THE ANALYTIC *S*-MATRIX

BY

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CHAPTER 1

INTRODUCTION

1.1 Survey of objectives

We begin by considering the motivation for developing an S-matrix theory of particle interactions. The purpose of the book is to indicate how such a theory may be developed from physical principles and to discuss some of the properties of the S-matrix. Particular attention will be given to its analyticity properties and our study of these will rest very largely, though not entirely, on an analysis of the corresponding properties of Feynman integrals.

At present it is believed that the forces between particles fall into four categories, depending on their strength. The most familiar of these is the electromagnetic force, which has been expressed in terms of a field since the work of Maxwell. The quantisation of the electromagnetic field finally resolved the old paradox of the wave and particle nature of light.

The quantisation procedure uses either a Lagrangian or a Hamiltonian whose form is taken from classical physics. A solution of the resulting equations can be achieved in the form of a perturbation series expansion in powers of the square of the electric charge which, in rationalised units, is

$$e^2 = \tfrac{1}{137}.$$

The two most important difficulties encountered in this perturbation solution arise from two types of divergence. One of these, the infra-red divergence, can be eliminated in principle by taking account of the fact that the zero mass of the photon makes it impossible for the number of zero-energy photons to be measured. The other, the ultraviolet divergence, is eliminated by renormalisation, although it may be thought that the manipulation of infinite constants is still an unsatisfactory feature of the theory. For an account of these methods the reader is referred to books on quantum field theory, for example Schweber (1961); here we remark only that the results are in very good agreement with experiment.

The type of interaction with which S-matrix theory is mainly concerned comprises all strong interactions (we do not distinguish between

these and the possible 'super-strong' interactions). These are responsible for nuclear forces and for the production of strange particles. Earlier formulations of a theory of strong interactions have proceeded analogously to electromagnetic theory. In the simplest form the strong interactions correspond to a field that is carried by the π-meson just as the electromagnetic field is carried by the photon. More generally a formal theory can be set up that involves the fields of all strongly interacting particles. However, there is a serious obstacle to the solution of the resulting equations since the only known methods of solution are based on a perturbation series in powers of the coupling constant and, in dimensionless units, so as to compare it with e^2, the square of this constant has a value about 15. Thus the perturbation series does not even begin to converge and a solution based on the first few terms is very unlikely to be useful.

In the last ten years a new approach to strong interactions has been developed which avoids the obvious defect of an expansion in the coupling constant, based on field theory. It is recognised that the fields themselves are of little interest, but that they are merely used to calculate transition amplitudes for interactions. These amplitudes are the elements of the S-matrix. The new approach is concerned with a direct study of the S-matrix, without the introduction of fields. It was first suggested by Heisenberg much earlier (Heisenberg, 1943; see also Møller, 1945, 1946) that the S-matrix might provide a means of avoiding the divergence difficulties of field theory, which at that time had not been solved by renormalisation. Heisenberg's formulation of S-matrix theory is in spirit very close to the formulation of a deductive S-matrix theory which will be described in chapter 4 of this book. However at that time (1943–52) progress was much more difficult, because a knowledge of the analytic properties of perturbation theory was not available to provide the guide-lines for applications of the S-matrix theory and for the formation of a deductive theory. The main parts of chapters 2 and 3 of this book will be concerned with the analytic properties of perturbation theory.

The deductive approach to S-matrix theory is based on the idea that one should try to calculate S-matrix elements directly, without the use of field quantities, by requiring them to have some general properties that ought to be valid, whether or not some underlying Lagrangian theory exists. (There is a tendency in the literature to call these properties 'axioms', but we do not use this term since it would suggest a degree of mathematical rigour that is lacking in the present

state of the subject.) A list of the important properties to be satisfied by the S-matrix would include:

(a) the superposition principle of quantum mechanics;

(b) the requirements of special relativity;

(c) the conservation of probability;

(d) the short-range character of the forces;

(e) causality and the existence of macroscopic time.

Notice that (d) actually excludes the electromagnetic interaction and there is at present no S-matrix theory which properly includes the presence of photons. The essential difficulty is the same one as leads to the infra-red divergence of perturbation theory, that the number of massless particles is not measurable. In practice one uses a combination of a perturbation series for the electromagnetic interaction and S-matrix theory for the strong interactions, but this procedure does not overcome the difficulty of principle where photons are involved.

The property (e) is the one whose consequences are most difficult to derive rigorously, and at the same time it is one of the most important. It is generally believed that the causality property requires the transition amplitudes to be the real-boundary values of analytic functions of complex variables. In view of the difficulties in deriving this result rigorously it is common to replace the property (e) by the assumption

(e') transition amplitudes are the real-boundary values of analytic functions.

This assumption is much more precise mathematically than the property of causality but its physical meaning is more obscure. We will illustrate the connections between the two, and indicate the nature of the difficulty of making it rigorous, by considering a simple example.

Let $A(z,t)$ be a wave packet travelling along the z-direction with velocity v:

$$A(z,t) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} d\omega\, a(\omega) \exp\left\{i\omega\left(\frac{z}{v} - t\right)\right\}. \tag{1.1.1}$$

Suppose this wave packet is scattered by a particle fixed at the origin $z = 0$. The scattered wave, in the forward direction, may be written as

$$G(r,t) = \frac{1}{r(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} d\omega f(\omega)\, a(\omega) \exp\left\{i\omega\left(\frac{z}{v} - t\right)\right\}. \tag{1.1.2}$$

The inverse of equation (1.1.1) is

$$a(\omega) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} dt\, A(0,t) \exp(i\omega t), \tag{1.1.3}$$

and this tells us that if the incident wave does not reach the scatterer before time $t = 0$, that is if

$$A(0,t) = 0 \quad \text{for} \quad t < 0, \tag{1.1.4}$$

then $a(\omega)$ is regular in the upper half of the plane of the variable ω now regarded as complex. For if, as we assume, the integral (1.1.3) converges for real ω, it will, by virtue of (1.1.4), converge even better for $\mathrm{Im}(\omega) > 0$.

We now impose a causality condition,

$$G(r,t) = 0 \quad \text{for} \quad vt - r < 0. \tag{1.1.5}$$

This expresses the requirement that no scattered wave reaches a point at distance r before a time v/r after the incident wave first reaches the scatterer. Then from the inverse of equation (1.1.2), using (1.1.5) we find that the product $a(\omega)f(\omega)$ is analytic in $\mathrm{Im}(\omega) > 0$. Hence the scattering amplitude $f(\omega)$ itself is analytic in $\mathrm{Im}(\omega) > 0$, except possibly at zeros of $a(\omega)$.

Arguments such as this about causality and analyticity can be made in various branches of classical physics, particularly in the theory of dispersion in optics (for a review see Hamilton (1959)). Thus the approach to high-energy physics that we describe in this book is often known as the 'dispersion relation' approach.

The difficulty in making the above discussion rigorous arises from the condition (1.1.4), which cannot actually be realised. This condition would imply a precise localisation in time (microscopic time) for the incident wave packet. But in S-matrix theory the quantity that we wish to know precisely is the energy, since this is one of the essential variables on which transition amplitudes depend. If we compromise, in accordance with the uncertainty principle, and set up our theory with only a partial knowledge of time and a partial knowledge of energy, our conclusions about analyticity are less precise. These problems relating to the use of macroscopic time persist in a relativistic formulation. They will be considered in more detail in chapter 4.

The causality condition in quantum field theory is usually assumed to correspond to the commutativity of the field operators for space-like separation of their arguments

$$[\phi(x), \phi(x')] = 0 \quad \text{for} \quad (x - x')^2 < 0, \tag{1.1.6}$$

where for four-vectors we write $x = (x_0, \mathbf{x})$ and use the metric

$$x^2 = x_0^2 - \mathbf{x}^2. \tag{1.1.7}$$

Only for the electromagnetic field is the field operator physically observable, so the condition (1.1.6) for any other field can only have an indirect relation to causality in physics. Even if this condition is accepted it is very difficult to make use of it to prove rigorously any analytic properties of transition amplitudes (see, for example Froissart (1964)), though an heuristic derivation can be given fairly simply (Gell-Mann, Goldberger & Thirring (1954)). Within the framework of quantum field theory, without using perturbation expansions, only very limited information about analyticity properties has been obtained.

If, however, the perturbation series for a transition amplitude is used as a means for obtaining analyticity properties, much more information becomes available. The procedure, which will be followed in chapter 2 of this book, is to examine the analytic properties of individual terms in the perturbation series. Although one does not believe the magnitude of the individual terms to be significant, it is hoped that their analytic properties will indicate the analytic properties of the transition amplitude itself, particularly when properties are derived that hold for every term in the series. In chapter 3 this method is extended to include some aspects (particularly asymptotic behaviour) of the analytic properties of partial infinite sums of series within the full perturbation series.

In this book our discussion of the analytic S-matrix is limited to strong interactions. One hopes that in time a method for dealing with massless particles can be found. In the meantime, apart from sum rules, which seem to have limited scope, it is necessary to incorporate electromagnetic effects by perturbation theory. A similar situation is met with the weak interactions, at least where neutrinos are involved, though there are difficulties of renormalisation. In most practical situations the weakness of the interactions (about 10^{-10} compared with electromagnetic $1/137$) permits the use of first-order perturbation theory. The fourth category of force, the gravitational force, has a strength of order 10^{-40}. It again is believed to be transferred by a massless particle, the graviton, so it cannot at present be incorporated into a dispersion approach.

1.2 The S-matrix and its unitary and kinematic properties

In using the S-matrix to describe a scattering experiment we will assume that the forces are of sufficiently short range that the initial and final states consist effectively of free particles. These states can then be specified by the momentum of each particle together with certain discrete quantum numbers such as the spin and isospin. Due to the finite size of any experiment there is some residual uncertainty in the momentum but we assume that this is unimportant in practice. The momentum eigenvalues form a continuous spectrum but for clarity of notation in this section we will begin by using a discrete symbol m, or n, to label the states.

Let $|n\rangle$ denote the initial state of two particles that subsequently come together, interact, and separate. The superposition principle in quantum mechanics tells us that the final state can be written $S|n\rangle$, where S is a linear operator. The probability that a measurement on the final state gives a result corresponding to the state $|m\rangle$ is obtained from the square of the modulus of the matrix element

$$\langle m|S|n\rangle. \tag{1.2.1}$$

The set of states $|n\rangle$ is assumed to be orthonormal and complete,

$$\langle m|n\rangle = \delta_{mn}, \quad \sum_m |m\rangle\langle m| = 1. \tag{1.2.2}$$

Thus any state can be expressed by a superposition of the states $|n\rangle$, and the quantum numbers denoted by n uniquely specify a state.

If the initial state in a scattering experiment is the normalised state $|\rangle$, the total probability of the system ending up in some other state must be unity. Hence, writing

$$|\rangle = \sum_n a_n|n\rangle, \tag{1.2.3}$$

where $\sum_n |a_n|^2 = 1$, we obtain

$$\begin{aligned} 1 &= \sum_m |\langle m|S|\rangle|^2 = \sum_m \langle|S^\dagger|m\rangle\langle m|S|\rangle \\ &= \langle|S^\dagger S|\rangle = \sum_{n,n'} a_{n'}^* a_n \langle n'|S^\dagger S|n\rangle. \end{aligned} \tag{1.2.4}$$

In order for this to hold for all choices of the a_n, it is necessary that

$$\langle n'|S^\dagger S|n\rangle = \delta_{n'n},$$

or

$$S^\dagger S = 1. \tag{1.2.5a}$$

In the same way, the condition that the total probability be unity for an arbitrary final state to arise from some initial state gives

$$SS^\dagger = 1. \tag{1.2.5b}$$

Thus the operator S is unitary.

We consider next the consequences of relativistic invariance. If L is any proper Lorentz transformation, and if

$$L|m\rangle = |m'\rangle, \tag{1.2.6}$$

we require that

$$|\langle m'|S|n'\rangle|^2 = |\langle m|S|n\rangle|^2. \tag{1.2.7}$$

in order that observable quantities be independent of the Lorentz frame. The definition of the S-matrix elements given above does not specify the phase uniquely. This permits us to replace (1.2.7) by the stronger condition

$$\langle m'|S|n'\rangle = \langle m|S|n\rangle. \tag{1.2.8}$$

For spinless particles this has the consequence that the matrix elements depend on the four-momenta only through their invariant scalar products. For example, the two-particle → two-particle matrix element

$$\langle p_3, p_4|S|p_1, p_2\rangle, \tag{1.2.9}$$

after removal of δ-functions specifying total energy-momentum conservation, can for the case of spinless particles be written as a function of the variables s, t, u, where

$$s = (p_1+p_2)^2, \quad t = (p_1-p_4)^2, \quad u = (p_1-p_3)^2. \tag{1.2.10}$$

Notice that as a consequence of total energy-momentum conservation and the mass shell condition for each particle, these variables are not independent. From

$$p_1+p_2 = p_3+p_4, \quad p_i^2 = m_i^2 \quad (i = 1, 2, 3, 4), \tag{1.2.11}$$

it follows that

$$s+t+u = \sum_{i=1}^{4} m_i^2. \tag{1.2.12}$$

The above form for the matrix elements applies only to spinless particles. However, even for these the elements of the S-matrix itself cannot be analytic, due to the occurrence of Dirac δ-functions. These occur in two ways. First, due to overall energy-momentum conservation, the matrix element (1.2.9) will contain a factor

$$\delta^{(4)}(p_1+p_2-p_3-p_4).$$

Secondly, since the state-vectors occurring in (1.2.9) are momentum eigenstates they can contain no information about the positions of the

particles in space. Hence they are overwhelmingly likely to be widely separated in space and not interact at all. When this happens, the four-momentum for each particle remains unchanged, and the S-matrix can therefore be separated usefully into two parts by writing (see Møller, 1945, 1946),

$$S = 1 + iR. \tag{1.2.13}$$

The relation between matrix elements of R and experimental cross-sections depends on the choice of normalisation for the free-particle states. These free-particle states are fully specified for spinless particles when the three-momentum of each particle is given, since the fourth component of p satisfies

$$p_0^2 = m^2 + \mathbf{p}^2. \tag{1.2.14}$$

We choose a covariant normalisation for the free particle states, so that the orthogonality and completeness relations, written symbolically in (1.2.2), become

$$\langle p' | p \rangle = (2\pi)^3 . 2p_0 \delta^{(3)}(\mathbf{p}' - \mathbf{p}), \tag{1.2.15}$$

$$\int |p'\rangle \frac{d^3\mathbf{p}'}{2p_0'(2\pi)^3} \langle p' | p \rangle = |p\rangle, \tag{1.2.16}$$

or equivalently

$$\int |p'\rangle \frac{d^4p' \delta^{(+)}(p'^2 - m^2)}{(2\pi)^3} \langle p' | p \rangle = |p\rangle. \tag{1.2.17}$$

Then the R-matrix element for two-particle scattering is related to a transition amplitude F by

$$\langle p_3, p_4 | R | p_1, p_2 \rangle = (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4) F. \tag{1.2.18}$$

The cross-section is obtained from $|R|^2$ with integration over all possible final states when the incident flux is normalised to unity. For two-particle scattering this gives for the cross-section σ,

$$\sigma = \frac{1}{(8\pi)^2 q W} \int d\Omega |F|^2 \frac{p}{W}, \tag{1.2.19}$$

where $\mathbf{q}$ is the centre of mass momentum for a particle in the initial state, $\mathbf{p}$ for the final state, W is the centre of mass energy, and Ω is the solid angle in the final state. The differential cross-section for scattering to an angle (θ, ϕ) in the centre-of-mass system is

$$\frac{d\sigma}{d\Omega} = \frac{p}{(8\pi)^2 q W^2} |F|^2, \tag{1.2.20}$$

with $d\Omega = \sin\theta \, d\theta \, d\phi$.

The transition amplitude F is the amplitude that is given by a series of Feynman integrals when the particles have spin zero. It is these Feynman amplitudes that form the main part of our discussion of analyticity in the remaining sections of this chapter and the whole of chapter 2.

If the scattered particles have spin the discussion of analytic properties does not apply directly to the scattering amplitudes, and in addition our remarks about Lorentz invariance made earlier in this section for spinless particles must be modified. We will do no more than illustrate the differences that arise, and for a more complete discussion of the scattering of particles with spin and charge, the reader is referred to the account by Jacob (Chew & Jacob, 1964).

For our example we consider pion-nucleon scattering (Chew, Goldberger, Low & Nambu, 1957). There the amplitude F that occurs in (1.2.18) can be expressed in the form

$$F = 2m\bar{u}(p_3)\{A - \tfrac{1}{2}iB\gamma_\mu(p_2^\mu + p_4^\mu)\}u(p_1), \tag{1.2.21}$$

where u denotes the Dirac spinor for the nucleon lines, for which p_3 and p_1 are the four-momenta. The quantities A and B are functions of two independent invariants chosen from s, t, u given by equations (1.2.10). The analytic properties of A and B are essentially the same as those of the amplitude F for scalar particles, to each order in perturbation theory. The other factors in equation (1.2.21) are often referred to as 'inessential complications'. Needless to say they are crucial in establishing relations between analyticity and experimental results.

It is frequently convenient in pion-nucleon scattering to make also a separation of the transition amplitude F into isospin amplitudes $F(\frac{1}{2})$ and $F(\frac{3}{2})$,

$$\left.\begin{aligned} F(\pi^+p \to \pi^+p) &= F(\tfrac{3}{2}), \\ F(\pi^-p \to \pi^-p) &= \tfrac{1}{3}F(\tfrac{3}{2}) + \tfrac{2}{3}F(\tfrac{1}{2}), \\ F(\pi^-p \to \pi^0n) &= \tfrac{1}{3}\sqrt{2}\{F(\tfrac{3}{2}) - F(\tfrac{1}{2})\}. \end{aligned}\right\} \tag{1.2.22}$$

In this book we are concerned with analytic properties of the S-matrix and not with the important considerations about spin and isospin which have been fully described elsewhere (Chew & Jacob, 1964). For most of our discussion we will therefore consider only the interactions of particles that have zero spin and isospin. Then the amplitude F is given by the series of Feynman integrals whose analyticity we will describe in chapters 2 and 3.

For the case of elastic scattering of two spinless particles a consequence of Lorentz invariance is the symmetry of the matrix element:

$$\langle m|S|n\rangle = \langle n|S|m\rangle, \tag{1.2.23}$$

that is $$\langle p_3, p_4|S|p_1, p_2\rangle = \langle p_1, p_2|S|p_3, p_4\rangle.$$

This is because in the centre-of-mass system (the Lorentz frame in which $\mathbf{p}_1+\mathbf{p}_2 = \mathbf{0} = \mathbf{p}_3+\mathbf{p}_4$) a rotation of π about the bisector of the angle between $\mathbf{p}_1$ and $\mathbf{p}_3$ interchanges these momenta, and it also interchanges $\mathbf{p}_2$ and $\mathbf{p}_4$ (see Fig. 1.2.1). So (1.2.23) follows as a result

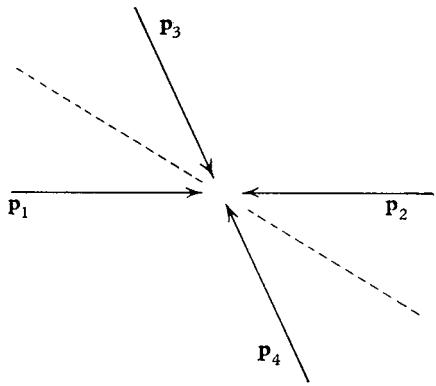

Fig. 1.2.1. The centre-of-mass picture in momentum space for two-particle scattering $1+2 \to 3+4$.

of (1.2.8). It need not, however, be true for other amplitudes, though it sometimes can be deduced from the invariance of strong interactions under the operation PT. This is true for two-particle → two-particle amplitudes.

1.3 Analyticity, crossing and dispersion relations

We now discuss in more detail some of the analyticity properties of the scattering amplitudes that will be derived in chapter 2. First we consider some consequences of unitarity.

Substitute from (1.2.13) into the unitarity relation (1.2.5), giving

$$R - R^\dagger = iR^\dagger R = iRR^\dagger, \tag{1.3.1}$$

or, in the notation of section (1.2), for two-particle scattering

$$\langle p_3, p_4|R|p_1, p_2\rangle - \langle p_1, p_2|R|p_3, p_4\rangle^* = i\langle p_3, p_4|R^\dagger R|p_1, p_2\rangle \tag{1.3.2a}$$

$$= i\langle p_3, p_4|RR^\dagger|p_1, p_2\rangle, \tag{1.3.2b}$$

where the star denotes complex conjugation, and the dagger hermitian conjugation. If the symmetry condition (1.2.23) is valid the left-hand side of (1.3.2) is just twice the imaginary part of the matrix element:

$$2i \operatorname{Im} \langle p_3, p_4 | R | p_1, p_2 \rangle. \tag{1.3.3}$$

Then the unitary relation (1.3.2) becomes

$$2 \operatorname{Im} \langle p_3, p_4 | R | p_1, p_2 \rangle = \sum_n \langle n | R | p_3, p_4 \rangle^* \langle n | R | p_1, p_2 \rangle \tag{1.3.4a}$$

$$= \sum_n \langle p_3, p_4 | R | n \rangle \langle p_1, p_2 | R | n \rangle^*, \tag{1.3.4b}$$

where the Σ denotes a sum and an integral over all intermediate states that are allowed by conservation of the total energy and momentum. Thus for total energies below the inelastic threshold the unitarity condition is, in terms of the amplitude F of (1.2.18)

$$\begin{aligned} 2 \operatorname{Im} \langle p_3 p_4 | F | p_1 p_2 \rangle = (2\pi)^{-2} \int & \frac{d^3\mathbf{k}_1 d^3\mathbf{k}_2}{W^2} \delta^{(4)}(p_1 + p_2 - k_1 - k_2) \\ & \times \langle p_3 p_4 | F | k_1 k_2 \rangle \langle p_1 p_2 | F | k_1 k_2 \rangle^* \\ = (2\pi)^{-2} \int & d^4k_1 d^4k_2 \delta^{(+)}(k_1^2 - m^2) \, \delta^{(+)}(k_2^2 - m^2) \\ & \times \delta^{(4)}(p_1 + p_2 - k_1 - k_2) \langle p_3 p_4 | F | k_1 k_2 \rangle \\ & \times \langle p_1 p_2 | F | k_1 k_2 \rangle^*, \end{aligned} \tag{1.3.5}$$

where W is the centre of mass energy.

Above the energy-threshold for inelastic scattering a new term must be added to the right-hand side of the unitarity relation (1.3.5) so as to include the extra intermediate states that are allowed by energy-conservation. This implies a change in the left-hand side, and suggests that the elastic scattering matrix-element has a singularity at each energy corresponding to a threshold for a new allowed physical process. This is our first encounter with an effect of unitarity on analyticity of the S-matrix; later, in chapter 4, we will consider these effects in more generality.

The thresholds are branch-points of the amplitude F (Eden, 1952), as we shall see in chapter 2, so we draw cuts in the complex energy-squared plane ($s = W^2$), attached to the branch-points and by convention running along the real axis. The purpose of the cuts is to make the amplitude single valued on a Riemann surface. If we do not cross the cuts in Fig. 1.3.1, we have a single sheet of this Riemann surface. This is called the 'physical sheet' if the physical scattering amplitude is a boundary value on the real cut of the amplitude on this sheet.

Other sheets of the Riemann surface, associated with the amplitude F as a function of s, are reached by burrowing through a branch cut or through several branch cuts, to reach another layer of this multi-layer surface. These other sheets are called unphysical sheets and they are to be distinguished from each other by the manner in which they are connected to the physical sheet, for example by specifying which branch cuts must be crossed to reach the physical sheet.

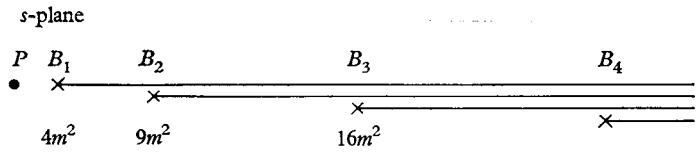

Fig. 1.3.1. Branch cuts for the scattering amplitude F in the complex s-plane arising from normal thresholds B_1, B_2, ... The point P denotes a pole in F.

For two-particle scattering, the elastic amplitude F is a function of two variables which, to be definite, we choose as s, t given by (1.2.10),

$$\langle p_3 p_4 | F | p_1 p_2 \rangle = F(s,t). \tag{1.3.6}$$

We have so far been considering F as a function of the invariant energy-squared s, keeping the momentum-transfer-squared t, fixed. The branch-points shown in Fig. 1.3.1 at $s = 4m^2$, $9m^2$, $16m^2$ are called 'normal thresholds' and correspond to the energies at which production of extra particles is possible. The leading normal threshold, $s = 4m^2$, is the least energy-squared at which a two-particle state can exist. We assume not only equal-mass particles in this example, but also that no conservation law except energy precludes the creation of any particular number of particles from a two-particle state. With this assumption, conservation laws (excluding energy) do not forbid going from a two-particle state to a one-particle state. It is assumed that such a state corresponds to a singularity of the amplitude $F(s,t)$, reached at an unphysical value of the variable s below the leading normal threshold, at

$$s = m^2. \tag{1.3.7}$$

This singularity is denoted P in Fig. 1.3.1, and in perturbation theory it is a pole, not a branch-point. Using unitarity and causality we will show more generally in § 4.5 that the singularity must be a pole.

The region in which $F(s,t)$ is the amplitude for the physical scattering process

$$A_1 + A_2 \to A_3 + A_4 \tag{1.3.8}$$

must have real positive energy $p_i^{(0)}$ for each particle, and real three-momentum $\mathbf{p}_i$. In the equal-mass case this gives

$$s \geqslant 4m^2, \quad t \leqslant 0, \quad u \leqslant 0. \tag{1.3.9}$$

This result can be obtained by expressing s, t, u in terms of the momentum $\mathbf{q}$ and the scattering angle θ in the centre-of-mass system, which gives

$$\left.\begin{aligned} s &= 4(m^2+q^2), \\ t &= -2q^2(1-\cos\theta), \\ u &= -2q^2(1+\cos\theta). \end{aligned}\right\} \tag{1.3.10}$$

When the masses are not equal the conditions are not quite so simple; they are derived by Kibble (1960) (see also §4.3).

So far we have varied only s in discussing analytic continuation, but in general both s and t can be regarded as complex variables in the amplitude $F(s,t)$. Then we can consider analytic continuation from the physical region (1.3.9) to the region

$$u \geqslant 4m^2, \quad s \leqslant 0, \quad t \leqslant 0. \tag{1.3.11}$$

It is assumed that the resultant function F, evaluated in a suitable limit on to the region (1.3.11), is the physical scattering amplitude for the process

$$A_1 + \bar{A}_3 \to \bar{A}_2 + A_4, \tag{1.3.12}$$

where $\bar{A}_i$ denotes the anti-particle of A_i. For this process the energy in the centre-of-mass frame for the initial (or final) state is just $\sqrt{u}$. It is further assumed that by analytic continuation to the region

$$t \geqslant 4m^2, \quad u \leqslant 0, \quad s \leqslant 0, \tag{1.3.13}$$

the function F, evaluated in a suitable limit, gives the physical scattering amplitude for the process

$$A_1 + \bar{A}_4 \to \bar{A}_2 + A_3 \tag{1.3.14}$$

for which the energy in the centre-of-mass frame is $\sqrt{t}$.

These important properties are called the 'crossing' properties. They state that the same analytic function can be used to describe the three different physical processes (1.3.8), (1.3.12), (1.3.14) by making an appropriate choice of physical values for the variables s and t (or u). These physical processes are often called different 'channels', and one refers to them as the s-channel, the t-channel and the u-channel when

s, t and u, respectively, are the energy variables. Remembering the relation (1.2.12), which here becomes

$$s+t+u = 4m^2, \tag{1.3.15}$$

we can draw the physical regions for the three channels using oblique axes as in Fig. 1.3.2 (Mandelstam, 1958).

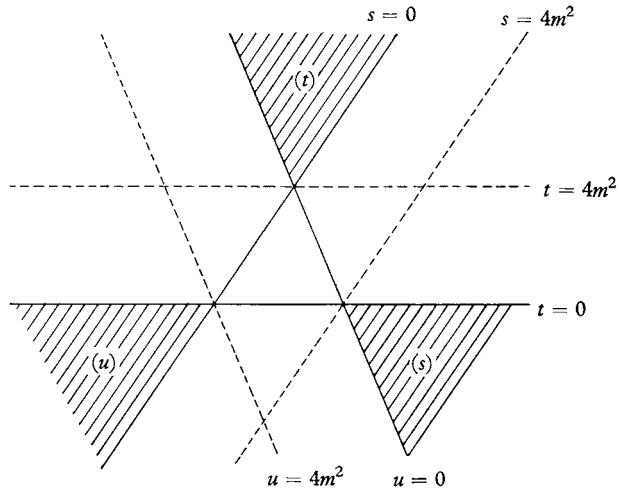

Fig. 1.3.2. The Mandelstam diagram using oblique axes showing the physical regions (shaded areas) in which s, or t, or u denotes the square of the centre-of-mass energy for equal-mass particles in collision.

Since we now have symmetry between the three variables s, t, u, it is convenient to change the signs of the four-momenta from those used in § 1.2 (equation (1.2.10)) so as to give

$$\left.\begin{aligned} s &= (p_1+p_2)^2 = (p_3+p_4)^2, \\ t &= (p_1+p_4)^2 = (p_2+p_3)^2, \\ u &= (p_1+p_3)^2 = (p_2+p_4)^2. \end{aligned}\right\} \tag{1.3.16}$$

This convention will often be used in the remainder of this book.

A further convention that is sometimes used in the literature is to write the amplitude $F(s,t,u)$ as a function of three variables, but with the constraint (1.3.15) relating s, t and u. In fact, F is defined only when (1.3.15) is satisfied so this formal achievement of symmetry is somewhat ambiguous, and in practice it is better to regard F as a function of two variables, $F(s,t)$ or $F(s,u)$, for example. Similarly, it is usually easier to work in the real s, t-plane with orthogonal axes,

rather than with the oblique axes shown in Fig. 1.3.2. We then obtain Fig. 1.3.3 which illustrates the physical regions for the three processes (1.3.8), (1.3.12) and (1.3.14).

If we commence from the u-channel, with process (1.3.12), and cross the particles A_1 and A_4 we obtain the process

$$\bar{A}_4 + \bar{A}_3 \to \bar{A}_1 + \bar{A}_2. \tag{1.3.17}$$

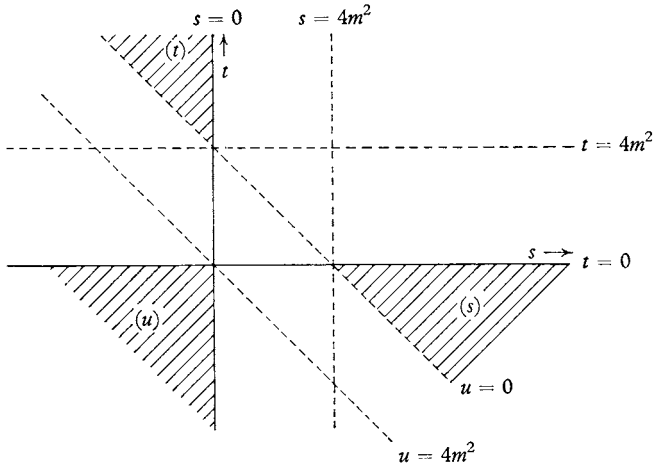

Fig. 1.3.3. Physical regions (shaded areas) for equal-mass scattering shown in the real (s, t)-plane.

The physical region for this process involving anti-particles is the same as that for the process (1.3.8) involving particles. The TCP theorem asserts that the amplitudes for these two processes are the same. Its proof is discussed in § 4.8.

The reactions (1.3.12) and (1.3.14) also have TCP-inverses,

$$\left.\begin{aligned} A_2 + \bar{A}_4 &\to \bar{A}_1 + A_3, \\ A_2 + \bar{A}_3 &\to \bar{A}_1 + A_4, \end{aligned}\right\} \tag{1.3.18}$$

so that altogether crossing and TCP relate the amplitudes for six physical processes to the same function $F(s,t)$. Combining this with the symmetry (1.2.23) [PT invariance] that is valid in strong interactions, we obtain a further six processes by reversing the direction of the above reactions.

Just as we were able to deduce the existence of singularities at the normal thresholds from unitarity, it is possible also to deduce the existence of further singularities from the assumption of crossing

symmetry. Since $\sqrt{t}$ and $\sqrt{u}$ represent energies in the t and u channels, they will yield branch points exactly corresponding to those drawn in B_1, B_2, B_3, ... for s in Fig. 1.3.1, that is at

$$t = 4m^2, \quad 9m^2, \quad 16m^2, \quad \ldots, \tag{1.3.19}$$

$$u = 4m^2, \quad 9m^2, \quad 16m^2, \quad \ldots. \tag{1.3.20}$$

If we fix, say, u at a real value u_0, then because of (1.3.15), the branch-points (1.3.19) will appear in the s-plane at

$$s = -u_0, \quad -u_0 - 5m^2, \quad -u_0 - 12m^2, \quad \ldots, \tag{1.3.21}$$

and the $t = m^2$ pole will appear at

$$s = -u_0 + 3m^2. \tag{1.3.22}$$

The resulting picture (for fixed $u = u_0$) in the complex s-plane is shown in Fig. 1.3.4. This figure depicts the physical sheet. As we remarked earlier this is called the physical sheet since the amplitude $F(s, u_0)$ becomes the physical amplitude for a suitably chosen value of s on this sheet. It is of course a matter of convention that the branch cuts are drawn along the real s-axis. Only their end-points are fixed and they can be distorted as desired without changing the value of the function F. The branch-points however are fixed and cannot be moved so long as the parameter u is kept fixed at u_0.

With real branch-cuts in the s-plane it is necessary to decide which limit on to the branch-cut gives the physical amplitude. For $u_0 < 0$, and with s real and $s > 4m^2 - u_0$, we have physical values of s, t, u that correspond to the s-channel shown in Fig. 1.3.2. We shall see in § 2.3 that perturbation theory shows the physical amplitude to be given by the limit on to this right-hand cut from the upper-half s-plane,

$$F(\text{physical}) = \lim_{\epsilon \to 0+} F(s + i\epsilon, u_0). \tag{1.3.23}$$

This result is obtained by showing it to be equivalent to Feynman's prescription for obtaining physical amplitudes by giving a small negative imaginary part $(-i\epsilon)$ to the mass of each particle in any internal line of a Feynman diagram. With this rule, each Feynman integral can be evaluated with real external four-momenta, that is real s.

This result (1.3.23) from perturbation theory is referred to as 'the $i\epsilon$-prescription'. Its derivation and significance outside the framework of perturbation theory is discussed in § 4.4.