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INTERNAL GROUPOIDS AND EXPONENTIABILITY

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Résumé. Nous étudions les objets et les morphismes exponentiables dans la 2-catégorie $\mathbf{Gpd}(\mathcal{C})$ des groupoides internes à une catégorie $\mathcal{C}$ avec sommes finies lorsque $\mathcal{C}$ est : (1) finiment complète, (2) cartésienne fermée et (3) localement cartésienne fermée. Parmi les exemples auxquels on s'intéresse on trouve, en particulier, (1) les espaces topologiques, (2) les espaces compactement engendrés, (3) les ensembles, respectivement. Nous considérons aussi les morphismes pseudo-exponentiables dans les catégories "pseudo-slice" $\mathbf{Gpd}(\mathcal{C})//B$. Comme ces dernières sont les catégories de Kleisli d'une monade T sur la catégorie "slice" stricte sur B, nous pouvons appliquer un théorème général de Niefield [17] qui dit que si TY est exponentiable dans une 2-catégorie $\mathcal{K}$, alors Y est pseudo-exponentiable dans la catégorie de Kleisli $\mathcal{K}_T$. Par conséquent, nous verrons que $\mathbf{Gpd}(\mathcal{C})//B$ est pseudo-cartésienne fermée, lorsque $\mathcal{C}$ est la catégorie des espaces compactement engendrés et chaque B_i est faiblement de Hausdorff, et $\mathbf{Gpd}(\mathcal{C})$ est localement pseudo-cartésienne fermée quand $\mathcal{C}$ est la catégorie des ensembles ou une catégorie localement cartésienne fermée quelconque.

Abstract. We study exponentiable objects and morphisms in the 2-category $\mathbf{Gpd}(\mathcal{C})$ of internal groupoids in a category $\mathcal{C}$ with finite coproducts when $\mathcal{C}$ is: (1) finitely complete, (2) cartesian closed, and (3) locally cartesian closed. The examples of interest include (1) topological spaces, (2) compactly generated spaces, and (3) sets, respectively. We also consider pseudo-exponentiable morphisms in the pseudo-slice categories $\mathbf{Gpd}(\mathcal{C})//B$. Since the latter is the Kleisli category of a monad T on the strict slice over B, we can apply a general theorem from Niefield [17] which states that if TY

is exponentiable in a 2-category $\mathcal{K}$, then Y is pseudo-exponentiable in the Kleisli category $\mathcal{K}_T$. Consequently, we will see that $\mathbf{Gpd}(\mathcal{C})//B$ is pseudo-cartesian closed, when $\mathcal{C}$ is the category of compactly generated spaces and each B_i is weak Hausdorff, and $\mathbf{Gpd}(\mathcal{C})$ is locally pseudo-cartesian closed when $\mathcal{C}$ is the category of sets or any locally cartesian closed category.
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1. Introduction

Suppose $\mathcal{C}$ is a category with finite limits. An object Y of $\mathcal{C}$ is *exponentiable* if the functor $- \times Y : \mathcal{C} \to \mathcal{C}$ has a right adjoint, usually denoted by $(\ )^Y$, and $\mathcal{C}$ is called *cartesian closed* if every object is exponentiable. A morphism $q : Y \to B$ is *exponentiable* if q is exponentiable in the slice category $\mathcal{C}/B$, and $\mathcal{C}$ is called *locally cartesian closed* if every morphism is exponentiable. Note that if $q : Y \to B$ is exponentiable and $r : Z \to B$, we follow the abuse of notation and write the exponential as $r^q : Z^Y \to B$.

It is well known that the class of exponentiable morphisms is closed under composition and pullback along arbitrary morphisms. For proofs of these and other properties of exponentiability, we refer the reader to Niefield [16].

An *internal groupoid* G in $\mathcal{C}$ is a diagram of the form

$$G_2 \xrightarrow{c} G_1 \overset{i}{\circlearrowleft} \underset{t}{\overset{s}{\rightrightarrows}}\ (\xleftarrow{u})\ G_0$$

where $G_2 = G_1 \times_{G_0} G_1$, making G an internal category in $\mathcal{C}$ in which every morphism is invertible. Unless otherwise stated, the morphism in the pullback is $t : G_1 \to G_0$ when G_1 appears on the left in $G_1 \times_{G_0} G_1$ and s when it is on the right. When $\mathcal{C}$ is the category of topological spaces, we say G is a topological groupoid.

Let $\mathbf{Gpd}(\mathcal{C})$ denote the 2-category whose objects are groupoids in $\mathcal{C}$, morphisms $\sigma : G \to H$ are "internal homomorphisms," i.e., morphisms $\sigma_0 : G_0 \to H_0$ and $\sigma_1 : G_1 \to H_1$ of $\mathcal{C}$ compatible with the groupoid structure, and 2-cells $\sigma \Rightarrow \sigma' : G \to H$ are "internal natural transformations," i.e, morphisms $\alpha : G_0 \to H_1$ of $\mathcal{C}$ such that the following diagram is defined and commutes

$$\begin{array}{ccc} G_1 & \xrightarrow{\langle \alpha s, \sigma_1' \rangle} & H_2 \\ {\scriptstyle \langle \sigma_1, \alpha t \rangle}\downarrow & & \downarrow{\scriptstyle c} \\ H_2 & \xrightarrow[c]{} & H_1 \end{array} \tag{1}$$

Note that for an object of a 2-category $\mathbb{C}$ to be 2-exponentiable, one requires that the 2-functor $- \times Y \colon \mathbb{C} \longrightarrow \mathbb{C}$ has a right 2-adjoint, i.e., there is an isomorphism of categories $\mathbb{C}(X \times Y, Z) \cong \mathbb{C}(X, Z^Y)$ natural in X and Z. One can similarly define 2-exponentiable morphisms of $\mathbb{C}$.

It is well known that the 2-category $\mathbf{Cat}(\mathcal{C})$ of internal categories in $\mathcal{C}$ is cartesian closed whenever $\mathcal{C}$ is, and the construction of exponentials restricts to $\mathbf{Gpd}(\mathcal{C})$ (see Bastiani/Ehresmann [1], Johnstone [10]). Since the construction of the exponentials H^G depends only on the exponentiability of G_0, G_1, and G_2 in $\mathcal{C}$, we will see that G is exponentiable in $\mathbf{Gpd}(\mathcal{C})$ whenever G_0, G_1, and G_2 are exponentiable in $\mathcal{C}$, for any merely finitely complete category $\mathcal{C}$. However, $\mathbf{Cat}(\mathcal{C})$ and $\mathbf{Gpd}(\mathcal{C})$ are not locally cartesian closed even when $\mathcal{C}$ is. In fact, $q \colon Y \longrightarrow B$ is exponentiable in $\mathbf{Cat}$ if and only if it satisfies a factorization lifting property (FLP) known as the Conduché-Giraud condition (see Conduché [3], Giraud [7]). In the groupoid case, q satisfies FLP if and only if it is a fibration in the sense of Grothendieck [8].

In [11], Johnstone characterized pseudo-exponentiable morphisms in the pseudo-slice $\mathbf{Cat}/\!/B$, where the morphisms commute up to specified natural transformation, as those satisfying a certain pseudo-factorization lifting property, and Niefield [17] later obtained this result as a consequence of a general theorem about pseudo-exponentiable objects in the Kleisli bicategory of a pseudo-monad on a bicategory. In a related note, Palmgren [18] showed that every groupoid homomorphism is pseudo-exponentiable, so that $\mathbf{Gpd}/\!/B$ is locally pseudo-cartesian closed. Although Palmgren includes a complete proof, we will see that his result follows from the characterization in [17].

The goal of this paper is to generalize these results so that we can eventually apply them to categories of topological groupoids arising in the study of orbifolds. We begin, in Section 2, by recalling a general construction from Niefield [15] of cartesian closed coreflective subcategories of the category $\mathbf{Top}$ of all topological spaces (see also Bunge/Niefield [2]), which includes compactly generated spaces as a special case, and leads to cartesian closed coreflective subcategories of $\mathbf{Top}$. In the next two sections, we consider exponentiable objects of $\mathbf{Gpd}(\mathcal{C})$ and its slices when $\mathcal{C}$ is not locally cartesian closed, and apply this to $\mathbf{Top}$ and its subcategories. In this process we will need the internal version of the notion of fibration. This has been developed in full detail for arbitrary 2-categories in [20]. However, for internal groupoids, the descriptions given in the literature for $q\colon G \longrightarrow B$ to be an internal cloven fibration are equivalent to the existence of a right inverse for the arrow $\langle s, q_1\rangle\colon G_1 \longrightarrow G_0 \times_{B_0} B_1$. The reason this naive internalization of the Grothendieck condition works is the fact that in groupoids all arrows of the domain of a fibration are cartesian. We conclude, in Section 5, with the construction of a pseudo-monad on $\mathbf{Gpd}(\mathcal{C})/B$, in the case where $\mathcal{C}$ also has finite coproducts, and thus obtain pseudo-cartesian closed slices of $\mathbf{Gpd}(\mathcal{C})$ when $\mathcal{C}/B$ is cartesian closed. This includes the case where $\mathcal{C} = \mathbf{Sets}$, giving another proof of Palmgren's result, as well as certain slices of $\mathbf{Top}$ considered in Section 2.

2. Exponentiability in Categories of Spaces

In this section, we recall some general results about cartesian closed coreflective subcategories of $\mathbf{Top}$ and their slices. It is well known that the exponentiable topological spaces Y are those for which the collection $\mathcal{O}(Y)$ is a continuous lattice, in the sense of Scott [19]. This is equivalent to local compactness for Hausdorff (or more generally sober [9]) spaces Y. The sufficiency of this condition goes back to R.H. Fox [6] and the necessity appeared in Day/Kelly [5]. A characterization of exponentiable morphisms of $\mathbf{Top}$ was established by Niefield in [15] and published in [16], where it was shown that the inclusion of a subspace Y of B is exponentiable if and only if it is locally closed, i.e., of the form $U \cap F$, with U open and F closed in B.

There are several general expositions of cartesian closed coreflective subcategories of $\mathbf{Top}$. One, we recall here, follows from a general construction presented in [15] and later included in Bunge/Niefield [2].

Let $\mathcal{M}$ be a class of topological spaces. Given a space X, let $\hat{X}$ denote the set X with the topology generated by the collection

$$\{f\colon M \longrightarrow X \mid M \in \mathcal{M}\}$$

of continuous maps. We say X is $\mathcal{M}$*-generated* if $X = \hat{X}$, and let $\mathbf{Top}_{\mathcal{M}}$ denote the full subcategory of $\mathbf{Top}$ consisting of $\mathcal{M}$-generated spaces. Then one can show that $\mathbf{Top}_{\mathcal{M}}$ is a coreflective subcategory of $\mathbf{Top}$ with coreflection $\widehat{\ }\colon \mathbf{Top} \longrightarrow \mathbf{Top}_{\mathcal{M}}$.

In particular, $\mathbf{Top}_{\mathcal{K}}$ and $\mathbf{Top}_{\mathcal{E}}$ are the categories of compactly generated and exponentiably generated spaces, when $\mathcal{K}$ and $\mathcal{E}$ are the classes of compact Hausdorff spaces and all exponentiable spaces, respectively. Moreover, it is not difficult to show that if $\mathcal{M} \subseteq \mathcal{N} \subseteq \mathbf{Top}_{\mathcal{M}}$, then $\mathbf{Top}_{\mathcal{M}} = \mathbf{Top}_{\mathcal{N}}$. Thus, since every locally compact Hausdorff space is known to be compactly generated, adding all such spaces to $\mathcal{K}$ does not increase $\mathbf{Top}_{\mathcal{K}}$.

The following theorem is a special case of the one in [15] and later included in [2]. We include a proof here for completeness.

Theorem 2.1. *If $\mathcal{M}$ is a class of exponentiable objects of $\mathbf{Top}$ such that $M \times N \in \mathbf{Top}_{\mathcal{M}}$, for all $M, N \in \mathcal{M}$, then $\mathbf{Top}_{\mathcal{M}}$ is cartesian closed.*

Proof. The product in $\mathbf{Top}_{\mathcal{M}}$ is given by

$$X \hat{\times} Y = \varinjlim_{L \to X \times Y} L = \varinjlim_{\substack{M \to X \\ N \to Y}} M \times N = \varinjlim_{N \to Y} ((\varinjlim_{M \to X} M) \times N) = \varinjlim_{N \to Y} X \times N$$

where the second equality holds since each $M \times N \in \mathbf{Top}_{\mathcal{M}}$ and the third since $- \times N$ preserves colimits as N is exponentiable. Thus,

$$\begin{aligned} \mathbf{Top}_{\mathcal{M}}(X \hat{\times} Y, Z) &= \mathbf{Top}(\varinjlim_{N \to Y} X \times N, Z) = \varprojlim_{N \to Y} \mathbf{Top}(X \times N, Z) \\ &= \varprojlim_{N \to Y} \mathbf{Top}(X, Z^N) = \mathbf{Top}_{\mathcal{M}}(X, \widehat{\varprojlim_{N \to Y} Z^N}) \end{aligned}$$

□

Although $\mathbf{Top}_{\mathcal{M}}$ is generally not locally cartesian closed, there are many cases of cartesian closed slices. In fact, we know of no nontrivial case (i.e., $\mathbf{Top}_{\mathcal{M}} \neq \mathbf{Sets}$) for which $\mathbf{Top}_{\mathcal{M}}$ is locally cartesian closed. The following general proposition leads to examples of such slices.

Proposition 2.2. *If Y is exponentiable in $\mathcal{C}$ and B is any object for which the diagonal $\Delta\colon B \longrightarrow B \times B$ is exponentiable, then every morphism $q\colon Y \longrightarrow B$ is exponentiable.*

Proof. Since the horizontal morphisms in the pullbacks

$$\begin{array}{ccc} Y & \xrightarrow{\langle id,q\rangle} & Y \times B \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q\times id} \\ B & \xrightarrow[\Delta]{} & B \times B \end{array} \qquad\qquad \begin{array}{ccc} Y \times B & \xrightarrow{\pi_2} & B \\ {\scriptstyle \pi_1}\downarrow & & \downarrow \\ Y & \longrightarrow & 1 \end{array}$$

are exponentiable, factoring $q = \pi_2\langle id, q\rangle$, yields the desired result. $\square$

Corollary 2.3. *If the diagonal $B \longrightarrow B \hat{\times} B$ is exponentiable in $\mathbf{Top}_{\mathcal{M}}$, then $\mathbf{Top}_{\mathcal{M}}/B$ is cartesian closed.*

For examples of spaces satisfying the hypotheses of Corollary 2.3, we use:

Proposition 2.4. *If $\mathbf{Top}_{\mathcal{M}}$ is closed under locally closed subspaces of all M in $\mathcal{M}$, then inclusions of locally closed subspaces are exponentiable in $\mathbf{Top}_{\mathcal{M}}$.*

Proof. Suppose B is $\mathcal{M}$-generated and $q\colon Y \longrightarrow B$ is the inclusion of a locally closed subspace. Then for all $p\colon X \longrightarrow B$ in $\mathbf{Top}_{\mathcal{M}}$, since $p^{-1}(Y)$ is locally closed, one can show that $X \hat{\times}_B Y = p^{-1}(Y) = X \times_B Y$ is the product in $\mathbf{Top}_{\mathcal{M}}/B$. Then $\mathbf{Top}_{\mathcal{M}}/B(X \hat{\times}_B Y, Z) = \mathbf{Top}/B(X \times_B Y, Z) = \mathbf{Top}/B(X, Z^Y) = \mathbf{Top}_{\mathcal{M}}/B(X, \widehat{Z^Y})$, since locally closed inclusions are exponentiable in $\mathbf{Top}$. $\square$

Corollary 2.5. *Locally closed inclusions are exponentiable in the categories $\mathbf{Top}_{\mathcal{K}}$ of compactly generated spaces and $\mathbf{Top}_{\mathcal{E}}$ of exponentially generated spaces.*

Proof. Locally closed subspaces of compact Hausforff spaces are compactly generated and locally closed subspaces of exponentiable space are exponentiable. $\square$

An $\mathcal{M}$-generated space X is called *$\mathcal{M}$-Hausdorff* (respectively, *locally $\mathcal{M}$-Hausdorff*) if the diagonal $B \longrightarrow B \hat{\times} B$ is closed (respectively, locally closed). A $\mathcal{K}$-Hausdorff space is also known as a weak Hausdorff compactly generated space or a k-space in the literature Lewis [12]. Note that weak Hausdorff compactly generated spaces also form a cartesian closed category but the only exponentiable morphisms there are the open maps [12].

Corollary 2.6. *If* $\mathbf{Top}_{\mathcal{M}}$ *is closed under locally closed subspaces of all* M *in* $\mathcal{M}$*, and* B *is* $\mathcal{M}$*-Hausdorff (more generally, locally* $\mathcal{M}$*-Hausdorff), then* $\mathbf{Top}_{\mathcal{M}}/B$ *is cartesian closed.*

Proof. Apply Corollary 2.3 and Proposition 2.4. □

In particular, we get:

Corollary 2.7. *If* B *is a weak Hausdorff space, then* $\mathbf{Top}_{\mathcal{K}}/B$ *is cartesian closed.*

3. Exponentiable Topological Groupoids

In this section, we consider exponentiable topological groupoids, but first some general results in $\mathbf{Gpd}(\mathcal{C})$, where $\mathcal{C}$ is a finitely complete category with finite coproducts. As noted in the introduction, G is exponentiable in $\mathbf{Gpd}(\mathcal{C})$, whenever G_0, G_1, and G_2 are exponentiable in $\mathcal{C}$. It is not true that $q\colon G \longrightarrow B$ is exponentiable in $\mathbf{Gpd}(\mathcal{C})/B$ whenever the $q_i\colon G_i \longrightarrow B_i$ are exponentable for i=0,1,2, since even when $\mathcal{C} = \mathbf{Sets}$, for $G \longrightarrow B$ to be exponentiable it is necessary that it is a fibration. Moreover, one cannot use Proposition 2.2 to obtain exponentiable morphisms of $\mathbf{Gpd}(\mathcal{C})$, since the diagonal $\Delta\colon B \longrightarrow B \times B$ is rarely exponentiable. In fact, when $\mathcal{C} = \mathbf{Sets}$, this is the case if and only if B is discrete.

When $\mathcal{C}$ is cartesian closed, the exponential H^G in $\mathbf{Gpd}(\mathcal{C})$ can be constructed as follows. The object of objects $(H^G)_0$ needs to encode triples of arrows $\langle \sigma_0\colon G_0 \longrightarrow H_0, \sigma_1\colon G_1 \longrightarrow H_1, \sigma_2\colon G_2 \longrightarrow H_2\rangle$ that fit in the appropriate commutative diagrams to form an internal functor $G \longrightarrow H$; i.e.,

$$
\begin{array}{ccc}
G_1 & \xrightarrow{\sigma_1} & H_1 \\
{\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s} \\
G_0 & \xrightarrow[\sigma_0]{} & H_0
\end{array}
\qquad
\begin{array}{ccc}
G_1 & \xrightarrow{\sigma_1} & H_1 \\
{\scriptstyle t}\downarrow & & \downarrow{\scriptstyle t} \\
G_0 & \xrightarrow[\sigma_0]{} & H_0
\end{array}
\qquad
\begin{array}{ccc}
G_1 & \xrightarrow{\sigma_1} & H_1 \\
{\scriptstyle u}\uparrow & & \uparrow{\scriptstyle u} \\
G_0 & \xrightarrow[\sigma_0]{} & H_0
\end{array}
$$

$$
\begin{array}{ccc}
G_2 & \xrightarrow{\sigma_2} & H_2 \\
{\scriptstyle c}\downarrow & & \downarrow{\scriptstyle c} \\
G_1 & \xrightarrow[\sigma_1]{} & H_1
\end{array}
\qquad
\begin{array}{ccc}
G_2 & \xrightarrow{\sigma_2} & H_2 \\
{\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_1} \\
G_1 & \xrightarrow[\sigma_1]{} & H_1
\end{array}
\qquad
\begin{array}{ccc}
G_2 & \xrightarrow{\sigma_2} & H_2 \\
{\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle \pi_2} \\
G_1 & \xrightarrow[\sigma_1]{} & H_1
\end{array}
$$

Hence, it is obtained as the equalizer

$$(H^G)_0 \rightarrowtail H_0^{G_0} \times H_1^{G_1} \times H_2^{G_2} \overset{f_0}{\underset{g_0}{\rightrightarrows}} H_0^{G_1} \times H_0^{G_1} \times H_1^{G_0} \times H_1^{G_2} \times H_1^{G_2} \times H_1^{G_2}$$

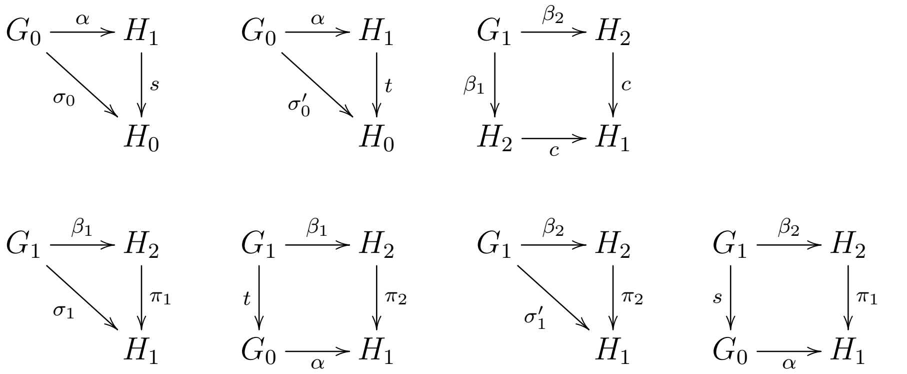
where

$$f_0 = \langle H_0^s \pi_1, H_0^t \pi_1, u^{G_0}\pi_1, H_1^c \pi_2, H_1^{\pi_1}\pi_2, H_1^{\pi_2}\pi_2 \rangle$$

and

$$g_0 = \langle s^{G_1}\pi_2, t^{G_1}\pi_2, H_1^u \pi_2, c^{G_2}\pi_3, \pi_1^{G_2}\pi_3, \pi_2^{G_2}\pi_3 \rangle$$

The object of arrows $(H^G)_1$ needs to encode internal natural tranformations $\alpha\colon \sigma \Rightarrow \sigma'$ between internal functors $\sigma, \sigma'\colon G \rightrightarrows H$. These are given by quintuples $\langle \sigma, \sigma', \alpha, \beta_1, \beta_2 \rangle$, where $\alpha\colon G_0 \to H_1$ and $\beta_1, \beta_2\colon G_1 \rightrightarrows H_2$, that make the following diagrams commute,



(Note that the last five encode commutativity of the naturality square (1).) Hence, it is obtained as the equalizer $(H^G)_1$ of the parallel pair,

$$(H^G)_0 \times (H^G)_0 \times H_1^{G_0} \times H_2^{G_1} \times H_2^{G_1} \overset{f_1}{\underset{g_1}{\rightrightarrows}} H_0^{G_0} \times H_0^{G_0} \times H_1^{G_1} \times H_1^{G_1} \times H_1^{G_1} \times H_1^{G_1} \times H_1^{G_1}$$

where

$$f_1 = \langle \pi_1\pi_1,\ \pi_1\pi_2,\ c^{G_1}\pi_4,\ \pi_2\pi_1,\ H_1^t\pi_3,\ \pi_2\pi_2,\ H_1^s\pi_3 \rangle$$

and

$$g_1 = \langle s^{G_0}\pi_3,\ t^{G_0}\pi_3,\ c^{G_1}\pi_5,\ \pi_1^{G_1}\pi_4, \pi_2^{G_1}\pi_4, \pi_2^{G_1}\pi_5, \pi_1^{G_1}\pi_5 \rangle$$

The source and target maps $(H^G)_1 \to (H^G)_0$ are given by first and second projection. The unit map $(H^G)_0 \to (H^G)_1$ has the identity map in the first and second coordinate and $u^{G_0}\pi_1$ in the third coordinate. We describe the last two coordinates using the transpose. Note that $(H^G)_0$ is a subobject of $H_0^{G_0} \times H_1^{G_1} \times H_2^{G_2}$. So consider

$$H_0^{G_0} \times H_1^{G_1} \times H_2^{G_2} \times G_1 \xrightarrow{\langle \pi_1,\pi_2,\Delta_{G_1}\pi_4\rangle} H_0^{G_0} \times H_1^{G_1} \times G_1 \times G_1$$

$$\xrightarrow{id_{H_0^{G_0}} \times id_{H_1^{G_1}} \times s \times id_{G_1}} H_0^{G_0} \times H_1^{G_1} \times G_0 \times G_1$$

$$\xrightarrow{\langle \mathrm{ev}\langle \pi_1,\pi_3\rangle, \mathrm{ev}\langle \pi_2,\pi_4\rangle\rangle} H_0 \times H_1$$

$$\xrightarrow{u \times id_{H_1}} H_1 \times H_1$$

When we take the subobject $(H^G)_0 \rightarrowtail H_0^{G_0} \times H_1^{G_1} \times H_2^{G_2}$, this restricts to a map

$$\tau\colon (H^G)_0 \times G_1 \longrightarrow H_1 \times_{H_0} H_1 \cong H_2$$

Its transpose $\hat{\tau}\colon (H^G)_0 \longrightarrow H_2^{G_1}$ is the projection of the fourth coordinate of the unit. The fifth coordinate is obtained in a similar fashion, but starting with the mapping

$$H_0^{G_0} \times H_1^{G_1} \times H_2^{G_2} \times G_1 \xrightarrow{\langle \pi_1,\pi_2,\Delta_{G_1}\pi_4\rangle} H_0^{G_0} \times H_1^{G_1} \times G_1 \times G_1$$

$$\xrightarrow{id_{H_0^{G_0}} \times id_{H_1^{G_1}} \times id_{G_1} \times t} H_0^{G_0} \times H_1^{G_1} \times G_1 \times G_0$$

$$\xrightarrow{\langle \mathrm{ev}\langle \pi_2,\pi_3\rangle, \mathrm{ev}\langle \pi_1,\pi_4\rangle\rangle} H_1 \times H_0$$

$$\xrightarrow{id_{H_1} \times u} H_1 \times H_1$$

Composition in $(H^G)_1$ can be expressed using projections in the first two coordinates and the appropriate composites in H_1 in the last three coordinates of the map. This makes H^G the "groupoid of homomorphisms"

$G \longrightarrow H$ and the adjunction can be established using only the exponentiability of G_0, G_1, and G_2. Thus:

Proposition 3.1. *If G_0, G_1, and G_2 are exponentiable in $\mathcal{C}$, then G is exponentiable in $\mathbf{Gpd}(\mathcal{C})$.*

To obtain a partial converse to Proposition 3.1, we use the left and right adjoints to $(\ )_0\colon \mathbf{Gpd}(\mathcal{C}) \longrightarrow \mathcal{C}$ which we recall are given by

$$L_0(X)\colon X \xrightarrow{\ id\ } X \overset{id}{\underset{id}{\overset{\longleftarrow id \longrightarrow}{\rightrightarrows}}} X \quad (\text{with } id\colon X \to X \text{ the inverse map}) \qquad \text{and}$$

$$R_0(X)\colon X \times X \times X \xrightarrow{\ \pi_{13}\ } X \times X \overset{\pi_1}{\underset{\pi_2}{\overset{\longleftarrow \Delta \longrightarrow}{\rightrightarrows}}} X \quad (\text{with } \langle \pi_2, \pi_1 \rangle\colon X \times X \to X \times X \text{ the inverse map})$$

respectively.

Proposition 3.2. *If G is exponentiable in $\mathbf{Gpd}(\mathcal{C})$, then G_0 is exponentiable in $\mathcal{C}$. The converse holds if s (or equivalently, t) is exponentiable.*

Proof. Suppose G is exponentiable in $\mathbf{Gpd}(\mathcal{C})$. Then G_0 is exponentiable in $\mathcal{C}$, since

$$\begin{aligned} \mathcal{C}(X \times G_0, Y) &\cong \mathcal{C}((L_0X \times G)_0, Y) \cong \mathbf{Gpd}(\mathcal{C})(L_0X \times G, R_0Y) \\ &\cong \mathbf{Gpd}(\mathcal{C})(L_0X, (R_0Y)^G) \cong \mathcal{C}(X, (R_0Y)^G_0) \end{aligned}$$

For the converse, suppose G_0 and $s\colon G_1 \longrightarrow G_0$ are exponentiable in $\mathcal{C}$. Then G_1 is exponentiable since composition preserves exponentiability. To see that G_2 is exponentiable, consider the pullback

$$\begin{array}{ccc} G_2 & \xrightarrow{\ \pi_2\ } & G_1 \\ {\scriptstyle \pi_1}\big\downarrow & & \big\downarrow{\scriptstyle s} \\ G_1 & \xrightarrow[\ t\ ]{} & G_0 \end{array}$$

where π_1 is exponentiable since s is and pullback preserves exponentiability, and so G_2 is exponentiable since G_1 is. Thus, G is exponentiable in $\mathbf{Gpd}(\mathcal{C})$ by Proposition 3.1. $\square$

Note that if G is an étale groupoid, in the sense of Moerdijk/Pronk [14], then s and t are local homeomorphisms in $\mathbf{Top}$, and we conjecture that H^G is étale when H is also étale and G_1/G_0 is compact. Thus, G is exponentiable in $\mathbf{Gpd}(\mathbf{Top})$ if and only if G_0 is exponentiable in $\mathbf{Top}$. Of course, all étale groupoids are exponentiable in $\mathbf{Gpd}(\mathbf{Top}_{\mathcal{K}})$, since $\mathbf{Top}_{\mathcal{K}}$ is cartesian closed.

An exponentiable internal groupoid of interest is the groupoid $\mathbb{I}$ with two objects and one nontrivial isomorphism. It is well known that $\mathbb{I}$ makes sense in $\mathbf{Gpd}(\mathcal{C})$, for any finitely complete $\mathcal{C}$ with finite coproducts, where $\mathbb{I}_0 = 1 + 1$ and $\mathbb{I}_1 = 1 + 1 + 1 + 1$. In particular, the exponentials $B^{\mathbb{I}}$ will play a role when we consider the pseudo-slices $\mathbf{Cat}//B$ in Section 5. We know that $B^{\mathbb{I}}$ is exponentiable whenever $B_0^{\mathbb{I}}$, $B_1^{\mathbb{I}}$, and $B_2^{\mathbb{I}}$ are.

Using our construction of exponentials, one can see that $B^{\mathbb{I}}$ can be described as follows. Since $B_0^{\mathbb{I}}$ can be thought of as the "object of homomorphisms $\mathbb{I} \to B$," i.e., morphisms $b_s \to b_t$ in B, we can take $B_0^{\mathbb{I}} = B_1$. Then $B_1^{\mathbb{I}}$ becomes $(B^{\mathbb{I}})_1 = B_2 \times_{B_1} B_2$ via the pullback

$$\begin{array}{ccc} B_2 \times_{B_1} B_2 & \xrightarrow{\pi_2} & B_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle c} \\ B_2 & \xrightarrow[c]{} & B_1 \end{array}$$

i.e., the "object of squares"

$$\begin{array}{ccc} b_s & \xrightarrow{\alpha} & b_t \\ {\scriptstyle \beta_s}\downarrow & & \downarrow{\scriptstyle \beta_t} \\ \bar{b}_s & \xrightarrow[\bar{\alpha}]{} & \bar{b}_t \end{array}$$

and $B_1^{\mathbb{I}} \underset{t}{\overset{s}{\rightrightarrows}} B_0^{\mathbb{I}}$ is given by $B_2 \times_{B_1} B_2 \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} B_2 \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} B_1$, i.e., $s(\beta_s \underset{\bar{\alpha}}{\overset{\alpha}{\rightrightarrows}} \beta_t) = \beta_s$ and $t(\beta_s \underset{\bar{\alpha}}{\overset{\alpha}{\rightrightarrows}} \beta_t) = \beta_t$. Finally, $B_2^{\mathbb{I}} = (B_2 \times_{B_1} B_2) \times_{B_1} (B_2 \times_{B_1} B_2)$ is the "object of commutative diagrams" with composition

$$\begin{array}{ccccc} b_s & \xrightarrow{\alpha} & b_t & \xrightarrow{\alpha'} & b_{t'} \\ {\scriptstyle\beta_s}\downarrow & & \downarrow{\scriptstyle\beta_t} & & \downarrow{\scriptstyle\beta_{t'}} \\ \bar{b}_s & \xrightarrow[\bar{\alpha}]{} & \bar{b}_t & \xrightarrow[\bar{\alpha}']{} & \bar{b}_{t'} \end{array} \quad \mapsto \quad \begin{array}{ccc} b_s & \xrightarrow{\alpha'\alpha} & b_{t'} \\ {\scriptstyle\beta_s}\downarrow & & \downarrow{\scriptstyle\beta_{t'}} \\ \bar{b}_s & \xrightarrow[\bar{\alpha}'\bar{\alpha}]{} & \bar{b}_{t'} \end{array}$$

Thus, we get the following corollary of Proposition 3.2.

Corollary 3.3. *If $B^{\mathrm{I\!I}}$ is exponentiable in $\mathbf{Gpd}(\mathcal{C})$, then B_1 is exponentiable in $\mathcal{C}$. The converse holds if the arrows $B_2 \overset{c}{\underset{\pi_1}{\rightrightarrows}} B_1$ are exponentiable in $\mathcal{C}$.*

Proof. The first part holds by Proposition 3.2, since $B_0^{\mathrm{I\!I}} = B_1$. So, assume that B_1 and $B_2 \overset{c}{\underset{\pi_1}{\rightrightarrows}} B_1$ are exponentiable in $\mathcal{C}$. Then $B_1^{\mathrm{I\!I}} = B_2 \times_{B_1} B_2 \xrightarrow{\pi_1} B_2$ is exponentiable being a pullback of $c\colon B_2 \longrightarrow B_1$, and so $B^{\mathrm{I\!I}}$ is exponentiable in $\mathbf{Gpd}(\mathcal{C})$ by Proposition 3.2, since $s\colon B_1^{\mathrm{I\!I}} \longrightarrow B_0^{\mathrm{I\!I}}$ is given by $B_2 \times_{B_1} B_2 \xrightarrow{\pi_1} B_2 \xrightarrow{\pi_1} B_1$. □

Recall [4] that a topological groupoid G is called an *orbigroupoid* if s and t are étale and $\langle s, t\rangle\colon G_1 \longrightarrow G_0 \times G_0$ is a proper map.

Proposition 3.4. *If B is an orbigroupoid, then so is $B^{\mathrm{I\!I}}$.*

Proof. Suppose B is an orbigroupoid. Since s is étale and

$$\begin{array}{ccc} B_2 & \xrightarrow{\pi_1} & B_1 \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle s} \\ B_1 & \xrightarrow[s]{} & B_0 \end{array}$$

is a pullback (as B is a groupoid), it follows that $c\colon B_2 \longrightarrow B_1$ and hence the projections $B_2 \times_{B_1} B_2 \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} B_2$ are étale. Thus, $B_1^{\mathrm{I\!I}} \overset{s}{\underset{t}{\rightrightarrows}} B_0^{\mathrm{I\!I}}$ are étale, as desired. To see that $\langle s,t\rangle\colon B_1^{\amalg} \longrightarrow B_0^{\amalg}\times B_0^{\amalg}$ is proper, consider the diagram

$$\begin{array}{ccc} B_1^{\amalg} & \xrightarrow{\langle s,t\rangle} & B_0^{\amalg}\times B_0^{\amalg} \\ \| & & \| \\ B_2\times_{B_1} B_2 & \longrightarrow & B_1\times B_1 \\ {\scriptstyle c\pi_1}\downarrow & & \downarrow{\scriptstyle s\times t} \\ B_1 & \xrightarrow[\langle s,t\rangle]{} & B_0\times B_0 \end{array}$$

which is a pullback as B is a groupoid. Since the bottom row is proper it follows that the top one is, and so $B^{\amalg}$ is an orbigroupoid. □

4. Exponentiable Morphisms of Groupoids

In this section, we consider exponentiable morphisms in $\mathbf{Gpd}(\mathcal{C})$. When $\mathcal{C} = \mathbf{Sets}$, or any topos, we know that these are precisely the fibrations. Though the categories $\mathcal{C}$ of spaces of interest are not even locally cartesian closed, we will see that if $q\colon G \longrightarrow B$ is a fibration (in the sense defined below) and each $q_i\colon G_i \longrightarrow B_i$ is exponentiable in $\mathcal{C}$, then q is exponentiable in $\mathbf{Gpd}(\mathcal{C})$.

Suppose $q\colon G \longrightarrow B$ is exponentiable in $\mathbf{Gpd}(\mathcal{C})/B$. Then, as in Proposition 3.2, we know $q_0\colon G_0 \longrightarrow B_0$ is exponentiable in $\mathcal{C}$, since

$$(\ )_0\colon \mathbf{Gpd}(\mathcal{C})/B \longrightarrow \mathcal{C}/B_0$$

has left and right adjoints, given by $(X \xrightarrow{p} B_0) \mapsto (L_0X \xrightarrow{L_0p} L_0B_0 \xrightarrow{\varepsilon} B)$, where ε is the counit of the adjunction $L_0 \dashv (\ )_0$, and $(X \xrightarrow{p} B_0) \mapsto (B \times_{R_0B_0} R_0X \xrightarrow{\pi_1} B)$, where $B \longrightarrow R_0B_0$ is the unit of the adjuncion $(\ )_0 \dashv R_0$.

Definition 4.1. *A morphism $q\colon G \longrightarrow B$ is a* fibration *in $\mathbf{Gpd}(\mathcal{C})$ if*

$$\langle s, q_1\rangle\colon G_1 \longrightarrow G_0 \times_{B_0} B_1$$

has a right inverse, or equivalently, $\langle q_1, t\rangle\colon G_1 \longrightarrow B_1 \times_{B_0} G_0$ has a right inverse in $\mathcal{C}$.

Remark 4.2. When $\mathcal{C}$ is the category of all topological spaces (or any concrete category), this says q is a fibration, in the sense of Grothendieck [8]; i.e., given a and $\beta\colon q_0a \longrightarrow \bar{b}$, there exists $\alpha\colon a \longrightarrow \bar{a}$ such that $q_1\alpha = \beta$, but our condition is stronger since $(a, \beta) \mapsto \alpha$ must be a morphism of $\mathcal{C}$.

Our notion is equivalent to the notion of a cloven strict internal fibration as given in [20] for the 2-category **Gpd**$(\mathcal{C})$. Note that the description for **Gpd**$(\mathcal{C})$ can be simplified this way because we do not need to worry about cartesian arrows: for a fibration between groupoids all arrows in the domain are cartesian.

Lemma 4.3. *An arrow $q\colon G \longrightarrow B$ in* **Gpd**$(\mathcal{C})$ *is a fibration in our sense precisely when it is representably a cloven strict internal fibration.*

Proof. Let $q\colon G \longrightarrow B$ be a fibration in **Gpd**$(\mathcal{C})$ with $\theta\colon G_0 \times_{B_0} B_1 \longrightarrow G_1$ a right inverse to $\langle s, q_1\rangle$. Let H be any groupoid in $\mathcal{C}$. We need to show that the induced functor

$$q_* = \mathbf{Gpd}(\mathcal{C})(H, q)\colon \mathbf{Gpd}(\mathcal{C})(H, G) \longrightarrow \mathbf{Gpd}(\mathcal{C})(H, B)$$

is a cloven strict fibration in **Cat**. So let $\varphi\colon H \longrightarrow G$ be an internal functor, viewed as object in **Gpd**$(\mathcal{C})(H, G)$ and let $\alpha\colon q\varphi \Rightarrow \psi$ be an internal natural transformation, viewed as an arrow in **Gpd**$(\mathcal{C})(H, B)$, Then α gives rise to a morphism $\alpha\colon H_0 \longrightarrow B_1$ in $\mathcal{C}$, with $s\alpha = q_0\varphi_0$. Hence this gives us $\langle\varphi_0, \alpha\rangle\colon H_0 \longrightarrow G_0 \times_{B_0} B_1$. It follows that the composition $\theta\langle\varphi_0, \alpha\rangle\colon H_0 \longrightarrow G_1$ is the required lifting. This defines a cleavage, because the internal categories here are groupoids. The fact that for any $f\colon H \longrightarrow H'$, the induced square is a morphism of fibrations follows immediately from the fact that we are working with groupoids.

Conversely, suppose that $q\colon G \longrightarrow B$ is representably a cloven internal fibration in **Gpd**$(\mathcal{C})$. This implies that

$$q_* = \mathbf{Gpd}(\mathcal{C})(H, q)\colon \mathbf{Gpd}(\mathcal{C})(H, G) \longrightarrow \mathbf{Gpd}(\mathcal{C})(H, B)$$

is a cloven strict fibration in **Cat** for each H in **Gpd**$(\mathcal{C})$. Now take H to be the strict comma square,

$$\begin{array}{ccc} H & \xrightarrow{\ r\ } & B \\ {\scriptstyle p}\downarrow & \overset{\alpha}{\Rightarrow} & \downarrow{\scriptstyle id_B} \\ G & \xrightarrow[\ q\ ]{} & B \end{array}$$

Then we may take H_0 to be the pullback

$$\begin{array}{ccc} H_0 & \longrightarrow & B_1 \\ \downarrow & & \downarrow{\scriptstyle s} \\ G_0 & \xrightarrow[q_0]{} & B_0 \end{array}$$

and $p_0 = \pi_1 \colon H_0 \longrightarrow G_0$ and $\alpha = \pi_2 \colon H_0 \longrightarrow B_1$.

Note that we have $\alpha\colon qp \Rightarrow r$, an arrow in **Gpd**$(\mathcal{C})(H, B)$, and p is such that $q_*(p) = qp$. Hence the cleavage gives us a lifting $\tilde{\alpha}\colon q \Rightarrow \tilde{r}$ in **Gpd**(H, G) represented by $\tilde{\alpha}\colon H_0 \longrightarrow G_1$ such that $s\tilde{\alpha} = p_0$ and $q_1\tilde{\alpha} = \alpha = \pi_2$. So we get that $\langle s, q_1\rangle\tilde{\alpha} = id_{G_0\times_{B_0}B_1}$ as required. □

Note that $B^{\mathbb{I}}$ becomes a groupoid over B via $B^{\mathbb{I}} \underset{t}{\overset{s}{\rightrightarrows}} B$ defined by $s_0 = s$, $t_0 = t$, $s_1\colon B_2 \times_{B_1} B_2 \xrightarrow{\pi_2} B_2 \xrightarrow{\pi_1} B_1$, and $t_1\colon B_2 \times_{B_1} B_2 \xrightarrow{\pi_1} B_2 \xrightarrow{\pi_2} B_1$, i.e., $s_1(\beta_s \underset{\bar{\alpha}}{\overset{\alpha}{\rightrightarrows}} \beta_t) = \alpha$, and $t_1(\beta_s \underset{\bar{\alpha}}{\overset{\alpha}{\rightrightarrows}} \beta_t) = \bar{\alpha}$.

Proposition 4.4. *The morphisms $B^{\mathbb{I}} \times_B G \xrightarrow{s\pi_1} B$ and $G \times_B B^{\mathbb{I}} \xrightarrow{t\pi_2} B$ are fibrations, for all $q\colon G \longrightarrow B$. In particular, $s\colon B^{\mathbb{I}} \longrightarrow B$ and $t\colon B^{\mathbb{I}} \longrightarrow B$ are fibrations, for all B.*

Proof. This result follows from the general theory on fibrations as spelled out in Theorem 14 [20] for instance, where it is shown that any span which is the comma object of some opspan is a split bifibration. However, in this particular case, there is also a short straightforward argument: For $s\pi_1$, the morphism $\langle s, (s\pi_1)_1\rangle\colon (B^{\mathbb{I}} \times_B G)_1 \longrightarrow (B^{\mathbb{I}} \times_B G)_0 \times_{B_0} B_1$ is given by

$$\begin{array}{ccc} b_s & \xrightarrow{\alpha} & b_t \\ {\scriptstyle \beta_s}\downarrow & & \downarrow{\scriptstyle \beta_t} \\ qa_s & \xrightarrow[q\gamma]{} & qa_t \end{array} \quad \mapsto \quad \begin{array}{ccc} b_s & \xrightarrow{\alpha} & b_t \\ {\scriptstyle \beta_s}\downarrow & & \\ qa_s & & \end{array}$$

and so

$$\begin{array}{ccc} b_s & \xrightarrow{\alpha} & b_t \\ {\scriptstyle \beta_s}\downarrow & & \\ qa_s & & \end{array} \quad \mapsto \quad \begin{array}{ccc} b_s & \xrightarrow{\alpha} & b_t \\ {\scriptstyle \beta_s}\downarrow & & \downarrow{\scriptstyle \beta_s\alpha^{-1}} \\ qa_s & \xrightarrow[qid]{} & qa_s \end{array}$$

is a right inverse to $\langle s, (s\pi_1)_1\rangle$. The proof for t is similar. □

Now, for "discrete" groupoids L_0B, we know

$$\mathbf{Gpd}(\mathcal{C})/L_0B \cong \mathbf{Gpd}(\mathcal{C}/B)$$

and so $q\colon G \longrightarrow L_0B$ is exponentiable in $\mathbf{Gpd}(\mathcal{C})$ if each $q_i\colon G_i \longrightarrow B$ is exponentiable in $\mathcal{C}$. Thus, if $\mathcal{C}$ is cartesian closed, then $\mathbf{Gpd}(\mathcal{C})/L_0B$ is cartesian closed whenever the diagonal on B is exponentiable in $\mathcal{C}$. In particular, $\mathbf{Gpd}(\mathbf{Top}_{\mathcal{M}})/L_0B$ is cartesian closed whenever B is $\mathcal{M}$-Hausdorff, e.g., weak Hausdorff in the case where $\mathcal{M} = \mathcal{K}$.

For the non-discrete case, given $q\colon G \longrightarrow B$ and $r\colon H \longrightarrow B$, to see how to define the exponentials $r^q\colon H^G \longrightarrow B$ when q is exponentiable in $\mathbf{Gpd}(\mathcal{C})/B$, consider the case where $\mathcal{C} = \mathbf{Sets}$. Recall that the fiber of $(H^G)_0$ over b in B is the set of homomorphisms $\sigma\colon G_b \longrightarrow H_b$ between the fibers of G and H over b. A morphism $\Sigma\colon \sigma \longrightarrow \sigma'$ over $\beta\colon b \longrightarrow b'$ in B is a family of morphisms $\Sigma_\alpha\colon \sigma a \longrightarrow \sigma' a'$ of H over β indexed by the morphisms $\alpha\colon a \longrightarrow a'$ of G over β such that the diagram

$$\begin{array}{ccc} \sigma\bar{a} & \xrightarrow{\sigma\bar{\alpha}} & \sigma a \\ {\scriptstyle \Sigma_{\alpha\bar{\alpha}}}\downarrow & \swarrow{\scriptstyle \Sigma_\alpha} & \downarrow{\scriptstyle \Sigma_{\bar{\alpha}'\alpha}} \\ \sigma' a' & \xrightarrow[\sigma'\bar{\alpha}']{} & \sigma'\bar{a}' \end{array} \qquad (2)$$

commutes, for all $\bar{a} \xrightarrow{\bar{\alpha}} a \xrightarrow{\alpha} a' \xrightarrow{\bar{\alpha}'} \bar{a}'$ such that $q(\bar{\alpha}) = id_b$ and $q(\bar{\alpha}') = id_{b'}$. Defining the morphisms s, t, u and i is straightforward, but for composition, one must assume q is a fibration. Then, let $r\colon G_0 \times_{B_0} B_1 \longrightarrow G_1$ be a right inverse of $\langle s, q_1\rangle$. Suppose $\sigma \xrightarrow{\Sigma} \sigma' \xrightarrow{\Sigma'} \sigma''$ is a composable pair over $b \xrightarrow{\beta} b' \xrightarrow{\beta'} b''$, and define $\sigma \xrightarrow{\Sigma'\Sigma} \sigma''$ as follows. Given $a \xrightarrow{\alpha''} a''$ over $b \xrightarrow{\beta'\beta} b''$, consider

$$\begin{array}{ccc} a & \xrightarrow{\alpha''} & a'' \\ & {\scriptstyle \alpha}\searrow \quad \nearrow{\scriptstyle \alpha'} & \\ & a' & \end{array}$$

where $\alpha = r(a, \beta)$ and $\alpha' = \alpha''\alpha^{-1}$, and define $(\Sigma'\Sigma)_{\alpha''} = \Sigma'_{\alpha'}\Sigma_\alpha$. Then it is not difficult to show that H^G is a groupoid over B and that this provides a right adjoint to the functor $- \times_B G\colon \mathbf{Gpd}/B \longrightarrow \mathbf{Gpd}/B$.

Theorem 4.5. *If $q\colon G \longrightarrow B$ is a fibration and $q_i\colon G_i \longrightarrow B_i$ is exponentiable in $\mathcal{C}$, for $i = 0, 1, 2$, then q is exponentiable in $\mathbf{Gpd}(\mathcal{C})/B$.*

Proof. Given $H \longrightarrow B$, define $(H^G)_0 \longrightarrow B_0$ by the equalizer

$$(H^G)_0 \rightarrowtail H_0^{G_0} \times_{B_0} (B_0 \times_{B_1} H_1^{G_1}) \times_{B_0} (B_0 \times_{B_2} H_2^{G_2}) \overset{f_0}{\underset{g_0}{\rightrightarrows}} X_0$$

in $\mathcal{C}/B_0$, capturing the fact that

$$(\sigma_0\colon G_0 \longrightarrow H_0, \sigma_1\colon G_1 \longrightarrow H_1, \sigma_2\colon G_2 \longrightarrow H_2)$$

is a "homomorphism of groupoids", where $H_0^{G_0} \longrightarrow B_0$, $H_1^{G_1} \longrightarrow B_1$ and $H_2^{G_2} \longrightarrow B_2$ are the exponentials,

$$\begin{array}{ccc} B_0 \times_{B_1} H_1^{G_1} & \xrightarrow{\pi_2} & H_1^{G_1} \\ {\scriptstyle \pi_1}\downarrow & & \downarrow \\ B_0 & \xrightarrow[u]{} & B_1 \end{array}$$

and

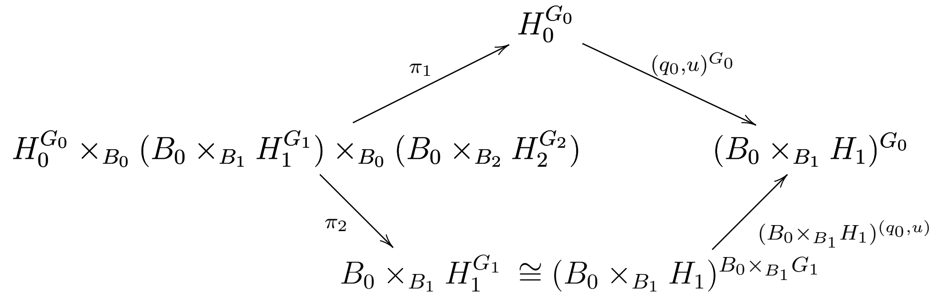
$$\begin{array}{ccc} B_0 \times_{B_2} H_2^{G_2} & \xrightarrow{\pi_2} & H_2^{G_2} \\ {\scriptstyle \pi_1}\downarrow & & \downarrow \\ B_0 & \xrightarrow[(u,u)]{} & B_2 \end{array}$$

are pullbacks in $\mathcal{C}$, and the morphisms f_0 and g_0 ensure that σ_0, σ_1 and σ_2 are compatible with s, t, u, c and the projections. In detail, for s, X_0 has a factor of the form $H_0^{B_0 \times_{B_1} G_1}$ whose projections of f_0 and g_0 are given by

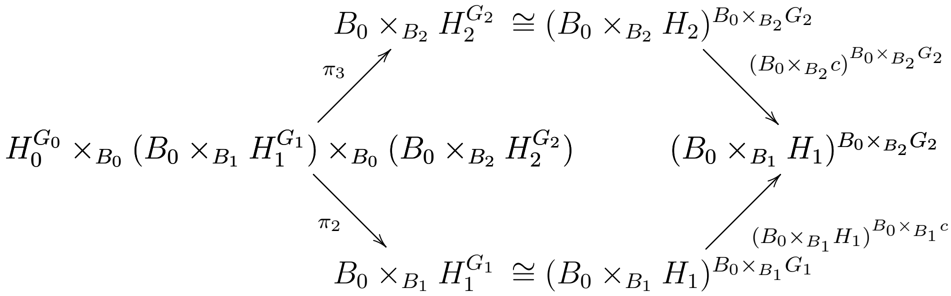
$$\begin{array}{ccccc} & & H_0^{G_0} & & \\ & {\scriptstyle \pi_1}\nearrow & & \searrow{\scriptstyle H_0^s} & \\ H_0^{G_0} \times_{B_0} (B_0 \times_{B_1} H_1^{G_1}) \times_{B_0} (B_0 \times_{B_2} H_2^{G_2}) & & & & H_0^{B_0 \times_{B_1} G_1} \\ & {\scriptstyle \pi_2}\searrow & & \nearrow{\scriptstyle (s\pi_2)^{B_0 \times_{B_1} G_1}} & \\ & & B_0 \times_{B_1} H_1^{G_1} \cong (B_0 \times_{B_1} H_1)^{B_0 \times_{B_1} G_1} & & \end{array}$$

The factor of X_0 for t is defined similarly: just replace both occurrences of s by t in this diagram.

The factor of X_0 for u is of the form $(B_0 \times_{B_1} H_1)^{G_0}$ and the projections of f_0 and g_0 for this factor are given by



The factor of X_0 for c is of the form $(B_0 \times_{B_1} H_1)^{B_0 \times_{B_2} G_2}$ and the projections of f_0 and g_0 for this factor are given by



The factors of X_0 for the commutativity with the two projections from the objects of composable pairs to the objects of arrows are given by two additional copies of $(B_0 \times_{B_1} H_1)^{B_0 \times_{B_2} G_2}$ and the projections of f_0 and g_0 are obtained by replacing c in this diagram by π_1 and π_2 respectively.

We conclude that

$$X_0 = H_0^{B_0 \times_{B_1} G_1} \times_{B_0} H_0^{B_0 \times_{B_1} G_1} \times_{B_0} (B_0 \times_{B_1} H_1)^{G_0} \times_{B_0} (B_0 \times_{B_1} H_1)^{B_0 \times_{B_2} G_2} \times_{B_0} (B_0 \times_{B_1} H_1)^{B_0 \times_{B_2} G_2} \times_{B_0} (B_0 \times_{B_1} H_1)^{B_0 \times_{B_2} G_2}$$

and the maps f_0 and g_0 are given by

$$f_0 = (H_0^s \pi_1,\ H_0^t \pi_1,\ (q_0, u)^{G_0} \pi_1,\ (B_0 \times_{B_2} c)^{B_0 \times_{B_2} G_2} \pi_3,\ (B_0 \times_{B_2} \pi_1)^{B_0 \times_{B_2} G_2} \pi_3,\ (B_0 \times_{B_2} \pi_2)^{B_0 \times_{B_2} G_2} \pi_3)$$

and

$$g_0 = ((s\pi_2)^{B_0\times_{B_1}G_1}\pi_2,\ (t\pi_2)^{B_0\times_{B_1}G_1}\pi_2,\ (B_0\times_{B_1}H_1)^{(q_0,u)}\pi_2,$$
$$(B_0\times_{B_1}H_1)^{B_0\times_{B_1}c}\pi_2,\ (B_0\times_{B_1}H_1)^{B_0\times_{B_1}\pi_1}\pi_2,\ (B_0\times_{B_1}H_1)^{B_0\times_{B_1}\pi_2}\pi_2)$$

To define $(H^G)_1 \longrightarrow B_1$ we use an equalizer over B_1 of the form

$$(H^G)_1 \rightarrowtail X_2 \underset{g_1}{\overset{f_1}{\rightrightarrows}} X_1$$

where X_2 is given by

$$((H^G)_0 \times_{B_0} H_1^{G_1} \times_{B_0} (H^G)_0) \times_{B_1} H_2^{(B_0\times_{B_1}G_1)\times_{G_0}G_1} \times_{B_1} H_2^{(B_0\times_{B_1}G_1)\times_{G_0}G_1}$$

and $H_1^{G_1} \longrightarrow B_1 \underset{t}{\overset{s}{\rightrightarrows}} B_0$ appear in the product over B_0 via the usual convention. The morphisms f_1 and g_1 are defined to encode the commutativity of the diagram (2) defining Σ in $\mathbf{Gpd}(\mathbf{Sets})$. The $H_2^{(B_0\times_{B_1}G_1)\times_{G_0}G_1}$ and $H_2^{(B_0\times_{B_1}G_1)\times_{G_0}G_1}$ components in X_2 have been added to be able to express commutativity of the top left triangle and bottom right triangle (respectively) in (2). To make our diagrams a bit more managable we will write G'_1 for $B_0 \times_{B_1} G_1$. Commutativity of the top left triangle is then expressed by commutativity of the following three diagrams:

$$\begin{array}{ccc}
H_2^{G'_1\times_{G_0}G_1} & \searrow^{\pi_1^{G'_1\times_{G_0}G_1}} & \\
\uparrow{\scriptstyle \pi_2} & & \\
((H^G)_0 \times_{B_0} H_1^{G_1} \times_{B_0} (H^G)_0) \times_{B_1} H_2^{G'_1\times_{G_0}G_1} \times_{B_1} H_2^{G'_1\times_{G_0}G_1} & & H_1^{G'_1\times_{G_0}G_1} \\
\downarrow{\scriptstyle \pi_2\pi_1\pi_1} & & \\
H_1^{G'_1} & \nearrow_{H_1^{\pi_1}} &
\end{array}$$

$$\begin{array}{ccc}
H_2^{G'_1\times_{G_0}G_1} & \searrow^{\pi_2^{G'_1\times_{G_0}G_1}} & \\
\uparrow{\scriptstyle \pi_2} & & \\
((H^G)_0 \times_{B_0} H_1^{G_1} \times_{B_0} (H^G)_0) \times_{B_1} H_2^{G'_1\times_{G_0}G_1} \times_{B_1} H_2^{G'_1\times_{G_0}G_1} & & H_1^{G'_1\times_{G_0}G_1} \\
\downarrow{\scriptstyle \pi_2\pi_1} & & \\
H_1^{G_1} & \nearrow_{H_1^{\pi_2}} &
\end{array}$$

$$\begin{array}{ccc}
H_2^{G'_1\times_{G_0}G_1} & \xrightarrow{\quad c^{G'_1\times_{G_0}G_1}\quad} & \\
\uparrow{\scriptstyle \pi_2} & & \\
((H^G)_0\times_{B_0}H_1^{G_1}\times_{B_0}(H^G)_0)\times_{B_1}H_2^{G'_1\times_{G_0}G_1}\times_{B_1}H_2^{G'_1\times_{G_0}G_1} & & H_1^{G'_1\times_{G_0}G_1} \\
\downarrow{\scriptstyle \pi_2\pi_1} & & \\
H_1^{G_1} & \xrightarrow{\quad H_1^c\quad} &
\end{array}$$

The diagrams for the commutativity of the bottom right triangle are constructed similarly.

So we need that

$$X_1 = H_1^{G'_1\times_{G_0}G_1}\times_{B_1}H_1^{G'_1\times_{G_0}G_1}\times_{B_1}H_1^{G'_1\times_{G_0}G_1}\times_{B_1}H_1^{G_1\times_{G_0}G'_1}$$
$$\times_{B_1}H_1^{G_1\times_{G_0}G'_1}\times_{B_1}H_1^{G_1\times_{G_0}G'_1},$$

and

$$f_1 = (\pi_1^{G'_1\times_{G_0}G_1}\pi_2,\pi_2^{G'_1\times_{G_0}G_1}\pi_2,c^{G'_1\times_{G_0}G_1}\pi_2,\pi_1^{G_1\times_{G_0}G'_1}\pi_3,\pi_2^{G_1\times_{G_0}G'_1}\pi_3,c^{G_1\times_{G_0}G'_1}\pi_3)$$
$$g_1 = (H_1^{\pi_1}\pi_2\pi_1\pi_1,H_1^{\pi_2}\pi_2\pi_1,H_1^{c}\pi_2\pi_1,H_1^{\pi_1}\pi_2\pi_1,H_1^{\pi_2}\pi_2\pi_3\pi_1,H_1^{c}\pi_2\pi_1)$$

Note that $s,t\colon (H^G)_1\longrightarrow(H^G)_0$ are given by the projections. The morphisms $i\colon (H^G)_1\longrightarrow(H^G)_1$ and $u\colon (H^G)_0\longrightarrow(H^G)_1$ are induced by $i^{G_1}\colon H_1^{G_1}\longrightarrow H_1^{G_1}$ and

$$\langle id,\varphi,id\rangle\colon (H^G)_0\longrightarrow(H^G)_0\times_{B_0}\times H_1^{G_1}\times_{B_0}(H^G)_0$$

respectively, where φ is the composition

$$(H^G)_0 \rightarrowtail H_0^{G_0}\times_{B_0}(B_0\times_{B_1}H_1^{G_1})\times_{B_0}(B_0\times_{B_2}H_2^{G_2}) \overset{\pi_2\pi_2}{\longrightarrow} H_1^{G_1}$$

To define composition, let $\theta\colon G_0\times_{B_0}B_1\longrightarrow G_1$ denote the right inverse of $\langle s,q_1\rangle$, which exists since q is a fibration, and consider the diagram

$$\begin{array}{ccc}
(H^G)_1\times_{(H^G)_0}(H^G)_1 & \longrightarrow & (H^G)_1 \\
\downarrow & & \downarrow \\
H_1^{G_1}\times_{B_1}B_2\times_{B_1}H_1^{G_1} & \longrightarrow & H_1^{G_1} \\
\downarrow & & \downarrow \\
B_2 & \xrightarrow{\quad c\quad} & B_1
\end{array}$$

where the vertical compositions are the "projections" and the unnamed horizontal morphisms are to be determined. It suffices to define a morphism $H_1^{G_1} \times_{B_1} B_2 \times_{B_1} H_1^{G_1} \longrightarrow H_1^{G_1}$ so that the bottom square commutes, since all other components can be derived from this map. Now, θ induces a morphism

$$\theta' \colon G_1 \times_{B_1} B_2 \xrightarrow{\langle \pi_1, s\pi_1, \pi_1\pi_2 \rangle} G_1 \times_{B_1} (G_0 \times_{B_0} B_1) \xrightarrow{\langle \theta\pi_2, c(i\theta\pi_2, \pi_1) \rangle} G_1 \times_{G_0} G_1$$

and hence, $(H_1^{G_1} \times_{B_1} B_2 \times_{B_1} H_1^{G_1}) \times_{B_1} G_1 \longrightarrow (H_1^{G_1} \times_{B_1} G_1) \times_{B_0} (H_1^{G_1} \times_{B_1} G_1) \longrightarrow H_1 \times_{H_0} H_1 \xrightarrow{c} H_1$, whose transpose gives the desired morphism. As in the case of $\mathcal{C} = \mathbf{Sets}$, this defines the exponential $H^G \longrightarrow B$. □

Remark 4.6. Since each one of our fibrations in $\mathbf{Gpd}(\mathcal{C})$ is a fibration in $\mathbf{Cat}(\mathcal{C})$ as used in [21], Theorem 4.5 describes a special case of Theorem 2.17 in that paper. We include the proof given here, because it gives an explicit construction of the exponential groupoid in the slice category and shows where each assumption is used.

By Theorem 4.5, a fibration $q \colon G \longrightarrow B$ is exponentiable in $\mathbf{Gpd}(\mathbf{Top})/B$, if each $q_i \colon G_i \longrightarrow B_i$ is exponentiable in $\mathbf{Top}$, for $i = 0, 1, 2$. Now, if $\mathcal{C}/B_i$ is cartesian closed, for $i = 0, 1, 2$, then every fibration is exponentiable in $\mathbf{Gpd}(\mathcal{C})/B$. This is the case when $\mathcal{C}$ is cartesian closed and each diagonal $\Delta \colon B_i \longrightarrow B_i \times B_i$ is exponentiable in $\mathcal{C}$, e.g., $\mathcal{C} = \mathbf{Top}_{\mathcal{M}}$ and the B_i are locally $\mathcal{M}$-Hausdorff. By the following lemma, we need not assume the $i = 2$ case.

Lemma 4.7. *Suppose $\mathcal{C}$ is a finitely complete category.*

(a) If X and Y have exponentiable diagonals, then so does $X \times Y$.

(b) If B is a groupoid in $\mathcal{C}$ and B_1 has an exponentiable diagonal, then so does B_2.

Proof. For (a), suppose X and Y have exponentiable diagonals. Then the diagonal on $X \times Y$ is exponentiable, since it can be factored

$$X \times Y \xrightarrow{id_X \times \Delta} X \times (Y \times Y) \xrightarrow{\Delta \times id_{Y \times Y}} (X \times X) \times (Y \times Y) \xrightarrow{\varphi} (X \times Y) \times (X \times Y)$$

where the first two morphisms are exponentiable being pullbacks of exponentiables and φ is an isomorphism.

For (b), suppose B_1 has an exponentiable diagonal. Then $B_1 \times B_1$ does, by (a). Since there is a monomorphism $\psi\colon B_2 \longrightarrow B_1 \times B_1$, we see that the diagram

$$\begin{array}{ccc} B_2 & \xrightarrow{\Delta} & B_2 \times B_2 \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\psi\times\psi} \\ B_1 \times B_1 & \xrightarrow[\Delta]{} & (B_1 \times B_1) \times (B_1 \times B_1) \end{array}$$

is a pullback, and it follows that B_2 has an exponentiable diagonal. □

Thus, we get the following corollaries to Theorem 4.5:

Corollary 4.8. *If G_0, G_1, and G_2 are exponentiable spaces, and B_0 and B_1 are locally Hausdorff, then every fibration $q\colon G \longrightarrow B$ is exponentiable in* $\mathbf{Gpd}(\mathbf{Top})$.

Corollary 4.9. *If B_0 and B_1 have exponentiable diagonals in a cartesian closed category $\mathcal{C}$, then every fibration $q\colon G \longrightarrow B$ is exponentiable in* $\mathbf{Gpd}(\mathcal{C})$.

Corollary 4.10. *Every fibration is exponentiable in* $\mathbf{Gpd}(\mathbf{Top}_{\mathcal{M}})/B$, *if B_0 and B_1 are locally $\mathcal{M}$-Hausdorff.*

Corollary 4.11. *The following are equivalent.*

(a) $s\colon B^{\mathbb{I}} \longrightarrow B$ is exponentiable in $\mathbf{Gpd}(\mathcal{C})$.

(b) $s\colon B_1 \longrightarrow B_0$ is exponentiable in $\mathcal{C}$.

(c) $t\colon B^{\mathbb{I}} \longrightarrow B$ is exponentiable in $\mathbf{Gpd}(\mathcal{C})$.

(d) $t\colon B_1 \longrightarrow B_0$ is exponentiable in $\mathcal{C}$.

Proof. Since $si = t$ and i is an isomorphism, we know (b) and (d) are equivalent. We will establish the equivalence of (a) and (b). The proof for (c) and (d) is similar.

First, (a) implies (b) follows from the remark at the beginning of this section. For the converse, it suffices to show that $s_1\colon B_1^{\mathbb{I}} \longrightarrow B_1$ and $s_2\colon B_2^{\mathbb{I}} \longrightarrow B_2$

are exponentiable in $\mathcal{C}$, since $s\colon B^{\mathbb{I}} \longrightarrow B$ is a fibration by Proposition 4.4. We know the first one is exponentiable, as it is given by

$$s_1\colon B_2 \times_{B_1} B_2 \xrightarrow{\pi_2} B_2 \xrightarrow{\pi_1} B_1$$

which is a composition of exponentiables when $s\colon B_1 \longrightarrow B_0$ is exponentiable, since the diagrams

$$\begin{array}{ccc} B_2 \times_{B_1} B_2 & \xrightarrow{\pi_2} & B_2 \\ {\scriptstyle \pi_1\pi_1}\downarrow & & \downarrow{\scriptstyle s\pi_1} \\ B_1 & \xrightarrow[s]{} & B_0 \end{array} \qquad\qquad \begin{array}{ccc} B_2 & \xrightarrow{\pi_2} & B_1 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle t} \\ B_1 & \xrightarrow[s]{} & B_0 \end{array}$$

are pullbacks in $\mathcal{C}$. To see that $s_2\colon B_2^{\mathbb{I}} \longrightarrow B_2$ is exponentiable, note that $s_2 = \pi_1\pi_2 \times \pi_2\pi_1$ and the square

$$\begin{array}{ccc} B_2^{\mathbb{I}} & \xrightarrow{\pi_1\times\pi_2} & B_2 \times_{B_0} B_2 \\ {\scriptstyle s_2}\downarrow & & \downarrow{\scriptstyle c(c\times c)} \\ B_2 & \xrightarrow[c]{} & B_1 \end{array}$$

is a pullback. Thus, it suffices to show that

$$B_2 \times_{B_0} B_2 = (B_1 \times_{B_0} B_1) \times_{B_0} (B_1 \times_{B_0} B_1) \xrightarrow{c\times c} B_1 \times_{B_0} B_1 \xrightarrow{c} B_1$$

is exponentiable. Since s is exponentiable and

$$\begin{array}{ccc} B_1 \times_{B_0} B_1 & \xrightarrow{\pi_1} & B_1 \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle s} \\ B_1 & \xrightarrow[s]{} & B_0 \end{array}$$

is a pullback, we know c is exponentiable. Since

$$\begin{array}{ccc} (B_1 \times_{B_0} B_1) \times_{B_0} (B_1 \times_{B_0} B_1) & \xrightarrow{\pi_2\pi_1\times\pi_1\pi_2} & B_1 \times_{B_0} B_1 \\ {\scriptstyle c\times c}\downarrow & & \downarrow{\scriptstyle s\pi_2} \\ B_1 \times_{B_0} B_1 & \xrightarrow[s\pi_2]{} & B_0 \end{array}$$

is a pullback and $s\pi_2$ is a composition of exponentiable morphisms, it follows that $c \times c$ is exponentiable. $\square$

Corollary 4.12. *If $s\colon B_1 \longrightarrow B_0$ (respectively, $t\colon B_1 \longrightarrow B_0$) and $q_i\colon G_i \longrightarrow B_i$ are exponentiable in $\mathcal{C}$, for $i = 0, 1, 2$, then $s\pi_1\colon B^{\mathbb{I}} \times_B G \longrightarrow B$ (respectively, $t\pi_2\colon G \times_B B^{\mathbb{I}} \longrightarrow B$) is exponentiable in $\mathbf{Gpd}(\mathcal{C})/B$.*

Proof. Since pullback and composition preserve exponentiability, the result follows from Proposition 4.4, Theorem 4.5, and Corollary 4.11. □

Corollary 4.13. *If B_0 and B_1 have exponentiable diagonals in a cartesian closed category $\mathcal{C}$, then $s\pi_1\colon B^{\mathbb{I}} \times_B G \longrightarrow B$ and $t\pi_2\colon G \times_B B^{\mathbb{I}} \longrightarrow B$ are exponentiable in $\mathbf{Gpd}(\mathcal{C})$, for all $q\colon G \longrightarrow B$.*

Proof. By Lemma 4.7(b), since B_1 has an exponentiable diagonal, so does B_2. Thus, applying Proposition 2.2, we see that every morphism $X \longrightarrow B_i$ is exponentiable in $\mathcal{C}$, for $i = 0, 1, 2$, and so the desired result follows from Corollary 4.12. □

Corollary 4.14. *If G_0, G_1, G_2, and B_1 are exponentiable spaces and B_0 and B_1 are locally Hausdorff, then $s\pi_1\colon B^{\mathbb{I}} \times_B G \longrightarrow B$ and $t\pi_2\colon G \times_B B^{\mathbb{I}} \longrightarrow B$ are exponentiable in $\mathbf{Gpd}(\mathbf{Top})$, for all $q\colon G \longrightarrow B$.*

Corollary 4.15. *If B_0 and B_1 are locally $\mathcal{M}$-Hausdorff, then $s\pi_1\colon B^{\mathbb{I}} \times_B G \longrightarrow B$ and $t\pi_2\colon G \times_B B^{\mathbb{I}} \longrightarrow B$ are exponentiable in $\mathbf{Gpd}(\mathbf{Top}_{\mathcal{M}})$, for all $q\colon G \longrightarrow B$.*

5. Pseudo-Exponentiability of Morphisms of Groupoids

In this section, we use a general theorem from Niefield [17] for monads and their Kleisli categories to show that $G \longrightarrow B$ is pseudo-exponentiable in $\mathbf{Gpd}(\mathcal{C})//B$ if $s\pi_1\colon B^{\mathbb{I}} \times_B G \longrightarrow B$ is exponentiable in $\mathbf{Gpd}(\mathcal{C})/B$, e.g., $s\colon B_1 \longrightarrow B_0$ and $G_i \longrightarrow B_i$ are exponentiable in $\mathcal{C}$, for $i = 0, 1, 2$. Consequently, $\mathbf{Gpd}(\mathcal{C})//B$ is pseudo-cartesian closed whenever B_0 and B_1 have exponentiable diagonals in a cartesian closed category $\mathcal{C}$. In particular, $\mathbf{Gpd}(\mathcal{C})$ is locally pseudo-cartesian closed when $\mathcal{C}$ is locally cartesian closed, e.g., $\mathcal{C} = \mathbf{Sets}$.

The general result in [17], i.e.,Theorem 3.4, was proved for pseudo-monads on a bicategory since one of the examples there was not a 2-category. Restricting to the strict case we get:

Theorem 5.1. *Suppose $\mathcal{K}$ is a 2-category with finite 2-products and T, μ, η is a 2-monad on $\mathcal{K}$ such that $\eta T \cong T\eta$ and the induced morphism*

$$\rho\colon T(X \times TY) \longrightarrow TX \times TY$$

is an isomorphism, for all X, Y in $\mathcal{K}$. If TY is 2-exponentiable in $\mathcal{K}$, then Y is pseudo-exponentiable in the Kleisli 2-category $\mathcal{K}_T$.

Before applying this theorem to $\mathcal{K} = \mathbf{Gpd}(\mathcal{C})/B$, we recall the definition of pseudo-exponentiability. First, a diagram

$$\begin{array}{ccc} & X \times Y & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ X & & Y \end{array}$$

is a *pseudo-product* in a 2-category $\mathcal{K}$ if the induced functor

$$\mathcal{K}(Z, X \times Y) \xrightarrow{\varphi_Z} \mathcal{K}(Z, X) \times \mathcal{K}(Z, Y)$$

is an equivalence of categories, for all Z. Since the definition of 2-product requires that φ_Z is an isomorphism, for all Z, it follows that every 2-product is necessarily a pseudo-product in $\mathcal{K}$. An object Y is *pseudo-exponentiable* if the pseudo-functor $- \times Y\colon \mathcal{K} \longrightarrow \mathcal{K}$ has a right pseudo-adjoint, i.e., for every object Z, there is an object Z^Y together with an equivalence

$$\mathcal{K}(X \times Y, Z) \xrightarrow{\theta_{X,Z}} \mathcal{K}(X, Z^Y)$$

which are pseudo-natural in X and Z.

As before, we are assuming that $\mathcal{C}$ is a finitely complete category with finite coproducts. Then there is an internal groupoid

$$B^{\mathbb{I}} \times_B B^{\mathbb{I}} \xrightarrow{c} B^{\mathbb{I}} \overset{i}{\circlearrowleft} \underset{t}{\overset{s}{\underset{\longleftarrow u \longrightarrow}{\longrightarrow}}} B$$

in $\mathbf{Gpd}(\mathcal{C})$, where as usual, we write $B^{\mathbb{I}}$ on the left of $\times_B$, when $t\colon B^{\mathbb{I}} \longrightarrow B$ and on the right when $s\colon B^{\mathbb{I}} \longrightarrow B$. Note that s and t are as in Section 4 and

c, i and u are defined analogously. Thus, as in [17] (see also Street [20]), we get a monad on $\mathbf{Gpd}(\mathcal{C})/B$ defined by

$$T(G \xrightarrow{q} B) = B^{\mathbb{I}} \times_B G \xrightarrow{s\pi_1} B \qquad \eta\colon G \xrightarrow{\langle uq, id\rangle} B^{\mathbb{I}} \times_B G$$

$$\mu\colon B^{\mathbb{I}} \times_B B^{\mathbb{I}} \times_B G \xrightarrow{c \times id} B^{\mathbb{I}} \times_B G$$

and it is not difficult to show that the 2-Kleisli category is (isomorphic to) the pseudo-slice $\mathbf{Gpd}(\mathcal{C})//B$ whose objects are homomorphism $q\colon G \longrightarrow B$, morphisms are triangles

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ & {}_{q}\searrow \;\overset{\varphi}{\to}\; \swarrow_{r} & \\ & B & \end{array} \qquad \text{or equivalently} \qquad \begin{array}{ccc} G & \xrightarrow{\langle \hat{\varphi}, f\rangle} & B^{\mathbb{I}} \times_B H \\ & {}_{q}\searrow \quad \swarrow_{s\pi_1} & \\ & B & \end{array}$$

and 2-cells $\theta\colon (f, \varphi) \longrightarrow (g, \psi)$ are 2-cells $\theta\colon f \longrightarrow g$ such that

$$\begin{array}{ccc} & q & \\ {}^{\varphi}\swarrow & & \searrow^{\psi} \\ rf & \xrightarrow[r\theta]{} & rg \end{array}$$

To show that $\rho\colon B^{\mathbb{I}} \times_B (G \times_B B^{\mathbb{I}} \times_B H) \longrightarrow (B^{\mathbb{I}} \times_B G) \times_B (B^{\mathbb{I}} \times_B H)$ is an isomorphism, note that $\pi_i \rho = \pi_i$, for $i = 1, 2, 4$, and

$$\begin{array}{ccc} B^{\mathbb{I}} \times_B (G \times_B B^{\mathbb{I}} \times_B H) & \xrightarrow{\rho} & (B^{\mathbb{I}} \times_B G) \times_B (B^{\mathbb{I}} \times_B H) \\ {\scriptstyle \langle \pi_1, \pi_3\rangle}\downarrow & & \downarrow{\scriptstyle \pi_3} \\ B^{\mathbb{I}} \times_B B^{\mathbb{I}} & \xrightarrow[c]{} & B^{\mathbb{I}} \end{array}$$

Then one can show that ρ is invertible with $\pi_i \rho^{-1} = \pi_i$, for $i = 1, 2, 4$, and

$$\begin{array}{ccc} (B^{\mathbb{I}} \times_B G) \times_B (B^{\mathbb{I}} \times_B H) & \xrightarrow{\rho^{-1}} & B^{\mathbb{I}} \times_B (G \times_B B^{\mathbb{I}} \times_B H) \\ {\scriptstyle \langle i\pi_1, \pi_3\rangle}\downarrow & & \downarrow{\scriptstyle \pi_3} \\ B^{\mathbb{I}} \times_B B^{\mathbb{I}} & \xrightarrow[c]{} & B^{\mathbb{I}} \end{array}$$

To show $\eta T \cong T\eta$, it suffices to show $(\eta T)_{B^{\mathbb{I}}} \cong T\eta_{B^{\mathbb{I}}}$, where $t\colon B^{\mathbb{I}} \longrightarrow B$, since $\eta_G = \eta_{B^{\mathbb{I}}} \times_B G$. Now, $(\eta T)_{B^{\mathbb{I}}}$ and $T\eta_{B^{\mathbb{I}}}$ are given by

$$B^{\mathbb{I}} \xrightarrow{\langle s, id\rangle} B \times_B B^{\mathbb{I}} \xrightarrow{\langle u, id\rangle} B^{\mathbb{I}} \times_B B^{\mathbb{I}} \quad \text{and} \quad B^{\mathbb{I}} \xrightarrow{\langle id, t\rangle} B^{\mathbb{I}} \times_B B \xrightarrow{\langle id, u\rangle} B^{\mathbb{I}} \times_B B^{\mathbb{I}}$$

Then one can show that the desired isomorphism is induced by the following morphism $\theta\colon (B^{\mathrm{II}})_0 \longrightarrow (B^{\mathrm{II}})_1 \times_{B_1} (B^{\mathrm{II}})_1$. First, recall that

$$(B^{\mathrm{II}})_0 \cong B_1 \quad \text{and} \quad (B^{\mathrm{II}})_1 \times_{B_1} (B^{\mathrm{II}})_1 \cong (B_2 \times_{B_1} B_2) \times_{B_1} (B_2 \times_{B_1} B_2)$$

Then $\pi_1\theta$ and $\pi_2\theta$ are given by

$$B_1 \xrightarrow{\langle us,id,us,id\rangle} (B_1 \times_{B_0} B_1) \times_{B_1} (B_1 \times_{B_0} B_1) \cong B_2 \times_{B_1} B_2$$

and

$$B_1 \xrightarrow{\langle id,ut,id,ut\rangle} (B_1 \times_{B_0} B_1) \times_{B_1} (B_1 \times_{B_0} B_1) \cong B_2 \times_{B_1} B_2$$

respectively.

Theorem 5.2. *If $s\colon B_1 \longrightarrow B_0$ and $q_i\colon G_i \longrightarrow B_i$ are exponentiable in $\mathcal{C}$, for $i = 0, 1, 2$, then $q\colon G \longrightarrow B$ is pseudo-exponentiable in $\mathbf{Gpd}(\mathcal{C})/\!\!/B$.*

Proof. Apply Corollary 4.12 and Theorem 5.1. □

In particular, we get the following corollaries:

Corollary 5.3. *If G_0, G_1, G_2, and B_1 are exponentiable (e.g., locally compact) and B_0 and B_1 are locally Hausdorff spaces, then every morphism $q\colon G \longrightarrow B$ is pseudo-exponentiable in $\mathbf{Gpd}(\mathbf{Top})$.*

Corollary 5.4. *If B_0 and B_1 have exponentiable diagonals in a cartesian closed category $\mathcal{C}$, then $\mathbf{Gpd}(\mathcal{C})/\!\!/B$ is pseudo-cartesian closed.*

Corollary 5.5. *If B_0 and B_1 are locally $\mathcal{M}$-Hausdorff, then $\mathbf{Gpd}(\mathbf{Top}_{\mathcal{M}})/\!\!/B$ is pseudo-cartesian closed.*

Corollary 5.6. *If B_0 and B_1 are compactly generated weak Hausdorff spaces, then $\mathbf{Gpd}(\mathbf{Top}_{\mathcal{K}})/\!\!/B$ is pseudo-cartesian closed.*

Corollary 5.7. *If $\mathcal{C}$ is locally cartesian closed, then $\mathbf{Gpd}(\mathcal{C})$ is locally pseudo-cartesian closed.*

Corollary 5.8. $\mathbf{Gpd}(\mathbf{Sets})$ *is locally pseudo-cartesian closed.*

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