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Embeddings of unitals such that each block is a subline

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Abstract

We consider unitals embedded in a pappian projective plane such that every block is a subline. We show that every such unital is a hermitian one, and that the embedding is standard.

The classical examples of unitals are the hermitian unitals, obtained as follows: Use a non-degenerate hermitian form in three variables over a finite field to describe a polarity of the projective plane over that field, take the set of absolute points of the polarity, and endow that set with its intersections with secant lines as blocks. Then the blocks are Baer sublines, and various results (see 3.2 below) have been obtained about unitals embedded in projective planes such that (some) blocks are Baer sublines.

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We take a more general point of view, and consider embeddings into a pappian projective space (coordinatized by some, possibly infinite, but commutative field F) where the blocks are sublines (coordinatized by an arbitrary subfield of F, not necessarily the fixed field of an involution, as used for Baer sublines). Using Wilbrink's characterization [7] of hermitian unitals and a recent result [3] about embeddings of hermitian unitals, we show that this condition alone suffices to characterize the hermitian unitals with their standard embeddings. We make no restriction on the order q of the unital apart from $q > 1$; in particular, we do not assume *a priori* that q is a prime power.

1 Sublines

Let $\mathbb{P} = (P, \mathcal{L})$ be a pappian projective plane, coordinatized by a commutative field F. A *subline* of $\mathbb{P}$ is a line of a subplane of $\mathbb{P}$, i.e., the intersection S of a subplane with a line of $\mathbb{P}$ joining two points of the subplane. The set of all cross ratios of quadruplets of points in S is then of the form $K \cup \{\infty\}$, where K is a subfield of F. If S is finite then this subfield is uniquely determined; it consists of the zeros of the polynomial $X^q - X$, where $q = |S| - 1$. This implies that each finite subline is determined uniquely by its size and three of its points.

If one introduces coordinates from $F \cup \{\infty\}$ for the points of a line of $\mathbb{P}$ in such a way that 0, 1 and ∞ are labels for points in S then the labels for arbitrary points of $S \setminus \{\infty\}$ form that subfield K. The subline S is called a *Baer subline* if F is a quadratic extension of K; i.e., if $\dim_K F = 2$.

The line of $\mathbb{P}$ containing the subline S will be denoted by $L(S)$.

1.1 Lemma. *Let B, B' be two sublines of the same finite size, and let p be a point of $\mathbb{P}$ that lies neither on $L(B)$ nor on $L(B')$. If the projection with center p from $L(B)$ onto $L(B')$ maps three points of B into B' then it maps B onto B'.*

PROOF: Let π denote the projection, and let Q be a subplane such that $B = Q \cap L(B)$. Applying a suitable elation with axis $L(B)$, we may assume that $p \in Q$. There is a homology α with center p and an axis through $L(B) \wedge L(B')$ mapping $L(B)$ to $L(B')$. This homology fixes each of the projecting lines, and $\alpha^{-1} \circ \pi$ clearly fixes each point in B. Now α maps B into the subline $\alpha(Q) \cap L(B')$, and the subline $\alpha(B) = \alpha \circ (\alpha^{-1} \circ \pi)(B) = \pi(B)$ coincides with B' because it shares three points with it (and has the right size). □

2 Embeddings

A unital $\mathbb{U} = (U, \mathcal{B})$ of order $q > 1$ is a 2-$(q^3 + 1, q + 1, 1)$-design. In other words, $\mathbb{U}$ is an incidence structure such that any two points in U are joined by a unique block in $\mathcal{B}$, there are $|U| = q^3 + 1$ points, and every block has exactly $q + 1$ points. It follows that every point is on exactly q^2 lines.

2.1 Theorem. *Let $\mathbb{U} = (U, \mathcal{B})$ be any unital of order q. Assume that $\mathbb{U}$ is embedded into a projective space such that U forms a subset of the point set of that space, and that every block $B \in \mathcal{B}$ is contained in a line $L(B)$ of the space. Then U is contained in a plane of the projective space. If U is not contained in a single line then mapping B to $L(B)$ is injective.*

PROOF: Pick any point $x \in U$ and a block $B \in \mathcal{B}$ such that $x \notin B$. Joining x with points on B, we obtain $q+1$ blocks. The union X of those blocks contains $(q+1)q+1 = q^2+q+1$ points.

Now assume that some point $z \in U$ is not contained in the subspace (a plane or a line) generated by X. Joining points of X with z (in the projective space) then gives an injective map into the line pencil of z, and thus into the set $\mathcal{B}_z$ of blocks (of the unital) through z. This contradicts the fact that $\mathcal{B}_z$ contains only q^2 blocks. □

2.2 Lemma. *In a unital of order q, there is no embedded projective plane of order q.*

PROOF: Aiming at a contradiction, assume that a projective plane of order q is contained in a unital of order q. Pick a point z of the unital outside the plane, and join each point of the plane to z. As every block with two points in the plane is entirely contained in the plane, this joining map is injective, and gives q^2+q+1 blocks through z — contradicting the fact that there are only q^2 blocks through z. □

2.3 Definition. Let F be a commutative field, and let $\mathrm{P}_2(F)$ be the projective plane over F. An embedding (as in 2.1) of a unital $\mathbb{U} = (U, \mathcal{B})$ of order q into $\mathrm{P}_2(F)$ is called *standard* if F has a subfield C of order q^2 such that — in coordinates with respect to a suitable quadrangle in $\mathrm{P}_2(F)$ — the point set U is identified with

$$\left\{ F(x_1, x_2, x_3) \,\middle|\, (x_1, x_2, x_3) \in C^3 \setminus \{(0,0,0)\},\ x_1^{q+1} + x_2^{q+1} + x_3^{q+1} = 0 \right\} .$$

In particular, the blocks of the unital are *Baer sublines* in the subplane $\mathrm{P}_2(C)$ of $\mathrm{P}_2(F)$.

3 Blocks that are sublines

Obviously, the hermitian unital (of order q, say) is embedded in a pappian projective plane such that every block is a subline; we may take the standard embedding (in the sense of Definition 2.3) into the plane over $\mathbb{F}_{q^2}$, and then make further embeddings into planes over commutative fields that contain $\mathbb{F}_{q^2}$. We are going to prove that, conversely, an embedding of a unital such that blocks are sublines is always a standard embedding of a hermitian unital.

Let $\mathbb{U} = (U, \mathcal{B})$ be a unital of order q, and consider a point x. Blocks $B, B' \in \mathcal{B}$ not through x are called *x-parallel* if every block through x meeting B also meets B'.

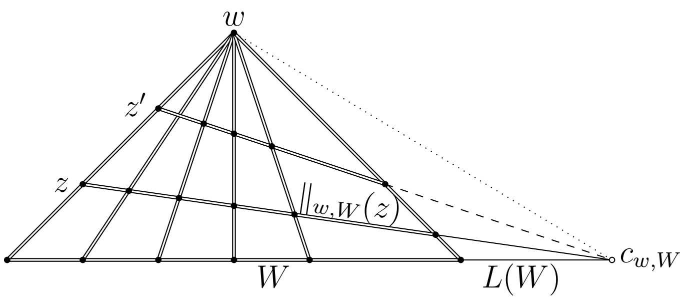

Figure 1: Constructing parallels (blocks are indicated by double strokes)

3.1 Theorem. *Assume that $\mathbb{U} = (U, \mathcal{B})$ is a unital of order q, embedded in a pappian projective plane $\mathbb{P}$ in such a way that every block is a subline of $\mathbb{P}$, and such that U is not contained in any line. Then $\mathbb{U}$ is the hermitian unital of order q, and the embedding is standard.*

PROOF: By our assumption, no two blocks of $\mathbb{U}$ are contained in the same line of $\mathbb{P}$; see 2.1. Thus any two intersecting blocks of $\mathbb{U}$ generate a projective subplane of $\mathbb{P}$.

To show that $\mathbb{U}$ is hermitian, we verify Wilbrink's three conditions ([7], see also [4]). We start with condition (II), which reads: if $W \in \mathcal{B}$, $w \in U \setminus W$, and $z \in U \setminus \{w\}$ is on a block joining w to a point of W, then there exists a block B_z through z which is w-parallel to W.

The union of all blocks obtained by joining w to a point on W forms a set D of size $(q+1)q+1 = q^2+q+1$. Joining the q^2 points of $D \setminus (w \vee z)$ to z we obtain at most $|\mathcal{B}_z \setminus \{w \vee z\}| = q^2 - 1$ many blocks through z. Therefore, there exists a block B_z through z that contains at least three points of D. From 1.1 we know that the projection with center w from $L(W)$ onto $L(B_z)$ maps W onto B_z. Thus a block through w meets B_z if, and only if, it meets W. In other words: The block B_z is w-parallel to W, and we have proved that Wilbrink's Condition (II) is satisfied.

We claim that the point $c := L(B_z) \wedge L(W)$ is not in U. Otherwise, we project the block $w \vee z$ from c onto the block $w \vee x$, for each $x \in W$. Then every point of the projective subplane generated by the sublines $w \vee z$ and $w \vee c$ belongs to the unital. This is impossible by 2.2.

Wilbrink's Condition (I) requires that there are no O'Nan configurations in $\mathbb{U}$; recall that such a configuration consists of four blocks B_0, B_1, B_3, B_4 and six points x_{jk}, where $\{j,k\}$ is any two-element subset of $\{0,1,2,3\}$, and x_{jk} lies on B_n precisely if $n \in \{j,k\}$. Aiming at a contradiction, we consider an O'Nan configuration in $\mathbb{U}$. Let w be one of the points, let W be one of the blocks not through w, and let z be one of the points not in $W \cup \{w\}$. Then B_z will be a block of the configuration, and $c = B_z \wedge W$ belongs to the configuration. This contradicts the observation of the previous paragraph that c is not in U.

As there are no O'Nan configurations, the w-parallel B_z to W through z is determined uniquely by (w, W) and z. We denote it by $\|_{w,W}(z)$.

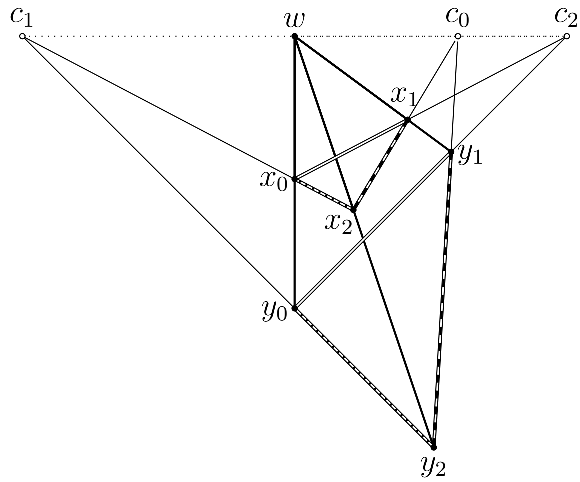

Figure 2: Desargues' theorem in $\mathbb{U}$

Let $a := W \wedge (w \vee z)$. For any $b \in W$, the projection with center c from $L(w \vee z)$ onto $L(w \vee b)$ maps $\{w, z, a\}$ into the block $w \vee b$. By 1.1, this projection maps the block $w \vee z$ onto the block $w \vee b$. For each $z' \in w \vee z$, we thus find $c \in L(\|_{w,W}(z'))$. The point c therefore depends only on (w, W), we denote it by $c_{w,W}$. See Figure 1.

For $b \in W \setminus \{a\}$, consider the block Y joining a with $x := (w \vee b) \wedge \|_{w,W}(z)$. We obtain w-parallels $\|_{w,Y}(y)$ for each $y \in w \vee z$, and the lines $L(\|_{w,Y}(y))$ all go through some point $c_{w,Y}$. The composition of the projections with centers $c_{w,W}$ and $c_{w,Y}$ from $L(w \vee z)$ onto $L(w \vee b)$ and then back to $L(w \vee z)$ is a projectivity of $L(w \vee z)$. It maps the block $w \vee z$ onto $w \vee b$ and then back onto $w \vee z$. It fixes w, and no other point in the subline $w \vee z$ is fixed. We introduce coordinates from a field F for the line $L(w \vee z)$ in such a way that 0 labels w and $1, \infty$ are labels of points in W. Then the block $w \vee z$ is the subline coordinatized by the subfield K of order q in F, and said projectivity is given as $x \mapsto kx + t$ with $k, t \in F$. As the subline $w \vee z$ is invariant under the projectivity, we obtain $K = kK + t$. This implies $k, t \in K$. For $k \neq 1$ the projectivity would fix $(1 - k)^{-1}t \in K$. So $k = 1$, the projectivity fixes no point of $L(w \vee z)$ apart from w, and we obtain that $c_{w,W}$ and $c_{w,Y}$ are collinear with w.

We thus observe that each w-parallel class is obtained by joining the points different from w on a block through w with a point (namely, $c_{w,Y}$, also different from w) on the line containing w and all points $c_{w,Y}$.

Wilbrink's Condition (III) is a version of Desargues' little affine theorem for unitals, as follows (see Figure 2). On three blocks M_0, M_1, M_2 through a common point w, let x_j, y_j be points different from w such that $x_j, y_j \in M_j$. Let A_{j+2} be the block joining x_j with x_{j+1}, and let B_{j+2} be the block joining y_j with y_{j+1}, for $j \in \mathbb{Z}/(3\mathbb{Z})$. If A_j and B_j are w-parallel for $j \in \{0, 1\}$ then A_2 and B_2 are w-parallel, as well. We abbreviate $c_j := c_{w,A_j}$, for the sake of readability.

For our embedded unital, the validity of Condition (III) follows from Desargues' Theorem, applied to the triangles (x_0, x_1, x_2) and (y_0, y_1, y_2) in $\mathbb{P}$ that lie in central position with respect to w, and in axial position with respect to the line containing $\{w, c_0, c_1, c_2\}$. So Wilbrink's three conditions are satisfied, and the unital $\mathbb{U}$ is the hermitian one according to [7]. Each embedding of a hermitian unital of order $q > 2$ in a pappian projective plane is standard, see [3, Thm. 5.1].

For $q = 2$, the unital is isomorphic to the affine plane $\mathrm{A}_2(\mathbb{F}_3)$ of order 3.
The embeddings of $\mathrm{A}_2(\mathbb{F}_3)$ into any pappian plane are known explicitly, see [6, 3.7]: such an embedding exists precisely if the coordinatizing field F contains a root of the polynomial $X^2 + X + 1$, and then any two embeddings are equivalent. The subline assumption implies that F has characteristic 2. Hence F has a subfield of size 4, and the embedding is standard. □

3.2 Remarks. A special case of our present result 3.1 (namely, the case where F is finite of square order $|F| = q^2$, and the blocks are *Baer* sublines) has been proved by Lefèvre-Percsy [5] and Faina and Korchmáros [2]. In fact, it suffices to assume only that a certain subset of the blocks consists of Baer sublines. See De Bruyn [1] for an overview of pertinent results, and a further generalization.

3.3 Remark. In 3.1, we require explicitly that U is not contained in a line. In fact, it is possible to "embed" the hermitian unital of order q into the projective line over $\mathbb{F}_{q^4}$ in such a way that every block is a subline.

In order to see this, start with the standard embedding of the unital into the projective plane over $\mathbb{F}_{q^2}$; the blocks are sublines. Take any proper extension field E of $\mathbb{F}_{q^2}$, and embed the plane $\mathbb{P}$ over $\mathbb{F}_{q^2}$ into the plane $\mathbb{P}'$ over E. Pick a tangent t to the unital, and choose a point z on t in $\mathbb{P}'$ but not in $\mathbb{P}$. Then no line through z meets the unital in more than one point. Projecting from z into a line of $\mathbb{P}'$ gives an injective map, preserving the fact that blocks are sublines.

3.4 Remark. The definition of embedding between incidence structures in [3, p. 939] must be amended: one has to require that the (injective) mapping between the point sets is accompanied by an *injective* mapping between the line sets.

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