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Sonification of multiple Fibonacci-related sequences*

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Abstract

Expanding on previous musical exploration involving the sonification of Fibonacci-related number sequences, five contrasting stereo electro-acoustic compositions utilizing multiple integer sequences simultaneously are presented and analyzed.

Keywords: music, sonification, musical composition

MSC: 00A65

1. Introduction

This article expands on previous research into the direct sonification of Fibonacci-related number sequences. Direct sonification, or pure sonification, is a technique in which the composer attempts to create a musical or sonic graph that represents as accurate a likeness of the mathematical object sonified as is possible. In two previous papers, a system of tunings based on the Fibonacci sequence and the golden ratio was introduced, and various sonifications of Fibonacci-related integer sequences and Zeckendorf representations were presented [1] [2]. In this paper, we focus for the first time on the sonification of multiple integer sequences simultaneously.

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1.1. Terminology and definitions

This article makes use of the terms *dynamic parameters* and *static parameters.* Dynamic parameters are those musical parameters that the integer sequence sonified controls – those musical parameters that change according to the values of the sequence. This can include parameters such as frequency, level of loudness, spatial location, etc. Static parameters are, in contrast, those musical parameters that do not change according to the sequence and remain the same throughout the sonification. This can include many things, such as the number of audio channels used, the timbre used or the duration attributed to each integer, etc. If a sonification utilizes more than one sequence simultaneously, then dynamic and static parameters can be either *individual* or *global.* Individual parameters are those that pertain only to a specific sequence, whereas global pertain to all sequences that are being simultaneously sonified.

In the following, let $\varphi = \frac{\sqrt{5}-1}{2}$ and $\Phi = \frac{\sqrt{5}+1}{2}$. Let F_n represent the Fibonacci sequence, where $F_0 = 0$ and $F_1 = 1$. Let G_n represent a generalized Fibonacci sequence. Sequences are enclosed in <> brackets.

2. Five compositions analyzed

In these analyses, we will utilize slightly modified and abridged versions of the scores of the works for practical purposes, as we have in previous papers. The original scores are Internet-based and in color, which can be very useful when graphing multiple sequences and in differentiating between individual and global parameters. It is highly recommended that the reader view and listen to the Quicktime files and see the original scores for each composition as well, to which links are provided. These pieces were all composed in 2011 and 2012 using the author's Objective-C++ program *Virahanka*, created for composing with various types of number sequences utilizing Csound [3] as an integrated sound synthesis engine.

2.1. φ and Φ signature sequences no. 3

If R is a positive irrational number and we arrange the set of all numbers $i + jR$ in order, where i and j are positive integers, $i_1 + j_1R$, $i_2 + j_2R$, $i_3 + j_3R$, $\ldots$, then $\langle i_1, i_2, i_3, \ldots \rangle$ is the signature sequence of R [4]. In the case of this composition, the two signature sequences used are those based on the smaller and larger golden ratio values, where R $= \varphi$ and R $= \Phi$, respectively (A084532 and A084531 in Sloane's OEIS [5]).

In this audio-visual work from *Collection XI* (B1237 in [6]), two signature sequences are sonified simultaneously with a synchronized point graph. The original score, Quicktime file (1920×1080 pixels) and other information can be found at `http://caseymongoven.com/b1237`.

The synthesis technique utilized in this work is based on the short-time Fourier transform (STFT), in which a monaural signal – in this case a wine glass from

Daniel Gehrs Winery being struck – is broken up into many smaller overlapping pieces of equal length and analyzed in the frequency domain as it changes over time in order to enable frequency shifting without time-stretching (among other possible manipulations). This means that each note articulated in this piece is derived from spectral data of the same single original sample of a wineglass being struck.

One of the challenges with signature sequences in particular, due to the relatively even distribution of the integers in such sequences, is finding a point of termination. Generally, a point of termination at either the highest point the sequence has reached, or at an occurrence of the integer 1, sounds best. Regardless of where one stops, the compositional result with signature sequences – and many other sequences with such uniform distribution of the integers – ends up sounding a bit as if one had suddenly torn it off at the end, reminiscent in a way of György Ligeti's instructions at the end of his *Continuum* for harpsichord: "Stop suddenly, as though torn off." In the case of this composition, 351 members of both sequences were utilized, beginning with the first members of the sequences. This resulted in a range of integers of 1-21 for the signature sequence where $R = \varphi$, and a range of 1-33 for the signature sequence where $R = \Phi$.

Each integer is attributed a static duration of .05 seconds, resulting in a piece duration of $351 * .05$ seconds $= 17.55$ seconds. Two dynamic parameters are used: frequency and simulated location. The latter is an individual dynamic parameter (each sequence has been given its own location),[1] the former is global. The unit interval of the tuning used in this piece is $\varphi^7 + 1$, and the pitch orientation is descending, meaning that higher integers are represented by lower pitches. Dynamic level (loudness) is a static parameter: each piece is attributed the dynamic level mezzo forte (medium loud). Attack and release values – also static parameters in this composition – of .0055 and .0089 seconds were used for each integer.[2]

Collection XI

φ and Φ Signature Sequences no. 3
Casey Mongoven
October 29, 2011

classification of work: `audio-visual`
synthesis engine: `Csound` `show Csound orchestra`
synthesis technique: `STFT-based phase vocoder`

description of sequences sonified

A084532 `Arranging the numbers s + j`φ `in increasing order, where s and j are positive integers, the sequence of s's is the signature sequence of` φ.

[1]The simulated location is given in degrees. 0 degrees represents straight ahead, while positive numbers represent sound sources emanating from the right and negative values represent those emanating from the left.

[2]Attack and release values are part of the so-called *envelope* of a note. Attack is the time in the very beginning of the note (often on a micro-sound time scale) where it becomes louder, and release is the part at the very end when it becomes quieter. Without attack and release values, it is possible that a click can result using certain synthesis techniques.

φ is equal to (-1 + sqrt(5))/2. 351 members used

A084531 Arranging the numbers s + jΦ in increasing order, where s and j are positive integers, the sequence of s's is the signature sequence of Φ. Φ is equal to (1 + sqrt(5))/2. 351 members used

global static parameters:

offset: 1
pitch orientation: descending
temperament: phi^7 + 1
number of channels: 2
note value: 0.05 seconds
piece length: 17.55 seconds
spectral data: gehrs glass seven
dynamic: mf
attack: .0055 seconds
release: .0089 seconds

global and individual dynamic parameters:

A084532

integer	approximate frequency Hz	simulated location degrees
1	873.0	-1.635
2	843.9	-2.394
3	815.8	-3.152
4	788.7	-3.910
5	762.4	-4.669
6	737.0	-5.427
7	712.5	-6.185
8	688.8	-6.944
⋮	⋮	⋮
19	474.6	-15.286
20	458.8	-16.044
21	443.5	-16.802

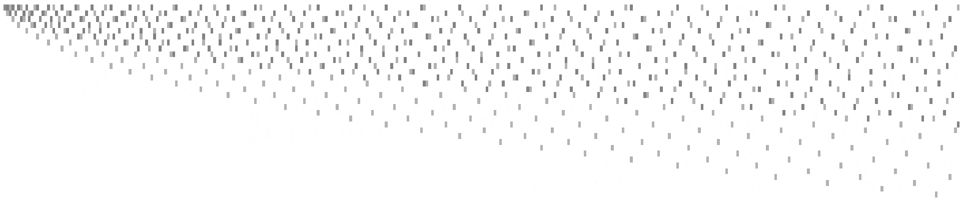
A084531

integer	approximate frequency Hz	simulated location degrees
1	873.0	25.903
2	843.9	25.144
3	815.8	24.386
4	788.7	23.628
5	762.4	22.869
6	737.0	22.111
7	712.5	21.353
8	688.8	20.594
⋮	⋮	⋮
31	316.1	3.152
32	305.6	2.394
33	295.4	1.635

351 values used of Sloane's A084532: 1, 1, 2, 1, 2, 1, 3, 2, 1, 3, 2, 4, 1, 3, 2, 4, 1, 3, 5, 2, 4, ..., 14, 6, 19, 11, 3, 16, 8, 21

351 values used of Sloane's A084531: 1, 2, 1, 3, 2, 4, 1, 3, 5, 2, 4, 1, 6, 3, 5, 2, 7, 4, 1, 6, 3, ..., 12, 25, 4, 17, 30, 9, 22, 1

graph of sequences: 1) A084532, 351 values; 2) A084531, 351 values.



Traditional European music theory has often expressed the aesthetic appeal of contrary motion (voices moving in opposite directions) in composition. If we zoom in and take a closer look at the beginning of the graph, we can see that the voices exhibit consistent contrary motion; after the first two members of the sequence, when one voice ascends, the other descends and vice versa:

In the original online version of the score, it is possible to hover over the individual integers in the graph with the cursor and view the dynamic parameters attributed to each integer.

2.2. Min and Max Fibbit Running no. 4

I learned about the sequences used in this work (B1359 in [6]) from Ron Knott. The first is based on the number of runs of equal bits in the Zeckendorf binary representations [7]. For example, 100010100 (the Zeckendorf representation of the integer $66 = 55 + 8 + 3$) has six runs of equal bits, namely 1, 000, 1, 0, 1, and 00. The following diagram shows the first 13 binary Zeckendorf representations with the number of runs of equal bits (A104324 in [5]):

integer	**Zeckendorf representation**	**number of runs**
1	1	1
2	10	2
3	100	2
4	101	3
5	1000	2
6	1001	3
7	1010	4
8	10000	2
9	10001	3
10	10010	4
11	10100	4
12	10101	5
13	10000	2
⇣	⇣	⇣

Ron Knott named the first composition written with this sequence *Min Fibbit Running* (B101 in [6]), as the Zeckendorf representation is sometimes called the *minimum* representation (in that it requires the smallest number of Fibonacci numbers), *bit* because of the binary representation, and *running* because the sequence counts *runs*.

The second sequence (A104325 in [5]) follows the same exact principle of counting runs, only using the dual Zeckendorf binary representations, described in [8]. Whereas the standard Zeckendorf binary representations contain no consecutive 1s, the dual Zeckendorf representations contain no consecutive 0s. The dual Zeckendorf binary representation can be created by starting with the standard Zeckendorf representation and applying a left-to-right algorithm recursively in which each occurrence of the bit string 100 is replaced with 011 until no consecutive zeros are found in the representation. The result is as follows:

integer	**dual Zeckendorf representation**	**number of runs**
1	1	1
2	10	2
3	11	1
4	101	3
5	110	2
6	111	1
7	1010	4
8	1011	3
9	1101	3
10	1110	2
11	1111	1
12	10101	5
13	10110	4
⇣	⇣	⇣

In this composition, wavetable synthesis was used, a technique based on the periodic reproduction of a single cycle of a waveform. $F_{15}-1 = 609$ members of each sequence were used; this number was chosen to preserve the symmetry of these self-similar sequences. In contrast to signature sequences, which are relatively uniform throughout, these sequences have clear "seams" at $F_n - 1$, which are convenient points to end a sonification. In addition, both sequences will always end with the same integer if the point of termination is $F_n - 1$, because there is only one possible representation of an integer $F_n - 1$ as a sum of Fibonacci numbers [10].

In this piece, the duration of each integer is slightly slower: .065 seconds. Location is a static parameter – one sequence is placed on each speaker. The wavetables were each derived from a single cycle of a wave from a viola built by Anne Cole; the viola was named "Bluebonnet" by its maker. The tuning used is the harmonic series with a fundamental frequency of 86 Hz, i.e. the integers of the sequence $a(n)$ were mapped directly to partials $a(n) \to a(n) * 86$ Hz, which makes the highest integer $13 \to 1118$ Hz. Each sequence is attributed its own wavetable and therefore its own timbre. Frequency, loudness, attack and release are the dynamic parameters utilized in this composition – all of them are global, applying to both sequences.

In the original online scores of Collection XIII, almost any element in the score can be hovered over with the cursor in order to gain more information and clarify meaning.

Collection XIII

Min and Max Fibbit Running no. 4
Casey Mongoven
March 13, 2012

classification of work: `audio-visual`
synthesis engine: `Csound` `show Csound orchestra`
synthesis technique: `wavetable with FFT resynthesis`

description of sequences sonified

A104324 `number of runs in the minimal Fibonacci (binary) representation of n 609 members used`
A104325 `number of runs in the maximal Fibonacci (binary) representation of n 609 members used`

global static parameters:

offset: `1`
temperament: `series of harmonic partials`
pitch orientation: `ascending`
number of channels: `2`
note duration: `0.065 seconds`
piece duration: `39.585 seconds`

individual static parameters:

location:
-30°
30°

wavetables:
cole bluebonnet 4
cole bluebonnet 5

global dynamic parameters:

A104324

integer	frequency Hz	loudness	attack s	release s
1	86.000000	pp	0.012300	0.014400
2	172.000000	pp	0.011908	0.013942
3	258.000000	p	0.011517	0.013483
4	344.000000	p	0.011125	0.013025
5	430.000000	p	0.010733	0.012567
6	516.000000	mp	0.010342	0.012108
7	602.000000	mp	0.009950	0.011650
8	688.000000	mp	0.009558	0.011192
9	774.000000	mf	0.009167	0.010733
10	860.000000	mf	0.008775	0.010275
11	946.000000	mf	0.008383	0.009817
12	1032.000000	f	0.007992	0.009358
13	1118.000000	f	0.007600	0.008900

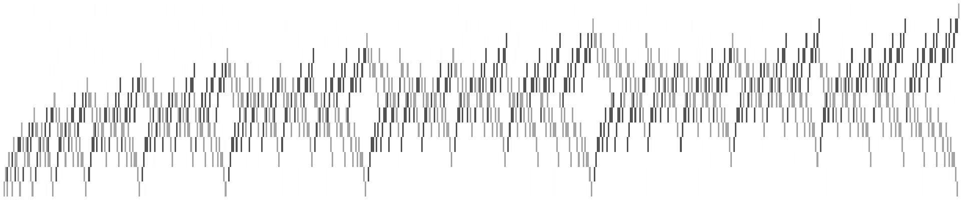
A104325

integer	frequency Hz	loudness	attack s	release s
1	86.000000	pp	0.012300	0.014400
2	172.000000	pp	0.011908	0.013942
3	258.000000	p	0.011517	0.013483
4	344.000000	p	0.011125	0.013025
5	430.000000	p	0.010733	0.012567
6	516.000000	mp	0.010342	0.012108
7	602.000000	mp	0.009950	0.011650
8	688.000000	mp	0.009558	0.011192
9	774.000000	mf	0.009167	0.010733
10	860.000000	mf	0.008775	0.010275
11	946.000000	mf	0.008383	0.009817
12	1032.000000	f	0.007992	0.009358
13	1118.000000	f	0.007600	0.008900

609 values used of Sloane's A104324: 1, 2, 2, 3, 2, 3, 4, 2, 3, 4, 4, 5, 2, 3, 4, 4, 5, 4, 5, 6, 2, ..., 10, 11, 12, 10, 11, 12, 12, 13

609 values used of Sloane's A104325: 1, 2, 1, 3, 2, 1, 4, 3, 3, 2, 1, 5, 4, 3, 4, 3, 3, 2, 1, 6, 5, ..., 4, 3, 4, 3, 3, 2, 1, 13

graph of sequences: 1) A104324, 609 values; 2) A104325, 609 values.



These two sequences combined exhibit particularly beautiful contrary motion.

2.3. Q(n) and U(n) rep sequences no. 1

The first sequence used in this work (B1361 in [6]) is based on the number of possible representations of an integer n as a sum of distinct elements of the Lucas sequence beginning $\langle 1, 3, 4, 7, 11, \dots \rangle$ (A003263 in [5]).[3] For example, the integer 8 can be represented in two ways: $7+1$ or $4+3+1$, so $a_8 = 2$. Integers that are not representable (e.g. 2 and 6) are sonified as silence. The second sequence sonified follows the exact same principle, but uses elements of the generalized Fibonacci sequence $\langle 1, 4, 5, 9, 14, \dots \rangle$ instead (A103344 in [5] – called U(n) here). 1363 members of the first sequence were used, and 1740 of the second. This means that in this case, the first sequence has an earlier point of termination than the second. This was done in order to highlight a certain relationship between these sequences: if one removes all of the 0s from both sequences, which are represented as silence in this sonification, then the sequences are identical. The musical form resulting from the combination of these sequences is therefore a unique type of mensural canon by augmentation.

The synthesis technique used in this composition was granular synthesis with a resonance filter. Granular synthesis is a general sound synthesis technique that operates on the microsound time scale in which small fragments of sound called *grains* (generally lasting between 1 to 50 milliseconds) are utilized. In the sonification of integer sequences, a grain can be used to represent an integer. This technique can be highly useful because, compared to other synthesis techniques, a much larger number of integers can be heard in a short timespan. If a resonance filter is used to filter a grain, as it is here, then each integer can be attributed its own resonance center frequency. Similarly, each grain can be attributed its own location according to the sequence or its own level of loudness. The grain used was from a different Anne Cole viola, named "1980" after the year it was made. The note duration here is a speedy .025 seconds. The unit interval of the tuning is again $\varphi^7 + 1$, this time with ascending frequency orientation (higher integers are represented by higher frequencies). The natural filtered attack of the grain was left unaltered, but a static release value of .0089 was used at the end of each note.

As in the last composition, location is a static parameter and each sequence is placed on an individual loudspeaker. Three global dynamic parameters are utilized: resonance filter center frequency, resonance filter bandwidth and loudness.

[3]This sequence was referred to as Q(n) in *Fibonacci and Related Number Theoretical Tables* [9].

Collection XIII

Q(n) and U(n) Rep Sequences no. 1
Casey Mongoven
March 18, 2012

classification of work: audio-visual
synthesis engine: Csound show Csound orchestra
synthesis technique: granular synthesis with resonance filter

description of sequences sonified

A003263 number of possible representations of n as a sum using distinct elements of the Lucas sequence beginning 1,3,4,7,11,... 1363 members used
A103344 number of possible representations of n as a sum using distinct elements of the Fibonacci-type sequence beginning 1,4,5,9,14,... 1740 members used

global static parameters:

offset: 1
temperament: phi^7 + 1
pitch orientation: ascending
number of channels: 2
note duration: 0.025 seconds
piece duration: 43.5 seconds
grain: cole 1980 24
release: 0.0089 seconds

individual static parameters:

location:
-30°
30°

global dynamic parameters:

A003263

integer	resonance center frequency Hz	resonance bandwidth Q factor	loudness
1	850.000000	8.333333	pp
2	879.275576	8.474576	pp
3	909.559456	8.620690	pp
4	940.886370	8.771930	p
5	973.292241	8.928571	p
⋮	⋮	⋮	⋮
24	1852.076217	13.513514	ff
25	1915.865155	13.888889	ff
26	1981.851103	14.285714	ff

A103344

integer	**resonance center frequency** Hz	**resonance bandwidth** Q factor	**loudness**
1	850.000000	8.333333	pp
2	879.275576	8.474576	pp
3	909.559456	8.620690	pp
4	940.886370	8.771930	p
5	973.292241	8.928571	p
⋮	⋮	⋮	⋮
24	1852.076217	13.513514	ff
25	1915.865155	13.888889	ff
26	1981.851103	14.285714	ff

1363 values used of Sloane's A003263: 1, 0, 1, 2, 1, 0, 2, 2, 0, 1, 3, 2, 0, 2, 3, 1, 0, 3, 3, 0, 2, ..., 6, 12, 6, 0, 7, 7, 0, 1

1740 values used of Sloane's A103344: 1, 0, 0, 1, 2, 1, 0, 0, 2, 2, 0, 0, 1, 3, 2, 0, 0, 2, 3, 1, 0, ..., 6, 0, 0, 7, 7, 0, 0, 1

graph of sequences: 1) A003263, 1363 values; 2) A103344, 1740 values.

The above graph of this work had to be truncated for practical reasons. The canonic principle inherent in this work was taken to an even higher level in the eight-channel work *Rep Sequences no. 1* (B1117 in [6]), in which eight such sequences are utilized at once, one per speaker.

2.4. Absent and unique residues no. 3

The two sequences sonified in this work (B1411 in [6]) are based on the Fibonacci sequence under a modulus. One sequence counts the number of unique residues absent in F_n modulo m, while the other counts the number of unique residues present. In the diagram below, F_n modulo m has been reduced to the length of its period.

m	F_n modulo m	**unique**	**absent**
1	$\langle 0\rangle$	1	0
2	$\langle 0,1,1\rangle$	2	0
3	$\langle 0,1,1,2,0,2,2,1\rangle$	3	0
4	$\langle 0,1,1,2,3,1\rangle$	4	0
5	$\langle 0,1,1,2,3,0,3,3,1,4,0,4,4,3,2,0,2,2,4,1\rangle$	5	0
6	$\langle 0,1,1,2,3,5,2,1,3,4,1,5,0,5,5,4,3,1,4,5,3,2,5,1\rangle$	6	0
7	$\langle 0,1,1,2,3,5,1,6,0,6,6,5,4,2,6,1\rangle$	7	0
8	$\langle 0,1,1,2,3,5,0,5,5,2,7,1\rangle$	6	2
9	$\langle 0,1,1,2,3,5,8,4,3,7,1,8,0,8,8,7,6,4,1,5,6,2,8,1\rangle$	9	0
10	$\langle 0,1,1,2,3,5,8,3,1,4,5,9,\ldots,9,5,4,9,3,2,5,7,2,9,1\rangle$	10	0
11	$\langle 0,1,1,2,3,5,8,2,10,1\rangle$	7	4
12	$\langle 0,1,1,2,3,5,8,1,9,10,7,5,0,5,5,10,3,1,4,5,9,2,11,1\rangle$	11	1
13	$\langle 0,1,1,2,3,5,8,0,8,8,3,11,\ldots,8,5,0,5,5,10,2,12,1\rangle$	9	4
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$

The sequence of unique residues is placed on the left speaker, and the sequence of absent residues on the right. A single grain created from the sound of pieces of raw lump charcoal colliding together was used as a sound source – a light percussive sound that is rich mostly in higher frequency components. The only dynamic parameter in this work is loudness, ranging from extremely quiet to loud (*pppp* to *f*). In both sequences, each member lasts .074 seconds, and $F_{16} = 987$ members are used of each sequence, resulting in a piece duration of 73.038 seconds. 0s in the absent residues sequence were sonified as silence, resulting in periodic gaps on the right speaker.

Collection XIV

Absent and Unique Residues no. 3
Casey Mongoven
May 18, 2012

classification of work: `audio-visual`
synthesis engine: `Csound` `show Csound orchestra`
synthesis technique: `granular synthesis`

description of sequences sonified

A066853 `number of unique residues in Fibonacci sequence mod n 987 members used`
A118965 `number of residues absent in Fibonacci sequence mod n 987 members used`

global static parameters:

sequence offset: `1`
number of channels: `2`
duration of single member of sequence: `0.074 seconds`
piece duration: `73.038 seconds`
grain: `raw charcoal 8`

individual static parameters:

location:

-30°

30°

global dynamic parameters:

A066853

integer	loudness
1	pppp
2	pppp
3	pppp
4	pppp
5	pppp
⋮	⋮
745	mf
750	mf
875	f

A118965

integer	loudness
1	pppp
2	pppp
4	pppp
5	pppp
7	pppp
⋮	⋮
928	ff
937	ff
966	ff

987 values used of Sloane's A066853: 1, 2, 3, 4, 5, 6, 7, 6, 9, 10, 7, 11, 9, 14, 15, 11, 13, 11, 12, 20, 9, ..., 555, 149, 614, 739, 61, 745, 94, 21

987 values used of Sloane's A118965: 0, 0, 0, 0, 0, 0, 0, 2, 0, 0, 4, 1, 4, 0, 0, 5, 4, 7, 7, 0, 12, ..., 425, 832, 368, 244, 923, 240, 892, 966

graph of sequences: 1) A003263, 987 values; 2) A103344, 987 values.

The graph at the end of the score had to be truncated for practical reasons. As can be seen, these sequences also display natural contrary motion when sonified together; however, the sequences' relatively erratic behavior obscures this to some degree, as does the fact that only a single dynamic parameter is used in the sonification, which is not frequency.

2.5. Stolarsky and Wythoff arrays no. 4

Unlike the preceding works, all of which utilized two integer sequences, four integer sequences based on the Wythoff and Stolarsky arrays are sonified simultaneously in *Stolarsky and Wythoff Arrays no. 4* (B1412 in [6]). The Wythoff array can be created by taking the non-negative integers $\mathbb{Z}_{\geq 0}$ and the Beatty sequence $\lfloor \Phi(\mathbb{Z}_{\geq 0} + 1) \rfloor$ as starting points for rows of G_n [11], as follows:

$\mathbb{Z}_{\geq 0}$	$\lfloor \Phi(\mathbb{Z}_{\geq 0} + 1) \rfloor$											
0	1	1	2	3	5	8	13	21	34	55	89	→
1	3	4	7	11	18	29	47	76	123	199	322	→
2	4	6	10	16	26	42	68	110	178	288	466	→
3	6	9	15	24	39	63	102	165	267	432	699	→
4	8	12	20	32	52	84	136	220	356	576	932	→
5	9	14	23	37	60	97	157	254	411	665	1076	→
6	11	17	28	45	73	118	191	309	500	809	1309	→
7	12	19	31	50	81	131	212	343	555	898	1453	→
8	14	22	36	58	94	152	246	398	644	1042	1686	→
9	16	25	41	66	107	173	280	453	733	1186	1919	→
10	17	27	44	71	115	186	301	487	788	1275	2063	→
11	19	30	49	79	128	207	335	542	877	1419	2296	→
12	21	33	54	87	141	228	369	597	966	1563	2529	→
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↘

The portion in black is the Wythoff array.

In the Stolarsky array, the first integer in each row k is the lowest that has not yet occurred in any row above. The integer that follows k is given by $[\Phi * k]$ [12]:

k	$[\Phi * k]$									
1	2	3	5	8	13	21	34	55	89	→
4	6	10	16	26	42	68	110	178	288	→
7	11	18	29	47	76	123	199	322	521	→
9	15	24	39	63	102	165	267	432	699	→
12	19	31	50	81	131	212	343	555	898	→
14	23	37	60	97	157	254	411	665	1076	→
17	28	45	73	118	191	309	500	809	1309	→
20	32	52	84	136	220	356	576	932	1508	→
22	36	58	94	152	246	398	644	1042	1686	→
25	40	65	105	170	275	445	720	1165	1885	→
27	44	71	115	186	301	487	788	1275	2063	→
30	49	79	128	207	335	542	877	1419	2296	→
33	53	86	139	225	364	589	953	1542	2495	→
35	57	92	149	241	390	631	1021	1652	2673	→
38	61	99	160	259	419	678	1097	1775	2872	→
41	66	107	173	280	453	733	1186	1919	3105	→
43	70	113	183	296	479	775	1254	2029	3283	→
46	74	120	194	314	508	822	1330	2152	3482	→
48	78	126	204	330	534	864	1398	2262	3660	→
51	83	134	217	351	568	919	1487	2406	3893	→
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↘

The sequence that gives the column in which an integer n occurs in the Wythoff array is called the horizontal para-Fibonacci sequence; the sequence that gives the row is called the vertical para-Fibonacci sequence (A035614 and A019586 in [5]). One can create two more similar sequences by applying the same principle to the Stolarsky array (A098861 and A098862 in [5]). All four of these sequences are sonified simultaneously in this composition.

In this work, wavetable synthesis is utilized. The purity of tone using this technique can increase clarity of representation when sonifying a significant number of sequences simultaneously. The wavetable used for all sequences was derived from the sound of an opening door. The dynamic parameters used here are frequency, attack and release, and simulated location. Of these parameters, all are global except for simulated location, which is individual for each sequence. $\varphi^7 + 1$ is the unit interval in the temperament used, and the frequencies in the work span from a soaring 2355 Hz to about 119.6 Hz (the frequency orientation is descending). A somewhat slower tempo of .12 seconds and 233 members of each sequence are utilized, resulting in a piece duration of 27.96 seconds. A static level of loudness of mezzo-forte (medium-loud) was chosen in order not to obscure either the lower or higher members of the sequence.

Collection XIV

Stolarsky and Wythoff Arrays no. 4
Casey Mongoven
May 20, 2012

classification of work: `audio-visual`
synthesis engine: `Csound` `show Csound orchestra`
synthesis technique: `wavetable with FFT resynthesis`

description of sequences sonified

A035614 gives number of column in Wythoff array that contains n 233 members used

A019586 gives number of row in Wythoff array that contains n 233 members used

A098862 gives number of column in Stolarsky array that contains n 233 members used

A098861 gives number of row in Stolarsky array that contains n 233 members used

global static parameters:

sequence offset: 1
temperament: $phi^7 + 1$
frequency orientation: descending
number of channels: 2
duration of single member of sequence: 0.12 seconds
piece duration: 27.96 seconds
wavetable: door sound 1
loudness: mf

global and individual dynamic parameters:

A035614

integer	frequency Hz	attack s	release s	simulated location degrees
0	2355.000000	0.012300	0.014400	-27.665
1	2276.590020	0.012247	0.014338	-27.565
2	2200.790708	0.012193	0.014275	-27.465
3	2127.515142	0.012140	0.014213	-27.365
4	2056.679295	0.012086	0.014150	-27.265
⋮	⋮	⋮	⋮	⋮
9	1736.345543	0.011819	0.013838	-26.765
10	1678.533730	0.011766	0.013775	-26.665
11	1622.646767	0.011713	0.013713	-26.565

A019586

integer	frequency Hz	attack s	release s	simulated location degrees
0	2355.000000	0.012300	0.014400	-12.155
1	2276.590020	0.012247	0.014338	-12.055
2	2200.790708	0.012193	0.014275	-11.955
3	2127.515142	0.012140	0.014213	-11.855
4	2056.679295	0.012086	0.014150	-11.755
⋮	⋮	⋮	⋮	⋮
86	128.017015	0.007707	0.009025	-3.554
87	123.754675	0.007653	0.008962	-3.454
88	119.634250	0.007600	0.008900	-3.354

A098862

integer	frequency Hz	attack s	release s	simulated location degrees
0	2355.000000	0.012300	0.014400	3.354
1	2276.590020	0.012247	0.014338	3.454
2	2200.790708	0.012193	0.014275	3.554
3	2127.515142	0.012140	0.014213	3.654
4	2056.679295	0.012086	0.014150	3.754
⋮	⋮	⋮	⋮	⋮
9	1736.345543	0.011819	0.013838	4.254
10	1678.533730	0.011766	0.013775	4.354
11	1622.646767	0.011713	0.013713	4.454

A098861

integer	frequency Hz	attack s	release s	simulated location degrees
0	2355.000000	0.012300	0.014400	18.864
1	2276.590020	0.012247	0.014338	18.964
2	2200.790708	0.012193	0.014275	19.064
3	2127.515142	0.012140	0.014213	19.164
4	2056.679295	0.012086	0.014150	19.264
⋮	⋮	⋮	⋮	⋮
86	128.017015	0.007707	0.009025	27.465
87	123.754675	0.007653	0.008962	27.565
88	119.634250	0.007600	0.008900	27.665

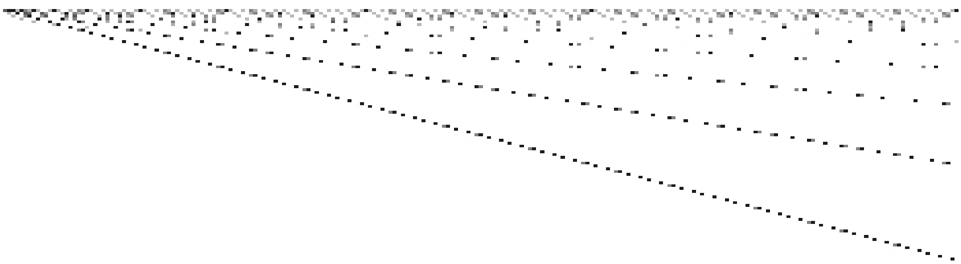
233 values used of Sloane's A035614: 0, 1, 2, 0, 3, 0, 1, 4, 0, 1, 2, 0, 5, 0, 1, 2, 0, 3, 0, 1, 6, 0, ..., 0, 1, 4, 0, 1, 2, 0, 11

233 values used of Sloane's A019586: 0, 0, 0, 1, 0, 2, 1, 0, 3, 2, 1, 4, 0, 5, 3, 2, 6, 1, 7, 4, 0, 8, ..., 86, 53, 12, 87, 54, 33, 88, 0

233 values used of Sloane's A098862: 0, 1, 2, 0, 3, 1, 0, 4, 0, 2, 1, 0, 5, 0, 1, 3, 0, 2, 1, 0, 6, 0, ..., 0, 1, 3, 0, 2, 1, 0, 11

233 values used of Sloane's A098861: 0, 0, 0, 1, 0, 1, 2, 0, 3, 1, 2, 4, 0, 5, 3, 1, 6, 2, 4, 7, 0, 8, ..., 86, 53, 20, 87, 33, 54, 88, 0

graph of sequences: 1) A035614, 233 values; 2) A019586, 233 values; 3) A098862, 233 values; 4) A098861, 233 values.



3. Ongoing research

New challenges are presented when composing sonifications of multiple integer sequences simultaneously. Which sequences can be sonified together and how? Which parameters should we allow to be individual and which global? When should such a sonification terminate? Currently, the author is working on theoretical criteria for the sonification of integer sequences that attempt to posit potential answers to such questions. In addition, an experiment involving more than 150 participants is being carried out that compares various sonified Fibonacci-related mathematical objects to analog mathematical objects more or less unrelated to F_n, in an attempt to gain some insight into the aesthetic value of F_n and the golden ratio in musical composition. This work is part of the author's doctoral dissertation in the Media Arts and Technology Department at UCSB.

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