$) in Y_{gl}. Then $\bar{g} = \mathrm{gl}_x(g^\sharp)$ for some $g^\sharp \in A(Y; \widehat{x}, x, c)$, where x is g restricted to $\pm S$. Necessarily, we define $\theta(\bar{g}) = \mathrm{gl}'_x(g^\sharp)$. Thus θ is unique if it exists.

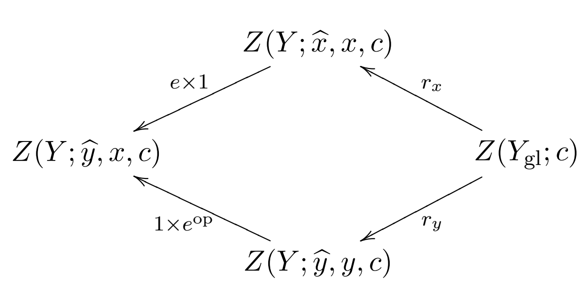
It remains to be shown that the above procedure for defining θ yields well-defined results. We must show that if g and h are equal in $A(Y_{\mathrm{gl}}; c)$, then $\mathrm{gl}'_x(g^\sharp) = \mathrm{gl}'_y(h^\sharp)$ (where y is h restricted to $\pm S$). If g and h are locally related with respect to a ball $B \subset Y_{\mathrm{gl}}$ disjoint from $\pm S$, then $g^\sharp = h^\sharp$, so *a fortiori* $\mathrm{gl}'_x(g^\sharp) = \mathrm{gl}'_x(h^\sharp)$. If g and h are related by a collar shift isotopy along $\pm S$, then $\mathrm{gl}'_x(g^\sharp) = \mathrm{gl}'_y(h^\sharp)$ by the assumed commutativity of the above diagram for W and gl'. Since collar shift isotopies plus isotopies supported in balls disjoint from $\pm S$ generate arbitrary isotopies on Y_{gl}, $\mathrm{gl}'_x(g^\sharp) = \mathrm{gl}'_y(h^\sharp)$ if g and h are related by an arbitrary isotopy. Finally, suppose g and h are locally related with respect to a ball B not disjoint from $\pm S$. Let φ be an isotopy of Y_{gl} which moves B off of $\pm S$. Note that $\varphi(g)^\sharp$ is related to $\varphi(h)^\sharp$ by a local relation supported in $\varphi(B)$. Then $\mathrm{gl}'_x(g^\sharp) = \mathrm{gl}'_w(\varphi(g)^\sharp) = \mathrm{gl}'_z(\varphi(h)^\sharp) = \mathrm{gl}'_y(h^\sharp)$, where w and z are the restrictions of $\varphi(g)$ and $\varphi(h)$ to $\pm S$. □

There is, of course, and dual version of the gluing theorem for Z:

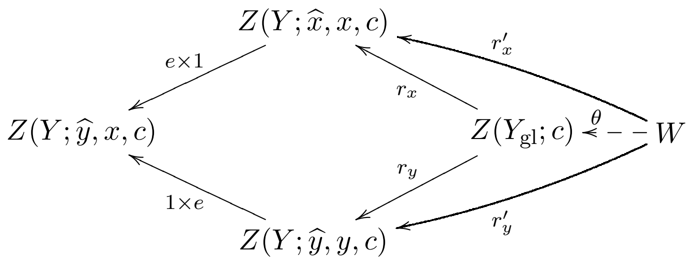
4.4.3 **Theorem.** *Let Y, S, Y_{gl}, c be as above.*
(a) For each object x of $A(S)$ there is a map

$$r_x : Z(Y_{\mathrm{gl}}; c) \to Z(Y; \widehat{x}, x, c).$$

(b) For each morphism $e : x \to y$ of $A(S)$ the following diagram commutes



(c) $Z(Y_{\mathrm{gl}};c)$ is the universal object with properties (a) and (b). In other words, given a W and maps $r'_x : W \to Z(Y;\widehat{x},x,c)$ (for all x) such that the diagram analogous to the one in (b) above commutes for all e, there is a unique $\theta : W \to Z(Y_{\mathrm{gl}};c)$ such that $r'_x = r_x \circ \theta$ for all x.



In other words, $Z(Y_{\mathrm{gl}};c)$ is the end (See Appendix A [need more specific reference]) of the action of $A(S)^{\mathrm{op}} \times A(S)$ on $Z(Y;\cdot,\cdot,c)$.

The proof is dual to the proof of (4.4.2). The map $r_x : Z(Y_{\mathrm{gl}};c) \to Z(Y;\widehat{x},x,c)$ is given by restricting a function in $Z(Y_{\mathrm{gl}};c)$ to the subset $\mathrm{gl}_x(\mathcal{C}(Y;\widehat{x},x,c))$ of $\mathcal{C}(Y_{\mathrm{gl}};c)$.

In the remainder of this section we state a number of special cases of the codimension-1 gluing theorem.

Corollary. *Let Y, S, Y_{gl}, c be as above. If the target category of A (on n-manifolds) is the category of vector spaces, then* 4.4.4

$$A(Y_{\mathrm{gl}};c) = \left(\bigoplus_{x\in\mathcal{C}(S)} A(Y;\widehat{x},x,c)\right) / \langle ev \sim ve\rangle.$$

By $\langle ev \sim ve\rangle$ we mean the subspace of $\bigoplus_{x\in\mathcal{C}(S)} A(Y;\widehat{x},x,c)$ generated by all $ev - ve$, for all morphisms $e : x \to y$ of $A(S)$ and all $v \in A(Y;y,x,c)$. Here we write the action of $A(S)$ as juxtaposition on the right and the action of $A(S)^{\mathrm{op}}$ as juxtaposition on the left.

Note that $\bigoplus$ above means finite linear combinations.

Corollary. *Let Y, S, Y_{gl}, c be as above. If the target category of Z (on n-manifolds) is vector spaces, then*

$$Z(Y_{\mathrm{gl}};c) = \{(v_x) \in \prod_{x\in\mathcal{C}(S)} Z(Y;\widehat{x},x,c) \mid v_x e = e v_y \text{ for all } e\}.$$ 4.4.5

Here $e : x \to y$ and $v_x e, e v_y \in Z(Y;\widehat{x},y,c)$.

Note that $\prod$ above means infinite linear combinations.

If $\partial Y_1 = (-R) \sqcup S$, then we can think of Y_1 as a bordism from R to S and $A(Y_1)$ as an $A(R)$-$A(S)$ bimodule (i.e. it has an action of $A(R)^{\mathrm{op}} \times A(S)$).

Corollary. *Let $\partial Y_1 = (-R) \sqcup S$ and $\partial Y_2 = (-S) \sqcup T$. Then*

$$A(Y_1 \cup_S Y_2) = A(Y_1) \otimes_S A(Y_2).$$

(See Appendix A for the definition of $\otimes_S$.) In other words, we have a functor from the category of n-dimensional bordisms to the category of bimodules.

(Some authors use the above functorial property as an axiom for the codimension-1 gluing properties of TQFTs, but note that without auxiliary axioms it fails to handle the case of gluing an n-manifold to itself, nor does it cover interrelationships between different bordisms with the same underlying manifold (i.e. moving a part of the incoming boundary to the outgoing boundary, or vice-versa).)

Let Y, S, R, Y_{gl}, c be as before. Assume now that $A(S)$ and $A(R)$ are semisimple categories. Let $\mathcal{L}$ be a complete set of irreps for $A(S)$ and $\mathcal{K}$ be a complete set of irreps for $A(R)$. Then

$$A(Y) = \bigoplus_{\alpha,\beta,\gamma} A(Y;\alpha^*,\beta,\gamma) \otimes \alpha^* \otimes \beta \otimes \gamma,$$

where the sum is over $\alpha, \beta \in \mathcal{L}$ and $\gamma \in \mathcal{K}$. Note that the dual representation α^* is a right irrep of $A(S)^{\text{op}} = A(-S)$, and so $\alpha^* \otimes \beta \otimes \gamma$ runs through a complete set of irreps for $A(\partial Y)$. (See (4.3.3) above.) Also

$$A(Y_{\text{gl}}) = \bigoplus_{\gamma \in \mathcal{K}} A(Y_{\text{gl}};\gamma) \otimes \gamma.$$

By (B.5.1), if α is an irrep then the coend of $\alpha^* \otimes \alpha$ is canonically isomorphic to $\mathbb{C}$, and if α and β are irreps and α is not isomorphic to β then the coend of $\alpha^* \otimes \beta$ is 0. It follows that

4.4.6 **Corollary.** *Retaining notation from above, there is a natural isomorphism*

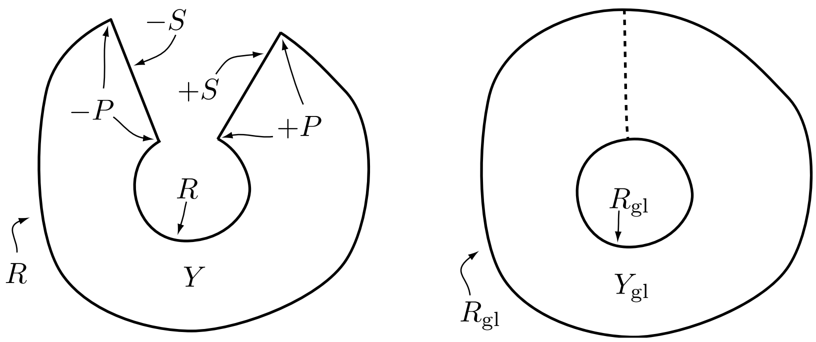
$$A(Y_{\text{gl}};\gamma) \cong \bigoplus_{\alpha \in \mathcal{L}} A(Y;\alpha^*,\alpha,\gamma).$$

Dually, we have that the end of $\alpha^* \otimes \alpha$ is canonically isomorphic to $\mathbb{C}$ and the end of $\alpha^* \otimes \beta$ is 0 if α is not isomorphic to β, for irreps α and β. Hence

4.4.7 **Corollary.** *Retaining notation from above, there is a natural isomorphism*

$$Z(Y_{\text{gl}};\gamma) \cong \bigoplus_{\alpha \in \mathcal{L}} Z(Y;\alpha^*,\alpha,\gamma).$$

The above corollary is the old-fashioned version of the codimension-1 gluing theorem (see e.g. [Walk91]).



4.4.8 Gluing n-manifolds with corners

Next we consider gluing with corners for n-manifolds.

Let Y be an n-manifold with identifications $\partial Y = (-S) \cup S \cup R$, $\partial S = P$, $\partial R = P \sqcup -P$ (see Figure (4.4.8)). Let Y_{gl} be Y with $-S$ glued to S. Let R_{gl} be R with $-P$ glued to P. Note that $\partial Y_{\mathrm{gl}} = R_{\mathrm{gl}}$.

Choose $c \in \mathcal{C}(R_{\mathrm{gl}})$ and assume that c is the gluing of some $c^\sharp \in \mathcal{C}(R; \widehat{z}, z)$, $z \in \mathcal{C}(P)$. Then $A(S; z)^{\mathrm{op}} \times A(S; z)$ acts on $A(Y; \cdot, \cdot, c^\sharp)$ and we have the following generalization of (4.4.2).

Theorem. *$A(Y_{\mathrm{gl}}; c)$ is the coend of the action of $A(S; z)^{\mathrm{op}} \times A(S; z)$ on $A(Y; \cdot, \cdot, c^\sharp)$.* 4.4.9

(a) For each object x of $A(S; z)$ there is a map

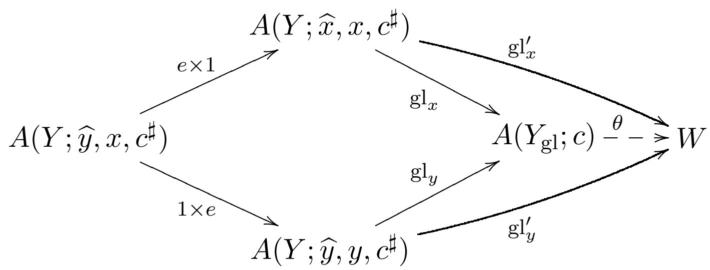
$$\mathrm{gl}_x : A(Y; \widehat{x}, x, c^\sharp) \to A(Y_{\mathrm{gl}}; c).$$

(b) For each morphism $e : x \to y$ of $A(S; z)$ the following diagram commutes

$$\begin{array}{ccccc} & & A(Y; \widehat{x}, x, c^\sharp) & & \\ & \overset{e\times 1}{\nearrow} & & \overset{\mathrm{gl}_x}{\searrow} & \\ A(Y; \widehat{y}, x, c^\sharp) & & & & A(Y_{\mathrm{gl}}; c) \\ & \underset{1\times e}{\searrow} & & \underset{\mathrm{gl}_y}{\nearrow} & \\ & & A(Y; \widehat{y}, y, c^\sharp) & & \end{array}$$

(c) $A(Y_{\mathrm{gl}}; c)$ is the universal object (vector space, set, or whatever flavor of A we're using) with properties (a) and (b). In other words, given a W and maps $\mathrm{gl}'_x : A(Y; \widehat{x}, x, c^\sharp) \to W$ (for all x) such that the diagram analogous to the one in (b) above commutes for all e, there is a unique $\theta : A(Y_{\mathrm{gl}}; c) \to W$ such that

$\mathrm{gl}'_x = \theta \circ \mathrm{gl}_x$ *for all* x.



The proof does not differ in any interesting way from the proof of (4.4.2). All of the other results in this section generalize easily to gluing n-manifolds with corners.

Note, however, that (4.4.9) is a somewhat unsatisfactory result. We want to know not only $A(Y_{\mathrm{gl}}; c)$ for various $c \in \mathcal{C}(\partial Y_{\mathrm{gl}})$, but also the action of $A(\partial Y_{\mathrm{gl}})$ on $A(Y_{\mathrm{gl}})$ (and also $Z(Y_{\mathrm{gl}})$). For gluing without corners, ∂Y_{gl} is a closed submanifold of ∂Y and knowledge of the action of $A(\partial Y)$ on $A(Y)$ translates easily into knowledge of the action of $A(\partial Y_{\mathrm{gl}})$ on $A(Y_{\mathrm{gl}})$. For gluing with corners, this is not the case. In order to state a more powerful result on gluing n-manifolds with corners, we first need to understand gluing $n-1$-manifolds without corners. That is, we need to be able to describe the category $A(R_{\mathrm{gl}})$ is terms of the categories $A(R; \widehat{z}, z)$ and the action of the 2-category $A(P)$ on them. This is done in Chapter 5.

---

still to do in this chapter:

- need to state cat conditions for general version of gluing
- YYYYY: insert examples into later sections (A and Z defs, cylinder cats, ...)
- need to be clearer about A maybe/maybe-not being linear (examples will help with this)
- make clear how empty boundary case works
- need to define “codim-k” near the start
- ?? retain old manifolds → cat → n-vect space thing? put it in some other chapter?

Chapter 5

Basic Constructions and Gluing II

This Chapter is analogous to Chapter 4, but one dimension lower and hence one category level higher. We show that $n-1$-manifolds with boundary give rise to collections of 1-categories; that these collections afford a representation of a 2-category associated to the $n-2$-dimensional boundary of the $n-1$-manifold; and that these 2-category actions can be used to state and prove gluing theorems for A and Z of 1-manifolds.

[need to finish this intro once chapter is complete]

[WARNING: need to replace a with $\widehat{a}$ and X with $-X$ in many places]

[WARNING: in order to be consistent with other chapters, need to switch right and left actions, and modify most of the figures]

5.1 2-Category Actions

By 2-category we mean a disk-like 2-category, as defined in (A.2.1). In particular, there are conjugations (corresponding to homeomorphisms of I and D^2) defined on the 1-morphisms and 2-morphisms, and these conjugations satisfy various identities.

Let P be a closed $n-2$-manifold. We define a 2-category $A(P)$ as follows:

- 0-morphisms (objects): $\mathcal{C}(P)$.
- 1-morphisms from a to b: $\mathcal{C}(P \times I; a, b)$.
- 2-morphisms from e to f: $A(P \times I \times I; e, f)$, where $P \times \partial I \times I \subset \partial(P \times I \times I)$ is pinched, so that $\partial(P \times I \times I)$ can be identified with $P \times I \times \partial I$.
- Composition of 1-morphisms: given by the gluing $(P \times I) \cup_P (P \times I) \cong P \times I$.

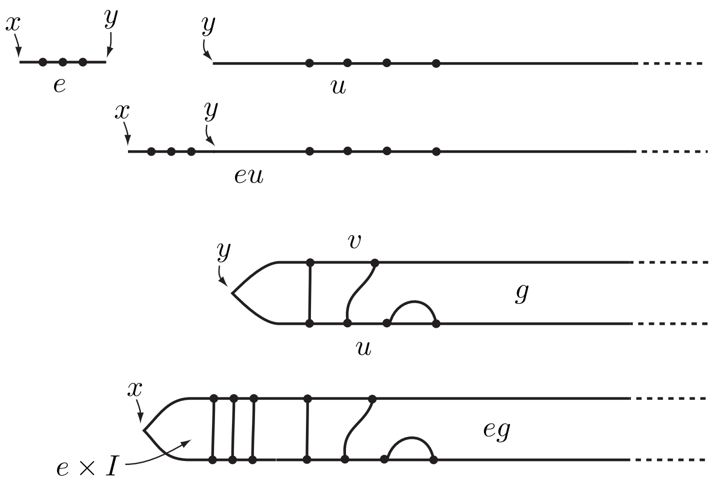
- Composition of 2-morphisms: given by the gluing $(P\times I\times I)\cup_{P\times I}(P\times I\times I)\cong P\times I\times I$.
- *[need to decide whether other type of composition of 2-mors is part of the data (as opposed to derived)]*
- Associativity 2-morphism for composable 1-morphisms e, f, g: the track of an isotopy from $(ef)g$ to $e(fg)$ (two distinct but isotopic fields on $P\times I$).
- Conjugation of 1-morphisms: given by an orientation-reversing homeomorphisms from I to itself.
- Conjugations of 2-morphsisms: given by various homeomorphisms from $I\times I$ to itself.

[maybe go over above in more detail at (A.2.1)]

A *left representation* W of a 2-category C consists of:

- For each $x\in C^0$, a 1-category W_x.
- For each $e\in C^1_{xy}$, a functor $W_e : W_y \to W_x$. (We usually simplify notation and denote W_e by e.)
- For each $e\in C^1_{xy}$ and $f\in C^1_{yz}$, an invertible natural transformation W^c_{ef} between W_eW_f (composition of two functors) and W_{ef} (functor associated to the composition of two 1-morphisms). In other words, composition is only preserved up to natural transformations, and these natural transformations are part of the data of the representation. (We usually simplify notation and denote W^c_{ef} by c_{ef}.)
- For each $h\in C^2_{ef}$, a natural transformation $W_h : W_e \to W_f$. (We usually simplify notation and denote W_h by h.)
- For each $e\in C^1_{xy}$, $u\in W^0_x$ and $v\in W^0_y$, an adjunction (natural bijection) between $\mathrm{mor}(ev, u)$ and $\mathrm{mor}(v, e^*u)$. If the $W_\cdot$'s are linear categories, the these adjunctions are required to be linear. *[is my use of "adjunction" here standard? if not, change or explain why I'm being slightly non-standard]*
- There are various relationships between conjugations in C^2 and the adjunctions, between the various c_{ef} and associator 2-morphisms, etc. *[be specific here?]*

A *right representation* is defined similarly, reversing the direction of the C^1 actions: for each $e\in C^1_{xy}$, a functor $W_e : W_x \to W_y$.



5.1.1 The functor for e

Let S be an $n-1$-manifold with boundary. For each $x \in \mathcal{C}(\partial S)$ we have the 1-category $A(S;x)$, and this collection of 1-categories affords a left representation of the 2-category $A(\partial S)$ as follows.

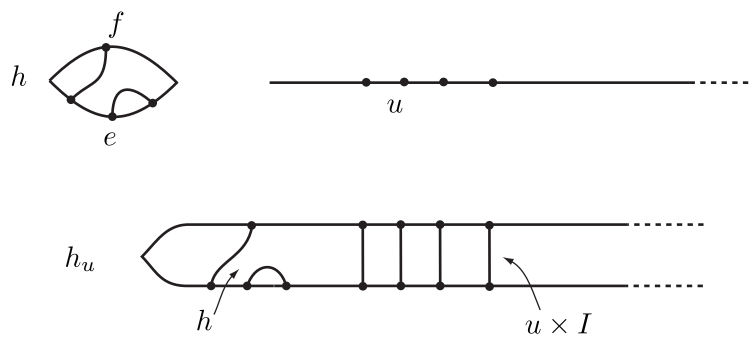
Let C denote $A(\partial S)$. For $e \in C^1_{xy} = \mathcal{C}(\partial S \times I; x, y)$ we need a functor (also denoted e) from $A(S, y)$ to $A(S, x)$. On objects this functor is given by gluing a copy of e to S along ∂S (i.e. we add a collar to ∂S containing the field e). On morphisms this functor is given by gluing a copy of $e \times I$ to $S \times I$ along $\partial S \times I$. (This involves unpinching parts of the boundaries; see *[need ref for this? or is it clear?]*.) It's clear that composition is preserved, so this defines a functor. (See Figure (5.1.1).)

For $e \in C^1_{xy}$ and $f \in C^1_{yz}$ we need an invertible natural transformation c_{ef} connecting the composition of the actions for e and f with the action of the composition ef. This is given by the track of an isotopy, supported in a collar neighborhood of ∂S, connecting the corresponding (compositions of) collaring homeomorphisms.

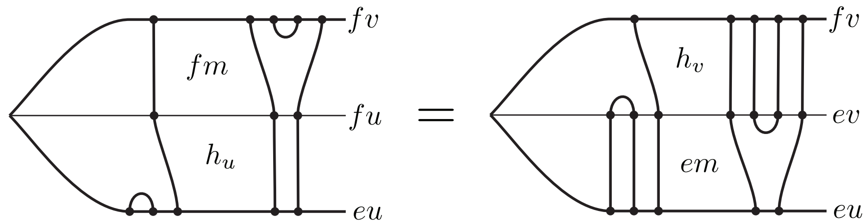
For $h \in C^2_{ef} = A(\partial S \times I \times I; e, f)$ we need a natural transformation between the functors for e and f. This is given, at $u \in \mathcal{C}(S;y)$, by gluing a copy of h to $u \times I$ (see Figure (5.1.2)). The proof that this collection of morphisms actually does comprise a natural transformation, that is, that this diagram

$$\begin{array}{ccc} eu & \xrightarrow{em} & ev \\ {\scriptstyle h_u}\downarrow & & \downarrow{\scriptstyle h_v} \\ fu & \xrightarrow[fm]{} & fv \end{array}$$

commutes for all $u, v \in \mathcal{C}(S;y)$ and $m \in A(S \times I; u, v)$, is illustrated in Figure (5.1.3). *[need to say why this works for general fields; cite specific properties from*



5.1.2 The natural transformation for h

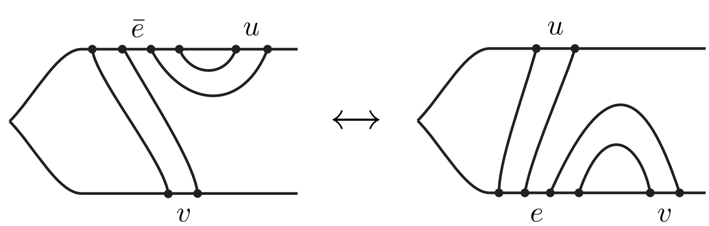


5.1.3 Verifying the natural transformation

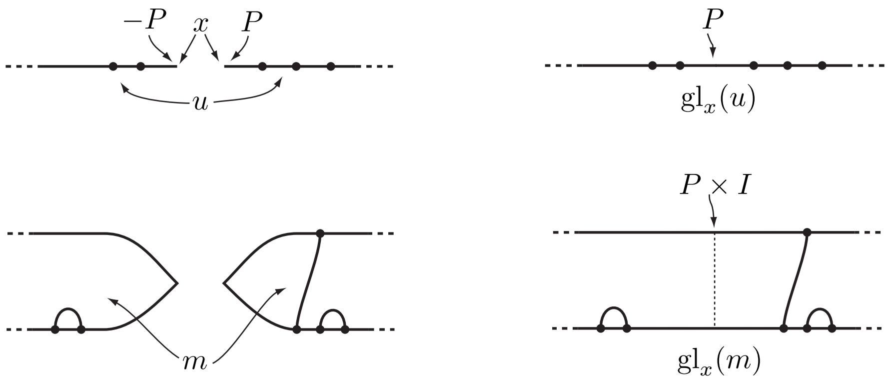
the field defs; clearly works for our basic examples (pictures and maps); or, better, do the proof in the fields section/chapter]

For $e \in C^1_{xy}$, $u \in \mathcal{C}(S;x)$ and $v \in \mathcal{C}(S;y)$. we need a bijection (isomorphism) between $A(S \times I; ev, u)$ and $A(S \times I; v, e^* u)$. This is given by an isotopy of $S \times I$ which shifts a collar of ∂S from $S \times \{0\}$ to $S \times \{1\}$; see Figure (5.1.4). (Note that this is one of the few times we consider isotopies of manifolds which do not fix the boundary.) This completes the proof that $\{A(S;\cdot)\}$ affords a left representation of $A(\partial S)$. We will denote the entire representation package by $A(S)$.

A similar argument shows that $\{Z(S;\cdot)\}$ affords a right representation of $A(\partial S)$. Denote this representation by plain $Z(S)$. If, for an $n-2$-manifold P, we define $Z(P)$



5.1.4 Adjunction isotopy



5.2.2 Gluing along P

to be the 2-category of all right representations of $A(P)$ *[need to say what sort of 2-cat this is and why]*, then we have, for all $n-1$-manifolds S,

$$Z(S) \in Z(\partial S) \tag{5.1.5}$$

This is a categorified version of (4.3.2) and a doubly categorified version of (1.2.1). *[also forward ref to combinatorial version of* $(n+1)$ *gluing?]*

5.2 Gluing

Next we use the above representations to prove gluing theorems for A and Z of $n-1$-manifolds. Suppose we have the usual gluing (without corners) scenario: an $n-1$-manifold S with an identification $\partial S = (-P) \sqcup P \sqcup Q$. For notational simplicity we will henceforth ignore Q. We have an action of the 2-category $A(-P \sqcup P) \cong A(P)^{\mathrm{op}} \times A(P)$ on $\{A(S; \cdot, \cdot)\}$ (equivalently, commuting right and left actions of $A(P)$), and we would like to compute the 1-category $A(S_{\mathrm{gl}})$ from this action.

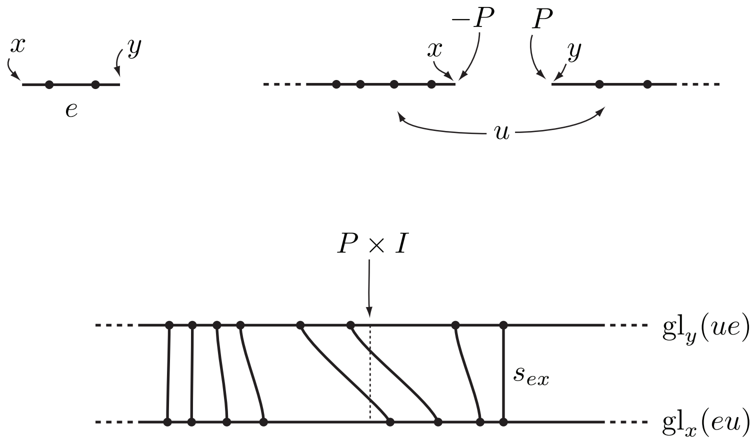
We will start by noting some relationships between $A(S_{\mathrm{gl}})$ and $A(S)$. We will then prove that $A(S_{\mathrm{gl}})$ is the universal 1-category with these properties (what one might (and we will) call a categorified coend or 2-coend construction).

To simplify notation (and also allow for its reuse), denote the 2-category $A(P)$ by A, denote the 1-category $A(S; x, y)$ by W_{xy} (where $x, y \in A^0 = \mathcal{C}(P)$), and denote the 1-category $A(S_{\mathrm{gl}})$ by C.

For each $x \in A^0$ we have a functor

$$\mathrm{gl}_x : W_{xx} \to C. \tag{5.2.1}$$

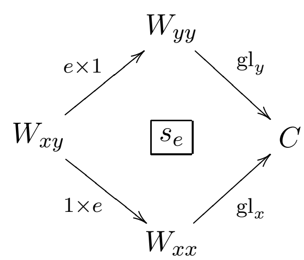
On objects this functor is given by gluing S along P to obtain S_{gl}. On morphisms it's given by gluing $S \times I$ along $P \times I$ to obtain $S_{\mathrm{gl}} \times I$. (See Figure (5.2.2).)



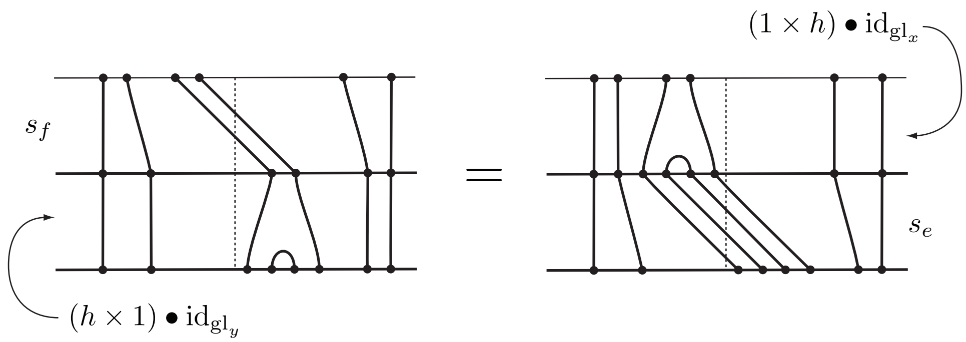
5.2.4 s_e at x

For each $e \in A^1_{xy}$ we can construct two functors from W_{xy} to C, $\mathrm{gl}_y \circ (e \times 1)$ and $(1 \times e) \circ \mathrm{gl}_x$, and there is an invertible natural transformation s_e between these two functors:

5.2.3

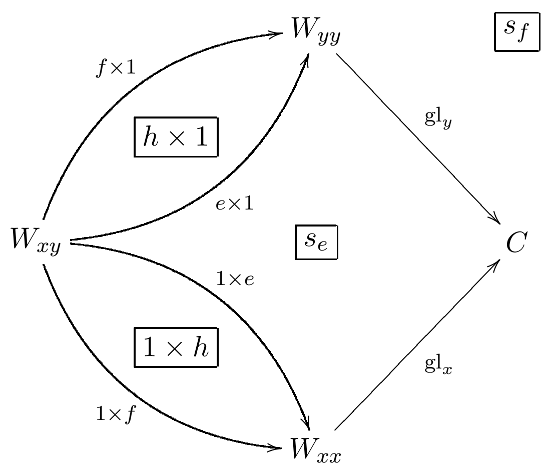


(This is a 2-dimensional diagram with each i-cell labeled by an i-morphism. Here and below we follow the convention of placing 2-morphisms (e.g. natural transformations) in the appropriate 2-cell of a diagram with a surrounding box instead of an accompanying arrow. This is in order to keep the diagrams from getting too cluttered. The range and domain of the 2-morphism is usually clear from context.) The morphisms for s_e are tracks of isotopies (in $S_{\mathrm{gl}} \times I$) which shift a copy of e across the gluing locus P. (See Figure (5.2.4).) It is easy to verify that this is a natural transformation.



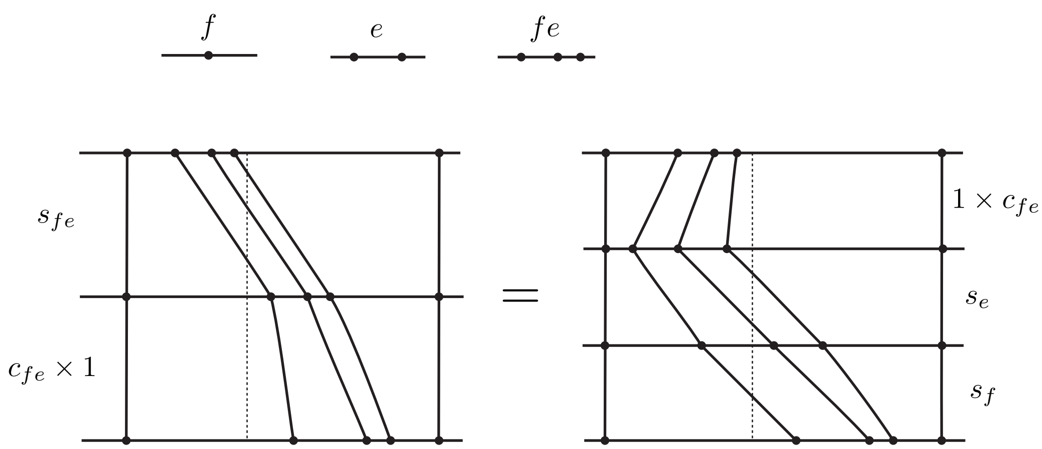
5.2.6 A commutativity relation

For all $h \in A^2_{ef}$, the following diagram commutes:



5.2.5

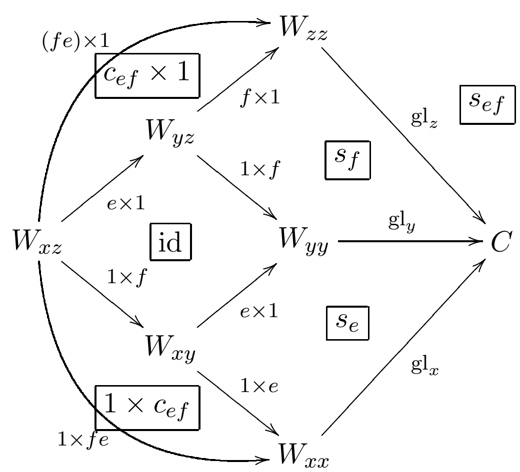
The above diagram should be thought of as a cell decomposition of the 2-sphere with four 2-cells. (The 2-cell "at infinity" is labeled by s_f.) There are two different ways of composing the natural transformations in the diagram, $((1 \times h) \bullet \mathrm{id}_{\mathrm{gl}_x}) \circ s_e$ and $s_f \circ ((h \times 1) \bullet \mathrm{id}_{\mathrm{gl}_y})$. (Here $\bullet$ denotes "horizontal" composition of natural transformations (see (A.2.2)) and $\mathrm{id}_{\mathrm{gl}_x}$ denotes the identity natural transformation from the functor gl_x to itself.) The commutativity of the diagram means that these give the same natural transformation between the functors $\mathrm{gl}_y \circ (e \times 1)$ and $\mathrm{gl}_x \circ (1 \times f)$. Roughly speaking, the shift morphisms s_e commute with the (gluings of) the 2-morphism actions h. The proof of this assertion is indicated in Figure (5.2.6).



5.2.8 Another commutativity relation

For all $e \in A^1_{xy}$ and $f \in A^1_{yz}$, the following diagram commutes:

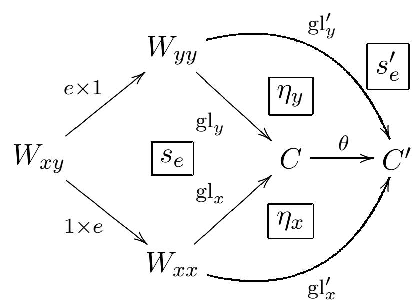
5.2.7



Again, the diagram should be thought of as a cell decomposition of the 2-sphere, this time with six 2-cells. The functors for the two sides of the 2-cell labeled "id" are both $e \times f$, and so are connected by the identity natural transformation. There are two ways of composing the natural transformations in the diagram to obtain a natural transformation between the functors $\mathrm{gl}_z \circ (fe \times 1)$ and $\mathrm{gl}_x \circ (1 \times fe)$, and the commutativity of the diagram means that these two natural transformations are equal. Roughly speaking, this means the the shift morphisms preserve composition of 1-morphisms; s_{ef} can be computed from s_e and s_f. The proof of this assertion is indicated in Figure (5.2.8).

It turns out that the above properties uniquely characterize $A(S_{\mathrm{gl}})$. To make this statement more precise, we need a definition.

Definition. *Let A be an arbitrary disk-like 2-category, and let $\{W_{xy}\}$ be a collection of 1-categories affording an $A^{\mathrm{op}} \times A$ action. A 1-category C, together with functors $\{\mathrm{gl}_x\}$ and invertible natural transformations $\{s_e\}$ satisfying (5.2.1), (5.2.3), (5.2.5) and (5.2.7), is called the 2-coend of the $A^{\mathrm{op}} \times A$ action if it is universal in the following sense. If C', $\{\mathrm{gl}'_x\}$ and $\{s'_e\}$ also satisfy (5.2.1) through (5.2.7), then there exists a functor $\theta : C \to C'$ and, for all $x \in A^0$, a natural transformation $\eta_x : \theta \circ \mathrm{gl}_x \to \mathrm{gl}'_x$, such that* 5.2.9



commutes for all $e \in A^1_{xy}$.

[need to talk about uniqueness of 2-coend]

Theorem. *$A(S_{\mathrm{gl}})$ is the 2-coend of the $A(P)^{\mathrm{op}} \times A(P)$ action on $\{A(S; x, y)\}$.* 5.2.10

Proof. We will introduce two new categories, $\mathcal{G}$ (G for generators and relations) and $\mathcal{P}$ (P for parallelogram). Both will have concrete algebraic descriptions. It will be easy to show that $\mathcal{G}$ is (up to natural isomorphism, of course) the 2-coend of the theorem, and that $\mathcal{P}$ is isomorphic to $A(S_{\mathrm{gl}})$. We then show that $\mathcal{G}$ and $\mathcal{P}$ are isomorphic. (So $\mathcal{G}$ and $\mathcal{P}$ provide two alternative, more concrete descriptions of the 2-coend.)

First we define $\mathcal{G}$. The objects of $\mathcal{G}$ are $\bigcup_{x \in A^0} W^0_{xx}$. The morphisms will be defined in terms of generators and relations. There are two types of generators, $\bigcup_{x \in A^0} W^1_{xx}$ (with the obvious choice of range and domain), and, for all $x, y \in A^0$, $e \in A^1_{xy}$ and $u \in W^0_{yx}$, morphisms

$$\sigma_{eu} : eu \to ue$$
$$\sigma^{-1}_{eu} : ue \to eu$$

The morphism of $\mathcal{G}$ corresponding to $f \in W^1_{xx}$ will be denoted $\widehat{f}$, unless there is no chance of confusion, in which case we denote it as plain f. The relations are

- $\widehat{f}\,\widehat{g} = \widehat{fg}$ 5.2.11
- $\sigma_{eu}\sigma^{-1}_{eu} = \mathrm{id}_{eu}$ and $\sigma^{-1}_{eu}\sigma_{eu} = \mathrm{id}_{ue}$ 5.2.12

5.2.13 • $\sigma_{e\cdot}$ is a natural transformation between the right and left actions of $e \in A^1_{xy}$: for all $f \in (W_{yx})^1_{uv}$, $\widehat{ef}\sigma_{ev} = \sigma_{eu}\widehat{fe}$

5.2.14 • for all $h \in A^2_{ee'}$, $\sigma_{eu}(h \times 1)_u = (1 \times h)_u \sigma_{e'u}$

5.2.15 • for all $e \in A^1_{xy}$ and $g \in A^1_{yz}$ and $u \in W^0_{zx}$, $\sigma_{e,gu}\sigma_{g,ue} = \alpha^{-1}\sigma_{eg,u}\alpha$, where α is the associator for e and g at u

Next we show that $\mathcal{G}$ is the 2-coend of the $A \times A^{\text{op}}$ action. There are obvious functors $\text{gl}_x : W_{xx} \to \mathcal{G}$ for all x (sending f to $\widehat{f}$), and, with $\sigma_{e\cdot}$ playing the role of s_e, the above relations for $\mathcal{G}$ guarantee that these data satisfy (5.2.1), (5.2.3), (5.2.5) and (5.2.7). Let C', $\{\text{gl}'_x\}$ and $\{s'_e\}$ be as in the definition of 2-coend. We must define a functor $\theta : \mathcal{G} \to C'$ satisfying the conditions of (5.2.9). On the objects of $\mathcal{G}$ define θ so that $\theta \circ \text{gl}_x = \text{gl}'_x$ for all x. On the generating morphisms coming from W_{xx} again define θ so that $\theta \circ \text{gl}_x = \text{gl}'_x$ for all x. Finally, define $\theta(\sigma_{eu}) = s'_{e,u}$ and $\theta(\sigma^{-1}_{eu}) = (s'_e)^{-1}_u$.

We must show that the above definition of θ on the generators of $\mathcal{G}^1$ respects the relations. Since the relations are just translations of the commutative diagrams defining the 2-coend, this is easy to do. (5.2.11) follows from the fact that gl'_x is a functor. (5.2.12) follows from the fact that s'_e and $(s'_e)^{-1}$ are mutually inverse. (5.2.13) follows from the fact that s'_e is a natural transformation. (5.2.14) follows from (5.2.5). (5.2.15) follows from (5.2.7). Thus θ is a well-defined functor.

Finally, we must define, for all x, an invertible natural transformation $\eta_x : \text{gl}'_x \to \theta \circ \text{gl}_x$. Since these two functors are equal, we can take η_x to be the identity natural transformation.

Next we define $\mathcal{P}$. The objects of $\mathcal{P}$ are again $\bigcup_{x \in A^0} W^0_{xx}$. The morphisms from $u \in W^0_{xx}$ to $v \in W^0_{yy}$ are

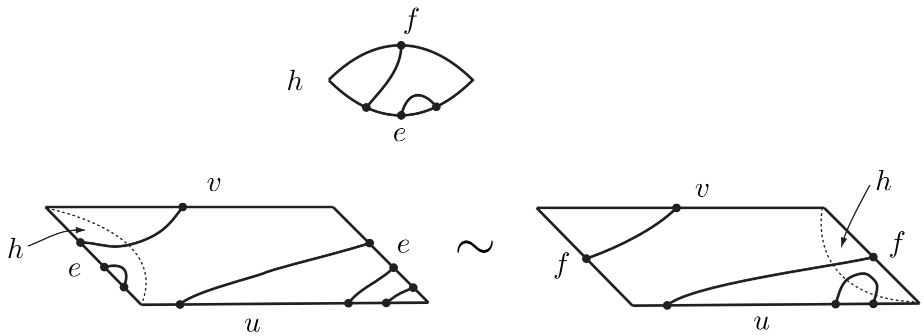
5.2.16
$$\left(\bigoplus_{e \in A^1_{yx}} \text{mor}(eu, ve)\right) \Big/ \Big\langle p \circ (h \bullet \text{id}_u) \sim (\text{id}_v \bullet h) \circ p \Big\rangle$$

Here $\text{mor}(eu, ve) = (W_{yx})^1_{eu,ve}$ and $h \in A^2_{ef}$. The equivalence class of $a \in \text{mor}(eu, ve)$ will be denoted $\text{gl}_e(a) \in \mathcal{P}^1_{uv}$. We think of a morphism of $\mathcal{P}$ as a parallelogram with short sides labeled by e and long sides labeled by u and v. See Figure (5.2.17). (More literally, a morphism of $\mathcal{P}$ can be thought of as a field on $S_{\text{gl}} \times I$ restricting to e on $P_\pm \times I$.)

[need to replace most (all?) occurrences of $\text{id}_{blah}\bullet$ *with simple juxtaposition]*

To complete the definition of $\mathcal{P}$ we must specify how to compose morphisms. Let $p \in \text{mor}(eu, ve)$ and $q \in \text{mor}(fv, wf)$, where $e \in A^1_{xy}$, $f \in A^1_{zx}$, $u \in W^0_{xx}$, $v \in W^0_{yy}$, $w \in W^0_{zz}$. Then define

$$\text{gl}_f(q) \circ \text{gl}_e(p) = \text{gl}_{ef}((qe) \circ (fp)).$$



5.2.17 Morphisms for $\mathcal{P}$

(There should be some associators inserted above, but they have been omitted for simplicity.) See Figure (5.2.18).

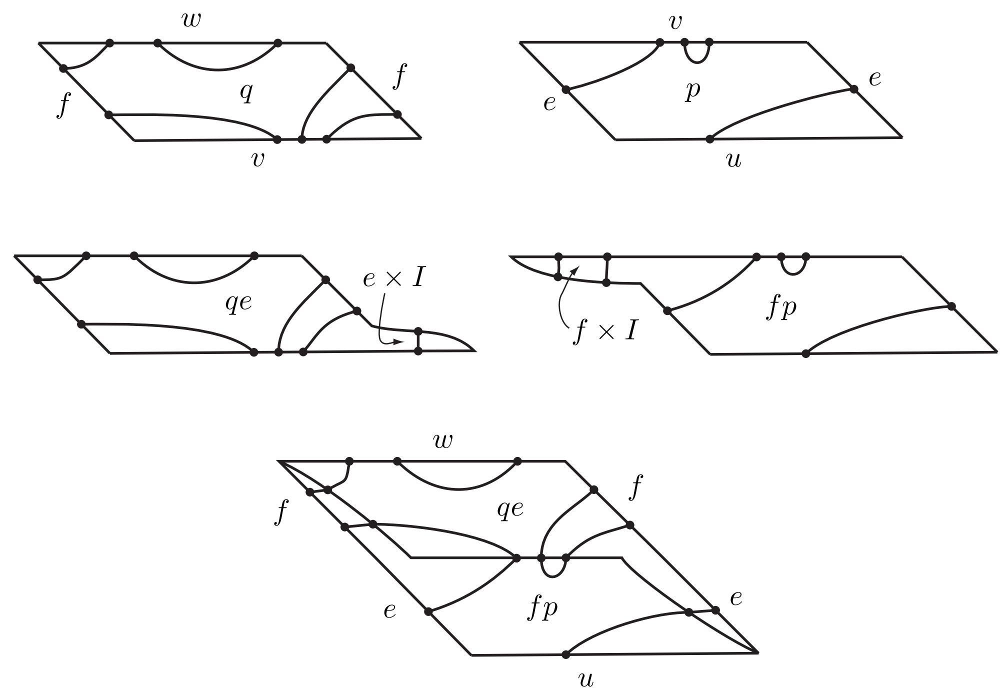
The objects of $\mathcal{P}$ are essentially all of the objects of $A(S_{\rm gl})$. (More precisely, they are all of the objects which are transverse to the gluing surface.) It follow from (4.4.4) that the morphisms of the two categories are identical. So all that remains is to show that the two composition rules coincide. This is illustrated in Figure (5.2.19). The basic idea is that the equivalence relation on morphisms of $\mathcal{P}$ allows us to eliminate id_e and id_f from the definition of composition in $\mathcal{P}$, yielding the result of composing in $A(S_{\rm gl})$.

Now we must show that $\mathcal{G}$ and $\mathcal{P}$ are isomorphic. Predictably, we define functors $\alpha : \mathcal{G} \to \mathcal{P}$ and $\beta : \mathcal{P} \to \mathcal{G}$ and show that $\alpha \circ \beta$ is the identity functor on $\mathcal{P}$ and $\beta \circ \alpha$ is the identity functor on $\mathcal{G}$. Note that $\mathcal{G}^0 = \mathcal{P}^0$, so we can (and do) define both α and β to be the identity on objects.

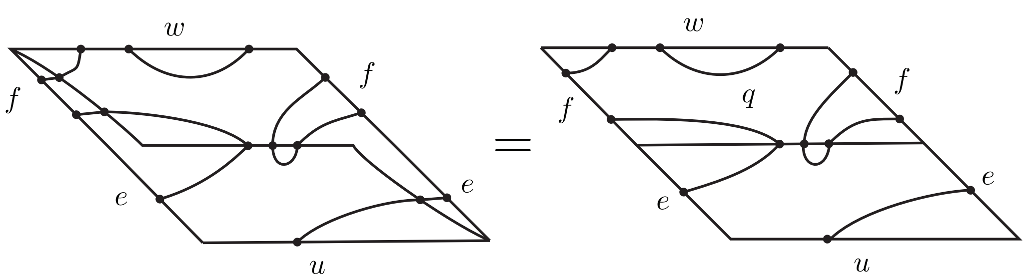
First some notation. For $e \in A^1_{xy}$, let $V_{ee^*} \in A^2$ be the 2-morphism conjugate to id_e with domain id_x and range ee^*. Let $\Lambda_{ee^*} \in A^2$ be the 2-morphism conjugate to id_e with domain ee^* and range id_x. See Figure (5.2.20). *[put these defs earlier and just recall them here??]*

Keeping the topological interpretation of the algebra in mind, it's easy to define α. Let $\alpha(\sigma_{eu}^{-1}) = \mathrm{gl}_e(\mathrm{id}_{eue})$ and $\alpha(\sigma_{eu}) = \mathrm{gl}_{e^*}(\Lambda_{e^*e} \bullet \mathrm{id}_u \bullet V_{ee^*})$. For $h \in (W_{xx})^1_{uv}$, let $\alpha(\widehat{h}) = \mathrm{gl}_{\mathrm{id}_x}(\tilde{h})$, where $\tilde{h}$ is the morphism from $\mathrm{id}_x \bullet u$ to $v \bullet \mathrm{id}_x$ guaranteed by the definition of 2-category action. *[need to make sure we included this in the def]* See Figure (5.2.21). We must verify that the above assignments of generators of $\mathcal{G}^1$ obey the relations. This straightforward exercise is left to the reader.

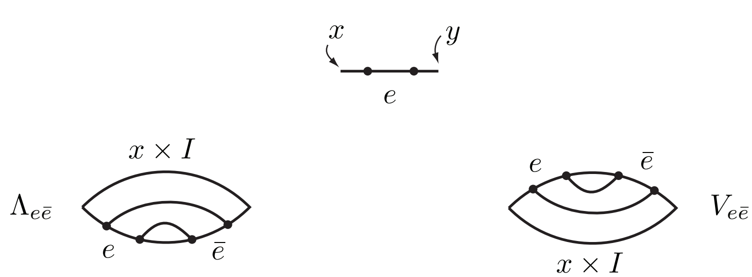
Defining $\beta : \mathcal{P} \to \mathcal{G}$ is a little more complicated. The key topological idea is that an arbitrary field on $S_{\rm gl} \times I$ is isotopic to a composition of fields (morphisms) which either: (a) are product fields near $P_\pm \times I \subset S_{\rm gl} \times I$, and so are gluings of fields on $S \times I$; or (b) are tracks of isotopies of $S_{\rm gl}$ which shift a collar neighborhood across $P_\pm$. The proof of this is illustrated in Figure (5.2.22). Accordingly, we define, for



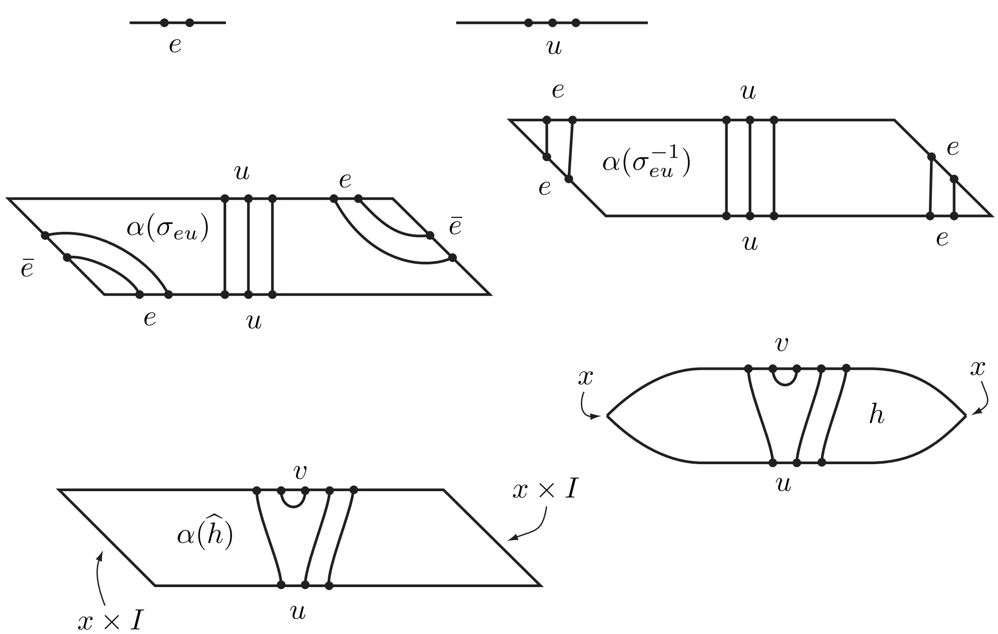
5.2.18 Composition in $\mathcal{P}$



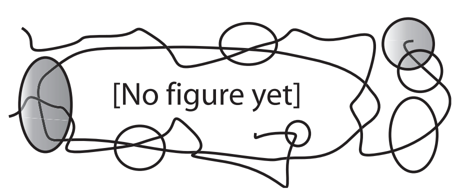
5.2.19 Proof that composition rules are the same



5.2.20 V_{ee^*} and Λ_{ee^*}



5.2.21 Definition of $\alpha : \mathcal{G} \to \mathcal{P}$



5.2.22 Decomposing a morphism of $\mathcal{P}$

$a \in (W_{yx})^1_{eu,ve}$,

$$\beta(\mathrm{gl}_e(a)) = (\widehat{\mathrm{id}_v \bullet \Lambda_{ee^*}}) \circ \sigma_{e^*,ve} \circ \widehat{a'},$$

where a' is the adjoint of a with domain u and range e^*ve. *[here we ignore some morphisms coming from the action of identity 1-morphisms of A; need to say this more precisely; or maybe include those morphisms]*

Now that α and β have been defined, we must verify that they are mutually inverse. *[relatively easy for $\mathcal{P} \to \mathcal{G} \to \mathcal{P}$; harder for $\mathcal{G} \to \mathcal{P} \to \mathcal{G}$; proofs illustrated in figs xxxx, yyyy, zzz]* □

There is a dual version of the codimension-2 gluing theorem (5.2.10). Retaining notation from above, we have

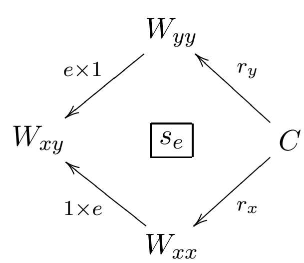
5.2.23 **Theorem.** *$Z(S_{\mathrm{gl}})$ is the 2-end of the $A(P)^{\mathrm{op}} \times A(P)$ action on $\{Z(S; x, y)\}$.* □

The $A(P)^{\mathrm{op}} \times A(P)$ action on $\{Z(S; x, y)\}$ needs no explanation. The proof of (5.2.23) is dual to the proof of (5.2.10). All that remains is to define the 2-end of the $A(P)^{\mathrm{op}} \times A(P)$ action. This dual to the definition of 2-coend above, but we repeat the (dualized) details below.

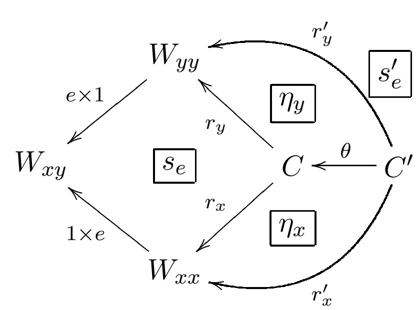
5.2.24 **Definition.** *Let A be an arbitrary disk-like 2-category, and let $\{W_{xy}\}$ be a collection of 1-categories affording an $A^{\mathrm{op}} \times A$ action. A 1-category C, together with functors*

$$r_x : C \to W_{xx}$$

(for all $x \in A^0$) and invertible natural transformations $\{s_e\}$

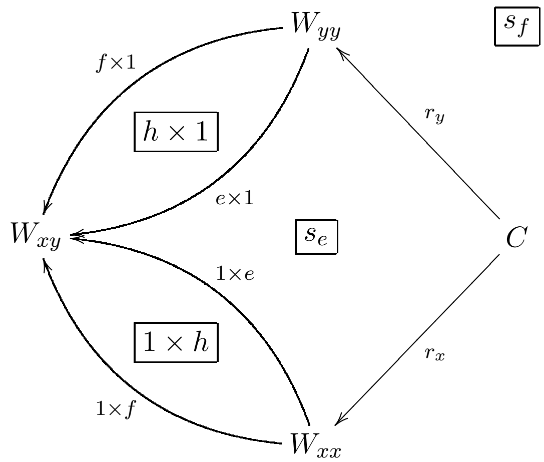


(for all $e \in A^1$) satisfying (5.2.25) and (5.2.26) below, is called the 2-end of the $A^{\mathrm{op}} \times A$ action if it is universal in the following sense. If C', $\{r'_x\}$ and $\{s'_e\}$ also satisfy (5.2.25) and (5.2.7), then there exists a functor $\theta : C' \to C$ and, for all $x \in A^0$, a natural transformation $\eta_x : r_x \circ \theta \to r'_x$, such that



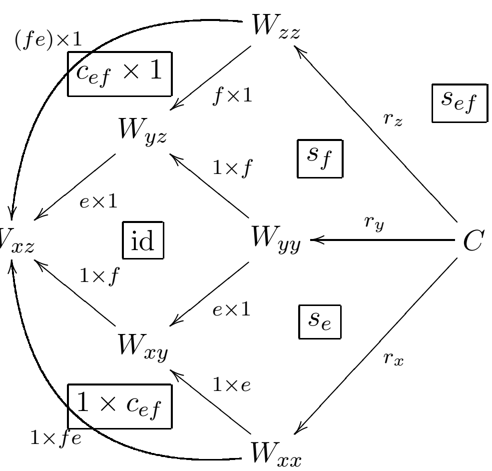
commutes for all $e \in A^1_{xy}$.

As referred to in the above definition, we have for all $h \in A^2_{ef}$,



5.2.25

Also, for all $e \in A^1_{xy}$ and $f \in A^1_{yz}$,



5.2.26

It will be useful to have a more concrete description of the 2-end. Suppose we have a (not necessarily universal) C', $\{r'_x\}$ and $\{s'_e\}$ satisfying the conditions above. To each object a of C' we can associate the collection of objects $\{a_x \stackrel{\text{def}}{=} r'_x(a) \in W^0_{xx}\}$ (indexed by $x \in A^0$) as well as the collection of morphisms $\{s_{e,a} : ea_y \to a_x e\}$ (indexed by $e \in A^1_{xy}$). *[need to fix: should be $s'_{e,a}$]* It follows from (5.2.25) that for all $h \in A^2_{ef}$ the following diagram commutes

$$\begin{array}{ccc} ea_y & \xrightarrow{s_{e,a}} & a_x e \\ {\scriptstyle h \bullet \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mathrm{id} \bullet h} \\ fa_y & \xrightarrow{s_{f,a}} & a_x f \end{array}$$

5.2.27

It follows from (5.2.26) that for all $e \in A^1_{xy}$ and $f \in A^1_{yz}$ the following diagram commutes

5.2.28
$$\begin{array}{ccc} (ef)a_z & \xrightarrow{\quad s_{ef,a} \quad} & a_x(ef) \\ \Big\downarrow{\scriptstyle c^L_{ef}} & & \Big\downarrow{\scriptstyle c^R_{ef}} \\ a(fa_z) \xrightarrow{es_{f,a}} & ea_yf & \xrightarrow{s_{e,a}f} (a_xe)f \end{array}$$

where c^L_{ef} $[c^R_{ef}]$ is the associator associated to the left [right] action of A on W.

For each morphism $m : a \to b$ of C' we have a collection of morphisms $\{m_x \stackrel{\text{def}}{=} r'_x(m) : a_x \to b_x\}$ (indexed by $x \in A^0$). Because s_e is a natural transformation, the following diagram commutes for all $e \in A^1$

5.2.29
$$\begin{array}{ccc} ea_y & \xrightarrow{s_{e,a}} & a_xe \\ {\scriptstyle em_y}\Big\downarrow & & \Big\downarrow{\scriptstyle m_xe} \\ eb_y & \xrightarrow{s_{e,b}} & b_xe \end{array}$$

We are thus led to define a 1-category $\mathcal{E}$ whose objects are collections $\{a_x \in W^0_{xx}\}$ and $\{s_{e,a} : ea_y \to a_xe\}$ satisfying (5.2.27) and (5.2.28). Morphisms of $\mathcal{E}$ are collections $\{m_x : a_x \to b_x\}$ satisfying (5.2.29). Composition is given by $\{m_x\} \circ \{n_x\} \stackrel{\text{def}}{=} \{m_x \circ n_x\}$. (It is easy to verify that $\{m_x \circ n_x\}$ satisfies (5.2.29).)

There are obvious functors $r_x : \mathcal{E} \to W_{xx}$ (for all $x \in A^0$), and the individual morphisms $s_{e,a}$ fit together to give natural transformations s_e as in (5.2.24). *[notational problem here: $s_{e,a}$ plays two roles, one from def of $\mathcal{E}$ and one from def of 2-end. should fix this.]* The above discussion motivating the definition of $\mathcal{E}$ can also be read as a proof that $\mathcal{E}$, $\{r_x\}$ and $\{s_e\}$ are universal in the appropriate sense. Thus we have the desired concrete description of the 2-end.

One of the simplest examples of a 2-end is when A above is a 2-category with only one object (i.e. a spherical tensor 1-category) and $A^{\text{op}} \times A$ acts on itself via left and right tensor multiplication. The 2-end of this action has a more familiar name: the Drinfeld center of the tensor 1-category A. Indeed, if we specialize the definition of $\mathcal{E}$ above to this case we obtain the usual definition of the Drinfeld center. (See for example [Kas95, p. 330]. Note however that we make no assumptions about strict associativity.) *[D. center works for general tensor 1-cat (don't need spherical). comment on this?]*

5.2.30 So a special case of the codimension-2 gluing theorem (5.2.23) is the following. Suppose P is a closed $n-2$-manifold such that the 2-category $A(P)$ has only one object, and thus can be thought of as a tensor 1-category. Then $Z(P)$, the representations of $A(P)$, can also be thought of as a tensor 1-category, and $Z(P \times S^1)$ is the Drinfeld center of $Z(P)$. *[need to say more here; need additional assumptions on $A(P)$]*

Note that for any closed P the category $Z(P\times S^1)$ has the structure of a braided tensor category. Let X^2 be a twice-punctured disk. Then $Z(P\times X)$ can be thought of as a functor $Z(P\times S^1)\times Z(P\times S^1)\to Z(P\times S^1)$, which gives a tensor (monoidal) structure to $Z(P\times S^1)$. The geometric braiding of X gives rise to an algebraic braiding of $Z(P\times S^1)$. *[need to show that in special case of D. center this def of tensor and braiding agrees with the usual one.] [need to go into more detail on above]*

Still to do:

- remark somewhere that we will (to deobfuscate notation) sometimes fail to distinguish between morphisms and their conjugates; e.g. add or subtract, without comment, actions of identity 1-morphisms of A, omit associators, etc.
- annularization
- gluing n-manifolds with corners (revisit)
- discuss higher codim case? or put it in separate chapter?
- refer forward to applications in example chapters
- in earlier (fields) section, prove a few relevant isotopy identities carefully, using field axioms (and refer to that from this chapter)

Chapter 6

From Local Relations to Path Integrals

[this an incomplete draft at the moment; also, it hasn't been proof-read]

In Chapter 1 we showed (non-rigorously) that topologically invariant path integrals lead to local relations. In this chapter we show (rigorously) how to reconstruct the path integral from local relations (Theorem (6.3.1)). In order for this reconstruction to work the local relations need to satisfy some relatively mild semi-simplicity and finiteness conditions.

The main ideas are that, as argued in Chapter 1, local relations carry essentially the same information as the path integral of the $n+1$-ball, and that by gluing (with corners) $n+1$-balls together and applying the gluing formula for the path integral, we can compute the path integral of any $n+1$-manifold. After these observations all that remains is to show that these computations yield consistent results.

[mention pairings too?]

6.1 Axioms for Path Integrals

By "path integral" we mean an invariant of $n+1$-manifolds (denoted as usual by Z) which relates to the rest of the TQFT as described below. (See Chapter 1 for an explanation of why one would expect these formal properties to arise from actual integrals.)

First, the path integral $Z(M^{n+1})$ is a function on fields on its boundary compatible with local relations, so 6.1.1

$$Z(M) \in Z(\partial M)$$

for all $n{+}1$-manifolds M. (If $\partial M = \emptyset$, this means that $Z(M) \in \mathbb{C}$, or whatever the ground ring is.) Equivalently, $Z(M)$ is a function

$$Z(M) : A(\partial M) \to \mathbb{C}.$$

6.1.2 Second, for all n-manifolds Y and boundary conditions $c \in \mathcal{C}(\partial Y)$, we have nondegenerate pairings

$$\begin{aligned} Z(Y;c) \otimes Z(-Y;\widehat{c}) &\to \mathbb{C} \\ x \otimes y &\mapsto \langle x, y\rangle \end{aligned}$$

and

$$\begin{aligned} A(Y;c) \otimes A(-Y;\widehat{c}) &\to \mathbb{C} \\ x \otimes y &\mapsto \langle x, y\rangle. \end{aligned}$$

The pairings should be compatible with the actions of $A(\partial Y)$: for all $x \in A(Y;c)$, $y \in A(-Y;\widehat{b})$ and $e \in A(\partial Y)^1_{cb} = A(\partial Y \times I; \widehat{c}, b)$,

6.1.3 $$\langle xe, y\rangle = \langle x, ey\rangle,$$

and similarly for $Z(Y;c)$. (Here we are using the identification $A(-\partial Y) = A(\partial Y)^{\mathrm{op}}$, so $A(\partial Y)$ has a left action on $A(-Y;\cdot)$.) More generally, we could replace ∂Y above with a codimension-0 submanifold of ∂Y.

Recall that associated to reversing the orientation of Y we have an isomorphism (conjugate linear if the ground ring is $\mathbb{C}$) $A(Y;c) \cong A(-Y;\widehat{c})$. Combining this isomorphism with the above pairings we get [sesquilinear] inner products

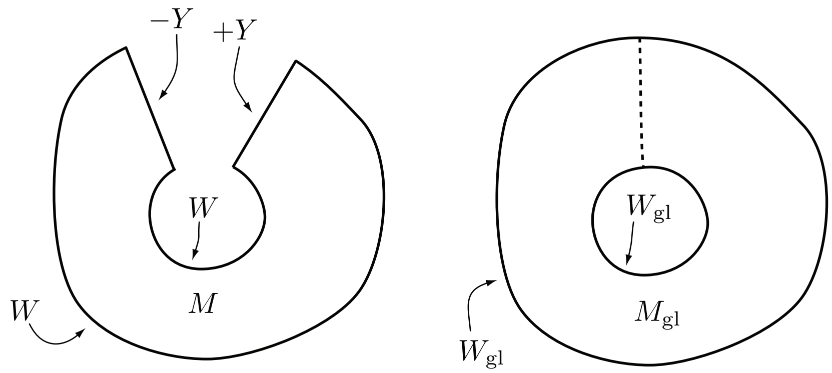
$$\begin{aligned} A(Y;c) \otimes A(Y;c) &\to \mathbb{C} \\ x \otimes y &\mapsto \langle x, y\rangle \stackrel{\text{def}}{=} \langle x, \widehat{y}\rangle \end{aligned}$$

and similarly

$$\begin{aligned} Z(Y;c) \otimes Z(Y;c) &\to \mathbb{C} \\ x \otimes y &\mapsto \langle x, y\rangle \stackrel{\text{def}}{=} \langle x, \widehat{y}\rangle. \end{aligned}$$

(Note that we have "overloaded" the angle brackets $\langle \cdot, \cdot \rangle$ to denote both the pairings and the inner products. Which meaning is intended can be deduced from what's being plugged into the angle brackets.) We require these inner products to be [skew] symmetric. The inner products induce isomorphisms between $A(Y;c)$ and $Z(Y;c)$ (recall that these spaces are mutually dual), and these isomorphisms should preserve the pairings.

(In all of the examples we will study, $A(Y;c)$ and $Z(Y;c)$ will be finite dimensional, so we won't worry about completeness.)



6.1.5 Yet another figure illustrating gluing

Finally, there is a gluing relation for $n+1$-manifolds. Let M be an $n+1$-manifold with boundary $\partial M = Y \cup -Y \cup W$ and let M_{gl} be M glued (with corners) along $\pm Y$ (see Figure (6.1.5)). Note that $\partial M_{\mathrm{gl}} = W_{\mathrm{gl}}$, where W_{gl} is W glued along $\partial(\pm Y)$. For each $c \in \mathcal{C}(\partial Y)$ we have maps 6.1.4

$$Z(\partial M) \xrightarrow{r_c} Z(Y;c) \otimes Z(-Y;\widehat{c}) \otimes Z(W;\widehat{c},c) \xrightarrow{\mathrm{tr}_c} Z(W;\widehat{c},c),$$

where r_c is a restriction map and tr_c comes from the pairing for $Z(Y;c)$. Call the composite map $\varphi_c : Z(\partial M) \to Z(W;\widehat{c},c)$. Recall from (4.4.5) that

$$Z(W_{\mathrm{gl}}) = \{(v_x) \in \prod_{x\in\mathcal{C}(\partial Y)} Z(Y;\widehat{x},x) \mid v_x e = e v_y \text{ for all } e \in A(\partial(Y))^1\}.$$

Then the gluing relation for $n+1$-manifolds states that $(\varphi_x(Z(M)))$ determines an element of $Z(W_{\mathrm{gl}})$ (i.e. $\varphi_x(Z(M))e = e\varphi_y(Z(M))$ for all $e \in A(\partial(Y))^1_{xy}$) and that $Z(M_{\mathrm{gl}})$ is equal to this element of $Z(W_{\mathrm{gl}}) = Z(\partial M)$. More succinctly,

$$Z(M_{\mathrm{gl}}) = \mathrm{tr}_Y(Z(M)), \qquad \text{6.1.6}$$

where $\mathrm{tr}_Y : Z(\partial M) \to Z(\partial M_{\mathrm{gl}})$ is induced by the various φ_c (see (6.2.2) below).

Here's a more concrete version of the gluing relation. Assume that $A(Y;c)$ is finite dimensional. Fix $c \in \mathcal{C}(\partial Y)$ and let $x \in A(W;\widehat{c},c)$. We want to evaluate the function $Z(M_{\mathrm{gl}})$ on the glued field $\mathrm{gl}_c(x) \in A(\partial M_{\mathrm{gl}})$ in terms of $Z(M)$ evaluated on elements of $A(\partial M) = A(Y \cup -Y \cup W)$. Let $\{e_i\}$ be a basis of $A(Y;c)$ and let $\{e^i\}$ be the dual basis of $Z(Y;c)$. Let $g^{ij} \stackrel{\mathrm{def}}{=} \langle e^i, e^j\rangle$ and $g_{ij} \stackrel{\mathrm{def}}{=} \langle e_i, e_j\rangle$. (Note that the matrices (g^{ij}) and (g_{ij}) are mutually inverse.) Then straightforward linear algebra shows that the above gluing relation is equivalent to 6.1.7

$$Z(M_{\mathrm{gl}})(\mathrm{gl}_c(x)) = \sum_{i,j} Z(M)(e_i \cup \widehat{e_j} \cup x) g^{ij}. \qquad \text{6.1.8}$$

If $\{e_i\}$ is an orthogonal basis then this becomes

6.1.9
$$Z(M_{\rm gl})({\rm gl}_c(x)) = \sum_i Z(M)(e_i \cup \widehat{e_i} \cup x)\frac{1}{\langle e_i, e_i\rangle},$$

and if $\{e_i\}$ is an orthonormal basis it becomes

6.1.10
$$Z(M_{\rm gl})({\rm gl}_c(x)) = \sum_i Z(M)(e_i \cup \widehat{e_i} \cup x).$$

6.2 Consequences of the Axioms

In this section we note some consequences of the above axioms. We will later use the reasoning of this section to prove that if we have defined a path integral for n+1-manifolds composed of handles of index less than k, then we can extend the definition to n+1-manifolds composed of handles of index less than $k+1$. Thus, starting with the path integral of the n+1-ball we construct inductively the path integral for any n+1-manifold.

The inner product on $A(Y;c)$ is determined by $Z(Y \times I)$. Indeed, applying (6.1.8) to the gluing $(Y\times I)\cup_Y (Y\times I) = Y\times I$ we have, for a basis $\{e_i\}$ of $A(Y;c)$,

$$\begin{aligned} Z(Y\times I)(\widehat{e_k}\cup e_l) &= \sum_{i,j} Z(Y\times I \sqcup Y\times I)(\widehat{e_k}\cup e_i\cup\widehat{e_j}\cup e_l)g^{ij} \\ &= \sum_{i,j} Z(Y\times I)(\widehat{e_k}\cup e_i)\cdot g^{ij}\cdot Z(Y\times I)(\widehat{e_j}\cup e_l), \end{aligned}$$

which implies that $Z(Y\times I)(\widehat{e_j}\cup e_i)$ gives the inverse of the matrix g^{ij}. In other words,

$$\langle e_i, e_j\rangle = Z(Y\times I)(\widehat{e_i}\cup e_j),$$

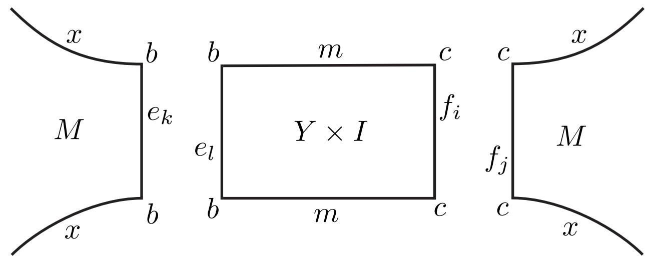
which implies (since the basis $\{e_i\}$ was arbitrary) that

6.2.1
$$\langle x, y\rangle = Z(Y\times I)(\widehat{x}\cup y)$$

for all $x, y \in A(Y;c)$. Note that if we use (6.2.1) to define the inner product then it is automatically compatible with the boundary category actions as in (6.1.3):

$$\begin{aligned} \langle xe, y\rangle &= Z(Y\times I)(\widehat{x}\widehat{e}\cup y) \\ &= Z(Y\times I)(\widehat{x}\cup ey) \quad \text{(because } \widehat{x}\widehat{e}\cup y = \widehat{x}\cup ey) \\ &= \langle x, ey\rangle. \end{aligned}$$

6.2.2 If (6.2.1) holds for Y then the gluing formulas of (6.1.4) automatically yield functions on $\mathcal{C}(\partial M_{\rm gl})$ which lie in $Z(\partial M_{\rm gl})$. Let $b, c \in \mathcal{C}(\partial Y)$, $m \in A(\partial Y\times I; \widehat{b}, c)$



6.2.3 Lots of boundary labels

and $x \in A(W; b, \widehat{c})$. Let $\{e_i\}$ be a basis of $A(Y; b)$ with dual inner product matrix g^{ij}. Let $\{f_i\}$ be a basis of $A(Y; c)$ with dual inner product matrix h^{ij}. (See Figure (6.2.3).) Then

$$\begin{aligned}
Z(M_{\text{gl}})(\text{gl}_c(xm)) &= \sum_{i,j} Z(M)(f_i \cup \widehat{f_j} \cup xm)h^{ij} \\
&= \sum_{i,j,k,l} Z(M)(e_k \cup \widehat{f_j} \cup x)g^{kl}\langle e_l, mf_i\rangle h^{ij} \\
&= \sum_{i,j,k,l} Z(M)(e_k \cup \widehat{f_j} \cup x)g^{kl}\langle e_l m, f_i\rangle h^{ij} \\
&= \sum_{k,l} Z(M)(e_k \cup \widehat{e_l} \cup mx)g^{kl} \\
&= Z(M_{\text{gl}})(\text{gl}_b(mx)).
\end{aligned}$$

Let R be an $n{-}1$-manifold and $c \in \mathcal{C}(\partial R)$. We have inner products on $A(R; c)^1_{ab} = A(R \times I; \widehat{a}, b)$ for all $a, b \in A(R; c)^0 = \mathcal{C}(R; c)$. Applying (6.1.3) to $R \times \{0\}$ and $R \times \{1\} \subset \partial(R \times I)$, and recalling the definition of x^* from (4.3.1), we have 6.2.4

$$\langle xy, z\rangle = \langle x, zy^*\rangle = \langle z^*x, y^*\rangle = \langle z^*, y^*x^*\rangle = \langle yz^*, x^*\rangle = \langle y, x^*z\rangle$$

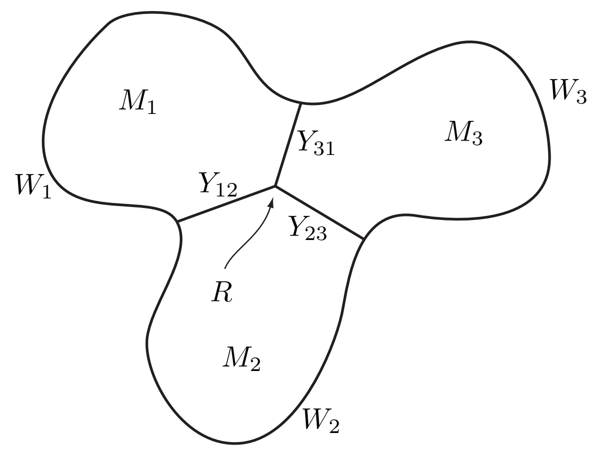
for all $x, y, z \in A(R; c)^1$ such that the above expression is defined.

Assume now that $A(R; c)$ is semisimple. Let α be an irrep of $A(R; c)$ and let $e_x \in A(R; c)^1_{xx}$ and $e_y \in A(R; c)^1_{yy}$ be two minimal idempotents for α (see (B.5.2)). Then there exists $u \in A(R; c)^1_{xy}$ such that $e_x = uu^*$ and $e_y = u^*u$. It follows that

$$\langle e_x, e_x\rangle = \langle uu^*, uu^*\rangle = \langle u, uu^*u\rangle = \langle u^*u, u^*u\rangle = \langle e_y, e_y\rangle.$$

It is also easy to see that if $e, f \in A(R; c)^1_{xx}$ are minimal idempotents for two non-isomorphic irreps, then $\langle e, f\rangle = 0$.

Next we describe how gluing n-manifolds affects the pairings. Let $\partial Y = R \cup S \cup -S$, and $b \in \mathcal{C}(R)$. (For simplicity we are suppressing from the notation labels for ∂S and $\partial R = \partial S \sqcup \partial(-S)$.) We want to compute the pairing for $A(Y_{\text{gl}}; \text{gl}(b))$ 6.2.5



6.2.7 Gluing three at a time

in terms of the pairings for $A(Y; b, \widehat{a}, a)$ for various $a \in \mathcal{C}(S)$. Assume that $A(R)$ is semisimple, and let $\{e_i\}$ be a complete set of minimal idempotents for $A(R)$. Let $\{f_{ij}\}$ be an orthogonal basis of $A(Y; b, e_i, e_i^*)$. *[need to introduce this notation somewhere]* It follows from *[need ref]* that $\{\mathrm{gl}(f_{ij})\}$ is a basis of $A(Y_{\mathrm{gl}}; \mathrm{gl}(b))$. To compute the inner product on $A(Y_{\mathrm{gl}}; \mathrm{gl}(b))$ we use (6.1.9) to compute $Z(Y_{\mathrm{gl}} \times I)$, which is obtained by gluing $Y \times I$ to itself along a copy of $R \times I$. Note that e_i is part of an orthogonal basis $\{g_k\}$ of $A(R \times I, \partial e_i)$ with the property that $e_i g_l e_i = 0$ unless $g_l = e_i$. It now follows from (6.1.9) that

$$\langle \mathrm{gl}(f_{ij}), \mathrm{gl}(f_{kl}) \rangle = \frac{\delta_{ik} \langle f_{ij}, f_{il} \rangle}{\langle e_i, e_i \rangle} = \frac{\delta_{ik} \delta_{jl} \langle f_{ij}, f_{ij} \rangle}{\langle e_i, e_i \rangle}.$$

In other words, $\{\mathrm{gl}(f_{ij})\}$ is an orthogonal basis of $A(Y_{\mathrm{gl}}, \mathrm{gl}(b))$, but we need to adjust by a correction factor $\langle e_i, e_i \rangle^{-1}$.

(We remark that these are the same inner product correction factors that appear in [Walk91]. One advantage of the present approach over that of [Walk91] is that here the correction factors arise naturally and explicitly.)

6.2.6 We can now derive a symmetric formula for gluing three $n+1$-manifolds together. Let $M = M_1 \cup M_2 \cup M_3$, $Y_{ij} = M_i \cap M_j$, and $\partial M_1 = Y_{12} \cup Y_{31} \cup W_1$ (and similarly for M_2 and M_3). Let $R = M_1 \cap M_2 \cap M_3$. Note that $\partial M = W_1 \cup W_2 \cup W_3$. (See Figure (6.2.7).) Choose fields $c_i \in \mathcal{C}(W_i)$ which can be glued to yield a field $c = c_1 \cup c_2 \cup c_3 \in \mathcal{C}(\partial M)$. Let $b \in \mathcal{C}(\partial R)$ be the common restriction of the c_i's to ∂R. Assume that $A(R; b)$ is semisimple. Let $\{e_i\}$ be a complete set of minimal idempotents for $A(R; b)$. Let $\{f_{ij}^{12}\}$ be an orthonormal basis of $A(Y_{12}; e_i)$, and define $\{f_{ik}^{31}\}$ and $\{f_{il}^{23}\}$ similarly. (We are suppressing from the notation the restriction of c to $\partial(Y_{12}) \setminus R$.) It follows from (6.1.10) that

$$Z(M_1 \cup M_2)(\widehat{f}_{ik}^{31} \cup f_{il}^{23}) = \sum_j Z(M_1)(\widehat{f}_{ik}^{31} \cup f_{ij}^{12}) \cdot Z(M_2)(\widehat{f}_{ij}^{12} \cup f_{il}^{23}).$$

(We continue to suppress from the notation c and its various restrictions.) It now follows from (6.2.5) that

$$\begin{aligned} Z(M_1 \cup M_2 \cup M_3)(c) &= \sum_{i,k,l} Z(M_1 \cup M_2)(\widehat{f}^{31}_{ik} \cup f^{23}_{il}) \cdot Z(M_3)(\widehat{f}^{23}_{il} \cup f^{31}_{ik}) \cdot \langle e_i, e_i \rangle^{-1} \\ &= \sum_{i,j,k,l} Z(M_1)(\widehat{f}^{31}_{ik} \cup f^{12}_{ij}) \cdot Z(M_2)(\widehat{f}^{12}_{ij} \cup f^{23}_{il}) \cdot Z(M_3)(\widehat{f}^{23}_{il} \cup f^{31}_{ik}) \cdot \langle e_i, e_i \rangle^{-1}. \end{aligned}$$

[need to adjust alignment above] Note that the above expression is symmetric in 1,2,3. In other words, if we applied the gluing formula in a different order (say by first gluing M_1 and M_3, then gluing M_2 to $M_1 \cup M_3$), the final answer for $Z(M_1 \cup M_2 \cup M_3)$ would not change. This self-consistency property of the gluing formula will be used in the proof of (6.3.1).

While the above expression might look complicated, the idea is simple: Orthogonally decompose the three spaces $A(Y_{xy}; \cdot)$ according to the minimal idempotents (irreps) of $A(R; b)$, then do the obvious tensorial contractions, but adjust them by a correction factor $\langle e_i, e_i \rangle^{-1}$. There is an obvious generalization for any number of M_i's glued together around a "corner" R.

[need to say something about total finiteness in an n-cat]

[need to revisit this section once the following section is complete]

6.3 Computing the Path Integral

Now, after much foreshadowing, we can state and prove

Theorem. *Suppose* 6.3.1

1. *there exists $z \in Z(S^n)$ such that the induced inner product $A(B^n; c) \otimes A(B^n; c) \to \mathbb{C}$ given by $a \otimes b \mapsto z(\widehat{a} \cup b)$ is positive definite for all $c \in \mathcal{C}(S^{n-1})$; and*

2. *$\dim A(Y^n; c) < \infty$ for all n-manifolds Y and all $c \in \mathcal{C}(\partial Y)$.*

Then there exists a unique path integral $Z(M^{n+1}) \in Z(\partial M)$ (for all n+1-manifolds M) satisfying the axioms of (6.1) and such that $Z(B^{n+1}) = z$.

(Note that we have assumed above standard identifications $S^n = \partial B^{n+1}$, $S^n = B^n \cup B^n$, and $S^{n-1} = \partial B^n$.)

[Need to replace assumption that $\dim A(Y^n; c) < \infty$ with some finiteness properties of the n-cat $Z(pt)$. (But the present version is also useful, so...??) Also, relax positive definite assumption??]

Proof. We will define $Z(M)$ by decomposing M into handles (each of which is homeomorphic to B^{n+1}) and applying the gluing formula (6.1.7). The challenge is to show that this computation is independent of the choice of handle decomposition. We will do this by inducting on the index of the handles.

First, some terminology. By a (k,l)-body we mean a k-dimensional manifold equipped with a handle decomposition with all handles of index $\le l$. We will use the same letter to denote both a (k,l)-body and the underlying k-dimensional manifold; which meaning is intended should be clear from context. If X is a (k,l)-body then $X \times I$ will denote the $(k+1,l)$-body given by thickening all the handles (increasing their dimension by one while leaving the index unchanged).

Our inductive hypotheses are

1. For each $(n+1,i)$-body M we have defined a $Z(M) \in Z(\partial M)$ which is unchanged by handle slides and handle cancellations of index $\le i$.

2. For each (n,i)-body Y, $A(Y;c)$ is finite-dimensional for all $c \in \mathcal{C}(\partial Y)$ and we have defined a positive definite inner product on $A(Y;c)$ which is unchanged by handle slides and handle cancellations of index $\le i$.

3. For each $(n-1,i)$-body R, $A(R;b)$ is semisimple with finitely many irreps for all $b \in \mathcal{C}(\partial R)$.

The hypotheses in the statement of the theorem imply the inductive hypotheses for $i = 0$. Note that a $(k,0)$-body is homeomorphic to a disjoint union of copies of B^k, and there are no handle slides or cancellations of dimension ≤ 0 to consider. $A(B^{n-1};c)$ is semisimple because of the positive definite inner product. It has finitely many irreps because otherwise $A(B^{n-1} \times S^1; c \times S^1)$ would be infinite dimensional *[refer to relevant gluing theorem]*.

We'll verify the inductive hypotheses for $i = 1$, then $i = 2$, then the general case.

Let M be an $(n+1,1)$-body. Choose an ordering of the 1-handles, then use this ordering to define $Z(M) \in Z(\partial M)$ by attaching each 1-handle in turn and applying (6.1.7). (Here we use the fact that $A(B^n \times S^0; c)$ is finite dimensional for all c.) Since the attaching targets of the 1-handles are disjoint, this calculation is clearly independent of the ordering of the 1-handles. Consider a slide of a 1-handle α over a 1-handle β. If α come before β in the ordering then it is clear that the calculation of $Z(M)$ using (6.1.7) is not affected, since if α is already attached then sliding β is simply an isotopy of the attaching target for β. Therefore $Z(M)$ is unaffected by handle slides. Next consider a 0-handle α which is canceled by a 1-handle β. Let α' be the 0-handle at the other end of β. Instead of gluing β simultaneously to α and α', we can first glue β to α', then glue α to β. Since the gluing targets are disjoint, the latter order of gluing yields the same computation of $Z(M)$ as the first. But each of the gluings in the latter order is equivalent to attaching a collar to a copy of B^n in the boundary of an n+1-maanifold, and thus has no effect on $Z(M)$.

Therefore $Z(M)$ is unaffected by the introduction or elimination of canceling pairs of 0- and 1-handles.

Next consider an $(n, 1)$-body Y. $A(Y; c)$ is finite-dimensional for all c by the hypotheses of the theorem. *[In the future, want to derive finite dimensionality inductively from properties of $Z(pt)$.]* Since $Y \times I$ is an $(n+1, 1)$-body, $Z(Y \times I)$ is defined and we can use (6.2.1) to define an inner product on $A(Y; c)$. Handle slides and cancelations for Y are mirrored by handle slides and cancelations for $Y \times I$, so the inner product is invariant under these changes of handle decomposition. By hypothesis the inner product is positive for the handles (copies of B^n) with any boundary conditions. The argument of (6.2.5) now shows that the inner product on $A(Y; c)$ is positive definite.

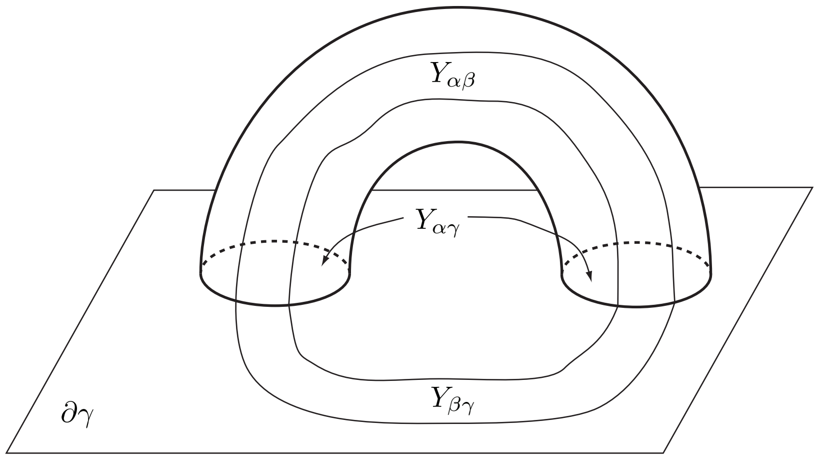
Next consider an $(n-1, 1)$-body R. Since $R \times I$ is an $(n, 1)$-body, the morphism spaces of $A(R; b)$ have positive definite inner products. We have $\langle ab, c\rangle = Z(R \times I)(\widehat{ab} \cup c) = Z(R \times I)(\widehat{b} \cup a^* c) = \langle b, a^* c\rangle$ (because $\widehat{ab} \cup c = \widehat{b} \cup a^* c$), so these inner products are compatible the action of $A(R; b)$ on itself. It follows from (B.5.3) that $A(R; b)$ is semisimple. $A(R; b)$ has finitely many irreps since otherwise $A(R \times S^1; b \times S^1)$ would be infinite dimensional.

We have now established the inductive hypotheses for $i = 1$. Next we establish them for $i = 2$, which will present one additional complication.

Let M be an $(n+1, 2)$-body. Choose an ordering of the 1- and 2-handles such that if the attaching map of a 2-handle α goes over a 1-handle β, then α follows β in the ordering. Use this ordering to define $Z(M) \in Z(\partial M)$ by attaching each handle in turn and applying (6.1.7). For 2-handles, this requires the use of the recently defined inner product on $A(B^{n-1} \times S^1; c)$. ($B^{n-1} \times S^1$ has a standard $(n, 1)$-body structure with one 0-handle and one 1-handle.) Because of disjointness properties of the attaching targets and the restrictions on the ordering, the calculation of $Z(M)$ does not depend on the choice of (restricted) ordering. As before, $Z(M)$ is not affected by handle slides, since for an appropriately chosen ordering the handle slide is merely an isotopy of the attaching target. Invariance under cancelation of 0- and 1-handles is proved as before.

Cancelation of 1- and 2-handles presents a new wrinkle because the attaching targets of a canceling 1-2 pair intersect. Let α be a 1-handle which is canceled by a 2-handle β. Assume that α immediately precedes β in the ordering, and let γ denote the $n+1$-manifold resulting from all gluings which precede α in the ordering. Attaching α and then β to γ results in an $n+1$-manifold which is homeomorphic to γ. We must show that doing these two gluings in this order has no effect on $Z(\gamma) \in Z(\partial\gamma)$. Define

$$\begin{aligned} Y_{\alpha\gamma} &\stackrel{\text{def}}{=} \alpha \cap \gamma \cong B^n \times S^0 \\ Y_{\alpha\beta} &\stackrel{\text{def}}{=} \alpha \cap \beta \cong B^{n-1} \times B^1 \\ Y_{\beta\gamma} &\stackrel{\text{def}}{=} \beta \cap \gamma \cong B^{n-1} \times B^1. \end{aligned}$$



6.3.2 Canceling 1- and 2-handles

(See Figure (6.3.2).) Note that if we attach β to γ along $Y_{\beta\gamma}$ this is equivalent to attaching a boundary collar to $Y_{\beta\gamma} \subset \partial\gamma$ and so has no effect on $Z(\gamma)$. *[need ref for this]* Similarly, attaching α to $\gamma \cup \beta$ along $Y_{\alpha\gamma} \cup Y_{\alpha\beta}$ is equivalent to attaching a boundary collar to $Y_{\alpha\gamma} \cup Y_{\alpha\beta} \subset \partial(\gamma \cup \beta)$, and so again does not affect the calculation of $Z(\gamma)$. ($Y_{\alpha\gamma} \cup Y_{\alpha\beta}$ here has an obvious $(n, 1)$-body structure, and we use this structure to define the inner product.) So it suffices to show that attaching α then β yields the same result for Z and attaching first β then α. This follows from (6.2.6). The proof that $Z(M)$ is independent of slides and cancelations is now complete.

Let Y be an $(n, 2)$-body. As above, since $Y \times I$ is an $(n + 1, 2)$-body, we can use (6.2.1) to define an inner product on $A(Y; c)$. Handle slides and cancelations for Y are mirrored by handle slides and cancelations for $Y \times I$, so the inner product is invariant under these changes of handle decomposition. By hypothesis the inner product is positive for the handles (copies of B^n) with any boundary conditions. The argument of (6.2.5) now shows that the inner product on $A(Y; c)$ is positive definite.

Let R be an $(n-1, 2)$-body. Since $R \times I$ is an $(n, 2)$-body, the morphism spaces of $A(R; b)$ have positive definite inner products. The same argument as before shows that these inner products are compatible with the action of $A(R; b)$ on itself. It follows from (B.5.3) that $A(R; b)$ is semisimple. Again, $A(R; b)$ has finitely many irreps since otherwise $A(R \times S^1; b \times S^1)$ would be infinite dimensional.

We have now established the inductive hypotheses for $i = 2$. The proof for arbitrary i is very similar. The only part worth commenting on is the cancelation of i- and $i-1$-handles.

Let α be an $i-1$-handle which is canceled by an i-handle β. Assume that α immediately precedes β in the ordering, and let γ denote the $n+1$-manifold resulting from all gluings which precede α in the ordering. Attaching α and then β to γ results in an $n+1$-manifold which is homeomorphic to γ. We must show that doing these two gluings in this order has no effect on $Z(\gamma) \in Z(\partial\gamma)$. Define

$$\begin{aligned} Y_{\alpha\gamma} &\stackrel{\text{def}}{=} \alpha \cap \gamma \cong B^{n-i+2} \times S^{i-2} \\ Y_{\alpha\beta} &\stackrel{\text{def}}{=} \alpha \cap \beta \cong B^{n-i+1} \times B^{i-1} \\ Y_{\beta\gamma} &\stackrel{\text{def}}{=} \beta \cap \gamma \cong B^{n-i+1} \times B^{i-1}. \end{aligned}$$

Note that if we attach β to γ along $Y_{\beta\gamma}$ this is equivalent to attaching a boundary collar to $Y_{\beta\gamma} \subset \partial\gamma$ and so has no effect on $Z(\gamma)$. *[need ref for this]* Similarly, attaching α to $\gamma \cup \beta$ along $Y_{\alpha\gamma} \cup Y_{\alpha\beta}$ is equivalent to attaching a boundary collar to $Y_{\alpha\gamma} \cup Y_{\alpha\beta} \subset \partial(\gamma \cup \beta)$, and so again does not affect the calculation of $Z(\gamma)$. ($Y_{\alpha\gamma} \cup Y_{\alpha\beta}$ here has a $(n, i-1)$-body structure, with $Y_{\alpha\beta}$ playing the role of an $i-1$-handle, and we use this structure to define the inner product.) So it suffices to show that attaching α then β yields the same result for Z and attaching first β then α. This follows from (6.2.6).

At this point we have defined the path integral $Z(M)$ for arbitrary handle-body structures on the $n+1$-manifold M and shown that it is invariant under handle slides and cancelations of any index. Thus $Z(M)$ is independent of the choice of handle decomposition and depends only on the underlying manifold. Similarly, we have well-defined positive definite inner products on $A(Y;c)$ for any n-manifold Y, and $A(R;b)$ is semisimple for all $n-1$-manifolds R.

All that remains to be done is to show that the gluing relation (6.1.4) holds. *[Sketch: Start with M, then construct M_{gl} by adding handles to $\pm Y$, mirroring a handle decomposition of Y. The effect of these handle additions is the same as taking inner products in $A(Y;c)$.]* □

The proof of (6.3.1) is also a recipe for producing a "state model" for the theory. If we assemble all the summations for all of the handle attachments into one big summation, we end up with a sum over labelings of the handles (the labels indexing the orthogonal bases for various handle attaching targets). If the handle decomposition is dual to a triangulation, then of course this is also a sum over labelings of the simplexes of the triangulation. Specific examples of this are given in later chapters. For 1+1-dimensional theories, we get the familiar *[need name]*. For 2+1-dimensional theories based on a spherical category, we get the Turaev-Viro model. For 3+1-dimensional theories based on a ribbon category (which, via decategorification, lead to 2+1 dimensional theories for "extended" manifolds), we get three different models depending on the type of handle decomposition. For handle decompositions with a single 0-handle and several 2-handles, we get the the Witten-Reshtikhin-Turaev surgery formula. For a handle decomposition dual to a triangulation, we get the Crane-Yetter model. For handle decompositions in the which the 2-skeleton has tetrahedral singularities, and which allow the the 2-handles to be disk bundles over higher genus surfaces, we get the Turaev shadow state sums. *[need section refs for all of the above]*

Still to do:

- need reference for handle stuff
- (is semi-simple case the only one that works? probably.)
- do general state model? only 4d? will need to make assumptions about general n-cat for general case. maybe just do 4-dim'l case as generally as possible and say that it's clear this could be generalized

Chapter 7

1+1-dimensional Examples

7.1 Generalities on 1+1-dimensional Theories

To do:

- Start with 1-category, derive Frobenius algebra etc.

7.2 Finite Group Theories in $1 + 1$ dimensions

To do:

- go over finite groups theories in detail, for 1+1-dimensional case

Chapter 8

2+1-dimensional Examples

8.1 Temperley-Lieb Theories

To do:

- determine all ideals
- idempotents
- quotients
- 6j?
- do 3-d part here, or wait until next section?

8.2 Spherical Categories in General

[put this section before Temperley-Lieb section?]

Outline (to be filled in later):

- A (disk-like) 2-category with only one object is what's known as a spherical tensor category. (Need to give details of equivalence and full definition of spherical category.)
- Given a spherical category C, with objects C^0 and morphisms C^1, we can define fields on surfaces consisting of embedded graphs with oriented edges labeled by C_0 and vertices ("coupons" in R-T terminology) labeled by C^1. (Graphs are allowed to have circular edges without vertices.) Frobenius reciprocity means that we don't need to distinguish range and domain at the coupons. Fields on a 1-manifold are oriented points (ends of oriented arcs), labeled by C^0. There is a unique (empty) field on a 0-manifold.

- The local relations are generated by

  1. isotopy,
  2. reversing the orientation of an arc and changing the label to its dual,
  3. replacing "identity" coupons with parallel arcs (note that this includes cups and caps because of Frobenius reciprocity),
  4. erasing arcs labeled by the trivial object (and adjusting labels of coupons if necessary),
  5. combining two adjacent coupons into one (using composition of morphisms in C), and
  6. replacing a diagram containing a coupon labeled by a linear combination of morphisms with the corresponding linear combination of fields.

- For this theory, $A(pt)$ is essentially C (thought of as a 2-category, of course). (Need to give more details on "essentially".)

- $A(S^1)$ is the annularization of C.

- $Z(S^1)$ is Drinfeld center of the category of representations of C. (See (5.2.30).)

- Note that any labeled graph in S^2 is equivalent, via the above local relations, to some multiple of the empty graph. We call this multiple the "standard evaluation" of a graph in S^2.

8.2.1
- $A(Y^2)$ can be described in terms of labelings of 0- and 1-skeleton of a fixed cell decomposition of the surface Y. An explicit list of relations corresponding to 2-cells of the cell decomposition can be given. (Give details.) Note that we already know that the vector space is independent of the cell decomposition, so we don't need a separate proof of independence.

- If C is semisimple, then we can restrict all coupons to be trivalent. We can also restrict all edge labels to be simple objects (or minimal idempotents or irreps; need to comment on equivalence between minimal idems, irreps and simple objects). We get the familiar description in terms of labeled, oriented trivalent graphs. (Plain "trivalent graphs" for short.)

- (Need to give details on "F" moves (a.k.a. recoupling), etc.)

8.2.2
- (Give refinement of description of $A(Y;c)$, including case when Y is a disk. Observe that in this case we have an orthogonal basis.)

Assume now that C is s semisimple spherical category with finitely many irreps. It follows from (8.2.1) that $A(Y;c)$ is finite dimensional for all 2-manifolds Y and $c \in \mathcal{C}(\partial Y)$. Note that $Z(S^2)$ is 1-dimensional, and the standard evaluation of trivalent graphs (which evaluates to $1 \in \mathbb{C}$ on the empty graph) is a basis. It follows from (8.2.2) that any non-zero element $z \in Z(S^2)$ determines a nondegenerate inner product on $A(D^2; c)$ for all c. Assume that these inner products are positive definite. *[comment on this assumption.]* We can now apply (6.3.1) to construct a path integral for 3-manifolds. Following the proof of (6.3.1) we will construct a state sum description of the path integral in terms of labelings of the cells of a cell decomposition of a 3-manifold. We will see that this state model turns out to be the Turaev-Viro state model [TV92, Tur94].

Choose $z \in Z(S^2)$; z is λ times the standard evaluation of trivalent graphs for some $\lambda \in \mathbb{C}$. *[for positive definiteness, need $\lambda \in \mathbb{R}_+$]* (Recall from (6.3.1) that we define $Z(B^3) = z$.) Our first task is the compute bases and inner products for $A(Y; c)$, where $(Y; c)$ runs through all attaching targets for handles that we will need below.

Recall that a field on 1-manifold R consists of a finite number of oriented points in R each labeled by an irrep. If R is contained in the boundary of a 2-manifold we will assume that all of the points are oriented inward. Fields will be denoted as a sequence of irreps, so, for example, $A(D^2; a, b, c)$ means a disk with boundary conditions given by three inward pointing points labeled by a, b and c. *[this remark on notation should go earlier]*

Let a and b be irreps (or equivalently minimal idempotents or simple objects) of C. Then $A(D^2; \widehat{a}, b)$ is 1-dimensional if a and b are equivalent and 0-dimensional otherwise. A basis for $A(D^2, \widehat{a}, a)$ is given by a single oriented arc in D^2 connecting the boundary points and labeled by a. Call this basis element $e_{\widehat{a}a}$. Then 8.2.3

$$\langle e_{\widehat{a}a}, e_{\widehat{a}a} \rangle = z(\widehat{e}_{\widehat{a}a} \cup e_{\widehat{a}a}) = \lambda d_a,$$

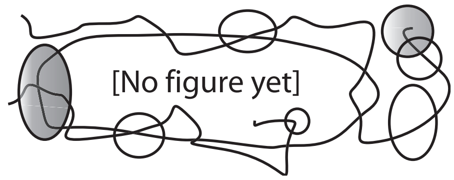
where d_a is the value of a loop labeled by a in the standard evaluation. *[need to introduce d_a above]*

Next consider $A(D^2; a, b, c) \cong \hom(\widehat{a}, bc) \cong \hom(\widehat{b}\widehat{a}, c) \cong \hom(1, abc)$ etc. If $x, y \in A(D^2, a, b, c)$ then $\langle x, y \rangle$ is given by evaluating a "theta" graph with arcs labeled by a, b, c and vertices labeled by $\widehat{x}$ and y, then multiplying by λ. By assumption this inner product is nondegenerate, so we can choose an orthogonal basis $\{e_{abci}\} \subset A(D^2, a, b, c)$. We have 8.2.4

$$\langle e_{abci}, e_{abcj} \rangle = z(\widehat{e}_{abci} \cup e_{abcj}) = \delta_{ij} \lambda \theta_{abci},$$

where θ_{abci} is the value of the standard evaluation on a theta graph labeled by a, b, c, e_{abci} and $\widehat{e}_{abci}$.

Next we consider $S^1 \times I$. We will see below that we will only need to know inner products on $A(S^1 \times I; c)$ when c is the empty boundary condition $\emptyset$. For a an irrep of C, consider the element $e_a \in A(S^1 \times I; \emptyset)$ represented by a loop $S^1 \times \{pt\} \subset S^1 \times I$ labeled by a. It follows from (4.4.6) that $\{e_a\}$, where a runs through a complete set of irreps of C, is a basis of $A(S^1 \times I; \emptyset)$. Recalling the definition of $e_{\widehat{a}a}$ from above, 8.2.5



8.2.6 Cutting a solid torus to get a ball

and applying (6.1.9), we see that

$$\begin{aligned}\langle e_a, e_a\rangle &= Z(S^1 \times I \times I; \widehat{e}_a, e_a)\\ &= z(e_{\widehat{a}a} \cup e_{\widehat{a}a} \cup e_{\widehat{a}a} \cup e_{\widehat{a}a})\langle e_{\widehat{a}a}, e_{\widehat{a}a}\rangle^{-1}\\ &= \lambda d_a(\lambda d_a)^{-1}\\ &= 1\end{aligned}$$

(see Figure (8.2.6)). A similar argument shows that $\langle e_a, e_b\rangle = 0$ if a and b are not equivalent. So $\{e_a\}$ is an orthonormal basis of $A(S^1 \times I; \emptyset)$.

8.2.7 Finally we consider S^2. Let $e_\emptyset \in A(S^2)$ be represented by the empty trivalent graph; $e_\emptyset$ spans $A(S^2)$. Let $\mathcal{L}$ be a complete set of irreps of C. Decomposing $S^2 \times I = (D^2 \times I) \cup (D^2 \times I)$ and applying (6.1.9) and (8.2.5) we have

$$\begin{aligned}\langle e_\emptyset, e_\emptyset\rangle &= Z(S^2 \times I)(\widehat{e}_\emptyset \cup e_\emptyset)\\ &= \sum_{a\in\mathcal{L}} z(\widehat{e}_a \cup \emptyset_{D^2} \cup \emptyset_{D^2})z(e_a \cup \emptyset_{D^2} \cup \emptyset_{D^2})\langle e_a, e_a\rangle^{-1}\\ &= \sum_{a\in\mathcal{L}}(\lambda d_a)(\lambda d_a)\cdot 1\\ &= \lambda^2 D,\end{aligned}$$

where $D \stackrel{\text{def}}{=} \sum_a d_a^2$.

Armed with the above computations, we can now calculate the path integral $Z(M)$ of a 3-manifold M in terms of a generic cell decomposition of M. By generic we mean dual to a triangulation, so that each 1-cell is incident to three 2-cells, and the 1- and 2-cells incident to a 0-cell form a tetrahedral pattern. By thickening the cells we get a handle decomposition of M. For simplicity, we will at first assume the M is closed. Later we will indicate how to extend the results to more general handles decompositions and to 3-manifolds with boundary.

We will start by analyzing the effect of adding 3-handles and work our way down to the 0-handles. Let $M_i \subset M$ be the union of all handles of index less than or equal to i. Let n_i be the number of i-handles, and let $\mathcal{H}_i$ denote the set of i-handles. Applying (6.1.9) and (8.2.7) to the decomposition $M = M_2 \cup \{\text{3-handles}\}$, we see that

$$\begin{aligned} Z(M) &= Z(M_2)(e_\emptyset^{n_3}) \prod_{\mathcal{H}_3} z(e_\emptyset)\langle e_\emptyset, e_\emptyset\rangle^{-1} \\ &= Z(M_2)(e_\emptyset^{n_3}) \prod_{\mathcal{H}_3} \lambda^{-1} D^{-1}. \end{aligned}$$

Of course, the product is equal to $(\lambda D)^{-n_3}$, but we prefer to write it as a product in order to be consistent with the treatment of the other handles below.

Let $\mathcal{L}_2$ denote the set of all labelings of the 2-handles by elements of $\mathcal{L}$. Applying (6.1.9) and (8.2.5) to the decomposition $M_2 = M_1 \cup \{\text{2-handles}\}$, we see that

$$Z(M_2)(e_\emptyset^{n_3}) = \sum_{\alpha\in\mathcal{L}_2} Z(M_1)(g_\alpha) \prod_{f\in\mathcal{H}_2} \lambda d(\alpha, f),$$

where g_α denotes is the graph on ∂M_1 consisting of a loop for the attaching target of each 2-handle labeled according to α, and $d(\alpha, f) = d_{\alpha(f)}$, the standard evaluation of a loop labeled by $\alpha(f)$.

Next we consider the decomposition $M_1 = M_0 \cup \{\text{1-handles}\}$. We want to evaluate $Z(M_1)$ on g_α. In applying (6.1.9) and (8.2.4) we will place a trivalent vertex (some e_{abci}) in each D^2 in $\partial(M_0 \sqcup \{\text{1-handles}\})$ which results from cutting M_1. *[need figure(?)]* The resulting field on the boundary of a 1-handle is a theta graph labeled according to α and the e_{abci}'s on the two attaching disks. If these two basis elements are not the same then the theta graph will evaluate to zero, so we may assume that they are the same. In other words, given α we have fixed arc labels a, b, c for the theta graph of each 1-handle, and we sum over i's so that e_{abci} runs through a basis of $A(D^2, a, b, c)$. Let $\mathcal{L}_{1,\alpha}$ denote the set of all such labelings of the 1-handles consistent with α. For $\beta \in \mathcal{L}_{1,\alpha}$ and $e \in \mathcal{H}_1$ let $\Theta(\alpha, \beta, e)$ denote the standard evaluation of the labeled theta graph on the boundary of e. For each 0-handle $v \in \mathcal{H}_0$ there is a labeled tetrahedral graph on the boundary of the 0-handle. (Labels of the edges of the tetrahedron come from α and labels of the vertices of the tetrahedron from from β.) Let $\mathrm{Tet}(\alpha, \beta, v)$ denote the standard evaluation of this graph. Note that the contribution of a 1-handle e to the $\langle e_*, e_*\rangle^{-1}$ factor of (6.1.9) is, by (8.2.4), $(\lambda\Theta(\alpha, \beta, e))^{-2}$, since we glue along two pairs of disks for each 1-handle. Putting this all together, we have

$$\begin{aligned} Z(M_1)(g_\alpha) &= \sum_{\beta\in\mathcal{L}_{1,\alpha}} \prod_{e\in\mathcal{H}_1} \lambda\Theta(\alpha, \beta, e)(\lambda\Theta(\alpha, \beta, e))^{-2} \prod_{v\in\mathcal{H}_0} \lambda\,\mathrm{Tet}(\alpha, \beta, v) \\ &= \sum_{\beta\in\mathcal{L}_{1,\alpha}} \prod_{e\in\mathcal{H}_1} (\lambda\Theta(\alpha, \beta, e))^{-1} \prod_{v\in\mathcal{H}_0} \lambda\,\mathrm{Tet}(\alpha, \beta, v) \end{aligned}$$

Combining all of the above, we have

$$\begin{aligned} Z(M) &= \sum_{\alpha\in\mathcal{L}_2}\sum_{\beta\in\mathcal{L}_{1,\alpha}}\prod_{\mathcal{H}_3}\lambda^{-1}D^{-1}\prod_{f\in\mathcal{H}_2}\lambda d(\alpha,f)\prod_{e\in\mathcal{H}_1}\lambda^{-1}\Theta(\alpha,\beta,e)^{-1}\prod_{v\in\mathcal{H}_0}\lambda\,\mathrm{Tet}(\alpha,\beta,v) \\ &= \lambda^{\chi(M)}\sum_{\alpha\in\mathcal{L}_2}\sum_{\beta\in\mathcal{L}_{1,\alpha}}\prod_{\mathcal{H}_3}D^{-1}\prod_{f\in\mathcal{H}_2}d(\alpha,f)\prod_{e\in\mathcal{H}_1}\Theta(\alpha,\beta,e)^{-1}\prod_{v\in\mathcal{H}_0}\mathrm{Tet}(\alpha,\beta,v). \end{aligned}$$

Here $\chi(M)$ denotes the Euler characteristic of M, which is always zero for a closed 3-manifold. We leave it in the formula because we want to indicate how to generalize to non-closed 3-manifolds, and because we want to emphasize the parallels with a similar expression for 4-manifolds. *[need forward ref]* The above expression for $Z(M)$ is essentially the Turaev-Viro state sum [TV92, Tur94]. But note that we do not need to show that it is invariant under Pachner moves on the triangulation — invariance follows from the easier and more general (6.3.1).

For a general cell decomposition of M (one not necessarily dual to a triangulation), The factors of Θ and above will be replaced with an evaluation of "multi-barred theta graph" reflecting the number of 2-cells incident to a 1-cell. *[need figure]* The factors of Tet will be replaced with an evaluation of the graph which is the link of the 2-skeleton of the decomposition around the vertex. If M has boundary then we first fix a graph g on ∂M, which we assume is in a neighborhood of the 1-skeleton of the cell decomposition restricted to the boundary. Then there is a similar state sum for $Z(M)(g)$. *[need to give more details]*

8.3 A2

(follow Kuperberg then extend some; include lots of detail)

8.4 G2

(follow Kuperberg then extend some; include lots of detail)

8.5 Finite Group Theories in 2+1 dimensions

8.6 Jones Planar Algebras

To do:

- Planar algebra is a disk-like 2-cat with two objects and singly generated 1-mors.
- Comment more generally on relation to subfactor point of view to this one? Provide translation table? Put this in another section/chapter?

Chapter 9

3+1-dimensional Examples

9.1 Theories From Ribbon Categories

Outline (to be filled in later):

- Note that this section is very similar to Section 8.2, so the reader might want to compare the two, or treat Section 8.2 as a warm-up for this section.

- A disklike 3-category with only one object (0-morphism) and only one 1-morphism is a ribbon category. *[need to give details]* Thus we expect that a from a ribbon category we can construct a 3+1-dimensional TQFT. We will see in the next section that (with some additional assumptions on the ribbon category) this 3+1-dimensional TQFT can be decategorified to yield the Witten-Reshtikhin-Turaev type 2+1-dimensional TQFT that can be constructed directly (but less cleanly) from the same ribbon category.

- Given a ribbon category C, with objects C^0 and morphisms C^1, we can define fields on 3-manifolds consisting of embedded ribbon graphs with oriented edges labeled by C_0 and vertices labeled by C^1. Following Reshetikhin and Turaev, we will call a labeled vertex a *coupon. [need to comment on pinning or cyclic ordering or ciliation at vertices]* (Graphs are allowed to have circular edges without vertices.) The dualities of the ribbon category mean that we don't need to distinguish range and domain at the coupons. Fields on a 2-manifold are oriented framed points (ends of oriented ribbons), labeled by C^0. There is a unique (empty) field on a 1-manifold. There is a unique (empty) field on a 0-manifold.

- The local relations are generated by

  1. isotopy,
  2. reversing the orientation of a ribbon and changing the label to its dual,

3. replacing "identity" coupons with parallel ribbons (note that this includes cups and caps because of Frobenius reciprocity),
4. erasing ribbons labeled by the trivial object (and adjusting adjacent labels of coupons as necessary),
5. combining two adjacent coupons into one (using composition of morphisms in C), and
6. replacing a diagram containing a coupon labeled by a linear combination of morphisms with the corresponding linear combination of fields.

- For this theory, $A(pt)$ is essentially C (thought of as a 3-category, of course). *[Need to give more details on "essentially".]*
- Note that any labeled ribbon graph in $(B^3, \emptyset)$ is equivalent, via the above local relations, to some multiple of the empty graph. We call this multiple the "standard evaluation" of a graph in $(B^3, \emptyset)$.

9.1.1
- $A(M^3; c)$ can be described in terms of labelings of 0- and 1-skeleton of a fixed cell decomposition of the 3-manifold M. An explicit list of relations corresponding to 2-cells of the cell decomposition can be given. *[Give details.]*
- *[Need to remark that $A(M^3; c)$ is the familiar (relative) skein module of M based on C.]*
- If C is semisimple, then we can restrict all coupons to be trivalent. We can also restrict all edge labels to be simple objects *[or minimal idempotents or irreps; need to comment on equivalence between minimal idems, irreps and simple objects]*. We get the familiar description in terms of labeled, oriented trivalent graphs. (Plain "trivalent graphs" for short.)
- (Need to give details on "F" moves (a.k.a. recoupling), etc.)
- Let $\mathcal{L}$ be a complete set of irreps of C.

9.1.2
- (Give refinement of description of $A(M; c)$, including case when M is a disk. Observe that in this case we have an orthogonal basis.)

Assume now that C is s semisimple ribbon category with finitely many irreps. It follows from (9.1.1) that $A(M; c)$ is finite dimensional for all 3-manifolds M and $c \in \mathcal{C}(\partial M)$. Note that $Z(S^3)$ is 1-dimensional, and the standard evaluation of trivalent ribbon graphs (which evaluates to $1 \in \mathbb{C}$ on the empty graph) is a basis. It follows from (9.1.2) that any non-zero element $z \in Z(S^3)$ determines a nondegenerate inner product on $A(B^3; c)$ for all c. Assume that these inner products are positive definite. *[comment on this assumption.]* We can now apply (6.3.1) to construct a path integral for 4-manifolds.

Following the proof of (6.3.1) we will construct a state sum description of the path integral in terms of labelings of the handles of a handle decomposition of a 4-manifold. *[need to be consistent about handle decomposition vs cell decomposition]* We will see below that this state sum specializes to: (a) the well-known Witten-Reshtikhin-Turaev framed link surgery formula, for handle decompositions consisting of a single 0-handle and several 2-handles; (b) the Crane-Yetter state sum, for handle decompositions which are dual to triangulations; and (c) the Turaev shadow state sum, for handle decompositions which have tetrahedral singularities in their 2-skeleton. *[need to be more precise here] [need refs for above]*

Choose $z \in Z(S^3)$; z is λ times the standard evaluation of trivalent graphs for some $\lambda \in \mathbb{C}$. *[for positive definiteness, need $\lambda \in \mathbb{R}_+$]* (Recall from (6.3.1) that we define $Z(B^4) = z$.) Our first task is the compute bases and inner products for $A(M;c)$, where $(M;c)$ runs through all attaching targets for handles that we will need below.

Recall that a field on 2-manifold Y consists of a finite number of oriented framed points in Y each labeled by an irrep. If Y is contained in the boundary of a 3-manifold we will assume that all of the points are oriented inward. Fields will be denoted as a sequence of irreps, so, for example, $A(B^3; a, b, c)$ means a 3-ball with boundary conditions given by three inward pointing points labeled by a, b and c. *[remark that the ordering of the irreps (points) in a connected component doesn't matter] [this remark on notation should go earlier]*

Let a and b be irreps (or equivalently minimal idempotents or simple objects) of C. Then $A(B^3; \widehat{a}, b)$ is 1-dimensional if a and b are equivalent and 0-dimensional otherwise. A basis for $A(B^3, \widehat{a}, a)$ is given by a single oriented ribbon in B^3 connecting the boundary points and labeled by a. Call this basis element $e_{\widehat{a}a}$. Then 9.1.3

$$\langle e_{\widehat{a}a}, e_{\widehat{a}a} \rangle = z(\widehat{e}_{\widehat{a}a} \cup e_{\widehat{a}a}) = \lambda d_a,$$

where d_a is the value of a 0-framed loop labeled by a in the standard evaluation. *[need to introduce d_a above]*

Next consider $A(B^3; a, b, c) \cong \hom(\widehat{a}, bc) \cong \hom(\widehat{b\widehat{a}}, c) \cong \hom(1, abc)$ etc. If 9.1.4
$x, y \in A(B^3, a, b, c)$ then $\langle x, y \rangle$ is given by evaluating a "theta" graph with ribbons labeled by a, b, c and vertices labeled by $\widehat{x}$ and y, then multiplying by λ. By assumption this inner product is nondegenerate, so we can choose an orthogonal basis $\{e_{abci}\} \subset A(B^3, a, b, c)$. We have

$$\langle e_{abci}, e_{abcj} \rangle = z(\widehat{e}_{abci} \cup e_{abcj}) = \delta_{ij} \lambda \theta_{abci},$$

where θ_{abci} is the value of the standard evaluation on a theta graph labeled by a, b, c, e_{abci} and $\widehat{e}_{abci}$.

More generally, if $c = (a_1, \ldots, a_n)$ $(n \geq 3)$, we can choose a trivalent ribbon 9.1.5
tree with boundary c, and labelings of the tree give an orthogonal basis of $A(B^3; c)$ (see (9.1.2)). The $n-3$ internal edges of the tree are labeled by irreps and the $n-2$

trivalent vertices of the tree are labeled by some orthogonal bases of the appropriate trivalent vertex spaces. Call these (multi) labels b and v respectively. Then

$$\langle e_{bv}, e_{bv}\rangle = \lambda \prod_{i\in V} \Theta(b,v,i) \prod_{j\in\mathcal{E}} d(b,j)^{-1},$$

where $\mathcal{V}$ denotes the set of vertices of the tree and $\mathcal{E}$ denotes the set of internal edges. *[need to give more detail and improve notation]*

9.1.6 Next we consider $S^1\times D^2$. We will see below that we will only need to know inner products on $A(S^1\times D^2;c)$ when c is the empty boundary condition $\emptyset$. For a an irrep of C, consider the element $e_a \in A(S^1\times D^2;\emptyset)$ represented by a loop $S^1\times\{pt\}\subset S^1\times D^2$ labeled by a. It follows from (4.4.6) that $\{e_a\}$, where a runs through a complete set of irreps of C, is a basis of $A(S^1\times D^2;\emptyset)$. Recalling the definition of $e_{\widehat{a}a}$ from above, and applying (6.1.9), we see that

$$\begin{aligned}
\langle e_a, e_a\rangle &= Z(S^1\times D^2\times I;\widehat{e}_a, e_a)\\
&= z(e_{\widehat{a}a}\cup e_{\widehat{a}a}\cup e_{\widehat{a}a}\cup e_{\widehat{a}a})\langle e_{\widehat{a}a}, e_{\widehat{a}a}\rangle^{-1}\\
&= \lambda d_a(\lambda d_a)^{-1}\\
&= 1
\end{aligned}$$

A similar argument shows that $\langle e_a, e_b\rangle = 0$ if a and b are not equivalent. So $\{e_a\}$ is an orthonormal basis of $A(S^1\times D^2;\emptyset)$.

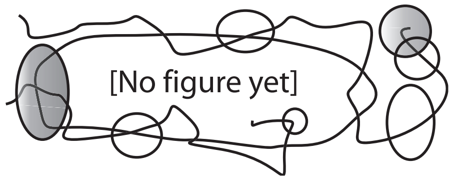
9.1.7 Next we consider $S^2\times I$. In applications we will only need to consider empty boundary conditions. $A(S^2\times I;\emptyset)$ is 1-dimensional and spanned by $e_\emptyset$, the empty field. Let $\mathcal{L}$ be a complete set of irreps of C. Decomposing $S^2\times I\times I = (D^2\times I\times I)\cup(D^2\times I\times I)$ and applying (6.1.9) and (9.1.6) we have

$$\begin{aligned}
\langle e_\emptyset, e_\emptyset\rangle &= Z(S^2\times I\times I)(\widehat{e}_\emptyset\cup e_\emptyset)\\
&= \sum_{a\in\mathcal{L}} z(\widehat{e}_a\cup\emptyset_{B^3}\cup\emptyset_{B^3})z(e_a\cup\emptyset_{B^3}\cup\emptyset_{B^3})\langle e_a, e_a\rangle^{-1}\\
&= \sum_{a\in\mathcal{L}}(\lambda d_a)(\lambda d_a)\cdot 1\\
&= \lambda^2 D,
\end{aligned}$$

where $D \stackrel{\text{def}}{=} \sum_a d_a^2$.

9.1.8 Finally we consider S^3. $A(S^3)$ is 1-dimensional and spanned by $e'_\emptyset$, the empty field. *[need better notation]* Decomposing $S^3\times I = (B^3\times I)\cup(B^3\times I)$ and applying (6.1.9) and (9.1.7) we have

$$\begin{aligned}
\langle e'_\emptyset, e'_\emptyset\rangle &= Z(S^3\times I)(\widehat{e}'_\emptyset\cup e'_\emptyset)\\
&= z(\emptyset)z(\emptyset)\langle e_\emptyset, e_\emptyset\rangle^{-1}\\
&= \lambda\lambda(\lambda^2 D)^{-1}\\
&= D^{-1}.
\end{aligned}$$



9.1.10 A figure which needs no caption

This completes the inner product calculations we will need for handle attachments.

Before doing general calculations we look at $S^2 \times I$ and $S^2 \times S^1$, and also show that frequently we can choose λ so that $Z(W^4)$ depends only on the bordism class of W.

Consider $A(S^2 \times I; \widehat{a}, b)$, where the two ribbon ends lie in different components of $\partial(S^2 \times I)$ and a and b are irreps (simple objects). It is easy to see that $A(S^2 \times I; \widehat{a}, b)$ is 0 if $a \not\cong b$, and that $A(S^2 \times I; \widehat{a}, a)$ is spanned by x_a, the obvious unknotted ribbon connecting the two ribbon ends. So we must determine whether $x_a = 0$ in $(S^2 \times I; \widehat{a}, a)$.

Let H_{ab} denote the standard evaluation of the (framed) Hopf link with labels a 9.1.9
and b. Note that $H_{1a} = d_a$. Let x_{ab} denote x_a plus a small linking circle labeled by b (see Figure (9.1.10)). Then $x_{ab} = (H_{ab}/d_a)x_a$. But in $S^2 \times I$ the linking circle is isotopic to an unlinked circle, so we also have $x_{ab} = d_b x_a$. So $(H_{ab}/d_a - d_b)x_a = 0$ for all $b \in \mathcal{L}$, which means that $x_a = 0$ unless $H_{ab} = d_a d_b$ for all $b \in \mathcal{L}$. By (9.1.1) this is also a sufficient condition for $x_a \neq 0$. Call such an $a \in \mathcal{L}$ *degenerate*. *[find a better name for this?]* Thus $A(S^2 \times I; \widehat{a}, a)$ is 1-dimensional when a is degenerate and 0-dimensional otherwise. Note that $1 \in \mathcal{L}$ is always degenerate.

Similar arguments show that a basis of $A(S^2 \times S^1)$ consists of $\{f_a\}$, where a is degenerate and $f_a \in A(S^2 \times S^1)$ denotes the ribbon graph $pt \times S^1 \subset S^2 \times S^1$ labeled by $a \in \mathcal{L}$.

Suppose that $\dim A(S^2 \times S^1) = 1$ (or equivalently 1 is the only degenerate irrep). *[in what follows, need to make clearer when this assumption is needed.]* We will show that in this case setting $\lambda^2 = 1/D$ yields a path integral $Z(W^4)$ which depends only on the bordism class of W. 4-dimensional bordism is generated by three types of surgery (and their inverses):

$$
\begin{aligned}
\emptyset &\leftrightarrow S^4 \\
B^4 \times S^0 &\leftrightarrow S^3 \times B^1 \\
B^3 \times S^1 &\leftrightarrow S^2 \times B^2.
\end{aligned}
$$

We have

$$Z(S^4) = Z(B^4)(e'_\emptyset) \cdot Z(B^4)(e'_\emptyset) \cdot \langle e'_\emptyset, e'_\emptyset \rangle^{-1} = \lambda^2 D,$$

so the first type of surgery does not affect $Z(W)$ if $\lambda^2 = 1/D$. We have

$$Z(B^4 \times S^0)(\emptyset) = \lambda^2,$$

while

$$Z(S^3 \times B^1) = \langle e'_\emptyset, e'_\emptyset \rangle = D^{-1},$$

so the second type of surgery does not affect $Z(W)$ if $\lambda^2 = 1/D$.

Finally, we consider $B^3 \times S^1 \leftrightarrow S^2 \times B^2$, with boundary $S^2 \times S^1$. With a eye toward future applications, we will temporarily abandon our assumption that $\dim A(S^2 \times S^1) = 1$. Let $f_a \in A(S^2 \times S^1)$ be as defined above. Note that f_1 is equivalent to the empty field. We have

$$Z(B^3 \times S^1)(f_a) = \dim A(B^3; a) = \delta_{1a}.$$

On the other hand, using the gluing formula for $S^2 \times B^2 = (B^2 \times B^2) \cup (B^2 \times B^2)$, we have

$$\begin{aligned} Z(S^2 \times B^2)(f_a) &= \sum_{b \in \mathcal{L}} \lambda H_{ab} \cdot \lambda d_b \\ &= \lambda^2 \sum_{b \in \mathcal{L}} (d_a d_b) \cdot d_b \\ &= d_a \lambda^2 D. \end{aligned}$$

(By (9.1.9) if $f_a \neq 0$ then $H_{ab} = d_a d_b$.) So $Z(W)$ is a bordism invariant if and only if $\dim A(S^2 \times S^1) = 1$ and $\lambda^2 = 1/D$.

[need to say more about the not-bordism-invariant case.]

Consider a 4-manifold W consisting of a single 0-handle and some 2-handles. Identify S^3 with the boundary of the 0-handle. The attaching curves of the 2-handles form a framed link L in S^3. Let $\mathcal{L}(L)$ denote the set of all labelings of L by irreps of $\mathcal{L}$. For $m \in \mathcal{L}(L)$ let $J(L, m)$ denote the standard evaluation of L labeled by m. *[remark that this is an evaluation of the generalized Jones polynomial of (L, m)]* Let $d(m)$ denote the product $d_{m_1} \cdots d_{m_k}$, where $m = (m_1, \ldots, m_k)$. It follows from the gluing formula and (9.1.6) that

$$Z(W)(\emptyset) = \sum_{m \in \mathcal{L}(L)} (\lambda J(L, m))(\lambda^k d(m)).$$

[remark that this is the well known Witten-Resheitkin-Turaev surgery formula] Note that we do not need to show that this expression is invariant under handle slides; this follows from (6.3.1). More generally, let K be a labeled ribbon graph (e.g. a framed link) in ∂W. By general position, we may assume that K lies in the complement of L in S^3. Let $J(L \cup K, m)$ denote the standard evaluation of $L \cup K$, with L labeled by m. Then

$$Z(W)(K) = \sum_{m \in \mathcal{L}(L)} (\lambda J(L \cup K, m))(\lambda^k d(m)).$$

9.1.11 Next consider a 4-manifold W equipped with a generic cell decomposition (i.e. a cell decomposition dual to a triangulation). This means that a neighborhood of each

0-cell looks like an open cone on the boundary of a 4-simplex, and a neighborhood of a point in a k-cell looks like a neighborhood of a point of a k-cell in this cone. (In other words, the open cone on the boundary of a 4-simplex is the local model for the cell decomposition.) We will use the gluing formula to derive a state sum description of $Z(W)$ in terms of labelings of the cells. *[refer to 2+1 dim'l case? need to say that this is very similar to 2+1/TV case, and so we'll use fewer words]*

For simplicity, assume that W is closed.

[say something about the fact that there is a closely related handle decomposition and we will treat the two as equivalent]

Let n_i be the number of i-handles, $\mathcal{H}_i$ be the set of i-handles, and let W_i denote the union of all handles of index less than or equal to i.

Applying the gluing formula (6.1.9) and the inner product calculation (9.1.8) to the decomposition $W = W_3 \cup \{4\text{-handles}\}$, we have

$$\begin{aligned} Z(W) &= Z(W_3)(\emptyset) \prod_{\mathcal{H}_4} z(e'_\emptyset)\langle e'_\emptyset, e'_\emptyset\rangle^{-1} \\ &= Z(W_3)(\emptyset)\lambda^{n_4} D^{n_4} \end{aligned}$$

Applying the gluing formula and the inner product calculation (9.1.7) to the decomposition $W_3 = W_2 \cup \{3\text{-handles}\}$, we have

$$\begin{aligned} Z(W_3)(\emptyset) &= Z(W_2)(\emptyset) \prod_{\mathcal{H}_3} z(\emptyset)\langle e_\emptyset, e_\emptyset\rangle^{-1} \\ &= Z(W_3)(\emptyset)\lambda^{-n_3} D^{-n_3} \end{aligned}$$

Let $\mathcal{L}_2$ denote the set of labelings of the 2-handles by elements of $\mathcal{L}$. Applying the gluing formula and the inner product calculation (9.1.6) to the decomposition $W_2 = W_1 \cup \{2\text{-handles}\}$, we have

$$Z(W_2)(\emptyset) = \sum_{\alpha \in \mathcal{L}_2} Z(W_1)(L_\alpha) \prod_{f \in \mathcal{H}_2} \lambda d(\alpha, f),$$

where L_α denotes the ribbon link in ∂W_1 consisting of an (appropriately framed) ribbon in the core of the attaching target of each 2-handle, labeled according to α, and $d(\alpha, f) = d_{\alpha(f)}$, the standard evaluation of an 0-framed unknot labeled by $\alpha(f)$.

Next we consider the decomposition $W_1 = W_0 \cup \{1\text{-handles}\}$. We want to evaluate $Z(W_1)$ on L_α. The attaching target of a 1-handle is a pair of 3-balls. The field L_α restricted to the boundary of one of these 3-balls consists of four labeled framed points (corresponding to the vertices of a tetrahedral graph in ∂B^3). The two ends of the 1-handle have the same four labels. For each 1-handle e choose an orthogonal basis of $A(B^3; c(\alpha, e))$, where $c(\alpha, e)$ denotes the four labeled points. Let $\mathcal{L}_1$ denote the set of all labelings of the 1-handles by these basis vectors. We are now ready to apply (6.1.9) to the gluing of the 1-handles to the 0-handles. The summation will be over $\beta \in \mathcal{L}_1$. Let $E(\alpha, \beta, e) = \langle f, f\rangle$, where f is the basis vector assigned by $\beta \in \mathcal{L}_2$ to the 1-handle e, and the inner product is computed using the standard evaluation ($\lambda = 1$). For each 1-handle e we have a factor of $\lambda E(\alpha, \beta, e)$ coming from the 1-handle and $(\lambda E(\alpha, \beta, e))^{-2}$ coming from the gluing correction factor, so the total contribution of a 1-handle is $(\lambda E(\alpha, \beta, e))^{-1}$. Each 0-handle v contributes $\lambda\Phi(\alpha, \beta, v)$, where $\Phi(\alpha, \beta, v)$ denotes the standard evaluation of the 1-skeleton of the boundary of a 4-simplex. The labels of the edges of this graph come from α, and the labels of the (4-valent) vertices come from β. Combining all of the above and applying the gluing formula we have

$$Z(W_1)(L_\alpha) = \sum_{\beta \in \mathcal{L}_2} \prod_{e \in \mathcal{H}_1} \lambda^{-1} E(\alpha, \beta, e)^{-1} \prod_{v \in \mathcal{H}_0} \lambda \Phi(\alpha, \beta, v)$$

If we choose for each 1-handle e a trivalent ribbon tree with boundary $c(\alpha, e)$ (equivalently, choose a partition of the four labeled points of $c(\alpha, e)$ into two groups of two), then labelings of the internal edge and two vertices of this tree give an orthogonal basis of $A(B^3; c(\alpha, e))$ (see (9.1.5)). The above expression becomes

$$Z(W_1)(L_\alpha) = \sum_{\beta \in \mathcal{L}_2} \prod_{e \in \mathcal{H}_1} \lambda^{-1} \Theta_1(\alpha, \beta, e)^{-1} \Theta_2(\alpha, \beta, e)^{-1} d(\beta, e) \prod_{v \in \mathcal{H}_0} \lambda \Phi(\alpha, \beta, v).$$

Here Θ_1 and Θ_2 are the θ factors for the two trivalent vertices corresponding to e, with labels coming from α and β. $d(\beta, e)$ is the loop value for the internal edge of the tree for e. $\Phi(\alpha, \beta, v)$ can now be interpreted as a "15j" symbol: it is the standard evaluation of a trivalent ribbon graph with 15 edges and 10 trivalent vertices.

Combining the expressions for all handles we have

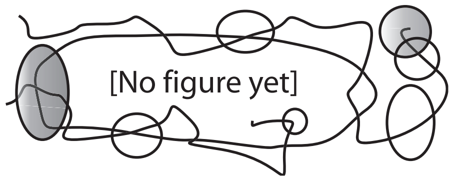
$$\begin{aligned} Z(W) = &\lambda^{\chi(W)} D^{n_4 - n_3} \sum_{\alpha \in \mathcal{L}_2} \sum_{\beta \in \mathcal{L}_2} \\ &\prod_{f \in \mathcal{H}_2} d(\alpha, f) \prod_{e \in \mathcal{H}_1} \Theta_1(\alpha, \beta, e)^{-1} \Theta_2(\alpha, \beta, e)^{-1} d(\beta, e) \prod_{v \in \mathcal{H}_0} \Phi(\alpha, \beta, v). \end{aligned}$$

(Here $\chi(W) = n_4 - n_3 + n_2 - n_1 + n_0$ is the Euler characteristic of W.) If $\lambda = 1$ this coincides with Crane-Yetter state sum [CY93], after taking into account different conventions for normalizing trivalent vertices. *[more specifically, the Crane-Yetter 15j symbol contributes a factor of $d(\beta, e)^2$ for each 1-handle e]*

Note that we do not need to show that this expression is independent of the choice of cell decomposition; this follow from (6.3.1).

[remark: the above can be adapted to 4-manifolds with boundary]

9.1.12 Next we consider a different class of cell decompositions of a 4-manifold and show that it leads to the Turaev shadow state sum [Tur94]. Consider a cell decomposition of W^4 where (a) the 2-skeleton near a 0-cell looks like a cone on the 1-skeleton of a 3-simplex, (b) each 1-cell is incident to three 2-cells, and (c) the "2-cells" are allowed to be arbitrary oriented surfaces with a single boundary component. (Such



9.1.13 Cutting a fat graph

a decomposition is called a *shadow* of W. Turaev allows slightly more generality (e.g. nonoriented 2-cells), but we will be content with the above.)

Before applying the gluing formula we need to compute the path integral of a generalized 2-cell. Let Y be an oriented surface with $\partial Y \cong S^1$. We want to compute $Z(Y \times D^2; e_a, \emptyset)$, where e_a resides on $(\partial Y) \times D^2$ (see (9.1.6)) and $\emptyset$ resides on $Y \times \partial D^2$. Y can be realized as thickening of a 1-complex G; see Figure (9.1.13). Making a cut in the middle of each 1-cell of G and applying the gluing formula we see that

$$\begin{aligned} Z(Y \times D^2; e_a, \emptyset) &= \prod_v \lambda d_a \prod_e (\lambda d_a)^{-1} \\ &= (\lambda d_a)^{\chi(Y)}, \end{aligned}$$

where v runs over 0-cells of G and e runs over 1-cells of G. If the disk bundle over Y is twisted with respect to the boundary trivialization determined by embedding of e_a, or equivalently the embedding of e_a is changed by a twist, this becomes

$$Z(Y \times D^2; e_a, \emptyset) = t_a^k (\lambda d_a)^{\chi(Y)},$$

where t_a is the twist factor for a and k is the number of twists (Euler number of the disk bundle). *[need to say this better; also refer back to def of t_a]*

Now we proceed as before. Gluing 3- and 4-handles presents nothing new:

$$Z(W) = (\lambda D)^{n_4 - n_3} Z(W_2)(\emptyset).$$

Gluing 2-handles to W_1 gives

$$Z(W_2)(\emptyset) = \sum_{\alpha \in \mathcal{L}_2} Z(W_1)(L_\alpha) \prod_{f \in \mathcal{H}_2} t(\alpha, f)^{k_f} (\lambda d(\alpha, f))^{\chi(f)},$$

where $t(\alpha, f)$ is the twist factor for the label that α assigns to f, and k_f is the euler number for f. *[comment on Turaev "gleams"]* Gluing 1-handles to W_0 gives

$$Z(W_1)(L_\alpha) = \sum_{\beta \in \mathcal{L}_2} \prod_{e \in \mathcal{H}_1} \lambda^{-1} \Theta(\alpha, \beta, e)^{-1} \prod_{v \in \mathcal{H}_0} \lambda \operatorname{Tet}(\alpha, \beta, v).$$

Here $\mathcal{L}_2$ is the set of all labelings of the 1-handles by orthogonal basis vectors associated to the three labels coming from α and the three 2-cells adjacent to the

1-handle. $\Theta(\alpha, \beta, e)$ is the standard evaluation of the theta graph corresponding to e, with edge labels coming from α and vertex labels coming from β. $\mathrm{Tet}(\alpha, \beta, v)$ is the standard evaluation of the tetrahedron associated to v with edge labels coming from α and vertex labels coming from β. Combining all this we have

$$\begin{aligned} Z(W) =& \lambda^{\chi(W)} D^{n_4 - n_3} \sum_{\alpha \in \mathcal{L}_2} \sum_{\beta \in \mathcal{L}_2} \\ & \prod_{f \in \mathcal{H}_2} t(\alpha, f)^{k_f} d(\alpha, f)^{\chi(f)} \prod_{e \in \mathcal{H}_1} \Theta(\alpha, \beta, e)^{-1} \prod_{v \in \mathcal{H}_0} \mathrm{Tet}(\alpha, \beta, v). \end{aligned}$$

If we set $\lambda^2 = 1/D$ this is essentially the Turaev shadow state sum for the cell decomposition of W. *[need to check normalization]* Note that we do not need to show that this expression is independent of the choice of cell decomposition; this follow from (6.3.1).

[need to remark that this can be generalized to manifolds with boundary]

[...]

9.2 Decategorification

Plan:

- choose λ to make Z a bordism invariant
- give details for case where S is singular

9.3 More on Chern-Simons Theories

Plan:

- use multiplication str to take square root
- cf K(G, 1) is H-space iff G is cyclic (U(1) theories)
- comment on lack of higher codim stuff [move this elsewhere]

9.4 Contact Structures

Plan:

- Tight contact structure is a local relation, so we can use TQFT machinery to study the set of tight contact structures on a 3-manifold
- Need to go into details of gluing smooth 3-men w/ corners
- Verify field axioms

- gluing
- note relation to Honda, Etnyre, etc.
- Eliashberg criterion for 3-ball
- algorithm for handlebody
- $A(S^1 \times I)$

Chapter 10

4+1-dimensional Examples

10.1 Theories From Khovanov Homology

Outline (to be filled in later):

- The main fact about Khovanov homology that we will use is that surface bordisms in $S^3 \times I$ give maps on Khovanov homology. (See [MW06] and [need ref for Jacobsson].)
- Recall from [MW06] the notion of disoriented 1-manifolds and 2-manifolds...
- We can give Khovanov homology the structure of a 4-category as follows.
  - The unique 0-morphism is an undecorated point.
  - The unique 1-morphism is an undecorated interval.
  - A 2-morphism is a collection of framed, oriented points in the interior D^2.
  - A 3-morphism is a framed, disoriented tangle in B^3. (The domain and range are the restriction to the southern and northern hemispheres.)
  - Given two 3-morphisms with the same boundary, we can glue them together and get a disoriented link in S^3. A 4-morphism is an element of the Khovanov homology of such a link.
  - Composition of i-morphsims ($i \leq 3$) is given by gluing.
  - Composition of 4-morphisms is given as follows. Let A, B and C be tangles with common boundary. Let AB, BC and AC denote the links formed by gluing the tangles. There is an obvious bordism W from $AB \sqcup BC$ to AC. (The underlying 4-manifold of W is B^4 with two smaller balls removed from its interior. The surface in W consists of $A \times I$, $B \times I$ and $C \times I$ with some identifications near the common boundary of A, B and C.) Choose a point p in the 3-ball containing B, disjoint from

the tangle. Let α be the corresponding arc in W which joins the two inner boundary components and is disjoint from the surface. Let W' be W with a neighborhood of α removed. W' is a bordism in $S^3 \times I$ from $AB \cup BC$ (a link in S^3 consisting of distant copies of AB and BC) to AC. Thus W' gives a map from $\mathrm{Kh}(AB) \otimes \mathrm{Kh}(BC) \to \mathrm{Kh}(AC)$. This is the definition of composition of 4-morphisms. It is not hard to show that this is independent of the choice of p.

  - The various conjugations needed to complete the definition of the 4-category have obvious geometric definitions.
  - [Need to also say something about (s)pin structures. Does this entail a modification of the definition of 4-category?]

- We can now define a 4+1-dimensional TQFT (for spin or pin 4-manifolds) based on this 4-category. The fields on a 2-manifold are collections of framed, oriented points. The fields on a 3-manifold are disoriented framed tangles. The fields on 4-manifolds are "4-dimensional spaghetti and meatballs pictures". (Standard, 2-dimensional spaghetti and meatballs pictures are the meat and potatoes of the planar algebra literature. This excellent terminology is due to Vaughan Jones. Here we double the dimensions of both the spaghetti and the meatballs.) The 2-dimensional spaghetti consists of framed disoriented surfaces. Each 4-dimensional meatball is a 4-ball which meets the spaghetti in a framed disoriented link L. It is labeled by an element of $\mathrm{Kh}(L)$. The local relations on these pictures are isotopy plus coalescing a subregion of spaghetti and/or meatballs into a larger meatball. The latter relation uses the composition of 4-morphisms defined above.
- In particular, $A(B^4; L) \cong \mathrm{Kh}(L)$ is this theory.

Chapter 11

$n+1$-dimensional Examples

11.1 Finite Group Theories

Plan:

- do twisted case too
- also mention original D-W approach via triangulations
- ? also Quinn's finitie total homotopy generalization?

[also twisted case]

11.2 Finite Total Homotopy Theories

(including theories based on homology; or maybe put these in a separate section?)

11.3 String Category

(Include this here? not very n-dimensional. relation to graph invariants)

Chapter 12

Reconstruction Results

In this Chapter we show how to construct various parts of a full TQFT from various sorts of combinatorial data (link invariants with certain properties, Moore-Seiberg data, modular tensor category, ...)

Maybe the point of this chapter should be to make contact with prior TQFT literature.

Maybe get rid of this chapter and instead put the various reconstruction results at the ends of the 1+1, 2+1, 3+1 chapters (?)

Plan:

- dehn surgery formula
- moore-seiberg equations
- TV model
- also dim 2 TV model (?)
- hopf algebra with properties blah blah blah

Chapter 13

[Other Chapters]

Some other chapters (or subchapters):

- summarize properties (axioms)
- (similarly) a summary chapter before the examples chapters
- local rels from Hopf algebras / quantum groups (and vice-versa?)
- lower bounds (both rank and norm stuff)
- various flavors of extended manifolds (p1, lagrangians, 2-framings)
- ?? tables of small theories
- ?? CS-U(1) theories
- ? general (disklike) n-cats
- (?) mu invariant as a TQFT
- need to put discussion of Frobenius-Schur stuff somewhere (this is a placeholder) 13.0.1
- ? more category-theoretic chapter

Appendix A

Categories

A.1 Definitions and Notation

A *category* C consists of the following data:

- Objects C^0.
- For all $a, b \in C^0$, a set of morphisms C^1_{ab} (also denoted $\mathrm{mor}(a, b)$). The collection of all morphisms is denoted $C^1 = \bigcup_{a,b} C^1_{ab}$.
- For all $a, b, c \in C^0$, a composition function $C^1_{ab} \times C^1_{bc} \to C^1_{ac}$. This is required to be associative. *[should I bother defining associativity?]*
- For all $a \in C^0$ an identity morphphsism $1_a \in C^1_{aa}$ (also denoted id_a). For all $a, b \in C^0$ and $f \in C^1_{ab}$ we require that $1_a f = f = f 1_b$.

Let k be a ring. A *k-category* is a category where each morphism set C^1_{ab} is a finitely generated k-module and the composition functions are bilinear in k. A.1.1

A.2 [Still to do]

(placeholder for definition of disk-like 2-category) A.2.1

(placeholder for discussion of "horizontal" composition of 2-morphisms (or maybe just natural transformations) A.2.2

[introduce (in appropriate places) convention that juxtaposition composes one way and $\circ$ and parens mean the other (more usual) way (arrows vs functions)]

- also 2-cats? or put them in a separate chapter?
- duality/conjugate stuff
- ends and coends

- relations to tensor over action, homomorphisms, invariants, coinvariants, ...
- representations = functors (2-cat reps also? if so need frob duality); "we'll use 'rep' instead of 'functor' because..."; also "action"; also "module" (maybe module should be the main term?)

Appendix B

Semisimple Categories

[need to complete transition from v-space to modules (seach for all 'vector', 'space', 'k')]

[use of "k-category" is inconsistent with the rest of the book (n-category)]

[need to switch from left reps to right reps (in order to be consistent with he rest of the book)]

In this appendix we prove structure theorems (see (B.4.5) and (B.4.6)) for semisimple k-categories. We then use this structure theorem to prove various things about these categories. When k is a field, some authors call k-categories "algebroids" (in analogy to groups and groupoids), and indeed most facts about semisimple algebras (or more generally semisimple rings) generalize easily to categories. The proofs given here follow Chapter XVII of [Lang xxxx] closely; in many cases they have been adapted almost verbatim. In what follows, the reader should keep in mind the special case of algebras, thought of as categories with only one object.

B.1 Definitions

In this appendix category will always mean k-category (see (A.1.1)), where k is a ring. An important special case is when k is a field. As in the rest of this book, we will write the composition of $f \in C^1_{ab}$ and $g \in C^1_{bc}$ as fg, not gf (think sticking arrows together, rather than composing functions).

For the remainder of this section let C be a k-category.

A *left ideal* $L \subset C$ is a subset of C^1 which is closed under left composition by morphsisms of C, and such that $L \cap C^1_{ab}$ is a submodule of C^1_{ab} for all $a, b \in C^0$. In other words, for all $f \in C^1$ and $g \in L$, $fg \in L$ whenever the composition is defined. Right ideals of C are defined similarly. Two-sided ideals are both left and right ideals.

A (left) *representation* of C (or *C-representation* or *C-module*) is a functor E from C to the category of k-modules. For every $a \in C^0$ we have a k-module E_a, and for every $(f : a \to b) \in C^1$ we have a linear map $E(f) : E_b \to E_a$. The map $E : C^1_{ab} \to \hom(E_b, E_a)$ is required to be k-linear. Composition of morphisms is preserved. We also say that C *acts* (via E) on the collection of k-modules $\{E_a\}_{a \in C^0}$. We sometimes drop E from the notation for morphisms and write the action as juxtaposition, e.g. for each $(f : a \to b) \in C^1$ we have a linear map $f : E_b \to E_a$.

Note that a left ideal of C is a C-representation.

A *subrepresentation* of E is a collection of submodules $F_a \subset E_a$, for all $a \in C^0$, which is preserved under the action of C (i.e. $fF_a \subset F_b$ for all f as above). We also denote this as $F \subset E$.

Given $F \subset E$ we can form the *quotient representation* E/F with $(E/F)_a = E_a/F_a$.

If E and E' are representations of C, the *direct sum* $E \oplus E'$ is defined by $(E \oplus E')_a = E_a \oplus E'_a$ and the obvious actions of C. If F and F' are subrepresentations of E, then the *sum* $F + F' \subset E$ is defined by $(F + F')_a = F_a + F'_a$ and the obvious actions of C. Note that $F + F'$ is direct if and only if $F_a \cap F'_a = 0$ for all $a \in C^0$.

A *free* C-representation is one of the form $\bigoplus_i C^1_{*a_i}$, where $a_i \in C^0$.

A *simple* representation is one which contains no nontrivial subrepresentations.

A *natural transformation* (or *intertwiner*) h between representations E and E' of C consists of linear maps $h_a : E_a \to E'_a$ (for all $a \in C^0$) such that for all $(f : a \to b) \in C^1$ $E(f)h_b = h_a E'(f)$.

For $a \in C^0$, let Ca denote the representation with $(Ca)_b = C^1_{ba}$ and the obvious action of C. For any representation E and $v \in E_a$, let Cv denote the subrepresentation of E generated by v, $(Cv)_b = C^1_{ba} v$. There is an obvious natural transformation $Ca \to Cv$.

B.2 Conditions Defining Semisimplicity

A representation is called *semisimple* if it satisfies the conditions in the following proposition.

B.2.1 **Proposition.** *The following three conditions on a representation E of C are equivalent:*

1. *E is a sum of simple representations.*
2. *E is a direct sum of simple representations.*
3. *Every subrepresentation $F \subset E$ is a direct summand: there exists $F' \subset E$ such that $E = F \oplus F'$.*

Proof. **1 ⇒ 2.** Let $E = \sum_{i \in I} F^i$, with each $F^i \subset E$ simple. (Recall that "$\sum$" here means finite sums.) Choose a maximal $J \subset I$ such that $\sum_{j \in J} F^j$ is direct (i.e. $\{F_a^j\}$ are independent subspaces of E_a for all $a \in C^0$). Let $E' = \bigoplus_{j\in J} F^j$. I claim that $E' = E$. It suffices to show that for all $i \in I$, $F^i \subset E'$. Consider the subrepresentation $(F^i \cap E') \subset F^i$. $F^i \cap E' = 0$ would contradict the maximality of J. Therefore, by the simplicity of F^i, $F^i \cap E' = F^i$ and $F^i \subset E'$.

2 ⇒ 3. Let $E = \bigoplus_{i\in I} F^i$ with each F^i simple and let $G \subset E$. Choose a maximal $J \subset I$ such that $G + (\sum_{j\in J} F^j)$ is direct (i.e. $G_a \cap (\sum_{j\in J} F_a^j) = 0$ for all $a \in C^0$). Let $E' = G + (\sum_{j\in J} F^j)$. Then for all $i \in I$, $F^i \cap E' = F^i$ (because $F^i \cap E' = 0$ would contradict the maximality of J), so $E' = E$.

3 ⇒ 1. First we show that any subrepresentation G of E contains a simple subrepresentation. Choose a non-zero $v \in G_a$, $a \in C^0$. Then Cv is a subrepresentation of G and $\ker(Ca \to Cv)$ is a left ideal $L \neq C$. Consider the partially ordered set of left ideals strictly smaller than Ca and containing L. By Zorn's Lemma there is a maximal ideal M in this set. It follows that Mv is a maximal subrepresentation of Cv. By condition 3, $E = Mv \oplus M'$ for some M'. Then $(Cv \cap M') \subset G$ is simple by the maximality of Mv. This is the desired simple subrepresentation of G.

Let E' be the sum of all the simple subrepresentations of E. If $E' \neq E$ then $E = E' \oplus F$ for some $F \neq 0$. But by the argument above F contains a simple subrepresentation, which is a contradiction. □

Proposition. *Every subrepresentation and every quotient representation of a semisimple representation E is semisimple.* B.2.2

Proof. Let $F \subset E$. For all $G \subset F$, $E = G \oplus G'$ for some G' and $F = G \oplus (G' \cap F)$. Hence F is semisimple.

Write $E = F \oplus F'$. Then F' is semisimple and the canonical natural transformation $E \to E/F$ induces an isomorphism $F' \to E/F$. Hence E/F is semisimple. □

B.3 The Density Theorem

Let E be a representation of C. Let $R = \mathrm{End}_C(E)$, the ring of natural transformations from E to itself. For each $a \in C^0$, E_a is an R-module. For each $f \in C^1_{ab}$, $f : E_b \to E_a$ is an R-map (i.e. f commutes with the R-actions). We can ask how much of $\mathrm{Hom}_R(E_a, E_b)$ is realized by morphisms of C.

Lemma. *Let C, E and R be as above and assume E is semisimple. Choose $a, b \in C^0$ and an R-map $\alpha : E_b \to E_a$. Choose $v \in E_a$. Then there exists $f \in C^1_{ab}$ such that $\alpha v = fv$.* B.3.1

Proof. Let $E = Cv \oplus E'$ and let $\pi : E \to Cv$ be the projection. Then $\pi \in R = \mathrm{End}_C(E)$, so $\alpha v = \alpha\pi_b v = \pi_a \alpha v$. Hence $\alpha v \in Cv$, which implies that there is an $f \in C^1_{ab}$ such that $\alpha v = fv$. □

Next we use a diagonal trick to enhance the above result to work for a finite number of $v_i \in E_a$.

B.3.2 **Theorem** (Density Theorem). *Let C, E and $R = \mathrm{End}_C(E)$ be as above and assume E is semisimple. Choose $a, b \in C^0$ and an R-map $\alpha : E_b \to E_a$. Choose $v_1, \ldots, v_m \in E_b$. Then there exists $f \in C^1_{ab}$ such that $\alpha v_i = f v_i$ for all i.*

Proof. Let $E^{(m)}$ be the direct sum of m copies of E. Then $\mathrm{End}_C(E^{(m)})$ is isomorphic
B.3.3 to $m \times m$ matrices with entries in R. (Proof: Clearly $\mathrm{Mat}_m(R) \subset \mathrm{End}(E^{(m)})$. Let $h \in \mathrm{End}(E^{(m)})$. Then $h_a : E_a^{(m)} \to E_a^{(m)}$ decomposes as a matrix (h_a^{ij}), where each h_a^{ij} is a linear map from E_a to itself. It is easy to see that for fixed i and j and varying a the collection $\{h_a^{ij}\}$ is in $\mathrm{End}_C(E) = R$.) Note that $\alpha^{(m)} : E_b^{(m)} \to E_a^{(m)}$ is an $\mathrm{End}_C(E^{(m)})$-map, and (v_i) can be thought of as an element of $E_a^{(m)}$. It now follows from the previous lemma that there exists $f \in C^1_{ab}$ such that $\alpha v_i = f v_i$ for all i. □

Next consider a family of C-representations E^j, $j \in J$. Let R denote the category of natural transformations amongst the E^j. We have $R^0 = J$ and R^1_{jl} are the natural transformations from E^l to E^j. Then each collection of k-modules $E_a = \{E_a^j \mid j \in J\}$ is a representation of R, and the actions of $f \in C^1_{ab}$ constitute a natural transformation from E_b to E_a. In other words, $\{E_a^j\}$ is a two dimensional array of k-modules, with C acting horizontally (say) and R acting vertically, and the C and R actions commute. As before, we can ask whether a general natural transformation from E_b to E_a is realized by a morphism of C.

B.3.4 **Theorem** (Generalized Density Theorem). *Let C, $\{E^j\}$ and R be as above and assume each E^j is semisimple. Choose $a, b \in C^0$ and an R-map (natural transformation) $\alpha : E_b \to E_a$. Choose a finite set $\{v_i^j\}$, where $v_i^j \in E_b^j$. Then there exists $f \in C^1_{ab}$ such that $\alpha_j v_i^j = f v_i^j$ for all i and j.*

Proof. Let $\widetilde{E} = \bigoplus_{i,j} E^j$, where i, j runs through the index set for $\{v_i^j\}$. (In general there are multiple copies of each E^j in the direct sum.) $\widetilde{E}$ is a C-representation, and $\mathrm{End}_C(\widetilde{E})$ consists of matrices whose $(i, j; i', j')$ entry is a natural transformation from E^j to $E^{j'}$. (This is similar to (B.3.3).) Note that α determines an $\mathrm{End}_C(\widetilde{E})$-map from $\widetilde{E}_b$ to $\widetilde{E}_a$, and (v_i^j) can be thought of as an element of $\widetilde{E}_a$. It now follows from (B.3.1) that there exists $f \in C^1_{ab}$ such that $\alpha_j v_i^j = f v_i^j$ for all i and j. □

B.3.5 **Corollary.** *Let C be a category and $\{E^i\}$ and collection of C-representations such that (a) for all non-zero $f \in C^1$ there is some E^i where f acts non-trivially, and (b) for each $a \in C^0$ there are only finitely many i such that $E_a^i \neq 0$. Let $R = \mathrm{End}_C(E^*)$, the category of natural transformations amongst $\{E^i\}$. Then there is a natural isomorphism $C \cong \mathrm{End}_R(E_*)$.*

Proof. Consider the natural map $C \to \mathrm{End}_R(E_*)$. By assumption this map is injective. By (B.3.4) it is surjective: Fix $b \in C^0$ and choose $\{v_i^j\}$ of (B.3.4) to be a collection of generators of the finitely many, finitely generated, non-zero E_b^j. □

B.4 Semisimple Categories

A category C is called *semisimple* if C^1_{*a} is a semisimple C-representation for all $a \in C^0$. In this section we prove structure theorems ((B.4.5) and (B.4.6)) for semisimple categories.

Proposition. *If C is semisimple, then every C-representation is semisimple.*

Proof. This follows from (B.2.2) and the observation that every C-representation is a quotient of a free C-representation. □

A left ideal of C is a C-representation, and is called simple if it is simple as a C-representation. Two ideals L and L' of C are called isomorphic if they are isomorphic as C-representations.

We will decompose C as a sum of its simple left ideals and thereby get a structure theorem for C.

Lemma. *Let L be a simple left ideal of C and let E be a simple representation of C. If L is not isomorphic to E, then $LE = 0$.* B.4.1

Proof. Since $CLE = LE$, LE is a subrepresentation of E. Since E is simple, $LE = E$ or $LE = 0$. If $LE = E$, choose $y \in E_a$ such that $Ly \neq 0$. Again by the simplicity of E, $Ly = E$. The map $h : l \mapsto ly$ is a C-map from L onto E. Since L is simple, $\ker(h) = 0$ and h is an isomorphism, contradicting our hypothesis. □

Let $\{L_i\}_{i \in I}$ be a collection of pair-wise nonisomorphic simple left ideals of C such that every simple left ideal of C is isomorphic to some L_i. For $i \in I$, define C_i to be the sum of all simple left ideals isomorphic to L_i. By (B.4.1), $C_i C_j = 0$ if $i \neq j$. Each C_i is a left ideal, and by (B.2.1)

$$C = \sum_{i \in I} C_i$$ B.4.2

Hence for all $i \in I$

$$C_i \subset C_i C = C_i C_i \subset C_i$$

(the first inclusion because C contains identity morphisms for all objects), so C_i is a two-sided ideal.

Choose $a \in C^0$ and write

$$1_a = \sum_{j \in I_a} e_{aj},$$

where I_a is a finite subset of I and $0 \neq e_{aj} \in C_j$. (This is possible by (B.4.2); recall the "$\sum$" means finite sums.) Note that e_{aj} is the identity morphism of C_j at a.

For any $x \in C^1_{ab}$,

$$x = 1_a x 1_b = \left(\sum_{j \in I_a} e_{aj} \right) x \left(\sum_{k \in I_b} e_{bk} \right) = \sum_{i \in I_a \cap I_b} e_{ai} x e_{bi}.$$

Furthermore, if $x = \sum_i x_i$, with $x_i \in C_i$, then necessarily $x_i = e_{ai} x e_{bi}$. It follows that C is a direct sum

$$C = \bigoplus_{i \in I} C_i.$$

A category is called *simple* if it has only one isomorphism class of non-zero simple left ideal. Clearly C_i above i simple for all i.

We have proved the following structure theorem for semisimple categories.

B.4.3 **Theorem.** *Let C be a semisimple category. Let $\{L_i\}_{i\in I}$ be a complete set of representatives for the simple left ideals of C, and let C_i be the sum of all left ideals isomorphic to L_i. Then $C = \bigoplus_{i\in I} C_i$ and each C_i is a simple category. For each $a, b \in C^0$, C^1_{iab} is non-zero for only fintely many i.* □

[C^1_{iab} needs to be defined? or is the meaning of this notation clear?]

It follows from (B.4.3) that the hypotheses of (B.3.5) apply to the collection of C-representations $\{L_i\}$. In order to better understand the natural transformations amongst the L_i we prove

B.4.4 **Lemma** (Schur Lemma)**.** *Let E, E' be simple representations of a category C and $h : E \to E'$ be a natural transformation. Then h is an isomorphism or zero. Assume now that k (the base ring for C) is an algebraically closed field. If h is an isomorphism and $h' : E \to E'$ is another natural transformation, then $h' = \lambda h$ for some $\lambda \in k$.*

Proof. Both $\ker(h)$ and $\mathrm{im}(h)$ are subrepresentations, so by the simplicity of E and E' h is either an isomorphism or zero.

Assume now that k (the base ring for C) is an algebraically closed field. Suppose h is an isomorphism and h' is another natural transformation. Choose $a \in C^0$ and let $\lambda \in k$ be an eigenvalue of $h_a^{-1} h'_a$. (Eigenvalues exist because k is algebraically closed.) Then $h' - \lambda h$ is a natural transformation with non-zero kernel. By the simplicity of E, the kernel is all of E and therefore $h' = \lambda h$. □

It follows that the only natural transformations amongst the L_i are automorphisms. Let R_i be the division ring of automorphisms of L_i. For each $a \in C^0$, L_{ia} is a left R_i-module. (B.3.5) now implies

B.4.5 **Theorem.** *Let C be a semisimple category. Let $\{L_i\}_{i\in I}$ be a complete set of representatives for the simple left ideals of C. To each object $a \in C^0$ we associate the collection of R_i-modules $\{L_{ia}\}_{i\in I}$. Then C is naturally isomorphic to the category with objects $\{L_{ia}\}$ and morphisms consisting of all (graded) R_i-maps from $\{L_{ib}\}$ to $\{L_{ia}\}$.* □

If k is an algebraically closed field, then by (B.4.4) $R_i = k$ for all i. A *complete category of graded vector spaces* is defined to be one where each object is a graded finite dimensional vector space and C^1_{ab} consists of all graded linear maps from a to b. We have proved

Theorem. *Let C be a semisimple category over an algebraically closed field k. Let $\{L_i\}_{i\in I}$ be a complete set of representatives for the simple left ideals of C. Then C is naturally isomorphic to the complete category of graded vector spaces with each object $a \in C^0$ corresponding to the graded vector space $\{L_{ia}\}_{i\in I}$.* □ B.4.6

Next we discuss representations...

B.5 To Do List

To do:

[placeholder] ends and coends of irreps B.5.1
[placeholder] definition of minimal idempotent for irrep B.5.2
[placeholder] positive definite inner product implies semisimplicity B.5.3

- compatible inner products
- representations
- idempotents
- ends and coends of irreps
- also tensor stuff?? Bratelli diagrams etc.

Bibliography

[NOTE] *[Note: Bibliography is not complete. At the moment I'm just testing different styles for entries.]*

[TO DO] *[To do: need to add Segal refs, Freed, FQ, Quinn, Barrett, Sawin, Frohman, JKB, Bullock, more Turaev, BHMV, Przytycki, Kirby-Melvin, Roberts, ...]*

[BFK03] D. Bullock, C. Frohman and J. Kania-Bartoszynska, The Yang-Mills measure in the Kauffman bracket skein module, *Comment. Math. Helv.* **78** (2003), 1–17.

[CY93]

[KL94] L. H. Kauffman and S. L. Lins., *Temperley-Lieb Recoupling Theory and Invariants of 3-Manifolds*, Princeton University Press, Princeton, NJ, 1994.

[Kas95] C. Kassel, *Quantum Groups*, Springer-Verlag, New York, 1995.

[Mac98] S. Mac Lane, *Categories for the Working Mathematician*, second edition, Springer, New York, 1998.

[MW06] S. Morrison and K. Walker, Fixing the functoriality of Khovanov homology, in preparation.

[Tur94] V. Turaev, *Quantum Invariants of Knots and 3-manifolds*, Walter de Gruyter, Berlin, 1994.

[TV92] V. Turaev and O. Viro, State sum invariants of 3-manifolds and quantum 6-j symbols, *Topology* **31** (1992), 862–902.

[Walk91] K. Walker, On Witten's 3-manifold invariants, preprint 1991.

[Wall69] C. T. C. Wall, Non-additivity of the signature, *Invent. Math.* **7** (1969), 269–274.

[Wit89a] E. Witten, Quantum field theory and the Jones polynomial, *Comm. Math. Phys.* **121** (1989), 351–399.

[Wit89b] E. Witten, Gauge theories and integrable lattice models, *Nucl. Phys. B* **322** (1989), 629–697.