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An unusual cubic representation problem

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Abstract

For a non-zero integer N, we consider the problem of finding 3 integers (a, b, c) such that

$$N = \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}.$$

We show that the existence of solutions is related to points of infinite order on a family of elliptic curves. We discuss strictly positive solutions and prove the surprising fact that such solutions do not exist for N odd, even though there may exist solutions with one of a, b, c negative. We also show that, where a strictly positive solution does exist, it can be of enormous size (trillions of digits, even in the range we consider).

Keywords: cubic representation, elliptic curve, rational points.

MSC: Primary 11D25 11G05, Secondary 11Y50

1. Introduction

Several authors have considered the problem of representing integers N (and in particular, positive integers N) by a homogeneous cubic form in three variables. See, for example, the papers of Bremner & Guy [1], Bremner, Guy, and Nowakowski [2], Brueggeman [3]. Analysis for cubic forms is made possible by the fact the the resulting equation is that of a cubic curve, and hence in general is of genus one.

In this note, we shall study the representation problem

$$N = \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}, \tag{1.1}$$

where a, b, c are rationals. Equivalently, by homogeneity, we may consider $a, b, c \in \mathbb{Z}$.

Studying numerical data, it was observed that, when N is odd, there seem never to exist "positive" solutions of (1.1), that is, solutions with $a, b, c > 0$, even though there may indeed exist solutions with one of a, b, c negative. This fact precludes a simple congruence argument to show the non-existence of positive solutions. In contrast, when N is even, there may or may not be positive solutions. The proof we give of non-existence of positive solutions, for N odd, depends on local considerations at judiciously chosen primes.

In investigating the existence of solutions to (1.1), and more specifically, existence of positive solutions, we discovered that on occasion solutions exist, but the smallest positive solution may be rather large. For example, when $N = 896$, the smallest positive solution has a, b, c with several trillion digits (we do not list it explicitly).

2. The cubic curve

We consider the following problem, that of representing integers N in the form

$$N = \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}$$

for rationals (or, by homogeneity, integers) a, b, c.

For fixed N, the homogenization is a cubic curve

$$C_N : \ N(a+b)(b+c)(c+a) = a(a+b)(c+a) + b(b+c)(a+b) + c(c+a)(b+c)$$

in projective 2-dimensional space which has a rational point, for example, $(a, b, c) = (1, -1, 0)$. The curve is therefore elliptic, and a cubic model is readily computed in the form

$$E_N : \ y^2 = x^3 + (4N^2 + 12N - 3)x^2 + 32(N+3)x.$$

Setting $s = a + b + c$, maps are given by

$$\frac{a}{s} = \frac{8(N+3) - x + y}{2(4-x)(N+3)}, \quad \frac{b}{s} = \frac{8(N+3) - x - y}{2(4-x)(N+3)}, \tag{2.1}$$

and

$$\frac{c}{s} = \frac{-4(N+3) - (N+2)x}{(4-x)(N+3)},$$

with inverse

$$x = \frac{-4(a+b+2c)(N+3)}{(2a+2b-c)+(a+b)N}, \quad y = \frac{4(a-b)(N+3)(2N+5)}{(2a+2b-c)+(a+b)N}.$$

The curve has discriminant

$$\Delta(E_N) = 2^{14}(N+3)^2(2N-3)(2N+5)^3$$

so $\Delta > 0$ for all integers N except $-3, -2, -1, 0, 1$. Thus, other than for these five values, the defining cubic has three real roots, and the elliptic curve has two components. There is an unbounded component with $x \geq 0$, and a bounded component with $x < 0$ (frequently referred to as the "egg").

Lemma 2.1. *The torsion subgroup of E_N is isomorphic to $\mathbb{Z}/6\mathbb{Z}$, except when $N = 2$, when it is isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$.*

Proof. The point $(0, 0)$ is clearly of order 2.

For there to be three rational points of order 2, necessarily there must be rational roots of

$$x^2 + (4N^2 + 12N - 3)x + 32(N + 3) = 0,$$

implying $(2N - 3)(2N + 5) = (2N + 1)^2 - 16 = \square$, with the only integer possibility $N = 2$.

Points of order 3 are points of inflexion of the curve E_N, and it is a standard exercise in calculus that

$$(4, \pm 4(2N + 5))$$

is such a point.

Points of order 2 and of order 3 imply a point (x, y) of order 6, which by the duplication formula, must satisfy

$$\frac{(x^2 - 32(N + 3))^2}{4(x^3 + (4N^2 + 12N - 3)x^2 + 32(N + 3)x)} = 4$$

giving the points $\pm T_0$ of order 6, where

$$T_0 = (\, 8(N + 3), 8(N + 3)(2N + 5) \,).$$

Note: the corresponding torsion point in $C_N(\mathbb{Q})$ is the point $(-1, 1, 1)$.

Further, there can be no point of order 12. For such can arise only when T_0 is divisible by 2, implying $8(N + 3) = \square$. Then from the duplication formula, the following equation

$$(U^2 - 32(N + 3))^2 = 32(N + 3)(U^3 + (4N^2 + 12N - 3)U^2 + 32(N + 3)U)$$

must have a rational root for U. Substituting $N = 2K^2 - 3$,

$$(U^2 + 8K(1 - 4K - 4K^2)U + 64K^2)(U^2 + 8K(-1 - 4K + 4K^2)U + 64K^2) = 0,$$

which demands

$$K(2K - 1)(2K + 1)(2K + 3)(2K - 3) = 0,$$

leading to singular curves.

Thus the torsion group is cyclic of order 6 when $N \neq 2$, and is isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$ when $N = 2$. $\square$

Remark 2.2. The torsion points themselves lead to singular solutions to the original problem, so we need points of infinite order for a finite non-trivial solution. Thus the rank of E_N must be at least 1. The first example, for positive N, is $N = 4$ with a generator for $E_4(\mathbb{Q})$ given by $G = (-4, 28)$. The formulae above give the integer solution $a = 11$, $b = 4$ and $c = -1$. We have $(-4, 28) + (0, 0) = (-56, -392)$ which gives $a = -5$, $b = 9$ and $c = 11$. Adding the other four torsion points gives cyclic permutations of these basic solutions. The point $9G$ is the smallest multiple of G that corresponds to a positive solution (in which $a, b, c \sim 10^{80}$).

Remark 2.3. The torsion points for $N > 0$ all lie on the unbounded component of E_N.

Remark 2.4. In the group $C_N(\mathbb{Q})$, the inverse of the point (a, b, c) is the point (b, a, c). Further, adding the torsion generator $(-1, 1, 1)$ to (a, b, c) gives rise to the order six automorphism ϕ of C given by

$$\begin{aligned}\phi(a,b,c) = \big(&a^2 + ab - ac + bc - 2c^2 - (b^2 - c^2)N,\\ &a^2 + 3ab + 2b^2 + 3ac + bc + 2c^2 - (b + c)(2a + b + c)N,\\ &(a + b)(a - 2b + c) + (b^2 - c^2)N\big).\end{aligned}$$

Then $\phi^2(a, b, c) = (b, c, a)$, $\phi^4(a, b, c) = (c, a, b)$.

Remark 2.5. The torsion group of $E_N(\mathbb{Q})$ is cyclic of order 6, and so there exist isogenies of E_N of degrees 2, 3, 6, which are readily computed from the formulae in Vélu [8] and which we record here in the following Lemma.

Lemma 2.6. *For $i = 2, 3, 6$, there are the following isogenies $\phi_i \colon E_N \to E_N^{(i)}$ of degree i.*

1. $E_N^{(2)} : Y^2 = X^3 - 2(4N^2 + 12N - 3)X^2 + (2N - 3)(2N + 5)^3 X,$
   $\phi_2(x, y) = (y^2/x^2, (x^2 - 32(N + 3))y/x^2);$
2. $E_N^{(3)} : Y^2 = X^3 + (4N^2 + 60N + 117)X^2 + 128(N + 3)^3 X,$
   $$\begin{aligned}\phi_3(x, y) = (&x(x - 8(N + 3))^2/(x - 4)^2,\\ &(x - 8(N + 3))(x^2 + 4(2N + 3)x + 32(N + 3))y/(x - 4)^3);\end{aligned}$$
3. $E_N^{(6)} : Y^2 = X^3 - 2(4N^2 + 60N + 117)X^2 + (2N - 3)^3(2N + 5)X,$
   $$\begin{aligned}\phi_6(x, y) = (&(x^2 + 4(2N + 3)x + 32(N + 3))^2 y^2/(x(x - 4)(x - 8(N + 3)))^2,\\ &p_1(x)p_2(x)p_3(x)y/(x^2(x - 4)^3(x - 8(N + 3))^3)),\end{aligned}$$

where

$$p_1(x) = x^2 - 32(N + 3), \quad p_2(x) = x^2 + 4(2N + 3)x + 32(N + 3),$$

$$\begin{aligned}p_3(x) = x^4 &- 32(N + 3)x^3 - 32(N + 3)(4N^2 + 12N - 1)x^2 - 1024(N + 3)^2 x\\ &+ 1024(N + 3)^2.\end{aligned}$$

3. Rational solutions for $N > 0$

We computed the rank of E_N in the range $1 \leq N \leq 1000$, and in the case of positive rank, attempted to compute a set of generators. The existence of 2-, 3-, and 6-isogenies was particularly helpful when treating the curves with generators of large height, in that we could focus on the curve where the estimated size of a generator was minimal.

In the range of N we consider, the curve with generator of largest height is E_{616}, where the rank is one, and a generator has height ~ 672.28. This was discovered by finding a point of height ~ 224.09 on a 3-isogenous curve. Most of these rank computations were feasible using programs written in Pari-GP; the very largest points were found with the help of Magma [5]. The rank results are summarized in the following table.

# rank 0	# rank 1	# rank 2	# rank 3
436	485	76	3

Table 1: Rank distribution for $1 \leq N \leq 1000$

Rank one examples occur for $N = 4, 6, 10, 12, \ldots$, rank two examples for $N = 34, 94, 98, 111, \ldots$, and rank three examples for $N = 424, 680, 975$.

4. Positivity

Henceforth, we assume that $N > 0$. A natural question is do positive solutions a, b, c of the original equation exist? In particular, how do we recognise points $(x, y) \in E_N(\mathbb{Q})$ that correspond to positive solutions of (1.1)?

Theorem 4.1. *Suppose $(a, b, c) \in C_N$ corresponds to $(x, y) \in E_N(\mathbb{Q})$. Then $a, b, c > 0$ if and only if either*

$$\tfrac{(3-12N-4N^2-(2N+5)\sqrt{4N^2+4N-15}}{2} < x < -2(N+3)(N+\sqrt{N^2-4}), \tag{4.1}$$

or

$$-2(N+3)(N-\sqrt{N^2-4}) < x < -4\left(\frac{N+3}{N+2}\right). \tag{4.2}$$

Proof. Suppose $a, b, c > 0$. From (2.1),

$$0 < \frac{ab}{s^2} = \tfrac{(4-x)\left(x^2+4N(N+3)x+16(N+3)^2\right)}{4(N+3)^2(4-x)^2}, \tag{4.3}$$

so necessarily $x < 4$; and then $c/s > 0$ implies

$$x < -4\left(\frac{N+3}{N+2}\right)$$

(and, in particular, the point (x, y) lies on the egg). By symmetry in a, b, we may suppose $y > 0$. From (2.1),

$$\frac{a}{s} = \frac{8(N+3) - x + y}{2(4-x)(N+3)} > 0.$$

It remains to ensure that $b/s = \frac{8(N+3)-x-y}{2(4-x)(N+3)} > 0$. But from (4.3), $b/s > 0$ precisely when

$$x^2 + 4N(N+3)x + 16(N+3)^2 > 0;$$

and this latter happens when

$$\text{either} \quad x < -2(N+3)(N + \sqrt{N^2-4}), \quad \text{or} \quad x > -2(N+3)(N - \sqrt{N^2-4}).$$

Putting these results together, necessary conditions for a, b, c to be positive are the following:

$$\frac{1}{2}(3 - 12N - 4N^2 - (2N+5)\sqrt{4N^2+4N-15}) < x < -2(N+3)(N + \sqrt{N^2-4}),$$

where the left inequality is automatic, arising from $y^2 > 0$, or

$$-2(N+3)(N - \sqrt{N^2-4}) < x < -4\left(\frac{N+3}{N+2}\right).$$

It is straightforward to see that these conditions on x, y are now also sufficient for the positivity of a, b, c. $\square$

It follows that positive solutions can only come from rational points on the egg component of the curve.

5. N odd

Analyzing solutions found from computation, it was observed that when N is odd (in contrast to the case N even) there seem never to be points on the curve E_N with $x < 0$. We show that this is indeed the case.

Theorem 5.1. *Suppose $N \equiv 1 \bmod 2$. Then $(x, y) \in E_N(\mathbb{Q})$ implies $x \geq 0$.*

Proof. Set $N + 3 = 2M$, $M \geq 2$, so that the curve E_N takes the form

$$E_M : \; y^2 = x(x^2 + (16M^2 - 24M - 3)x + 64M).$$

A point $(x, y) \in E_M(\mathbb{Q})$ satisfies $x = dr^2/s^2$, for $d, r, s \in \mathbb{Z}$, $(r, s) = 1$, with $d \mid 64M$, and without loss of generality, d squarefree. Then

$$dr^4 + (16M^2 - 24M - 3)r^2s^2 + \frac{64M}{d}s^4 = \square.$$

The claim is that this quartic can have no points r, s when $d < 0$.

On completing the square

$$\left(2dr^2 + (16M^2 - 24M - 3)s^2\right)^2 - (4M-1)^3(4M-9)s^4 = 4d\,\square, \qquad d \mid 2M.$$

Case I: $d < 0$, d odd.
Let $d = -u$, $u > 0$, u odd, with $M = um$. We now have

$$-ur^4 + (16M^2 - 24M - 3)r^2s^2 - 64ms^4 = \square,$$

equivalently,

$$(-2ur^2 + (16M^2 - 24M - 3)s^2)^2 - (4M-1)^3(4M-9)s^4 = -4u\square.$$

Note that the Jacobi symbol

$$\begin{aligned}\left(\frac{-u}{4M-1}\right) = \left(\frac{-1}{4M-1}\right)\left(\frac{u}{4M-1}\right) &= -\left(\frac{4M-1}{u}\right)(-1)^{(u-1)/2}\\ &= -\left(\frac{-1}{u}\right)(-1)^{(u-1)/2} = -1.\end{aligned}$$

However, if every prime p dividing $4M-1$ satisfies $\left(\frac{-u}{p}\right) = +1$, then the Jacobi symbol $\left(\frac{-u}{4M-1}\right) = +1$ by multiplicativity of the symbol. In consequence, there exists a prime p dividing $4M-1$ satisfying $\left(\frac{-u}{p}\right) = -1$. Then for such a prime p,

$$\begin{aligned}-2ur^2 + (16M^2 - 24M - 3)s^2 &\equiv 0 \bmod p,\\ -2ur^2 - 8s^2 &\equiv 0 \bmod p,\\ 4s^2 &\equiv -ur^2 \bmod p,\end{aligned}$$

forcing $r \equiv s \equiv 0 \bmod p$, contradiction.

Case II: $d < 0$, d even.
Let $d = -2u$, $u > 0$, u odd, with $M = um$. We now have

$$-2ur^4 + (16M^2 - 24M - 3)r^2s^2 - 32ms^4 = \square,$$

equivalently

$$(-4ur^2 + (16M^2 - 24M - 3)s^2)^2 - (4M-1)^3(4M-9)s^4 = -8u\,\square.$$

Subcase (i): M even.
Now

$$\left(\frac{-2u}{4M-1}\right) = \left(\frac{-2}{4M-1}\right)\left(\frac{4M-1}{u}\right)(-1)^{(u-1)/2} = -\left(\frac{-1}{u}\right)(-1)^{(u-1)/2} = -1,$$

and arguing as above, there exists a prime p dividing $4M-1$ with $\left(\frac{-2u}{p}\right) = -1$. Then

$$\begin{aligned} -4ur^2 + (16M^2 - 24M - 3)s^2 &\equiv 0 \bmod p, \\ -4ur^2 - 8s^2 &\equiv 0 \bmod p, \\ 4s^2 &\equiv -2ur^2 \bmod p, \end{aligned}$$

forcing $r \equiv s \equiv 0 \bmod p$, contradiction.

Subcase (ii): M odd (so in particular, m odd).
In this case, the equation is 2-adically unsolvable, as follows. We have

$$-2ur^4 + (16M^2 - 24M - 3)r^2s^2 - 32ms^4 = \square$$

implying s is odd. Modulo 4, r cannot be odd, and thus r is even. Then

$$-3(r/2)^2 \equiv \square \bmod 8$$

so that $r/2$ is even; and now

$$-3(r/4)^2 - 2m \equiv \square \bmod 4,$$

$$-3(r/4)^2 - 2 \equiv \square \bmod 4,$$

impossible. □

Corollary 5.2. *If N is odd and E_N is of positive rank, then generators for $E_N(\mathbb{Q})$ lie on the unbounded component of E_N.*

Consequently, in the situation of Corollary 5.2, there are no rational points on the egg, so no positive solutions of (1.1) exist. This happens when the rank is one for $N = 19, 21, 23, 29, \ldots$, when the rank is two, for $N = 111, 131, 229, 263, \ldots$, and when the rank is three, for $n = 975$. It can also occur that when N is even, all generators for $E_N(\mathbb{Q})$ lie on the unbounded component of E_N, so that there are no rational points on the egg. This situation occurs for rank one examples $N = 40, 44, 50, 68, \ldots$, rank two examples $N = 260, 324, 520, 722, \ldots$, and the rank three example $N = 680$.
Hence there exist even N, namely $N = 40, 44, 50, 68, \ldots$ where there exist solutions to (1.1), but there do not exist positive solutions. In contrast, we have the following result.

Theorem 5.3. *There exist infinitely many positive even integers N such that (1.1) has positive solutions.*

Proof. The proof is immediate, using the parameterization $N = t^2 + t + 4$ with the point on the egg of E_N given by

$$(x, y) = \left(-4(t^2 + t + 1)^2,\ 4(2t + 1)(t^2 + t + 1)(3t^2 + 3t + 7)\right). \qquad \square$$

Remark 5.4. It is straightforward to show that this point corresponds to

$$a = (t^2+1)(3t^3+8t^2+14t+11), \quad b = -(t^2+2t+2)(3t^3+t^2+7t-2),$$

$$c = t^6+3t^5+11t^4+17t^3+20t^2+12t-1;$$

with no (real) value of t making $a, b, c > 0$; so some multiple of the point will be needed to obtain a positive solution.

6. Size of positive solutions

A positive solution of (1.1) demands the existence of a point in $E_N(\mathbb{Q})$ that lies on the egg; and in particular not all generators for $E_N(\mathbb{Q})$ can lie on the unbounded branch of the curve. For a positive solution, therefore, a generator in $E_N(\mathbb{Q})$ must lie on the egg.

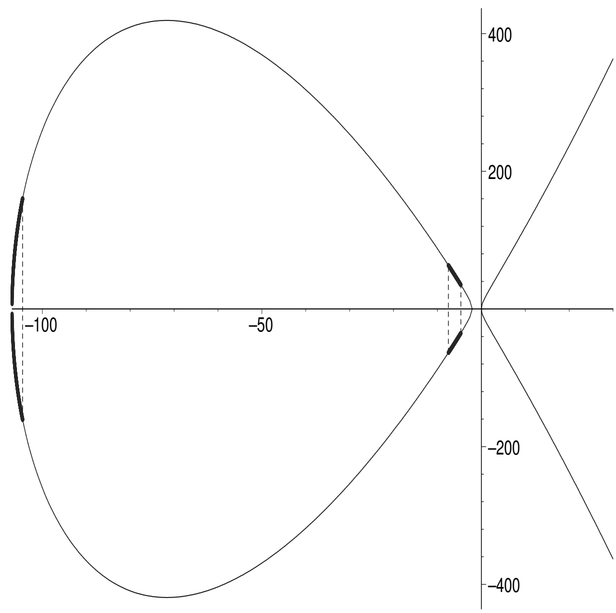

Figure 1: Region for $a, b, c > 0$ on $E_4(\mathbb{Q})$

Such a point may not satisfy the inequalities (4.1), (4.2), of course, but a result of Hurwitz [4] implies that rational points on E_N are now dense on both components of E_N, so that there will indeed exist points in $E_N(\mathbb{Q})$ that satisfy (4.1), (4.2). The width of the interval at (4.1) tends to 1 as $N \to \infty$, and the width of the interval at (4.2) tends to 0. The width of the egg however is $O(N^2)$. Thus if the rank of the curve E_N is equal to one, with a generator P on the egg regarded as lying essentially at random on the egg, then the smallest integer m such that mP satisfies (4.1), (4.2), may be very large. If we assume equidistribution of random points on the egg, then a crude estimate of arc length shows that there is probability $O(\frac{1}{N})$ that a random point of the egg lies within the region defined by (4.1), (4.2). In Figure 1 we sketch the graph for $N = 4$, indicating the region corresponding to positive a, b, c, representing the intervals $-106.9046 < x < -104.4974$, $-7.5026 < x < -4.6667$. We consider curves E_N of rank one in the range $1 \leq N \leq 1000$, where there is a generator P of $E_N(\mathbb{Q})$ lying on the egg. For these curves, we computed the smallest integer m such that one of the points $mP + T$, $T \in \mathrm{Tor}(E_N(\mathbb{Q}))$, satisfies (4.1), (4.2).

We then computed the maximum number of digits in a, b, c. The results for $1 \leq N \leq 200$ are given in Table 2.

N	m	# digits	N	m	# digits	N	m	# digits
4	9	81	48	311	418086	136	65	26942
6	11	134	58	221	244860	146	307	259164
10	13	190	60	61	9188	156	353	12046628
12	35	2707	66	107	215532	158	1211	15097279
14	47	1876	76	65	23662	162	457	1265063
16	11	414	82	157	85465	178	2945	398605460
18	49	10323	92	321	252817	182	853	2828781
24	107	33644	102	423	625533	184	851	20770896
28	121	81853	112	223	935970	186	643	5442988
32	65	14836	116	101	112519	196	701	11323026
38	659	1584369	126	75	196670	198	121	726373
42	419	886344	130	707	8572242	200	2957	71225279
46	201	198771	132	461	3607937			

Table 2: The maximum number of digits in a, b, c in the range $1 \leq N \leq 200$

For comparison, the twenty volume second edition of the Oxford English Dictionary is estimated to contain 350 million printed characters (see [6]), a little less than the number of digits in each of a, b, c when $n = 178$.

It is not practical to compute points on elliptic curves with heights that begin to exceed those of the previous table. For example, when $N = 896$, the curve E_{896} has rank one, and the smallest multiple of the generator P (which itself has height 128.76) such that mP corresponds to a positive solution at (1.1), is given by $m = 161477$.

Remark 6.1. Such computations were performed using high-precision real arithmetic. Computing a multiple mP takes $O(\log_2(m))$ operations, so in the computed range where $m \leq 161477$, precision is not a problem. For safety however, and because the computation took only slightly longer, we worked with 10^6 digits of precision.

7. Size bounds on positive solutions

We proceed to determine a crude lower bound for the number of digits in the positive solution a, b, c from a knowledge of the canonical height of the corresponding point on E_N.

Suppose that $P(x, y)$, where $x < 0$, is a point on E_N giving rise to a positive a, b, c. So one of the inequalities (4.1), (4.2) holds, and in particular x is negative (in fact, $x < -4$).

Theorem 7.1. *Let $(a, b, c) \in C_N(\mathbb{Q})$ correspond to $P(x, y) \in E_N(\mathbb{Q})$, and suppose that $a, b, c > 0$. Then*

$$\max(\log(a), \log(b)) > \frac{3}{2}h(P) - 6\log(N) - 10,$$

where h denotes the canonical height function on E_N.

Proof. The mapping $E_N \to C_N$ is given by

$$a : b : c = -x + y + 8(N + 3) : -x - y + 8(N + 3) : -2x(N + 2) - 8(N + 3).$$

Write $x = -u/w^2$, $y = v/w^3$, where $u > 0$, $w > 0$, and $(u, w) = (v, w) = 1$. Since $x < -1$, we have $u > w^2$, and the naive height $H(P)$ of P is equal to u.

Either inequality (4.1), (4.2), implies

$$u/w^2 < \frac{1}{2}(-3 + 12N + 4N^2 + (2N + 5)(2N + 1)) = 4N^2 + 12N + 1 < (2N + 3)^2.$$

Write

$$\begin{aligned} ah &= uw + v + 8(N + 3)w^3, \\ bh &= uw - v + 8(N + 3)w^3, \\ ch &= 2uw(N + 2) - 8(N + 3)w^3, \end{aligned}$$

where h is the greatest common divisor of the three expressions on the right.

Now $ah + bh = 2(u + 8(N + 3)w^2)w$ so that

$$\begin{aligned} ah + bh = 2(u + 8(N + 3)w^2)w &> 2(1 + \frac{8(N + 3)}{(2N + 3)^2})uw \\ &= \frac{2(4N^2 + 20N + 33)}{(2N + 3)^2}uw \\ &> \frac{2(4N^2 + 20N + 33)}{(2N + 3)^3}u^{3/2} \\ &> \frac{1}{N}H(P)^{3/2}. \end{aligned}$$

Thus necessarily either ah or bh (and by choice of sign of y, we may assume this is ah) is at least equal to $\frac{1}{2N}H(P)^{3/2}$.

We now estimate h. We have $(a-b)h = 2v$, $(a+b)h = -2uw + 16(N+3)w^3$, $(a+b+2c)h = -2uw(2N+5)$, so that $((N+2)(a+b)-c)h = 8(N+3)(2N+5)w^3$. Thus $h \mid 8(N+3)(2N+5)w^3$. Now if p is a prime dividing (h,w), necessarily $p \mid 2v$, so that $p=2$, since $(v,w)=1$. But w even implies v odd, so that $2\|h$. Moreover, in the case that w is odd, then $(h,w)=1$. It follows that $h \mid 8(N+3)(2N+5)$, and in particular, $h \leq 8(N+3)(2N+5)$. This bound is best possible, in that in our range of computation, there are several instances where $h = 8(N+3)(2N+5)$. Consequently, a is at least equal to $H(P)^{3/2}/(16N(N+3)(2N+5))$.

There are known bounds on the difference between the canonical height and the logarithm of the naive height, in the form

$$c_1 \leq \log H(P) - h(P) \leq c_2,$$

for constants c_1, c_2. The following estimate for c_1 is taken from Silverman [7, Theorem 1.1], where $\Delta(E_N)$ and $j(E_N)$ denote the discriminant and j-invariant of E_N, respectively.

$$\begin{aligned}
c_1 &= -\frac{1}{12}\log|\Delta(E_N)j(E_N)| - \frac{1}{2}\log|\frac{4N^2+12N-3}{3}| - \frac{1}{2}\log(2) - 1.07 \\
&= -\frac{1}{4}\log|\frac{(2N+3)(8N^3+36N^2+6N-93)(4N^2+12N-3)^2}{9}| - \frac{3}{2}\log(2) - 1.07 \\
&> -\frac{1}{4}\log(226N^8) - \frac{3}{2}\log(2) - 1.07 \quad (\text{for } N \geq 4) \\
&> -2\log(N) - 3.47.
\end{aligned}$$

Thus,

$$\begin{aligned}
\log a &> \frac{3}{2}\log H(P) - \log(16N(N+3)(2N+5)) \\
&> \frac{3}{2}h(P) + \frac{3}{2}c_1 - \log(16N(N+3)(2N+5)) \\
&> \frac{3}{2}h(P) - 3\log(N) - 5.20 - \log(92N^3) \quad (\text{for } N \geq 4) \\
&> \frac{3}{2}h(P) - 6\log(N) - 10. \qquad \square
\end{aligned}$$

Remark 7.2. In the case where (4.2) holds, the above bound may be improved to $\log a > \frac{3}{2}h(P) - 4\log(N) - 9$.

When $N = 896$, with multiple $m = 161477$, then $x(161477P) \sim -4.0133512$, so that (4.2) holds. Now $h(P) = 3357394890723.0389$ and the above estimate gives $\log a > 5036092336048.36658$. That is, a has in excess of 2.187 trillion digits (which amounts to about 6250 OED units).

Remark 7.3. This estimate is very crude. For example, when $N = 178$, with multiple 2945, then $h(P) = 265736973.117$ and the above estimate gives $\log(a) >$ 398605418.5847, that is, a has in excess of 173112134 digits. From the table in section 4, we see that the actual number of digits is equal to 398605460.

8. Rational solutions for $N < 0$

The motivation has been to study the case $N > 0$ because of our interest in positive solutions. But in the course of the investigation we also computed ranks and generators for all $-1 \geq N \geq -1000$ ($N \neq -3$). The rank results are summarized in the following table.

# rank 0	# rank 1	# rank 2	# rank 3
393	471	126	9

Table 3: Rank distribution for $-1000 \leq N \leq -1$, $N \neq -3$

Rank one examples occur for $N = -5, -8, -9, \ldots$, rank two examples for $N = -17, -29, -38, \ldots$, and rank three examples for $N = -181, -365, -369, \ldots$. The generator of greatest height occurs for $N = -994$, where the height is ~ 690.84.

References

[1] Bremner, A., Guy, R.K., Two more representation problems, *Proc. Edin. Math. Soc.* Vol. 40 (1997), 1–17.

[2] Bremner,A., Guy, R.K., Nowakowski, R.J., Which integers are representable as the product of the sum of three integers with the sum of their reciprocals, *Math. Comp.* Vol. 61 (1993), no. 203, 117–130.

[3] Brueggeman, S., Integers representable by $(x + y + z)^3/xyz$, *Internat. J. Math. Math. Sci.* Vol. 21 (1998), no. 1, 107—116.

[4] Hurwitz, A., Über ternäre diophantische Gleichungen dritten Grades, *Vierteljahrschrift d. Naturforsch. Ges. Zürich* Vol. 62 (1917), 207–229.

[5] Bosma, W., Cannon, J., Playoust, C. The Magma algebra system. I. The user language, *J. Symbolic Comput.* Vol. 24 (1997), 235–265.

[6] `http://public.oed.com/history-of-the-oed/dictionary-facts/`

[7] Silverman, J.H., The difference between the Weil height and the canonical height on elliptic curves, *Math. Comp.* Vol. 55 (1990), no. 192, 723—743.

[8] Vélu, J., Isogénies entre courbes elliptiques, *C.R. Acad. Sci. Paris (A)*, Vol. 273, (1971), 238-241.