parallel circumferences of the two circles. Similarly a two-sided figure may be made to change gradually into a one-sided one.

In particular, if s is a plane curve the solution is evidently the portion of the plane enclosed by it. If s lies on a closed convex surface, S is essentially two-sided. The functions $v(M)$, $V(M)$ are in general not necessarily continuous on s, for some of the points of s may be irregular boundary points for the conductor potential $V(M)$.

In two dimensions a similar problem has been considered in particular cases as a generalization of a well-known theorem of Koebe on conformal mapping. Pólya and Szegö consider it as a problem in transfinite diameter for a two point boundary in the plane, where the solution is the segment joining them.[2] I am indebted to Professor Szegö for the citations[3] with respect to the problem in the plane. In particular Grötzsch demonstrates by methods of conformal mapping the uniqueness of the solution for an arbitrary finite number of points in the plane.

[1] O. D. Kellogg, "Foundations of Potential Theory," p. 335 (1920).

[2] G. Pólya and G. Szegö, "Transfiniter Durchmesser ebener und räumlicher Punktmengen," *Jour. für die reine und angewandte Mathematik*, **165**, 4–49 (1931).

[3] L. Bieberbach, "Ueber die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln," *Akademie der Wissenschaften, Berlin, Zitzungsberichte*, 940–955 (1916); G. Pólya, "Beitrag zur Verallgemeinerung des Verzerrungssatzes auf mehrfach zusammenhängende Gebiete," *Ibid.*, 228–232 and 280–282 (1928–2), 55–62 (1929–2); H. Grötzsch, "Ueber ein Variations problem der konformen Abbildung," *Akademie der Wissenschaften, Leipzig, Berichte*, **82**, 251–263 (1930).

THE POLYNOMIAL OF MITTAG-LEFFLER

BY H. BATEMAN

NORMAN BRIDGE LABORATORY OF PHYSICS, CALIFORNIA INSTITUTE OF TECHNOLOGY

Communicated July 13, 1940

1. The polynomial $g_n(z) = 2zF(1 - n, 1 - z; 2; 2)$ occurs as a coefficient in the expansions

$$(1 + t)^z(1 - t)^{-z} = 1 + \sum_{n=0}^{\infty} g_n(z)t^n, \ |t| < 1 \tag{1}$$

$$2ze^t F(1 - z; 2; -2t) = \sum_{n=1}^{\infty} g_n(z)t^{n-1}/(n - 1)! \tag{2}$$

It was used by Mittag-Leffler[1] in a study of the analytical representation of the integrals and invariants of a linear homogeneous differential equation in which he made use of a conformal mapping of the t-plane on the w-plane by means of a relation

$$w = w_0(1 + t)^z(1 - t)^{-z} \tag{3}$$

in which the index z was an imaginary quantity $2b/i\pi$. The first expansion was used later in his researches on the analytical representation of a uniform branch of a monogenic function and was connected with some other expansions. The second expansion is new only in notation being merely a particular case of a well-known expansion in which the coefficients are hypergeometric functions.[2]

Pidduck[3] used an expansion equivalent to (1) in his researches on the propagation of a disturbance in a fluid acted upon by gravity. He gave the recurrence relation

$$g_n(z + 1) - g_{n-1}(z + 1) = g_n(z) + g_{n-1}(z) \tag{4}$$

which is an immediate consequence of the fact that the generating function $G(z, t) = (1 + t)^z(1 - t)^{-z}$ satisfies the functional equation

$$(1 - t)G(z + 1, t) = (1 + t)G(z, t). \tag{5}$$

A second recurrence relation

$$ng_n(z) = (n - 2)g_{n-2}(z) + 2zg_{n-1}(z) \tag{6}$$

has been given by Belorizky;[4] it is a consequence of the fact that

$$(1 - t^2)dG/dt = 2zG. \tag{7}$$

When $|u|$ is sufficiently small the relation

$$\sum_{n=0}^{\infty} u^n(1 + t)^n(1 - t)^{-n} = 1 + 2u\sum_{m=1}^{\infty} t^m(1 + u)^{m-1}(1 - u)^{-m-1} \tag{8}$$

shows that if $n > 0$

$$2u(1 + u)^{n-1}(1 - u)^{-n-1} = \sum_{m=0}^{\infty} u^m g_n(m) \tag{9}$$

and so
$$\begin{aligned} 2g_m(n + 1) &= g_n(m - 1) + 2g_n(m) + g_n(m + 1) \\ &= g_{n+1}(m + 1) - g_{n+1}(m - 1). \end{aligned}$$

Consequently, if $n \geq 1$

$$2g_m(n) = g_n(m + 1) - g_n(m - 1). \tag{10}$$

2. The polynomial $g_n(z)$ may be generalized by writing

$$(1 + t)^{z+r}(1 - t)^{-z} = \sum_{n=0}^{\infty} t^n g_n(z, r). \tag{11}$$

If $(1 + t)^r$ is expanded by the binomial theorem

$$(1 + t)^r = \sum_{n=0}^{\infty}(r/, n)t^n \qquad (12)$$

it is found that

$$g_n(z, r) = \sum_{s=0}^{n}(r/, s)g_{n-s}(z), \qquad (13)$$

the series terminating earlier when r is a positive integer $< n$.

The expansion (11) is a particular case of the more general expansion of W. Gordon[5] and J. L. Lagrange[6]

$$(1 + t)^{b-c}[1 + (1 - z)t]^{-b} = \sum_{n=0}^{\infty}t^n(-c/, n)F(-n, b; c; z) \quad |t| < 1, |t - tz| < 1. \qquad (14)$$

It follows from this expansion that when $r = 0, 1, \ldots n - 1$

$$g_n(z, r) = (r/, n)F(-n, z; -r; 2). \qquad (15)$$

Pidduck considered the case $r = -1$ and so his second coefficient is with a different notation

$$g_n(z + 1, -1) = (-)^n F(-n, z + 1; 1; 2) = F(-n, -z; 1; 2). \qquad (16)$$

A second expression for $g_n(z, r)$ in terms of the hypergeometric series is a consequence of Euler's relation

$$F(a, b; c; x) = (1 - x)^{-a}F\left(a, c - b; c; \frac{x}{x - 1}\right) \qquad (17)$$

which gives the formula

$$g_n(z, r) = (-)^n(r/, n)F(-n, -z - r; -r; 2). \qquad (18)$$

3. The relation

$$F(-n, -z - r; -r; t) = [(z + r/, n)/(r/, n)]\,(-t)^n F(r - n + 1, -n; z + r - n + 1; t^{-1}) \qquad (19)$$

indicates also that

$$g_n(z, r) = (z + r/, n)2^n F(r - n + 1, -n; z + r - n + 1; {}^1/_2) \qquad (20)$$

and Euler's relation

$$F(a, b; c; x) = (1 - x)^{c-a-b}F(c - a, c - b; c; x) \qquad (21)$$

indicates that

$$g_n(z, r) = (z + r/, n)2^{-z}F(z, r + z + 1; r + z - n + 1; {}^1/_2). \qquad (22)$$

This formula gives an estimate of $g_n(z, r)$ for large positive values of n when $r + z$ is not one of the numbers 0, 1, 2, ... $n - 1$ and is also independent of n. The hypergeometric series actually gives a convergent expansion in a series of inverse factorials when $g_n(z, r)$ is regarded as a function of n. Similarly

$$F(-n, b;\ c;\ x) = x^n[(b, n)/(c, n)]\ (-)^n F(1 - n - c, -n;\ 1 - n - b;\ z) \tag{23}$$

where $z = 1/x$ and (21) gives for $|x| > 1$

$$F(-n, b;\ c;\ x) = x^{b-c}(x-1)^{n+c-b}(b, n)/(c, n)(-)^n F(c - b, 1 - b;\ 1 - n - b;\ z). \tag{24}$$

Hence when $|x| > 1$ and n is large

$$F(-n, b;\ c;\ x) \sim (-)^n x^{b-c}(x-1)^{n+c-b}(b, n)/(c, n). \tag{25}$$

Negative integral values of c which made $(c, n) = 0$ should be excepted. Also negative integral values of $b - 1$ should be excepted unless $b = c$.

4. Definite integrals for $g_n(z)$ and $g_n(z, r)$ may be derived from the well-known definite integrals for the hypergeometric function. In particular, if $-1 < z < 1, n > 0$

$$B(z, 1 - z)g_n(z) = \int_{-1}^{1} t^{n-1}(1 + t)^z(1 - t)^{-z}dt. \tag{26}$$

This result may be written in the alternative form

$$g_n(z) = (1/\pi)\sin(\pi z)\int_{-\infty}^{\infty} e^{uz}(\tanh {}^1/_2 u)^n du/sh\ u \tag{27}$$

Differentiating m times with respect to z we find that

$$g_n^{(m)}(z) = \left(\frac{1}{2\pi i}\right)\int_{-\infty}^{\infty}[e^{z(u+i\pi)}(u + i\pi)^m - e^{z(u-i\pi)}(u - i\pi)^m] (\tanh {}^1/_2 u)^n du/sh\ u \tag{28}$$

Putting $z = 0$ we obtain the formula

$$g_n^{(m)}(0) = \left(\frac{1}{2\pi i}\right)\int_{-\infty}^{\infty}[(u + i\pi)^m - (u - i\pi)^m](\tanh {}^1/_2 u)^n\ du/sh\ u \tag{29}$$

for the coefficient of t^n in Mittag-Leffler's expansion

$$[2Q_0(t)]^m = \left[\log\frac{1+t}{1-t}\right]^m = \sum_{n=m}^{\infty} t^n g_n^{(m)}(0). \tag{30}$$

When $|R(z)| < 1$ there is a formula

$$g_n(z) = (1/\pi)\int_0^{\pi}(\cot {}^1/_2 u)^z \cos({}^1/_2\pi z - nu)du \tag{31}$$

which may be established with the aid of the recurrence relation and may be regarded as holding for negative integral values of n as well as for positive integral values. Since the recurrence relation gives $g_n(z) = 0$ when n is a negative integer we have for all positive integral values of n

$$g_n(z) = (2/\pi)\int_0^{\pi}(\cot {}^1/_2u)^z \cos ({}^1/_2\pi z) \cos (nu)du \qquad (32)$$

$$g_n(z) = (2/\pi)\int_0^{\pi}(\cot {}^1/_2u)^z \sin ({}^1/_2\pi z) \sin (nu)du \qquad (33)$$

If $n > 0$ and $|R(z)| < 1$ the formula

$$g_n(z) = (1/\pi)\int_0^{2\pi}(1 + e^{ia})^z(2 + e^{ia})^{n - 1}e^{-ina}da \qquad (34)$$

may be derived from the series for $g_n(z)$ in terms of binomial coefficients. A corresponding formula may be obtained for $g_n(z, r)$. Another type of formula for $g_n(z)$ is obtained by starting from the expansion of Liouville[7] and Lerch[8]

$$\exp\left(x\frac{t - 1}{t + 1}\right) = \sum_{n=0}^{\infty} k_{2n}(x)t^n \qquad (35)$$

in which $k_{2n}(x) = e^{-x}(-)^{n - 1}(2x)F(1 - n;\ 2;\ 2x)$ for $n > 1$
$k_0(x) = e^{-x}$

The formula in question is

$$\Gamma(z)g_n(z) = \int_0^{\infty} x^{z - 1}k_{2n}(x)dx,\ R(z) > -1 \text{ for } n > 0 \qquad (36)$$

A corresponding formula for $g_n(z, r)$ is

$$\Gamma(z)g_n(z, -m - 1) = (-)^n\int_0^{\infty} e^{-x}L_n^m(2x)x^{z - 1}dx,\ R(z) > 0 \quad (37)$$

where $L_n^m(u)$ is the generalized polynomial of Laguerre. With the notation of Sonine's polynomial

$$g_n(z, -m - 1)\Gamma(z) = \Gamma(m + n + 1)\int_0^{\infty} e^{-x}T_m^n(2x)x^{z - 1}dx. \quad (38)$$

Another expression for $g_n(z)$ is obtained from the expansion

$$(1 - u)^{-2} \exp\left[-x\left(\frac{1 + u}{1 - u}\right)^2\right] = \sum_{n=0}^{\infty} u^nT_n^*(x)$$

which indicates that

$$\int_0^\infty T_n^*(x)x^{z-1}dx = \frac{\Gamma(z)}{2-4z}(n+1)g_{n+1}(1-2z),\ R(z) > 0 \quad (39)$$

TABLE OF $g_n(m)$

n / m	0	1	2	3	4	5	6	7	8	9	10
0	1	0	0	0	0	0	0	0	0	0	0
1	1	2	2	2	2	2	2	2	2	2	2
2	1	4	8	12	16	20	24	28	32	36	40
3	1	6	18	38	66	102	146	198	258	326	402
4	1	8	32	88	192	360	608	952	1408	1992	2720
5	1	10	50	170	450	1002	1970	3530	5890	9290	14002
6	1	12	72	292	912	2364	5336	10836	20256	35436	58728
7	1	14	98	462	1666	4942	12642	28814	59906	115598	209762
8	1	16	128	688	2816	9424	27008	68464	157184	332688	658048
9	1	18	162	978	4482	16722	53154	148626	374274	864146	1854882
10	1	20	200	1340	6800	28004	97880	299660	822560	2060980	4780008

$g_0(z) = 1$, $g_1(z) = 2z$, $3g_3(z) = 4z^3 + 2z$, $3g_4(z) = 2z^4 + 4z^2$, $g_2(z) = 2z^2$, $15g_5(z) = 4z^5 + 20z^3 + 6z$, $45g_6(z) = 4z^6 + 40z^4 + 46z^2$.

$$g_n(-z) = (-)^n g_n(z)$$

$$g_{2n}(1/2) = g_{2n+1}(1/2) = \frac{1.3.5\ (2n-1)}{2.4.6.\ 2n} = (n - 1/2/, n)$$

[1] Mittag-Leffler, G., *Acta Math.*, **15,** 1–32 (1891); **24,** 183–245 (1901).

[2] Appell, P., *Ann. école norm.*, (2) **9,** 119–144 (1880).

[3] Pidduck, F. B., *Proc. Roy. Soc.* (*London*), **A83,** 347–356 (1910); **86,** 396–405 (1912).

[4] Belorizky, D., *Comptes Rendus*, **195,** 1222–1224 (1932).

[5] Gordon, W., *Ann. der Physik,* (5) **2,** 1031–1056 (1929).

[6] Lagrange, J. L., *Oeuvres*, **2,** 173–234 (1770–1773), see p. 220.

[7] Liouville, J., *J. de Math.*, (2) **2,** 433–440 (1857).

[8] Lerch, M., *J. f. Math.*, **130,** 47–65 (1905).