Algorithms Seminar 2001–2002,
F. Chyzak (ed.), INRIA, (2003), pp. 97–100.

Available online at the URL
`http://algo.inria.fr/seminars/`.

Irrationality of the ζ Function on Odd Integers

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February 5, 2001

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Abstract

The ζ function is defined by $\zeta(s) = \sum_n 1/n^s$. This talk is a study of the irrationality of the zeta function on odd integer values > 2.

1. Introduction

The sum $\sum_n 1/n^2$ was first studied by Bernoulli, who proved around 1680 that it converged to a finite limit less than 2. Euler proved in 1735 that it is equal to $\pi^2/6$, and studied the more general function $\zeta(s) = \sum_n 1/n^s$. He also showed that on even integers the ζ function has a closed form, namely $\zeta(2n) = C_n \pi^{2n}$ where the coefficients C_n are rational numbers that he wrote in terms of Bernoulli numbers. A century later Riemann studied this function on the whole complex plane, and he stated a conjecture on the location of the zeroes of the zeta function, that is known as the Riemann hypothesis, and is still unproved.

The first result on the irrationality of the ζ function on odd integers is due to Apéry, who proved in 1978 that $\zeta(3)$ is irrational [1]. Recently Tanguy Rivoal showed that the ζ function takes infinitely many irrational values on the odd integers [4, 5], and that there exists an odd integer j with $5 \leq j \leq 21$ such that $\zeta(j)$ is irrational [5]. Zudilin [6] refined this result and proved it for $5 \leq j \leq 11$.

2. Irrationality of $\zeta(3)$

Theorem 1 (Apéry(1978))**.** *The number $\zeta(3)$ is irrational.*

The following proof is due to Nesterenko [3], after ideas by Beukers. The theorem is proved using the following generating function

$$S_n(z) = \sum_{k=1}^{\infty} \frac{\partial}{\partial k} \left(\frac{(k-1)^2(k-2)^2 \dots (k-n)^2}{k^2(k+1)^2 \dots (k+n)^2} \right) z^{-k}$$

The decomposition of the coefficient of z^{-k} in partial fractions gives the equality

$$S_n(z) = P_{0,n}(z) + P_{1,n}(z)\,\mathrm{Li}_2(1/z) + P_{2,n}(z)\,\mathrm{Li}_3(1/z) \tag{1}$$

where $\mathrm{Li}_s(z) = \sum_{n\geq 1} \frac{z^n}{n^s}$ is a polylogarithm function, and $P_{k,n}$ are polynomials of degree n such that $P_{1,n}(1) = 0$. When Equation (1) is specialized at $z = 1$, it becomes

$$S_n(1) = P_{0,n}(1) + P_{2,n}(1)\zeta(3),$$

with the additionnal properties that $P_{2,n}(1) \in \mathbb{Z}$ and $d_n^3 P_{0,n}(1) \in \mathbb{Z}$ where $d_n = \mathrm{ppcm}(1, 2, \dots, n)$.

The value $S_n(1)$ is bounded by using an integral representation.

$$S_n(1) = \frac{1}{2i\pi}\int_L \left(\frac{\Gamma(n+1-s)\Gamma(s)^2}{\Gamma(n+1+s)}\right)^2 ds, \tag{2}$$

where L is the vertical line $\Re(z) = c,\ \ 0 < c < n+1$, oriented from top to bottom. From this integral, the bounds $0 < S_n(1) \le c(\sqrt{2}-1)^{4n}$ are obtained.

The inequalities $0 < d_n^3 P_{0,n}(1) + d_n^3 P_{2,n}(1)\zeta(3) < cr^n$, where c is a constant, and $r < 1$ prove that $\zeta(3)$ is irrational; because if $\zeta(3)$ is rational and equal to p/q, then $qd_n^3 P_{0,n}(1) + qd_n^3 P_{2,n}(1)\zeta(3)$ is an integer greater than 0 and bounded by qcr^n that converges to 0.

3. The ζ Function Has Infinitely Many Irrational Values on Odd Integers

Tanguy Rivoal in fact proved a stronger result, that is:

Theorem 2. *Let a be an odd integer greater than 3 and $\delta(a)$ be the dimension of the $\mathbb{Q}$-vector space spanned by $1, \zeta(3), \ldots, \zeta(a)$, then*

$$\delta(a) \ge \frac{1}{3}\log a.$$

This implies directly that infinitely many $\zeta(2n+1)$ are irrational.

To prove Theorem 2, we introduce the series

$$S_{n,a,r}(z) = n!^{a-2r}\sum_{k=1}^{\infty}\frac{(k-rn)_{rn}(k+n+1)_{rn}}{(k)_{n+1}^a} z^{-k},$$

where $(k)_n = k(k+1)\ldots(k+n-1)$ is the Pochhammer symbol, and n, r, and a are integers satisfying $n \ge 0,\ \ 1 \le r < a/2$, so that $S_{n,a,r}(z)$ exists when $|z| \ge 1$. As for the proof of the irrationality of $\zeta(3)$, an equality between the series studied and values of ζ is found, namely

$$S_{n,a,r}(1) = P_{0,n}(1) + \sum_{l=2}^{a} P_{l,n}(1)\zeta(l),$$

moreover, if $(n+1)a + l$ is odd then $P_{l,n}(1) = 0$. For n odd and a odd greater than 3, $P_{l,n}(1) = 0$ if l is even, so that $S_{n,a,r}(1)$ is a linear combination of values of ζ on odd integers.

The dimension of the vector space spanned by $1, \zeta(3), \ldots, \zeta(a)$ is based on the following theorem:

Theorem 3 (Nesterenko's criterion)**.** *Let $\theta_1, \theta_2, \ldots, \theta_N$ be N real numbers, and suppose that there exist N sequences $(p_{l,n})_{n\ge 0}$ such that*

1. $\forall\ i = 1, \ldots, N,\ \ p_{l,n} \in \mathbb{Z}$*;*
2. $\alpha_1^{n+o(n)} \le \left|\sum_{l=1}^N p_{l,n}\theta_l\right| \le \alpha_2^{n+o(n)}$*, with* $0 < \alpha_1 \le \alpha_2 < 1$*;*
3. $\forall l = 1, \ldots, N,\ \ |p_{l,n}| \le \beta^{n+o(n)}$ *with* $\beta > 1$.

Then

$$\dim_{\mathbb{Q}}(\mathbb{Q}\,\theta_1 + \mathbb{Q}\,\theta_2 + \cdots + \mathbb{Q}\,\theta_N) \ge \frac{\log(\beta) - \log(\alpha_1)}{\log(\beta) - \log(\alpha_1) + \log(\alpha_2)}.$$

This criterion, applied to the real numbers $\theta_i = \zeta(2i+1),\ \ i \le (a-1)/2$, with the sequences $p_{l,n}$ defined by $p_{0,n} = d_{2n}^a P_{0,2n}(1)$ and $p_{l,n} = d_{2n}^a P_{2l+1,2n}(1)$ if $1 \le l \le (a-1)/2$ yields the inequality

$$\delta(a) \ge \frac{\log(r) + \frac{a-r}{a+1}\log(2)}{1 + \log(2) + \frac{2r+1}{a+1}\log(r+1)}, \tag{3}$$

for all $1 \le r < a/2$.

For each odd integer $a > 1$, there exists an r (that can be made explicit) such that the inequality (3) reduces to $\delta(a) \geq \log(a)/3$.

The proof of this property can be adapted to show that $\delta(169) > 2$, which means that there exists an integer j, $5 \leq j \leq 169$, such that 1, $\zeta(3)$, and $\zeta(j)$ are linearly independent over $\mathbb{Q}$.

4. At Least One Number Amongst $\zeta(5)$, $\zeta(7)$, ..., $\zeta(21)$ Is Irrational

The linear independence of $1, \zeta(3), \zeta(j)$ for some $j \leq 169$ implies the irrationality of $\zeta(j)$, but is stronger. The bound 169 is improved in this section by only seeking the irrationality.

Theorem 4. *There exists an integer j, $5 \leq j \leq 21$, such that $\zeta(j)$ is irrational.*

The proof of this theorem follows the same directions as the two previous ones. First an adequate generating function $S_n(z)$ is considered, that gives a linear equation implying the zeta function on odd integers when specialized. The coefficients of this equation are studied, and their denominator bounded; a saddle-point method gives asymptotic results on $S_n(1)$. These lemmas, combined with the Nesterenko criterion finally give the result.

The generating function $S_n(z)$ is

$$S_n(z) = n!^{a-6} \sum_{k=1}^{\infty} \frac{1}{2} \frac{d^2}{dk^2} \left(\left(k + \frac{n}{2}\right) \frac{(k-n)_n^3 (k+n+1)_n^3}{(k)_{n+1}^a} \right) z^{-k},$$

where a is an integer ≥ 6. This sum is convergent when $|z| \geq 1$. This sum is expanded in simple elements, and then specialized at $z = 1$ to give a relation between values of ζ on odd integers, $\zeta(3)$ excluded, namely

$$S_n(1) = P_{0,n}(1) + \sum_{j=2}^{a/2} j(2j-1) P_{2j-1,n}(1) \zeta(2j+1). \tag{4}$$

The coefficients $P_{l,n}$ satisfy $2d_n^{a+2} P_{0,n}(1) \in \mathbb{Z}$ and $2d_n^{a-l} P_{l,n}(1) \in \mathbb{Z}$ for $1 \leq l \leq a$.

The next step of the proof is to get an asymptotic result on $S_n(1)$, using a saddle-point method. We do not know of any integral representation similar to (2) for $S_n(1)$, but we can express $S_n(1)$ as the real part of a complex integral. First we introduce $R_n(k)$,

$$R_n(k) = n!^{a-6} \left(k + \frac{n}{2}\right) \frac{(k-n)_n^3 (k+n+1)_n^3}{(k)_{n+1}^a}.$$

So that $S_n(z) = \sum_{k=1}^{\infty} \frac{1}{2} \frac{d^2}{dk^2} R_n(k) z^{-k}$. We also define

$$J_n(u) = \frac{n}{2i\pi} \int_L R_n(nz) \left(\frac{\pi}{\sin(n\pi z)} \right)^3 e^{nuz} \, dz,$$

where L is a vertical line from $i\infty$ to $-i\infty$ with a real part between 0 and 1. With those notations, the property $S_n(1) = \Re\big(J_n(i\pi)\big)$ holds.

The quantity $J_n(i\pi)$ is rewritten in terms of the Γ function, using the complement formula $\Gamma(t)\Gamma(1-t) = \pi/\sin(\pi t)$, and is then approximated using the Stirling formula. This gives

$$J_n(i\pi) = \left(i(-1)^{n+1} (2\pi)^{a/2-1} n^{2-a/2} \int_L g(z) e^{nw(z)} \, dz \right) \big(1 + O(1/n)\big),$$

where $g(z) = (z + 1/2) \frac{\sqrt{1-z}^3 \sqrt{2+z}^3}{\sqrt{z}^{a+3} \sqrt{z+1}^{a+3}}$ and $w(z) = (a+3) z \log(z) - (a+3)(z+1) \log(z+1) + 3(1-z) \log(1-z) + 3(z+2) \log(z+2) + i\pi z$. The variable a is now specialized to 20 in order to have a relation between $\zeta(5)$, ..., $\zeta(21)$. The saddle-point method, see [2, pp. 279–285], now applies to the point z_0, the only root of $w'(z) = 0$ such that $0 < \Re(z) < 1$. The numerical value of z_0 is $0.992 - 0.012i$. The estimation of $J_n(i\pi)$ obtained is

$$J_n(i\pi) = u_n r(-1)^{n+1} n^{-8} e^{nw(z_0)+i\beta},$$

with r and β real constants and u_n a sequence of complex numbers converging to 1. We define $v_0 = \Im\big(w(z_0)\big)$. The real part of this expression is

$$r(-1)^{n+1} n^{-8} e^{\Re\left(nw(z_0)\right)} \big(\Re(u_n)\cos(nv_0+\beta) - \Im(u_n)\sin(nv_0+\beta)\big).$$

Since $v_0 \sim 3.104$ is not a multiple of π, there exists an increasing sequence $\phi(n)$ such that $\cos\big(\phi(n)v_0+\beta\big)$ tends to a limit $l \neq 0$. As a direct consequence

$$\lim_{n\to\infty} \Re J_{\phi(n)}(i\pi) = K(-1)^{\phi(n)+1}\phi(n)^{-8} e^{\Re\left(\phi(n)w(z_0)\right)},$$

where K is a constant. So $\lim_{n\to\infty}\big|S_{\phi(n)}(1)\big|^{1/\phi(n)} = e^{\Re\left(w(z_0)\right)}$.

This result, combined with Equation (4) proves Theorem 4 as follows. Equation (4) tells that $l_n = 2d_n^{22}S_n(1)$ is a linear combination of $\zeta(5)$, …, $\zeta(21)$ with integer coefficients. The paragraph above shows that l_n satisfies $\lim_{n\to\infty}\big|l_{\phi(n)}\big|^{1/\phi(n)} \in (0,1)$. So one of the values $\zeta(5)$, …, $\zeta(21)$ is irrational.

This result has been refined by Zudilin [6], who proved that at least one of the four numbers $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, and $\zeta(11)$ is irrational, by using a general hypergeometric construction of linear forms in odd zeta values.

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