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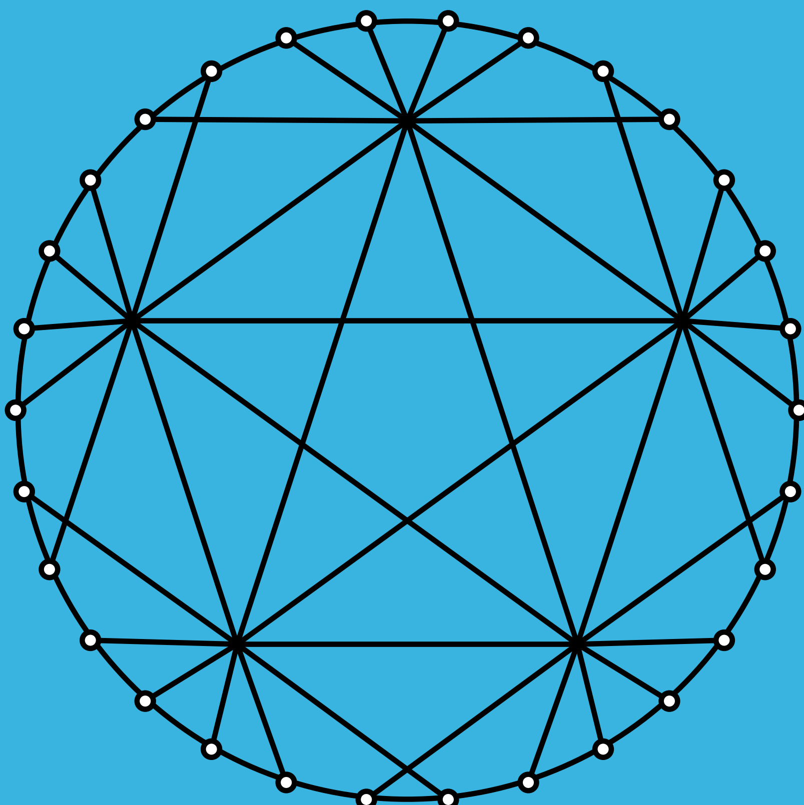

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Some matrix constructions of L_2-type Latin square designs

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Abstract. Several methods of construction of L_2-type Latin square designs by various authors are scattered over literature, see Clatworthy [2] and elsewhere. We have constructed L_2-type Latin square designs from combinatorial matrices including Hadamard matrices, Generalized Bhaskar Rao designs, circulant matrices, mutually orthogonal Latin squares and others. These constructions yield solutions of all L_2-type Latin square designs listed in Clatworthy [2] except one.

1 Introduction

Several methods of constructions of L_2-type Latin square design may be found in Dey [3, 4], Raghavarao [7] and Raghavarao and Padgett [8]. By using matrix approaches, solutions of all the L_2-type Latin square designs listed in Clatworthy [2] are obtained except one. Some of the series obtained here may be new as these are not found in Dey [3, 4], Raghavarao [7] and Raghavarao and Padgett [8]. This paper is in sequel to the paper by Saurabh and Singh [10]. We recall some relevant definitions in the context of the paper.

A *balanced incomplete block design* (BIBD) or a 2-(v, k, λ) *design* is an arrangement of v treatments in b blocks, each of size k $(< v)$ such that

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every treatment occurs in exactly r blocks and any two distinct treatments occur together in λ blocks. v, b, r, k, λ are called parameters of the BIBD. A BIBD is *symmetric* if $v = b$ and is *self-complementary* if $v = 2k$.

An $n \times n$ matrix $H = [H_{ij}]$ with entries H_{ij} as ± 1 is called a *Hadamard matrix* if $HH' = H'H = nI_n$, where H' is the transpose of H and I_n is the identity matrix of order n. A Hadamard matrix is in normalized form if its first row and first column contain only +1s.

A *generalized Bhaskar Rao design* GBRD $(v, b, r, k, \lambda; G)$ over a group G is a $v \times b$ array with entries from $G \cup \{0\}$ such that:

1. Every row has exactly r group element entries;
2. Every column has exactly k group element entries;
3. For every pair of distinct rows $(x_1, x_2, \ldots, x_b)$ and $(y_1, y_2, \ldots, y_b)$, the multi-set $\{x_i y_i^{-1} : i = 1, 2, \ldots, b; x_i, y_i \neq 0\}$ contains each group element exactly $\lambda/|G|$ times.

When $|G| = 2$, such a design is a *Bhaskar Rao design.* A *generalized Bhaskar Rao design* GBRD$(v, b, r, k, \lambda; G)$ with $v = b$ and $r = k$ is also known as a balanced *generalized weighing matrix* BGW$(v, k, \lambda; G)$. A *generalized Hadamard matrix* GH(λ, g) over a group G of order g is a balanced generalized weighing matrix with $v = b = k = r = \lambda$. For details we refer to Ionin and Kharghani [6], Abel et al. [1]) and Tonchev [11].

A Latin square of order n is an $n \times n$ array on n symbols such that each of the n symbols occurs exactly once in each row and each column. The *join of two Latin squares* $A = \left[a_{ij}\right]_{1 \leq i,j \leq n}$ and $B = \left[b_{ij}\right]_{1 \leq i,j \leq n}$ is the $n \times n$ array whose (i, j)-th entry is the ordered pair (a_{ij}, b_{ij}). Two Latin squares are *orthogonal* if the join of A and B contains every ordered pair exactly once. A set of Latin squares are *mutually orthogonal Latin squares* (MOLS) if they are pairwise orthogonal.

Let $v = n^2$ treatments be arranged in an $n \times n$ array. An L_2-type Latin square design is an arrangement of the $v = n^2$ treatments in b blocks each of size k such that:

1. Every treatment occurs at most once in a block;
2. Every treatment occurs in r blocks;
3. Every pair of treatments, which are in the same row or in the same column of the $n \times n$ array, occur together in λ_1 blocks; while every other pair of treatments occur together in λ_2 blocks.

The parameters of the L_2-type Latin square design are $v = n^2$, b, r, k, λ_1 and λ_2 and they satisfy the relations: $bk = vr$; $2(n-1)\lambda_1 + (n-1)^2\lambda_2 = r(k-1)$. Let N be the incidence matrix of an L_2-type Latin square design with $v = n^2$ treatments then the structure of NN' is:

$$NN' = \begin{bmatrix} (r-\lambda_1)I_n + \lambda_1 J_n) & (\lambda_1-\lambda_2)I_n + \lambda_2 J_n) & \cdots & (\lambda_1-\lambda_2)I_n + \lambda_2 J_n) \\ (\lambda_1-\lambda_2)I_n + \lambda_2 J_n) & (r-\lambda_1)I_n + \lambda_1 J_n) & \cdots & (\lambda_1-\lambda_2)I_n + \lambda_2 J_n) \\ \vdots & \vdots & \ddots & \vdots \\ (\lambda_1-\lambda_2)I_n + \lambda_2 J_n) & (\lambda_1-\lambda_2)I_n + \lambda_2 J_n) & \cdots & (r-\lambda_1)I_n + \lambda_1 J_n) \end{bmatrix}.$$

Notations: I_n is the identity matrix of order n, $J_{v\times b}$ is the $v\times b$ matrix all of whose entries are 1, $J_{v\times v} = J_v$, $[A|B]$ is the juxtaposition of two matrices A and B; $A \otimes B$ is the Kronecker product of two matrices A and B and LSX numbers are from Clatworthy [2].

2 Matrix constructions

Method I: From I and J matrices.

The following Theorem is the same as the Theorem 4.4.18 of Dey [4, p. 110]. Here the proof is given using a matrix approach.

Theorem 2.1. *There exists a symmetric L_2-type Latin square design with parameters:*

$$v = b = n^2,\ r = k = 2n-1,\ \lambda_1 = n,\ \lambda_2 = 2. \tag{1}$$

Proof. $N = I_n \otimes J_n + (J-I)_n \otimes I_n$ is the incidence matrix of the symmetric L_2-type Latin square design with parameters (1). □

Theorem 2.2. *There exists a symmetric L_2-type Latin square design with parameters:*

$$v = b = n^2,\ r = k = 2(n-1),\ \lambda_1 = (n-2),\ \lambda_2 = 2. \tag{2}$$

Proof. $N = I_n \otimes (J-I)_n + (J-I)_n \otimes I_n$ is the incidence matrix of the symmetric L_2-type Latin square design with parameters (2) □

Theorem 2.3. *There exists an L_2-type Latin square design with parameters:*

$$\begin{array}{lll} v = n^2, & b = 2n^2, & r = 2(n-1), \\ k = n-1, & \lambda_1 = n-2, & \lambda_2 = 0. \end{array} \tag{3}$$

Proof. $N = [I_n \otimes (J-I)_n | (J-I)_n \otimes I_n]$ is the incidence matrix of the L_2-type Latin square design with parameters (3). □

Method II: From 2-(v, k, λ) designs and Hadamard Matrices.

Theorem 2.4. *The existence of a* 2-(v, k, λ) *design implies the existence of an L_2-type Latin square design with parameters:*

$$\begin{array}{lll} v^* = v^2, & b^* = b^2, & r^* = r^2, \\ k^* = k^2, & \lambda_1 = \lambda r, & \lambda_2 = \lambda^2. \end{array} \tag{4}$$

Proof. Let $N_{v\times b}$ be the incidence matrix of a 2-(v, k, λ) design. Since the inner product of any two distinct rows of the 2-(v, k, λ) design is λ, $N_{v\times b} \otimes N_{v\times b}$ is the incidence matrix of L_2-type Latin square design with parameters (4) □

The following Theorem is the same as the Theorem 4.4.17 of Dey [4, p. 109]. Here the proof is given using matrix approach.

Theorem 2.5. *The existence of a* 2-(v, k, λ) *design implies the existence of an L_2-type Latin square design with parameters:*

$$\begin{array}{lll} v^* = v^2, & b^* = 2vb, & r^* = 2r, \\ k^* = k, & \lambda_1 = \lambda, & \lambda_2 = 0. \end{array} \tag{5}$$

Proof. Let be the incidence matrix of a 2-(v, k, λ) design. Then

$$N = [I_v \otimes M_{v\times b} | M_{v\times b} \otimes I_v]$$

is the incidence matrix of the L_2-type Latin square design with parameters(5) □

The following Theorem is the same as the Theorem 4.4.16 of Dey [4, p. 109]. Here the proof is given using matrix approach.

Theorem 2.6. *The existence of a* 2-(v,k,λ) *design implies the existence of an* L_2*-type Latin square design with parameters:*

$$\begin{array}{lll} v^* = v^2, & b^* = 2b, & r^* = 2r, \\ k^* = vk, & \lambda_1 = r+\lambda, & \lambda_2 = 2\lambda. \end{array} \tag{6}$$

Proof. Let M be the incidence matrix of a 2-(v,k,λ) design. Let $R_i = (\ldots 11 \ldots 000 \ldots 1 \ldots)$, $i \le i \le v$ be the i^{th} row of M with 1 at i_1^{th} , i_2^{th} , i_ℓ^{th} positions and 0 elsewhere. Then corresponding to each R_i we form a $v \times b$ matrix Γ_i whose i_1^{th} , i_2^{th} , i_ℓ^{th} columns have entries ones and zero elsewhere. Since each row sum of M is r and any pair of distinct rows have 1 at λ positions we have: $\Gamma_i \Gamma_j' = \begin{cases} rJ_v, & i=j \\ \lambda J_v, & i \neq j \end{cases}$. Then

$$\begin{bmatrix} M & \Gamma_1 \\ M & \Gamma_2 \\ \vdots & \vdots \\ M & \Gamma_v \end{bmatrix}$$

represents an L_2-type Latin square design with parameters (6) □

Example 1: Consider a 2-$(4,2,1)$ design whose incidence matrix M is:

$$M = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

Then using Theorem 2.6 the incidence matrix of LS98: $v = 16$, $b = 12$, $r = 6$, $k = 8$, $\lambda_1 = 4$, $\lambda_2 = 2$ is given as:

$$N = \begin{bmatrix} M & \Gamma_1 \\ M & \Gamma_2 \\ M & \Gamma_3 \\ M & \Gamma_4 \end{bmatrix} = \left[\begin{array}{c|cccccc} & 1 & 0 & 1 & 0 & 1 & 0 \\ M & 1 & 0 & 1 & 0 & 1 & 0 \\ & 1 & 0 & 1 & 0 & 1 & 0 \\ & 1 & 0 & 1 & 0 & 1 & 0 \\ \hline & 1 & 0 & 0 & 1 & 0 & 1 \\ M & 1 & 0 & 0 & 1 & 0 & 1 \\ & 1 & 0 & 0 & 1 & 0 & 1 \\ & 1 & 0 & 0 & 1 & 0 & 1 \\ \hline & 0 & 1 & 1 & 0 & 0 & 1 \\ M & 0 & 1 & 1 & 0 & 0 & 1 \\ & 0 & 1 & 1 & 0 & 0 & 1 \\ & 0 & 1 & 1 & 0 & 0 & 1 \\ \hline & 0 & 1 & 0 & 1 & 1 & 0 \\ M & 0 & 1 & 0 & 1 & 1 & 0 \\ & 0 & 1 & 0 & 1 & 1 & 0 \\ & 0 & 1 & 0 & 1 & 1 & 0 \end{array}\right]$$

Theorem 2.7. *The existence of a Hadamard matrix of order 4t and a self-complementary* 2-(v, k, λ) *design satisfying* $(4t-1)\lambda = (2t-1)r$ *and* $4tv$ *a perfect square implies the existence of an* L_2*-type Latin square design with parameters:*

$$\begin{array}{lll} v^* = 4tv, & b^* = (4t-1)b, & r^* = (4t-1)r, \\ k^* = 4tk, & \lambda_1 = (4t-1)\lambda, & \lambda_2 = 2tr - \lambda. \end{array} \tag{7}$$

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Proof. Let N be the incidence matrix of a self-complementary 2-(v, k, λ) design and H be a $4t \times (4t-1)$ matrix obtained by deleting the first column of a normalized Hadamard matrix of order $4t$. Then replacing 1 by N and -1 by $J - N$ in H, we obtain the incidence matrix of an L_2-type Latin square design with parameters (7). □

Example 2: For $t = 2$ and a 2-$(8, 4, 3)$ design, Theorem 2.7 yields L_2-type Latin square design with parameters: $v = 64$, $b = 98$, $r = 49$, $k = 32$, $\lambda_1 = 21$, $\lambda_2 = 25$.

Method III: From mutually orthogonal Latin squares.

Here we define B_i^j, $1 \leq i, j \leq q$, as a $q \times q$ matrix whose j^{th} row is $(\cdots 011 \cdots 1)$ with 0 at the (j, i)-th position, 1 elsewhere and the entries of remaining rows are zero. Then

(i) $B_i^j(B_i^k)' = (q-1)$ at (j,k)-th position irrespective of j and k;

(ii) For $i \neq j$; $B_i^k\big(B_j^\ell\big)' = (q-1)$ at (k,ℓ)-th position irrespective of k and ℓ.

Example 3: For $q=4$, $B_1^1 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$; $B_1^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Theorem 2.8. *There exists an L_2-type Latin square design with parameters:*

$$\begin{array}{lll} v = q^2, & b = q^2(q-1), & r = (q-1)^2, \\ k = q-1, & \lambda_1 = 0, & \lambda_2 = q-2, \end{array} \tag{8}$$

where q is a prime or prime power.

Proof. Consider a set of $q\times q$ matrices $S = \{B_i^j : 1 \leq i,j \leq q\}$. Let $\mathrm{GF}(q) = \{1,2,3,\ldots,q\}$ be a finite field of order q after renaming the elements. It is known, see Furino et al. [5, p. 10] that there exist $q-1$ MOLS of order q.

Let

$$L_i = \begin{bmatrix} 1 & 2 & \cdots & q \\ q & 1 & \cdots & q-1 \\ \vdots & \vdots & \ddots & \vdots \\ 2 & 3 & \cdots & 1 \end{bmatrix}$$

be one of the $q-1$ MOLS. Corresponding to each L_i we form a $q\times q$ matrix N_i as:

$$N_i = \begin{bmatrix} B_1^1 & B_1^2 & \cdots & B_1^q \\ B_2^q & B_2^1 & \cdots & B_2^{q-1} \\ \vdots & \vdots & \ddots & \vdots \\ B_q^2 & B_q^3 & \cdots & B_q^1 \end{bmatrix}.$$

Then $N = [N_1|N_2|\cdots|N_{q-1}]$ represents an L_2-type Latin square design with parameters (8). $\square$

Example 4: Consider a set of MOLS of order 4:

$$M = \left[\begin{array}{cccc|cccc|cccc} 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 & 3 & 4 & 1 & 2 & 2 & 1 & 4 & 3 \\ 2 & 1 & 4 & 3 & 4 & 3 & 2 & 1 & 3 & 4 & 1 & 2 \\ 3 & 4 & 1 & 2 & 2 & 1 & 4 & 3 & 4 & 3 & 2 & 1 \end{array}\right]$$

Then

$$N = \left[\begin{array}{cccc|cccc|cccc} B_1^1 & B_1^2 & B_1^3 & B_1^4 & B_1^1 & B_1^2 & B_1^3 & B_1^4 & B_1^1 & B_1^2 & B_1^3 & B_1^4 \\ B_2^4 & B_2^3 & B_2^2 & B_2^1 & B_2^3 & B_2^4 & B_2^1 & B_2^2 & B_2^2 & B_2^1 & B_2^4 & B_2^3 \\ B_3^2 & B_3^1 & B_3^4 & B_3^3 & B_3^4 & B_3^3 & B_3^2 & B_3^1 & B_3^3 & B_3^4 & B_3^1 & B_3^2 \\ B_4^3 & B_4^4 & B_4^1 & B_4^2 & B_4^2 & B_4^1 & B_4^4 & B_4^3 & B_4^4 & B_4^3 & B_4^2 & B_4^1 \end{array}\right]$$

represents LS20: $v = 16$, $b = 48$, $r = 9$, $k = 3$, $\lambda_1 = 0$, $\lambda_2 = 2$.

Method IV: From E_i, $1 \le i \le n$, and I matrices.

On page 229 of [12] van Lint and Wilson used E_i-matrices, $1 \le i \le 3$, in the construction of a 2-$(9, 3, 1)$ design where E_i denotes a 3 by 3 matrix with ones in column i and zero elsewhere. Here we have used such types of matrices in the construction of L_2-type Latin square designs.

Theorem 2.9. *There exists an L_2-type Latin square design with parameters:*

$$\begin{array}{lll} v = n^2, & b = 2n, & r = 2, \\ k = n, & \lambda_1 = 1, & \lambda_2 = 0. \end{array} \tag{9}$$

Proof. Let E_i, $1 \le i \le n$, denote an $n \times n$ matrix whose i^{th} column contains only $+1$s and 0 elsewhere. Then

$$N = \begin{bmatrix} E_1 & I_n \\ E_2 & I_n \\ \vdots & \vdots \\ E_n & I_n \end{bmatrix}$$

represents the incidence matrix of the L_2-type Latin square design with parameters (9). □

Method V: From generalized Bhaskar Rao designs over EA(g).

As GH(g, λ) is a special case of GBRD$(v, b, r, k, \lambda; G)$, the following Theorem follows from the method in Section 2 of Sarvate and Seberry [9] with slight modification.

Theorem 2.10. *There exists an L_2-type Latin square design with parameters:*

$$\begin{array}{lll} v^* = g^2, & b^* = g[t(g-1)+2s], & r^* = t(g-1)+2s, \\ k^* = g, & \lambda_1 = s, & \lambda_2 = t, \end{array} \tag{10}$$

where g is a prime or prime power.

Proof. Let M be a $g \times (g-1)$ matrix obtained by deleting the first column of a normalized Generalized Hadamard Matrix, GH(g,g) over elementary abelian group, EA(g). Then replacing the elements of an EA(g) by the corresponding right regular $g \times g$ permutation matrices and 0 entry by $g \times g$ null matrix in M we obtain an L_2-type Latin square design with parameters:

$$\begin{array}{lll} v^* = g^2, & b^* = g(g-1), & r^* = g-1, \\ k^* = g, & \lambda_1 = 0, & \lambda_2 = 1. \end{array} \tag{11}$$

Let N_1 be the matrix obtained by taking t copies of the incidence matrix of an L_2- type Latin square design with parameters (11). Let $N_2 = s$ copies of the block matrix

$$\begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_g \end{bmatrix}$$

and $N_3 = s$ copies of the block matrix $\begin{bmatrix} I_g \\ I_g \\ \vdots \\ I_g \end{bmatrix}$ arranged column-wise where E_i, $1 \leq i \leq g$, denote a $g \times g$ matrix whose i^{th} column contains only +1s and 0 elsewhere. Then $N = [N_1|N_2|N_3]$ represents an L_2-type Latin square design with generalized parameters (10). □

3 Tables of designs

This section contains Tables 1 and 2 of L_2-type Latin square designs listed in Clatworthy [2] constructed using the present Theorems. Designs obtained by taking m copies or the complement of a design are not included in the Tables.

Table 1: Symmetric L_2-type Latin square designs

No.	LS:$(v, k, \lambda_1, \lambda_2)$	Source
1	LS26: $(9, 4, 1, 2)$	Th. 2.2;$n = 3$
2	LS83: $(16, 7, 4, 2)$	Th. 2.1;$n = 4$
3	LS101:$(25, 8, 3, 2)$	Th. 2.2;$n = 5$
4	LS117:$(25, 9, 5, 2)$	Th. 2.1;$n = 5$
5	LS118:$(49, 9, 3, 1)$	Th. 2.4;2-$(7, 3, 1)$ design
6	LS136:$(36, 10, 4, 2)$	Th. 2.2;$n = 6$

4 Conclusion

In this paper we have constructed some series of L_2-type Latin square designs using matrix approaches. These series yield patterned constructions of all the L_2-type Latin square designs listed in Clatworthy [2] except one. The series (10)) for a prime g may be found in Saurabh and Singh [10]. The series (11)) may be found in Dey [4, p. 109]. Here the proof is given using matrix approach. The series (2)), (3)), (4)), (7)), (8)) and (9)) obtained above may be new as these are not found in Dey [3, 4], Raghavarao [7] and Raghavarao and Padgett [8].

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Table 2: Asymmetric L_2-type Latin square designs

No.	LS:$(v, r, k, b, \lambda_1, \lambda_2)$	Source
1	LS1: (9, 4, 2, 18, 1, 0)	Th. 2.3; $n = 3$
2	LS3: (16, 6, 2, 48, 1, 0)	Th. 2.5; 2-$(4, 2, 1)$ design
3	LS4: (16, 9, 2, 72, 0, 1)	Unknown
4	LS5: (25, 8, 2, 100, 1, 0)	Th. 2.5; 2-$(5, 2, 1)$ design
5	LS6: (36, 10, 2, 180, 1, 0)	Th. 2.5; 2-$(6, 2, 1)$ design
6	LS7: (9, 2, 3, 6, 1, 0)	Th. 2.9; $n = 3$
7	LS12: (9, 6, 3, 18, 1, 2)	Saurabh and Singh [10]
8	LS13: (9, 8, 3, 24, 1, 3)	Saurabh and Singh [10]
9	LS14: (9, 10, 3, 30, 2, 3)	Saurabh and Singh [10]
10	LS15: (9, 10, 3, 30, 1, 4)	Saurabh and Singh [10]
11	LS16: (16, 6, 3, 32, 2, 0)	Th. 2.3; $n = 4$
12	LS20: (16, 9, 3, 48, 0, 2)	Th. 2.8; $q = 4$
13	LS23: (36, 10, 3, 120, 2,	Th. 2.5; 2-$(6, 3, 2)$ design
14	LS24: (49, 6, 3, 98, 1, 0)	Th. 2.5; 2-$(7, 3, 1)$ design
15	LS25: (81, 8, 3, 216, 1, 0)	Th. 2.5; 2-$(9, 3, 1)$ design
16	LS28: (16, 2, 4, 8, 1, 0)	Th. 2.9; $n = 4$
17	LS36: (16, 3, 4, 12, 0, 1)	Th. 2.10; GH (4, 4); $s = 0$, $t = 1$
18	LS37: (16, 7, 4, 28, 2, 1)	Th. 2.10; GH (4, 4); $s = 2$, $t = 1$.
19	LS39: (16, 9, 4, 36, 3, 1)	Th. 2.4; 2-$(4, 2, 1)$ design
20	LS42: (16, 8, 4, 32, 1, 2)	Th. 2.10; GH (4, 4); $s = 1$, $t = 2$
21	LS45: (25, 8, 4, 50, 3, 0)	Th. 2.3; $n = 5$
22	LS47: (169, 8, 4, 338, 1,	Th. 2.5; 2-$(13, 4, 1)$ design
23	LS48: (256 10, 4, 640, 1, 0)	Th. 2.5; 2-$(16, 4, 1)$ design
24	LS51: (25, 2, 5, 10, 1, 0)	Th. 2.9; $n = 5$
25	LS61: (25, 4, 5, 20, 0, 1)	Saurabh and Singh [10]
26	LS62: (25, 8, 5, 40, 2, 1)	Saurabh and Singh [10]
27	LS63: (25, 10, 5, 50, 3, 1)	Saurabh and Singh [10]
28	LS66: (25, 10, 5, 50, 1, 2)	Saurabh and Singh [10]
29	LS67: (36, 10, 5, 72, 4, 0)	Th. 2.3; $n = 6$
30	LS70: (121, 10, 5, 242, 2, 0)	Th. 2.5; 2-$(11, 5, 2)$ design
31	LS71: (441, 0, 5, 442, 1, 0)	Th. 2.5; 2-$(21, 5, 1)$ design
32	LS74: (36, 2, 6, 12, 1, 0)	Th. 2.9; $n = 6$
33	LS84: (49, 2, 7, 14, 1, 0)	Th. 2.9; $n = 7$
34	LS97: (49, 10, 7, 70, 2, 1)	Saurabh and Singh [10]
35	LS98: (16, 6, 8, 12, 4, 2)	Th. 2.6; 2-$(4, 2, 1)$ design
36	LS100: (16, 9, 8, 18, 3, 5)	Th. 2.7; H_4 and 2-$(4, 2, 1)$ design
37	LS102: (64, 2, 8, 16, 1, 0)	Th. 2.9; $n = 8$
38	LS119: (81, 2, 9, 18, 1, 0)	Th. 2.9; $n = 9$
39	LS135: (25, 8, 10, 20, 5, 2)	Th. 2.6, 2-$(5, 2, 1)$ design
40	LS137: (100, 2, 10, 20, 1, 0)	Th. 2.9; $n = 10$

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