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Representation of integers as terms of a linear recurrence with maximal index

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Abstract. For sequences $H_n(a,b)$ of positive integers, defined by $H_0=a$, $H_1=b$ and $H_n=H_{n-1}+H_{n-2}$, we investigate the problem: for a given positive integer N find positive integers a and b such that $N=H_n(a,b)$ and n is as large as possible. Denoting by $R(N)=r$ the largest integer, for which $N=H_r(a,b)$ for some a and b, we give bounds for $R(N)$ and a polynomial time algorithm for computing it. Some properties of $R(N)$ are also proved.

Introduction

Let $H_n(a,b)$ be a sequence of positive integers defined by $H_0=a$, $H_1=b$ and $H_n=H_{n-1}+H_{n-2}$ where a and b are arbitrary positive integers (the parameters). The sequence $H_n(a,b)$ occurs in a problem of Cohn [1]: *Given a positive integer N, find positive integers a and b such that $N=H_n(a,b)$ and n is as large as possible.*

Cohn [1] actually formulated the problem slightly differently, replacing 'n is as large as possible' by '$a+b$ is as small as possible'. However this makes little difference. We shall consider the problem as stated above.

Let $R=R(N)$ be the largest integer R such that $N=H_R(a,b)$ for some $a,b\geq 1$. The function R is well defined. For any $N\geq 1$, there exist integers a,b and n such that $N=H_n(a,b)$, $1\leq a$, $1\leq b$. Since $N=H_1(1,N)$, we can let $n=1$, $a=1$ and $b=N$. If $2\leq N$, we can also let $a=1$, $b=N-1$ and $n=2$ so $N=H_2(1,N-1)$. Thus there always exist integers n, a and b such that $N=H_n(a,b)$, $1\leq a$ and $1\leq b$.

It is also easy to see that there exist a,b and r such that $N=H_r(a,b)$, $1\leq a$, $1\leq b$ and r is maximal. If $N=H_r(a,b)$, $1\leq a$ and $1\leq b$, then $r\leq H_r(a,b)$. Hence for all such r,a and b, $r\leq N$. Thus all possible values of r are bounded above by N. In fact this argument shows that $R(N)\leq N$ for all N.

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The first few values of R are given by $R(1) = 1$, $R(2) = 2$, $R(3) = 3$, $R(4) = 3$, $R(5) = 4$, $R(6) = 3$, $R(7) = 4$, $R(8) = 5$, $R(9) = 4$, $R(10) = 4$, $R(11) = 5$, $R(12) = 4$ and $R(13) = 6$. Note that the function R is not increasing. That is, $N \leq M$ does not imply $R(N) \leq R(M)$.

Since $R(N)$ is well defined, Cohn's problem becomes one of giving an algorithm to compute $R(N)$. In this paper we shall give a simple algorithm which solves this problem. We shall also show that this algorithm is polynomial time, that is the time to find $R(N)$ is less than a polynomial in $\ln(N)$. We also prove some theorems about the number of N such that $R(N) = r$ and about the number of pairs (a, b) such that $H_r(a, b) = N$. First we need some lemmas.

1. Representation of N in the form $N = H_r(a, b)$ with r maximal

We use $\lfloor x \rfloor$ to denote the floor of x, (integer part of x). $\lceil x \rceil$ denotes the ceiling of x, $\lceil x \rceil = -\lfloor -x \rfloor$. F_n denotes the n^{th} Fibonacci number, where $F_0 = 0$, $F_1 = 1$ and $F_{i+2} = F_i + F_{i+1}$. L_n denotes the n^{th} Lucas number, defined by $L_0 = 2$, $L_1 = 1$ and $L_{i+2} = L_i + L_{i+1}$. We define $H_{-n}(a, b)$ by $H_{-n}(a, b) = (-1)^{n+1} H_n(-a, b - a)$.

Below we shall use many elementary identities and inequalities such as $L_{n+1} = 2F_n + F_{n+1}$, $L_n + 1 \leq F_{n+2}$ for $1 \leq n$ and $F_{n+1} < L_n$, for $2 \leq n$. We shall also need the following well known identity due to Horadam [3].

Lemma 1.1. *For all integers n, a and b, $H_n(a, b) = aF_{n-1} + bF_n$.*

Proof. By induction on n using $F_{i+2} = F_i + F_{i+1}$. The result can also be seen to hold for negative n since $H_{-n}(a, b) = (-1)^n (aF_{n+1} - bF_n)$.

Lemma 1.2. $H_n(a, b) = H_n(a + F_n, b - F_{n-1})$ *and* $H_n(a, b) = H_n(a - F_n, b + F_{n-1})$.

Lemma 1.3. *For all integers n, k, a and b, we have*

(*i*) $$H_n(a, b) = H_{n-1}(b, a + b).$$

(*ii*) $$H_n(a, b) = H_{n-k}(H_k(a, b), H_{k+1}(a, b)),$$

(*iii*) $$H_n(a, b) = H_{n+1}(b - a, a).$$

(*iv*) $$H_n(a, b) = H_{n+k}(H_{-k}(a, b), H_{1-k}(a, b)).$$

Proof. They follow from the definitions.

Lemma 1.4. *If $N = H_r(a, b)$, $1 \leq a$, $1 \leq b$ and $R(N) = r$, then $b \leq a$.*

Proof. Suppose $R(n) = r$ and $N = H_r(a, b)$. If $a < b$, then by Lemma 1.3 we would have $N = H_r(a, b) = H_{r+1}(b - a, a)$ so that $r + 1 \leq R(N)$, contradicting $r = R(N)$.

Earlier we saw that $n = 1$ is realizable as a value of n such that $N = H_n(a, b)$ for $a \geq 1$, $b \geq 1$. In the next lemma we shall show that all values of $n \leq R(N)$ are realizable as values of n such that $N = H_n(a, b)$. We shall call this the Intermediate Value Lemma (IVL).

Lemma 1.5. (I.V.L.) *If $n \leq R(N)$, then there exist a, b such that $N = H_n(a, b)$, $1 \leq a$ and $1 \leq b$.*

Proof. Suppose $r = R(N)$ and $n \leq r$. There exist $a \geq 1$, $b \geq 1$ such that $N = H_r(a, b)$. Let $k = r - n$. Then $0 \leq k$. By Lemma 1.3 (ii), $N = H_r(a, b) = H_{r-k}(H_k(a, b), H_{k+1}(a, b)) = H_n(H_k(a, b), H_{k+1}(a, b))$ where $1 \leq H_k(a, b)$ and $1 \leq H_{k+1}(a, b)$, since $0 \leq k$ and $a, b \geq 1$.

Lemma 1.6. *If $n \geq 1$ then $R(F_{n+1}) = n$.*

Proof. Let $r = R(F_{n+1})$. Since $F_{n+1} = F_{n-1} + F_n = H_n(1, 1), n \leq r$. Conversely, $F_{n+1} = H_r(a, b) = aF_{r-1} + bF_r \geq F_{r-1} + F_r = F_{r+1}$. Hence $n \geq r$. Therefore $n = r$.

Lemma 1.7. *If $n \geq 2$, then $R(L_{n+1}) = n + 1$.*

Proof. Here we need the inequality $L_{n+1} + 1 \leq F_{n+3}$. Let $r = R(L_{n+1})$. Since L_n may be defined by $L_0 = 2, L_1 = 1$ and $L_{n+2} = L_n + L_{n+1}$, we have $L_n = H_n(2, 1)$ and so $L_{n+1} = H_{n+1}(2, 1)$. Hence $n + 1 \leq r$. Conversely, $L_{n+1} = H_r(a, b) = aF_{r-1} + bF_r \geq F_{r-1} + F_r = F_{r+1}$. Hence $F_{r+1} \leq L_{n+1}$. Therefore $F_{r+1} + 1 \leq L_{n+1} + 1 \leq F_{n+3}$ and so $F_{r+1} < F_{n+3}$. Therefore $r + 1 < n + 3$. Hence $r < n + 2$. Therefore $r \leq n + 1$. So $r = n + 1$ and $R(L_{n+1}) = n + 1$.

Lemma 1.8. *If $N < F_{n+1}$, then $R(N) < n$.*

Proof. Let $R(N) = r$. Then there exist $a, b \geq 1$ such that $N = H_r(a, b)$. Hence we have $F_{n+1} > N = H_r(a, b) = aF_{r-1} + bF_r \geq F_{r-1} + F_r = F_{r+1}$. Thus $F_{n+1} > F_{r+1}$. Hence we have $n+1 > r+1$ so that $n > r$. In otherwords $n > R(N)$.

Corollary 1.9. *If $1 \leq n$ and $N \leq F_{n+1}F_{n+2}$, then $R(N) \leq 2n$.*

Proof. If $1 \leq n$, then $F_{n+2} < L_{n+1}$. Hence $F_{n+1}F_{n+2} < F_{n+1}L_{n+1} = F_{2n+2}$. Therefore $N < F_{2n+2}$. Hence by Lemma 1.8, $R(N) < 2n + 1$. Therefore $R(N) \leq 2n$.

Lemma 1.10. *Let A be an arbitrary positive integer and suppose $0 \leq n$. Then*

(i) $$n = R(AF_{n+1}) \quad \text{if} \quad A \leq F_n,$$

(ii) $$n < R(AF_{n+1}) \quad \text{if} \quad F_n < A.$$

Proof. $AF_{n+1} = A(F_{n-1} + F_n) = AF_{n-1} + AF_n = H_n(A, A)$ implies $n \le R(AF_{n+1})$. For (i) suppose $A \le F_n$ and $n + 1 \le R(AF_{n+1})$. By the Intermediate Value Lemma there exist $c \ge 1$ and $d \ge 1$ such that $AF_{n+1} = cF_n + dF_{n+1}$. Then $F_{n+1} \mid cF_n$ and $(F_n, F_{n+1}) = 1$ imply $F_{n+1} \mid c$. Hence $F_{n+1} \le c$ so that $d = 0$. Hence $R(AF_{n+1}) = n$. For (ii) suppose $F_n < A$. Then there exist b and t such that $A = tF_n + b$, $1 \le b$ and $1 \le t$. Let $a = tF_{n+1}$. Then $AF_{n+1} = (tF_{n+1})F_n + bF_{n+1} = H_{n+1}(tF_{n+1}b) = H_{n+1}(a, b)$, $1 \le a$ and $1 \le b$. Hence $n + 1 \le R(AF_{n+1})$ so that $n < R(AF_{n+1})$.

Corollary 1.10. *For all $n \ge 0$, $R(F_nF_{n+1}) = n$.*

Lemma 1.11. *If $F_nF_{n+1} < N$, then $n < R(N)$.*

Proof. Suppose $F_nF_{n+1} < N$. We shall show that $n + 1 \le R(N)$ by finding a and b such that $N = H_{n+1}(a, b) = aF_n + bF_{n+1}$, $1 \le a$ and $1 \le b$. Let b be the least positive solution to the congruence $N \equiv bF_{n+1} \pmod{F_n}$, (taking $b = F_n$, if $F_n \mid N$, so that $b \ge 1$). We claim

$$bF_{n+1} + F_n \le N. \tag{1.12}$$

This inequality (1.12) will be proved by considering two cases:

Case 1. $N \equiv 0 \pmod{F_n}$. Then $b = F_n$. Since $F_n \mid N$ and $F_nF_{n+1} < N$, we have $F_n(F_{n+1} + 1) \le N$. So we have $F_{n+1}b + F_n = F_{n+1}F_n + F_n = F_n(F_{n+1} + 1) \le N$, and so (1.12) holds.

Case 2. $N \not\equiv 0 \pmod{F_n}$. Then $1 \le b < F_n$, so $b \le F_n - 1$. Therefore $bF_{n+1} + F_n \le (F_n - 1)F_{n+1} + F_n = F_nF_{n+1} + (F_n - F_{n+1}) \le F_nF_{n+1} < N$ and so again (1.12) holds.

Now that (1.12) is established, let $a = (N - bF_{n+1})/F_n$. Then a is an integer, $N = aF_n + bF_{n+1} = H_{n+1}(a, b)$ and (1.12), implies $1 \le a$.

Corollary 1.13. *If $1 < n$ and $F_{2n} \le N$, then $n < R(N)$.*

Proof. By Lemma 1.11. If $1 < n$, then $F_{n+1} < L_n$ and $F_nF_{n+1} < F_nL_n = F_{2n} \le N$.

Lemma 1.14. *If $1 \le N$, then $R(N) \le (\lfloor 1 + 2.128 \cdot \ln(N))\rfloor$.*

Proof. Let $r = R(N)$. The inequality holds for $N = 1$, since $R(1) = 1$. Suppose $N \ge 2$. Then $2 \le r$. Let $k = r + 1$. Then $3 \le k$ so that we can use the inequality

$$(8/5)^{k-2} < F_k \qquad (3 \le k). \tag{1.15}$$

(This inequality, which is well known, is easy to prove by induction on $k \ge 3$, using the fact that if $x = 8/5$, then $x^2 < x + 1$). Using the inequality with $k = r+1$, by Lemma 1.8 we get $(8/5)^{r-1} < F_{r+1} \le N$. Taking logs of both sides we have $(r-1)\ln(8/5) \le \ln(N)$. Hence we have $r - 1 \le \ln(N)/\ln(8/5) < \ln(N)/(47/100) < \ln(N) \cdot 2.128$, proving the lemma.

Lemma 1.16. *If* $1 \le N$, *then* $\lceil 1.5 + .893 \cdot \ln(N) \rceil \le R(N)$.

Proof. Let $r = R(N)$. Lemma 1.11 implies $N \le F_r F_{r+1}$. If $N \le 6$, the inequality can be checked by cases. Suppose $7 \le N$. Then $4 \le r$. We will use the following elementary inequality which is easy to prove using the fact that $x^2 > x + 1$ for $x = 7/4$.

$$F_k < (7/4)^{k-2} \qquad (3 < k). \tag{1.17}$$

Using the inequality twice, with $k = r$ and $k = r+1$, we get

$$N \le F_r F_{r+1} < (7/4)^{r-2}(7/4)^{r-1} = (7/4)^{2r-3}. \tag{1.18}$$

Taking logs of both sides, $\ln(N) < (2r-3)\ln(7/4)$. Hence $\ln(N)/\ln(7/4) < 2(r-1.5)$. Therefore $2^{-1} \cdot \ln(N)/\ln(7/4) < r - 1.5$. Consequently $1.5 + 2^{-1} \cdot \ln(N)/\ln(7/4) < r$. Hence $\lceil 1.5 + 2^{-1} \cdot \ln(N)/\ln(7/4) \rceil \le r$. Therefore $\lceil 1.5 + .893 \cdot \ln(N) \rceil \le r$.

Corollary 1.19. *For* $N \ge 1$,

$$\lceil 1.5 + .893 \cdot \ln(N) \rceil \le R(N) \le \lfloor 1 + 2.128 \cdot \ln(N) \rfloor.$$

Proof. By Lemma 1.14 and Lemma 1.15.

Corollary 1.20. *If* $R(N) = r$, *then* $F_{r+1} \le N \le F_r F_{r+1}$.

Proof. Suppose $R(N) = r$. By Lemma 1.8, $F_{r+1} \le N$. By Lemma 1.11, $N \le F_r F_{r+1}$.

The equation $N = H_r(a,b)$ sometimes has two solutions (a,b) in positive integers with $r = R(N)$. E.g. if $N = 6$, then $R(6) = 3$, $6 = H_3(2,2)$ and $6 = H_3(4,1)$.

Definition 1.21. N is called a *double number* if there exist $a, b, c, d \ge 1$ such that $N = H_r(a,b) = H_r(c,d), a \ne c$ or $b \ne d$, (equivalently if $a \ne c$ and $b \ne d$), where $r = R(N)$. If N is not a double number, then N is called a *single number*.

Examples 1.22. Some representations of N in the form $N = H_r(a,b)$ with $R = R(N)$:

$N = 1$,	$R = 1$,	$a = 1$,	$b = 1$,
$N = 10$,	$R = 4$,	$a = 2$,	$b = 2$,

$N = 100,$	$R = 7,$	$a = 6,$	$b = 4,$
$N = 1,000,$	$R = 12,$	$a = 8,$	$b = 2,$
$N = 10,000,$	$R = 12,$	$a = 80,$	$b = 20,$
$N = 100,000,$	$R = 14,$	$a = 269,$	$b = 99,$
$N = 1,000,000,$	$R = 19,$	$a = 154,$	$b = 144,$
$N = 10,000,000,$	$R = 19,$	$a = 1540,$	$b = 1440,$
$N = 100,000,000,$	$R = 23,$	$a = 5143,$	$b = 311,$
$N = 1,000,000,000,$	$R = 23,$	$a = 51430,$	$b = 3110.$

$N = 1,000,000,000$ happens to be an example of a double number. For we have $N = H_r(c, d)$ also for $c = 22773$ and $d = 20821$, besides $a = 51430$ and $b = 3110$. Other examples of double numbers are $15 = H_4(6, 1) = H_4(3, 3)$ and $32 = H_5(9, 1) = H_5(4, 4)$.

In the next section we shall prove that the equation $N = H_r(a, b)$ never has three solutions (a, b) in positive integers with $r = R(N)$. (Of course it may have other solutions when $r < R(N)$. E.g. for $N = 6$ and $r = 3$ we have $6 = H_2(1, 5) = H_2(2, 4) = H_2(3, 3)$ where $2 < r$.) Thus there is no concept of a triple number.

2. An algorithm for $R(N)$

In this section we shall show that there exists an algorithm for computing $R(N)$. In fact we shall prove that there is a polynomial-time algorithm for computing $R(N)$. We give a procedure which finds, given N, the value of $R(N)$ and also a and b. Since the number of steps in the procedure will be less then a polynomial in $\ln(N)$, the number of bit operations needed to compute $R(N)$ will be less than a polynomial in $\ln(N)$.

Suppose N is given. To compute $R(N)$, begin with any sufficiently large value of r, satisfying $r \geq R(N)$. For example by Corollary 1.19 we can take $r = \lfloor 1 + 2.128 \cdot \ln(N) \rfloor$. Then proceed as follows.

Step 1: Find a positive solution b to the congruence

$$N \equiv bF_r \pmod{F_{r-1}}, \qquad (1 \leq b). \tag{2.1}$$

This congruence is solvable in natural numbers since $(F_r, F_{r-1}) = 1$. Hence there is a solution b in the range $1 \leq b \leq F_{r-1}$. Take the least such b in this range.

Step 2: Check whether

$$bF_r < N. \tag{2.2}$$

If this is the case, put $a = (N - bF_r)/F_{r-1}$. Then a is an integer by (2.1). Also we have $N = aF_{r-1} + bF_r$ and condition (2.2) implies $1 \leq a$. In this case the algorithm terminates and $R(N) = r$. If (2.2) does not hold, then we decrease r by 1 and return to Step 1. We iterate steps 1 and 2, decreasing r until (2.2) holds. Since initially $R(N) \leq \lfloor 1 + 2.128 \cdot \ln(N) \rfloor \leq r$, the algorithm must terminate after at most $\lfloor 1 + 2.128 \cdot \ln(N) \rfloor$ iterations.

We claim that this computation is polynomial time. Certainly the number of operations needed at each step is less than or equal to a polynomial in $\ln(N)$. Calculation of F_r requires time exponential in $\ln(r)$, i.e. proportional to a polynomial in r. However r is less than or equal to a polynomial in $\ln(N)$, since $F_r \leq N$. So this is polynomial time.

In addition to finding r, the algorithm also finds (a, b) such that $H_r(a, b) = N$. The pair (a, b) is not uniquely dependent upon N. There is sometimes a second pair (c, d) such that $H_r(c, d) = N$. As sketched above the algorithm finds the pair (a, b) with least b. It can easily be extended also to find the second pair (c, d), when that exists. After (a, b) has been found, let $d = b + F_{r-1}$ and $c = a - F_r$. Then $N = H_r(c, d)$ by Lemma 1.2. d is positive. If $dF_r < N$, then c will also be positive and (c, d) will be a second pair. If not, then there is no second pair, i.e. N is a single.

The algorithm can be simplified to yield a more explicit formula for $r = R(N)$ and explicit formulas for a, b, c and d. For this we shall use an old identity of Lucas [4]:

$$F_{r-1}^2 - F_{r-2} \cdot F_r = (-1)^r. \tag{2.3}$$

Multiplying both sides of (2.3) by $(-1)^r N$ and rearranging terms we get

$$(-1)^r F_{r-1} N \cdot F_{r-1} - (-1)^r F_{r-2} N \cdot F_r = N. \tag{2.4}$$

Equation (2.4) provides a solution to the linear diophantine equation $aF_{r-1} + bF_r = N$. It shows that $AF_{r-1} + BF_r = N$ will hold if we put $A = A_r(N)$ and $B = B_r(N)$, where

$$A_r(N) = (-1)^r F_{r-1} N \quad \text{and} \quad B_r(N) = -(-1)^r F_{r-2} N. \tag{2.5}$$

Thus $a = A_r(N)$ and $b = B_r(N)$ is a particular solution of the equation $aF_{r-1} + bF_r = N$. Since $(F_r, F_{r+1}) = 1$, from a particular solution we may obtain all solutions (a, b) by

$$a = A_r(N) - tF_r, \quad b = B_r(N) + tF_{r-1}, \quad (t = 0, \pm 1, \pm 2, \pm 3, \ldots). \tag{2.6}$$

Then by Lemma 1.1 $H_r(a, b) = N$ for all integers t. Now define $g_r(N)$ and $h_r(N)$ by

$$g_r(N) = \frac{(-1)^r F_{r-2} N + 1}{F_{r-1}} \tag{2.7}$$

and

$$h_r(N) = \frac{(-1)^r F_{r-1} N - 1}{F_r}. \tag{2.8}$$

Then $g_r(N)$ and $h_r(N)$ are reals. For a and b as in (2.6), we have $1 \le a$ iff $t \le h_r(N)$ and $1 \le b$ iff $g_r(N) \le t$. Hence (a, b) is a positive solution of $aF_{r-1} + bF_r = N$ iff

$$g_r(N) \le t \le h_r(N). \tag{2.9}$$

Since t is integer valued, condition (2.9) is equivalent to

$$g_r(N) \le \lceil g_r(N) \rceil \le t \le \lfloor h_r(N) \rfloor \le h_r(N). \tag{2.10}$$

Condition (2.9) is in turn equivalent to $\lceil g_r(N) \rceil \le h_r(N)$ and also to $g_r(N) \le \lfloor h_r(N) \rfloor$.

From (2.3), (2.7) and (2.8), it is easy to see that

$$h_r(N) - g_r(N) = \frac{N - F_{r+1}}{F_{r-1} F_r}. \tag{2.11}$$

The functions $g_r(N)$ and $h_r(N)$ give us a new algorithm to compute $R(N)$. We have

Theorem 2.12. *Suppose $N > 1$. Then $R(N)$ is the largest integer $r > 1$ such that*

$$\left\lceil \frac{(-1)^r F_{r-2} N + 1}{F_{r-1}} \right\rceil \le \left\lfloor \frac{(-1)^r F_{r-1} N - 1}{F_r} \right\rfloor. \tag{2.12}$$

Furthermore, the set of $r > 1$ satisfying (2.12) is the set $\{2, 3, \dots, R(N)\}$. Hence (2.12) can be used as an algorithm to calculate $R(N)$.

Proof. By Lemma 1.8, if $r \le R(N)$, then $F_{r+1} < N$ and hence by (2.11), $g_r(N) \le h_r(N)$. Thus

$$r \le R(N) \Rightarrow g_r(N) \le h_r(N). \tag{2.13}$$

By (2.9) and the IVL, for all $r \le R(N)$, there exist $t\,(g_r(N) \le t \le h_r(N))$, and this implies $\lceil g_r(N) \rceil \le \lfloor h_r(N) \rfloor$. On the other hand, by (2.9), when $R(N) < r$, there is no integer t such that $g_r(N) \le t \le h_r(N)$ and so we have not $\lceil g_r(N) \rceil \le \lfloor h_r(N) \rfloor$.

This shows that the set of $r > 1$ satisfying (2.12) is an interval.

This approach to $R(N)$, thru $g_r(N)$ and $h_r(N)$, also gives a new algorithm to decide whether N is a single or a double. From (2.9) and (2.10) we have

$$N \quad \text{is a single iff} \quad \lceil g_r(N) \rceil = \lfloor h_r(N) \rfloor. \tag{2.14}$$

Also

$$N \quad \text{is a double iff} \quad \lceil g_r(N) \rceil < \lfloor h_r(N) \rfloor. \tag{2.15}$$

From (2.5), (2.6), (2.7) and (2.8) we can obtain explicit formulas for a, b, c and d:

$$a = A_r(N) - \lceil g_r(N) \rceil F_r \quad \text{and} \quad b = B_r(N) + \lceil g_r(N) \rceil F_{r-1}, \tag{2.16}$$

$$c = A_r(N) - \lfloor h_r(N) \rfloor F_r \quad \text{and} \quad d = B_r(N) + \lfloor h_r(N) \rfloor F_{r-1}. \tag{2.17}$$

If N is a single, $c = a$ and $d = b$. If N is a double, $c = a - F_r$ and $d = b + F_{r-1}$. Thus when $r = R(N)$, formulas (2.16) and (2.17) can be used as definitions of a, b, c and d. The ratio on the right side of (2.11) is not always less than 2 however, even when $r = R(N)$. In this case, when $r = R(N)$, we have only the weak inequality

$$R(N) \leq r \Longrightarrow \frac{N - F_{r+1}}{F_{r-1}F_r} < \alpha + 1. \tag{2.18}$$

Here $\alpha = (1+\sqrt{5})/2 = 1.61803\ldots$ so that $\alpha + 1 = 2.61803\ldots$. The idea of the proof is the following: From Lemma 1.11 we see that $R(N) \leq r$ implies $N \leq F_r F_{r+1}$. Then $(F_r F_{r+1} - F_{r-1})F_{r-1}F_r < \alpha + 1$ can be shown using $F_r^2 < \alpha F_1 F_{r-1} + F_{r+1}$.

Next we shall prove that there are no triples. The following lemmas will be used.

Lemma 2.19. *If* $1 < r$, *then* $F_r F_{r+1} < (1 + 2F_{r+1})F_{r-1} + F_r$.

Proof. If $1 < r$, then $F_r < 1 + 2F_{r-1}$. Hence

$$\begin{aligned} F_r F_{r+1} &< (1 + 2F_{r-1})F_{r+1} = F_{r+1} + 2F_{r-1} \cdot F_{r+1} \\ &= F_{r-1} + 2F_{r-1} \cdot F_{r+1} + F_r = (1 + 2F_{r+1})F_{r-1} + F_r. \end{aligned}$$

Lemma 2.20. *Suppose* $1 < N$, $N = H_r(a,b)$, $R(N) = r$ *and* $1 \leq b$. *Then* $a \leq 2F_r$.

Proof. Let $r = R(N)$ and $N = H_r(a,b)$. Since $1 < N$ and $r = R(N)$, we have $1 < r$. We claim

$$a < b + 2F_{r+1}. \tag{2.21}$$

If not, then $b+2F_{r+1} \le a$. Since $1 \le b$ and $N = H_r(a,b)$, by Lemma 2.19 and Lemma 1.11 we have

$$N = aF_{r-1}+bF_r \ge (b+2F_{r+1})F_{r-1}+bF_r \ge (1+2F_{r+1})F_{r-1}+F_r > F_rF_{r+1}.$$

But this contradicts Lemma 1.11 which says that $N \le F_rF_{r+1}$, since $r = R(N)$. Hence (2.21) holds. Now it is easy to see that

$$N = aF_{r-1} + bF_r = (b + 2F_{r+1} - a)F_r + (a - 2F_r)F_{r+1}.$$

Supposing $2F_r < a$ and using (2.21), we get the contradiction $R(N) \ge r+1$. So $a \le 2F_r$.

Theorem 2.22. *If $R(N) = r$, then the equation $N = H_r(a,b)$ has at most two solutions in positive integers a, b. There are no triples.*

Proof. Suppose the equation $N = H_r(a,b)$ has three solutions in positive integers, say $(a,b), (c,d)$ and a third solution (x,y). Then $c = a - F_r$, $d = b+F_{r-1}$, $x = a-2F_r$ and $y = b+2F_{r-1}$. But by Lemma 2.20, $a \le 2F_r$. Hence $x \le 0$, a contradiction.

From Theorem 2.22, if $r = R(N)$, then $\lfloor h_r(N)\rfloor \le \lceil g_r(N)\rceil + 1$. And so in (2.15), when $\lceil g_r(N)\rceil < \lfloor h_r(N)\rfloor$, we have $\lceil g_r(N)\rceil + 1 = \lfloor h_r(N)\rfloor$.

Following F_nF_{n+1} there is a very long interval consisting entirely of singlels.

Suppose $R(N) = r$. Recall from Corollary 1.20 that if $R(N) = r$, then N must lie in the interval $F_{r+1} \le N \le F_rF_{r+1}$. We can show that most N in this interval are singles.

Theorem 2.23. *If $F_nF_{n+1} < N < F_nF_{n+1} + F_n^2 + F_{n+2}$, then N is a single.*

We won't prove this result, (Theorem 2.23.). However it will be clear how to do so after we have proved Lemma 3.1 in the next section.

Taking a limit as $n \to \infty$, one finds that the interval $[F_nF_{n+1}, F_nF_{n+1} + F_n^2 + F_{n+2}]$ occupies some 38% of the interval $[F_nF_{n+1}, F_{n+1}F_{n+2}]$. ($\beta^2 = \left((1-\sqrt{5})/2\right)^2 = (-.61803)^2 = .381966\ldots$) Thus on average more than 28% of N are singles. Actually, in the next section, we shall prove that 92.7% of N are singles.

3. The number of N such that $R(N) = r$

In this section we consider the problem of the number of N such that $R(N) = r$. Here r is a fixed positive integer. The number of such N must be finite. By Lemma 1.11, the number of such N must be less than or equal to F_rF_{r+1}. We shall give an exact formula for this number. First we need some lemmas.

Lemma 3.1. *Suppose $R(N) = r$ and a, b, c, d are as defined in (2.16) and (2.17). Then $N = H_r(a, b) = H_r(c, d)$. If N is a single, then $c = a, d = b$,*

$$(i) \qquad 1 \le b \le a \le F_r \quad \text{and} \quad 1 \le b \le F_{r-1}.$$

If N is a double, then we have $c = a - F_r,\ d = b + F_{r-1},\ b \le a$,

$$(ii) \qquad F_{r-1} < d \le c \le F_r, \qquad F_{r+1} < a \le 2F_r \quad \text{and} \quad 1 \le b \le F_{r-2}.$$

Proof. Suppose a, b, c and d are as above and $N > 1$. Let $r = R(N)$. Then $N = H_r(a, b) = H_r(c, d)$. Suppose first N is a single. By (2.16) and (2.17), $c = a, d = b$, $1 \le a$ and $1 \le b$. By Lemma 1.2, $N = H_r(a, b) = H_r(a+F_r, b-F_{r-1})$. Hence $b \le F_{r-1}$, else N would be a double. By Lemma 1.4, $b \le a$. By Lemma 1.2 we know $N = H_r(a, b) = H_r(a - F_r, b + F_{r-1})$. Hence $a \le F_r$, else N would be a double. Therefore (i) holds.

Next suppose N is a double. Then by (2.15), (2.16) and (2.17), $c = a-F_r$, $d = b+F_{r-1}$, $1 \le a, b, c, d$. By Lemma 2.20, $a \le 2F_r$. Since $c = a-F_r$, this implies $c \le F_r$. By Lemma 1.4, since $N = H_r(c, d)$, $d \le c$. Hence $d \le F_r$. Since $d \le F_r$ and $d = b+F_{r-1}$, $b+F_{r-1} \le F_r$, so that $b \le F_r-F_{r-1} = F_{r-2}$, i.e. $b \le F_{r-2}$. Since $0 < b$ and $d = b + F_{r-1}$, $F_{r-1} < d$. Since $F_{r-1} < d$ and $d \le c$, $F_{r-1} < c$. Since $a = c + F_r$, this implies that $F_{r+1} < a$. Hence statement (ii) holds.

Lemma 3.2. *If $R(N) = r$, then there exist unique positive integers x and y satisfying*

$$(3.2) \qquad N = H_r(x, y) \quad \text{and} \quad 1 \le y \le x \le F_r.$$

Proof. By Lemma 3.1. If N is a single, then we can let $x = a$ and $y = b$. If N is a double, then we can let $x = c$ and $y = d$ and we will have $\le y \le x \le F_r$. x and y are unique by Theorem 2.22, to the effect that $N = H_r(x, y)$ has at most two solutions. Every N is either a single or a double. Note that if N is a double, then $x = a$ and $y = b$ won't satisfy $1 \le y \le x \le F_r$ since $F_{r+1} < a$.

Lemma 3.3. *Suppose $R(N) = r$. Then all solutions (x, y) of $N = H_r(x, y)$ in positive integers satisfy either*

$$(3.3.1) \qquad 1 \le y \le x \le F_r$$

or

(3.3.2) $$F_{r+1} < x \le 2F_r, \quad 1 \le y \le F_{r-2} \quad \textit{and} \quad y \le x.$$

But not both.

Proof. By Theorem 2.22, N is either a double or a single. Hence there are only two cases to consider. If N is a single, then $(x,y) = (a,b)$ and condition (3.3.1) holds by Lemma 3.1. (i). If N is a double, then $(x,y) = (a,b)$ or $(x,y) = (c,d)$. In the first case, by Lemma 3.1 (ii) (3.3.2) holds. In the sceond case, by Lemma 3.1 (ii) (3.3.1) holds.

Lemma 3.4. *Suppose $R(N) = r$. Then all solutions of $N = H_r(x,y)$ in positive integers (x,y) satisfy the conditions $x \le 2F_r$ and $y \le F_r$.*

Proof. By Lemma 3.3, either (3.3.1) holds or (3.3.2) holds. (3.3.1) implies $x \le F_r \le 2F_r$ and $y \le F_r$. (3.3.2) implies $x \le 2F_r$ and $y \le F_{r-2} \le F_r$. Hence $x \le 2F_r$ and $y \le F_r$.

Lemma 3.5. *If $0 < k$, then for all positive integers a and b,*

$$0 < H_k(a,b) < H_{k+1}(a,b).$$

Proof. From the definition it follows that $H_n(a,b)$ is a strictly increasing sequence of positive integers.

Theorem 3.6. *There exist integers x and y such that*

(3.6) $$N = H_n(x,y) \quad \textit{and} \quad 1 \le y \le x \le F_n$$

iff $n = R(N)$. Furthermore x and y are unique.

Proof. To prove the first part of the theorem suppose $R(N) = n$. Then by Lemma 3.2 there exist unique integers x and y suct that $N = H_n(x,y)$ and $1 \le y \le x \le F_n$, i.e. (3.6). To prove the second part suppose x and y are integers satisfying (3.6). Then $n > 0$. Let $R(N) = r$. Then $n \le r$. Let $k = r - n$. By definition of $R(N)$ there are positive integers a and b such that $N = H_r(a,b)$. By Lemma 1.3 (ii), since $n = r - k$, we have $N = H_r(a,b) = H_n(H_k(a,b), H_{k+1}(a,b))$ so that $N = H_n(H_k(a,b), H_{k+1}(a,b))$.

Thus $x = H_k(a,b)$ and $y = H_{k+1}(a,b)$ are particular solutions to the linear diophantine equation $N = xF_{n-1} + yF_n$. Since $(F_n, F_{n+1}) = 1$, all solutions to the equation are given by

$$x = H_k(a,b) - tF_n \quad \text{and} \quad y = H_{k+1}(a,b) + tF_{n-1},$$

where t is an integer. Since $y \le x$, we have for some t the inequality $H_{k+1}(a,b) + tF_{n-1} \le H_k(a,b) - tF_n$. This implies

$$t \le (H_k(a,b) - H_{k+1}(a,b))/F_{n+1},$$

so that

$$t \leq H_k(a,b) - H_{k+1}(a,b).$$

Since $x \leq F_n$, we also have the inequality $H_k(a,b) - tF_n \leq F_n$, which implies

$$H_k(a,b)/F_n \leq t+1.$$

If $0 < k$, then by Lemma 3.5 we have $t < 0$ and $0 < t+1$ so that $-1 < t < 0$. This is a contradiction since t is an integer. Hence $k = 0$. Thus $r = n$ and hence $R(N) = n$.

Remark. Condition (3.6) cannot be replaced by the weaker condition $N = H_n(x,y)$ and $1 \leq y \leq x$, This condition is not strong enough to imply $n = R(N)$. For example if $N = 96$, then $R(N) = 6$ but $N = H_5(17,9)$ and $9 \leq 17$. Also $N = H_5(12,12)$ and $12 \leq 12$.

Theorem 3.7. *Let r be fixed nonnegative integer. Then the number of N such that $R(N) = r$ is exactly*

$$\frac{F_r\,(F_r+1)}{2}.$$

Proof. Let r be fixed nonnegative integer. We will use Theorem 3.6 to count the number of N such that $R(N) = r$. We will count pairs (x,y) such that $1 \leq y \leq x \leq F_r$. For each such pair, we put $N = H_r(x,y)$. For each N there is only one pair (x,y) satisfying $N = H_r(x,y)$ and $1 \leq y \leq x \leq F_r$, by Theorem 2.6. How many pairs (x,y) are there such that $1 \leq x \leq F_r$? For each such x, there are x choices of y such that $1 \leq y \leq x$. Hence the number of N such that $R(N) = r$ is given by the sum

$$\sum_{x=1}^{F_r} x = \frac{F_r\,(F_r+1)}{2}.$$

Example 3.7. The number of N such that $R(N) = 5$ is $F_5(F_5+1)/2 = 5 \cdot 6/2 = 15$. By Corollary 1.20, these 15 N all lie in the interval $8 = F_6 \leq N \leq F_5F_6 = 40$. They are the 15 values $N = 8, 11, 14, 16, 17, 19, 20, 22, 24, 25, 27, 30, 32, 35$ and 40.

4. Double numbers

In this section we first prove that there are infinitely many double numbers. Then we give a combinatorial formula for the number of double numbers N having a fixed value of R. Last we give an asymptotic estimate for the number of double numbers up to F_nF_{n+1}.

Lemma 4.1. *For all $n > 2$, F_nF_{n+1} is a double number.*

Proof. Suppose $2 < n$. Recall that by Corollary 1.10, $R(F_nF_{n+1}) = n$. We have $F_nF_{n+1} = F_n\,(F_{n-1} + F_n) = F_nF_{n-1} + F_nF_n = H_n(F_n, F_n)$. On the other hand,

$$\begin{aligned} F_nF_{n+1} &= (F_n + F_n)F_{n-1} + (F_n - F_{n-1})F_n \\ &= H_n(2F_n, F_n - F_{n-1}) = H_n(2F_n, F_{n-2}). \end{aligned}$$

$0 < F_{n-2}$ since $n > 2$. The two representations of F_nF_{n+1} are distinct since $F_n \neq F_{n-2}$.

Lemma 4.2. *For $n > 4$, if $N = F_n(F_{n+1} - 1)$, then $R(N) = n$ and N is a double number.*

Proof. By an argument similar to that in the proof of Lemma 4.1 it is easy to see that

$$N = H_n(F_n, F_n - 1) = H_n(2F_n, F_{n-2} - 1). \tag{4.2}$$

To prove that $R(N) = n$ we will use the IVL. Obviously $n \leq R(N)$. Suppose that $n + 1 \leq R(n)$. Then by the IVL there exist $a \geq 1$ and $b \geq 1$ such that $N = H_{n+1}(a, b)$. Hence $F_n(F_{n+1} - 1) = aF_n + bF_{n+1}$. Then $F_n \mid b$, since $(F_n, F_{n+1}) = 1$. Let $b = eF_n$, where $1 \leq e$. Then we have $a+(e-1)F_{n+1} < 0$, a contradiction. Thus $R(N) = n$.

We give next a formula for the number of double numbers N with a fixed R value r. For this it is necessary first to characterise double numbers. From section 2 we have the following result.

Lemma 4.3. *Suppose $R(N) = r$. Then N is a double number iff*

$$\left\lceil \frac{(-1)^rF_{r-2}N + 1}{F_{r-1}} + 1 \right\rceil = \left\lfloor \frac{(-1)^rF_{r-1}N - 1}{F_r} \right\rfloor.$$

Proof. See the remark following Theorem 2.22 that N is a double iff $\lceil g_r(N)\rceil + 1 = \lfloor h_r(N)\rfloor$.

Theorem 4.4. *N is a double number and $R(N) = r$ iff there exist unique positive integers x and y such that*

$$N = H_r(x, y) \quad \text{and} \quad F_{r-1} < y \leq x \leq F_r. \tag{4.4}$$

Proof. For the proof of one part of the theorem, suppose N is a double number and $R(N) = r$. By Lemma 3.1, there exist positive integers c and

d such that $N = H_r(c, d)$ and $F_{r-1} < d \leq c \leq F_r$. Let $x = c$ and $y = d$. Then (4.4) holds. Also since the condition $F_{r-1} < y \leq x \leq F_r$ implies $1 < y \leq x \leq F_r$, x and y are unique by Lemma 3.1. For the proof of the second part, suppose (4.4) for some positive integers x and y. Then since $1 \leq r$, $1 \leq y \leq x \leq F_r$. Hence $R(N) = r$ by Theorem 3.6. N cannot be a single since in that case, by Lemma 3.1, we would have $x = a, y = b$ and $b \leq F_{r-1}$. Hence N is a double.

Note that if (x, y) satisfies $F_{r-1} < y \leq x \leq F_r$, then $(x + F_r, y - F_{r-1})$ satisfies $F_{r+1} < x \leq 2F_r$ and $1 \leq y \leq F_{r-2}$. Also if (x, y) satisfies $F_{r+1} < x \leq 2F_r$ and $1 \leq y \leq F_{r-2}$, then $(x - F_r, y + F_{r-1})$ satisfies $F_{r-1} < y \leq x \leq F_r$. So one could also prove a version of Theorem 4.4, with condition (4.4) replaced by

$$N = H_r(x, y), \quad F_{r+1} < x \leq 2F_r \quad \text{and} \quad 1 \leq y \leq F_{r-2}.$$

Theorem 4.5. *Let $r \geq 3$. The number of N such that N is a double number and $R(N) = r$ is exactly*

$$\frac{F_{r-2}\,(F_{r-2} + 1)}{2}.$$

Proof. Suppose r is a fixed positive integer. To count the number of double numbers N such that $R(N) = r$ we will use representation (4.4) of Theorem 4.4. We can determine the number of double numbers N such that $R(N) = r$ by counting pairs of integers (x, y) such that $F_{r-1} < y \leq x \leq F_r$. For each such pair (x, y) we can let $N = H_r(x, y)$ since N depends uniquely on (x, y). How many pairs of integers (x, y) are there such that $F_{r-1} < y \leq x \leq F_r$? Since $F_r - F_{r-1} = F_{r-2}$, there are F_{r-2}, there are F_{r-2} choices for x such that $F_{r-1} < x \leq F_r$. For each choice of x, there are x choices for y such that $F_{r-1} < y \leq x$. Therefore the numbers N such that $R(N) = r$ is given by the sum

$$\sum_{x=1}^{F_{r-2}} x = \frac{F_{r-2}\,(F_{r-2} + 1)}{2}.$$

Example. The number of N such that N is a double and $R(N) = 6$ is $F_4(F_4 + 1)/2 = 3 \cdot 4/2 = 6$. By Corollary 1.20 and Theorem 2.23 with $n = 5$ these N lie in the interval $18 = 5 \cdot 8 + 5^2 + 13 = F_5F_6 + F_5^2 + F_1 \leq N \leq F_6F_7 = 8 \cdot 13 = 104$. They are $N = 78, 83, 88, 91, 96$ and 104.

Lemma 4.6. *For all double numbers $N, N \leq F_nF_{n+1}$ iff $R(N) \leq n$.*

Proof. The first part of the lemma is the contrapositive of Lemma 1.11, if $R(N) \leq n$ then $N \leq F_nF_{n+1}$. For the proof of the second part suppose N is a double and $N \leq F_nF_{n+1}$. Let $r = R(N)$. We will show that $r \leq n$. Suppose not. Suppose $n < r$. Let $N = H_r(a,b)$ where a and b are as in (2.16). By Lemma 3.1 (ii), since N is a double, $F_{r+1} < a$. Hence $N = H_r(a,b) = aF_{r-1} + bF_r \geq F_{r+1}F_{r-1} + F_r > F_{n+2}F_n \geq F_n \cdot F_{n+1}$ contradicting $N \leq F_nF_{n+1}$. Therefore $r \leq n$.

Theorem 4.7. *For $n \geq 1$, the number of double numbers $N \leq F_nF_{n+1}$ is equal to*

$$\frac{F_{n-1}F_{n-2} + F_n - 1}{2}.$$

Proof. By Lemma 4.6 and Theorem 4.5, the number of double numbers $N \leq F_nF_{n+1}$ is

$$\sum_{r=3}^{n} \frac{F_{r-2}(F_{r-2}+1)}{2} = \frac{1}{2}\sum_{r=3}^{n}(F_{r-2}^2 + F_{r-2})$$
$$= \frac{1}{2}\left(\sum_{i=1}^{n-2} F_i^2 + \sum_{i=1}^{n-2} F_i\right) = \frac{1}{2}\left(F_{n-2}F_{n-1} + F_n - 1\right).$$

What proportion of integers N are double numbers? We shall show that on average approximately 7.3% of numbers are doubles. We shall show this by proving that for n sufficiently large, approximately $\beta^4/2$ of the numbers N up to F_nF_{n+1} are doubles. Here $\beta = (1-\sqrt{5})/2 = -61803\ldots$ so that $\beta^4/2 = .072949016\ldots$.

Theorem 4.8. *The probability that N is a double number is asymptotic to $\beta^4/2$.*

Proof. Let $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$. Then $\alpha\beta = -1$. It is known that F_n is asymptotic to $\alpha^n/\sqrt{5}$, i.e. that $\lim F_n/\alpha^n \approx 1/\sqrt{5}$. By Lemma 4.6 and Theorem 4.7, the number of double numbers N up to F_nF_{n+1}, divided by the number of N up to F_nF_{n+1} is equal to

$$(F_{n-1}F_{n-2} + F_n - 1)/2F_nF_{n+1} \approx F_{n-1}F_{n-2}/2F_nF_{n+1}$$
$$\approx \Big((\alpha^{n-1}/\sqrt{5})(\alpha^{n-2}/\sqrt{5})\Big) \,/\, \Big(2(\alpha^n/\sqrt{5})(\alpha^{n+1}/\sqrt{5})\Big)$$
$$= \alpha^{n-1}\alpha^{n-2}/2\alpha^n\alpha^{n+1} = 1/2\alpha^4 = \beta^4/2.$$

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