AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **77(3)** (2020), Pages 355–372

On independent triples and vertex-disjoint chorded cycles in graphs

Ronald J. Gould

Department of Mathematics
Emory University, Atlanta, GA 30322
U.S.A,
rg@emory.edu

Kazuhide Hirohata

Department of Industrial Engineering, Computer Science
National Institute of Technology
Ibaraki College, Hitachinaka, 312-8508
Japan
hirohata@ece.ibaraki-ct.ac.jp

Ariel Keller Rorabaugh

Department of Electrical Engineering and Computer Science
University of Tennessee
Knoxville, 37996
U.S.A.
ariel.keller@gmail.com

Abstract

Let G be a graph, and let $\sigma_3(G)$ be the minimum degree sum of three independent vertices of G. We prove that if G is a graph of order at least $8k+5$ and $\sigma_3(G) \geq 9k-2$ with $k \geq 1$, then G contains k vertex-disjoint chorded cycles. We also show that the degree sum condition on $\sigma_3(G)$ is sharp.

1 Introduction

The study of cycles in graphs is a rich and important area. One question of particular interest is to find conditions that guarantee the existence of k vertex-disjoint cycles. In 1963, Corrádi and Hajnal [3] proved that if $|G| \geq 3k$ and the minimum degree

ISSN: 2202-3518

$\delta(G) \geq 2k$, then G contains k vertex-disjoint cycles. For an integer $t \geq 1$, let

$$\sigma_t(G) = \min \left\{ \sum_{v \in X} d_G(v) \mid X \text{ is an independent vertex set of } G \text{ with } |X| = t \right\},$$

and $\sigma_t(G) = \infty$ when the independence number $\alpha(G) < t$. Enomoto [4] and Wang [11] independently extended the Corrádi and Hajnal result showing that, if $|G| \geq 3k$ and $\sigma_2(G) \geq 4k-1$, then G contains k vertex-disjoint cycles. Fujita et al. [6] proved that if $|G| \geq 3k+2$ and $\sigma_3(G) \geq 6k-2$, then G contains k vertex-disjoint cycles, and in [9], this result was extended to $\sigma_4(G) \geq 8k-3$.

A *chord* of a cycle is an edge between two non-adjacent vertices of the cycle, and a *chorded cycle* is a cycle with at least one chord. In 2008, Finkel improved Corrádi and Hajnal's result for chorded cycles.

Theorem 1.1. (Finkel [5]) *Let $k \geq 1$ be an integer. If G is a graph of order at least $4k$ and $\delta(G) \geq 3k$, then G contains k vertex-disjoint chorded cycles.*

In 2010, Chiba et al. proved Theorem 1.2 which is a stronger result than Theorem 1.1, since $\sigma_2(G) \geq 2\delta(G)$.

Theorem 1.2. (Chiba, Fujita, Gao, Li [1]) *Let $k \geq 1$ be an integer. If G is a graph of order at least $4k$ and $\sigma_2(G) \geq 6k-1$, then G contains k vertex-disjoint chorded cycles.*

In this paper, we consider a similar extension for chorded cycles, as Fujita et al. [6] proved the existence of k vertex-disjoint cycles under the condition $\sigma_3(G)$. In particular, we first show the following.

Theorem 1.3. *If G is a graph of order at least 7 and $\sigma_3(G) \geq 7$, then G contains a chorded cycle.*

Remark 1. We define the following graphs: $G_1 = K_2 \cup K_2$, $G_2 = K_2 \cup K_3$, and $G_3 = K_3 \cup K_3$, where $H_1 \cup H_2$ denotes the union of two disjoint graphs H_1 and H_2. Then for each $1 \leq i \leq 3$, G_i satisfies the $\sigma_3(G)$ condition of Theorem 1.3, since the independence number $\alpha(G_i) = 2$. However, G_i for each $1 \leq i \leq 3$ does not contain a chorded cycle. Thus $|G| \geq 7$ is necessary.

Our main result is the following theorem.

Theorem 1.4. *Let $k \geq 1$ be an integer. If G is a graph of order at least $8k+5$ and $\sigma_3(G) \geq 9k-2$, then G contains k vertex-disjoint chorded cycles.*

Remark 2. Theorem 1.4 is sharp with respect to the degree sum condition. Consider the complete bipartite graph $G = K_{3k-1,n-3k+1}$, where large $n = |G|$. Then $\sigma_3(G) = 3(3k-1) = 9k-3$. However, G does not contain k vertex-disjoint chorded cycles, since any chorded cycle must contain at least 3 vertices from each partite set. Thus $\sigma_3(G) \geq 9k-2$ is necessary. Also, since $\sigma_3(G) \geq 3\sigma_2(G)/2$, when the order of G is sufficiently large, Theorem 1.4 is a stronger result than Theorem 1.2.

For other related results on vertex-disjoint chorded cycles in graphs and bipartite graphs, we refer the reader to see [2, 7, 10].

In this paper, all graphs are simple. Let G be a graph, H a subgraph of G and $X \subseteq V(G)$. For $u \in V(G)$, the set of neighbors of u in G is denoted by $N_G(u)$, and we denote $d_G(u) = |N_G(u)|$. For $u \in V(G)$, we denote $N_H(u) = N_G(u) \cap V(H)$ and $d_H(u) = |N_H(u)|$. Also we denote $d_H(X) = \sum_{u \in X} d_H(u)$. If $H = G$, then $d_G(X) = d_H(X)$. The subgraph of G induced by X is denoted by $\langle X \rangle$. Let $G - X = \langle V(G) - X \rangle$ and $G - H = \langle V(G) - V(H) \rangle$. If $X = \{x\}$, then we write $G - x$ for $G - X$. If there is no fear of confusion, then we use the same symbol for a graph and its vertex set. For a graph G, comp(G) is the number of components of G. If G is one vertex, that is, $V(G) = \{x\}$, then we simply write x instead of G. For an integer $r \geq 1$ and two vertex-disjoint subgraphs A, B of G, we denote by $(d_1, d_2, \ldots, d_r)$ a degree sequence from A to B such that $d_B(v_i) \geq d_i$ and $v_i \in V(A)$ for each $1 \leq i \leq r$. In this paper, since it is sufficient to consider the case of equality in the above inequality, when we write $(d_1, d_2, \ldots, d_r)$, we assume $d_B(v_i) = d_i$ for each $1 \leq i \leq r$. For two disjoint $X, Y \subseteq V(G)$, $E(X, Y)$ denotes the set of edges of G connecting a vertex in X and a vertex in Y. Let Q be a path or a cycle with a given orientation and $x \in V(Q)$. Then x^+ denotes the first successor of x on Q and x^- denotes the first predecessor of x on Q. If $x, y \in V(Q)$, then $Q[x, y]$ denotes the path of Q from x to y (including x and y) in the given direction. The reverse sequence of $Q[x, y]$ is denoted by $Q^-[y, x]$. We also write $Q(x, y] = Q[x^+, y]$, $Q[x, y) = Q[x, y^-]$ and $Q(x, y) = Q[x^+, y^-]$. If Q is a path (or a cycle), say $Q = x_1, x_2, \ldots, x_t(, x_1)$, then we assume an orientation of Q is given from x_1 to x_t. If P is a path connecting x and y of $V(G)$, then we denote the path P as $P[x, y]$. A cycle of length ℓ is called a *ℓ-cycle*. For terminology and notation not defined here, see [8].

2 Preliminaries

Definition 2.1. Suppose $C_1, \ldots, C_r$ are r vertex-disjoint chorded cycles in a graph G. We say $\{C_1, \ldots, C_r\}$ is *minimal* if G does not contain r vertex-disjoint chorded cycles $C'_1, \ldots, C'_r$ such that $|\cup_{i=1}^r V(C'_i)| < |\cup_{i=1}^r V(C_i)|$.

Definition 2.2. Let $C = v_1, \ldots, v_t, v_1$ be a cycle with chord $v_i v_j$, $i < j$. We say a chord $vv' \neq v_i v_j$ is *parallel* to $v_i v_j$ if either $v, v' \in C[v_i, v_j]$ or $v, v' \in C[v_j, v_i]$. Note if two distinct chords share an endpoint, then they are parallel. We say two distinct chords are *crossing* if they are not parallel.

Definition 2.3. Let $u_i v_j$ and $u_\ell v_m$ be two distinct edges between two vertex-disjoint paths $P_1 = u_1, \ldots, u_s$ and $P_2 = v_1, \ldots, v_t$. We say $u_i v_j$ and $u_\ell v_m$ are *parallel* if either $i \leq \ell$ and $j \leq m$, or $\ell \leq i$ and $m \leq j$. Note if two distinct edges between P_1 and P_2 share an endpoint, then they are parallel. We say two distinct edges between two vertex-disjoint paths are *crossing* if they are not parallel.

Definition 2.4. Let $v_i v_j$ and $v_\ell v_m$ be two distinct edges between vertices of a path $P = v_1, \ldots, v_t$, with $j \geq i + 2$ and $m \geq \ell + 2$. We say $v_i v_j$ and $v_\ell v_m$ are *nested* if either $i \leq \ell < m \leq j$ or $\ell \leq i < j \leq m$.

Definition 2.5. Let $P = v_1, \dots, v_t$ be a path. We say a vertex v_i on P has a *left edge* if there exists an edge v_iv_j for some $j < i - 1$. We also say v_i has a *right edge* if there exists an edge v_iv_j for some $j > i + 1$.

3 Lemmas

Lemma 3.1. *Let $r \geq 1$ be an integer, and let $\mathscr{C} = \{C_1, \dots, C_r\}$ be a minimal set of r vertex-disjoint chorded cycles in a graph G. For any $1 \leq i \leq r$, C_i cannot have two or more parallel chords.*

Proof. This follows easily from the minimality of $\mathscr{C}$. □

Lemma 3.2. *Let $r \geq 1$ be an integer, and let $\mathscr{C} = \{C_1, \dots, C_r\}$ be a minimal set of r vertex-disjoint chorded cycles in a graph G. If $|C_i| \geq 7$ for some $1 \leq i \leq r$, then C_i has at most two chords. Furthermore, if C_i has two chords, then these chords must be crossing.*

Proof. Let $|C_i| \geq 7$ for some $1 \leq i \leq r$. Suppose C_i contains at least three chords. By Lemma 3.1, no two of them can be parallel. Thus they are all mutually crossing. Label the endpoints of these three chords $v_1, v_2, \dots, v_6$ in that order on C_i. Since the chords are mutually crossing, the three chords are given by v_1v_4, v_2v_5, v_3v_6. These six endpoints partition C_i into six intervals $C_i[v_j, v_{j+1}), 1 \leq j \leq 6$, where $v_7 = v_1$. Since $|C_i| \geq 7$, some interval contains at least one vertex of C_i which is not an endpoint of the three chords. Without loss of generality, we may assume $C_i[v_1, v_2)$ contains some vertex of C_i other than v_1. Then $C_i[v_2, v_4], v_1, C_i^-[v_1, v_5], v_2$ is a shorter cycle with chord v_3v_6. Thus C_i has at most two chords. If the C_i has two chords, then these chords must be crossing by Lemma 3.1. □

Lemma 3.3. *Let $r \geq 1$ be an integer, and let $\mathscr{C} = \{C_1, \dots, C_r\}$ be a minimal set of r vertex-disjoint chorded cycles in a graph G. Then $d_{C_i}(x) \leq 4$ for any $1 \leq i \leq r$ and any $x \in V(G) - \cup_{i=1}^r V(C_i)$. Furthermore, for some $C \in \mathscr{C}$ and some $x \in V(G) - \cup_{i=1}^r V(C_i)$, if $d_C(x) = 4$, then $|C| = 4$, and if $d_C(x) = 3$, then $|C| \leq 6$.*

Proof. Suppose $d_C(x) \geq 5$ for some $C \in \mathscr{C}$ and some $x \in V(G) - \cup_{i=1}^r V(C_i)$. Let $v_j \in N_C(x)$ with $1 \leq j \leq 5$, and let $v_1, v_2, \dots, v_5$ be in that order on C. Then $x, C[v_1, v_3], x$ is a shorter cycle with chord xv_2, contradicting the minimality of $\mathscr{C}$. Thus $d_{C_i}(x) \leq 4$ for any $1 \leq i \leq r$ and any $x \in V(G) - \cup_{i=1}^r V(C_i)$.

Next suppose $d_C(x) = 4$ for some $C \in \mathscr{C}$ and some $x \in V(G) - \cup_{i=1}^r V(C_i)$. Let $v_i \in N_C(x)$ with $1 \leq i \leq 4$, and let v_1, v_2, v_3, v_4 be in that order on C. Let $X = \{v_1, v_2, v_3, v_4\}$. These neighbors define four intervals $C[v_i, v_{i+1}), 1 \leq i \leq 4$, where $v_5 = v_1$. Assume $|C| \geq 5$. Then a vertex of $C - X$ lies in one of the intervals. Without loss of generality, we may assume there exists a vertex of $C - X$ in $C[v_1, v_2)$. Then $x, C[v_2, v_4], x$ is a shorter cycle with chord xv_3, contradicting the minimality of $\mathscr{C}$. Thus $|C| = 4$.

Finally, suppose $d_C(x) = 3$ for some $C \in \mathscr{C}$ and some $x \in V(G) - \cup_{i=1}^{r} V(C_i)$. Let $v_i \in N_C(x)$ with $1 \leq i \leq 3$, and let v_1, v_2, v_3 be in that order on C. Let $X = \{v_1, v_2, v_3\}$. These neighbors define three intervals $C[v_i, v_{i+1}), 1 \leq i \leq 3$, where $v_4 = v_1$. If $|C| \geq 7$, then some interval contains at least two vertices of $C - X$. Without loss of generality, we may assume $C[v_1, v_2)$ contains them. Then $x, C[v_2, v_1], x$ is a shorter cycle with chord xv_3, contradicting the minimality of $\mathscr{C}$. Thus $|C| \leq 6$. □

Lemma 3.4. *Suppose there exist at least five edges connecting two vertex-disjoint paths P_1 and P_2. Then there exist at least three mutually parallel edges or at least three mutually crossing edges.*

Proof. Let $x_i y_i \in E(P_1, P_2)$ for each $1 \leq i \leq 5$. Without loss of generality, let $x_1, x_2, \ldots, x_5$ appear in that order on P_1. Also we may assume that y_1, y_5 are in that order on P_2, otherwise, we consider the reverse orientation of P_2. Let $P_2 = u_1, u_2, \ldots, u_s$ $(s \geq 1)$. If $s = 1$, then all the edges connecting P_1 and P_2 are mutually parallel. Thus we may assume that $s \geq 2$. Now we claim that $y_1 \neq u_1$. Suppose not. Then there exist at least two parallel edges in $\{x_i y_i \mid 2 \leq i \leq 5\}$, otherwise, the lemma holds. Let $x_{i_1} y_{i_1}, x_{i_2} y_{i_2}$ for $2 \leq i_1 < i_2 \leq 5$ be the parallel edges. Then $x_1 y_1, x_{i_1} y_{i_1}, x_{i_2} y_{i_2}$ are three mutually parallel edges. Thus the claim holds. By symmetry, $y_5 \neq u_s$. If $y_i \in P_2[y_1, y_5]$ for some $2 \leq i \leq 4$, then $x_1 y_1, x_i y_i, x_5 y_5$ are three mutually parallel edges. Thus $y_i \notin P_2[y_1, y_5]$ for each $2 \leq i \leq 4$. Then $|P_2[u_1, y_1) \cap \{y_2, y_3, y_4\}| \geq 2$ or $|P_2(y_5, u_s] \cap \{y_2, y_3, y_4\}| \geq 2$. By symmetry, we may assume that $|P_2[u_1, y_1) \cap \{y_2, y_3, y_4\}| \geq 2$. Let i_1, i_2 be integers such that $2 \leq i_1 < i_2 \leq 4$ and $y_{i_1}, y_{i_2} \in P_2[u_1, y_1)$. If y_{i_1}, y_{i_2} are in that order on P_2, then $x_{i_1} y_{i_1}, x_{i_2} y_{i_2}$ are parallel edges, and $x_{i_1} y_{i_1}, x_{i_2} y_{i_2}, x_5 y_5$ are three mutually parallel edges. On the other hand, if y_{i_2}, y_{i_1} are in that order on P_2, then $x_{i_1} y_{i_1}, x_{i_2} y_{i_2}$ are crossing edges, and $x_1 y_1, x_{i_1} y_{i_1}, x_{i_2} y_{i_2}$ are three mutually crossing edges. Thus the lemma holds. □

Lemma 3.5. *Suppose there exist at least three mutually parallel edges or at least three mutually crossing edges connecting two vertex-disjoint paths P_1 and P_2. Then there exists a chorded cycle in $\langle P_1 \cup P_2 \rangle$.*

Proof. If there exist at least three mutually crossing edges connecting the paths P_1 and P_2, then we consider the reverse orientation of P_2. Then the edges are all mutually parallel. Thus we have only to consider the case where all the edges are mutually parallel. Now let $x_1 y_1, x_2 y_2, x_3 y_3$ be the edges. Without loss of generality, let x_1, x_2, x_3 appear in that order on P_1. Note that the endpoints y_1, y_2, y_3 appear in that order on P_2. Then $P_1[x_1, x_3], y_3, P_2^-[y_3, y_1], x_1$ is a cycle with chord $x_2 y_2$. □

Lemma 3.6. *Suppose there exist at least five edges connecting two vertex-disjoint paths P_1 and P_2 with $|P_1 \cup P_2| \geq 7$. Then there exists a chorded cycle in $\langle P_1 \cup P_2 \rangle$ not containing at least one vertex of $\langle P_1 \cup P_2 \rangle$.*

Proof. By Lemma 3.4, there must be at least three mutually parallel edges or at least three mutually crossing edges. Then by Lemma 3.5, there exists a chorded cycle C in $\langle P_1 \cup P_2 \rangle$. If $V(C) \neq V(P_1 \cup P_2)$, then the lemma holds. Thus suppose $V(C) = V(P_1 \cup P_2)$. Let C' be a cycle obtained from C by removing all chords. Since $|E(\langle P_1 \cup P_2 \rangle) - E(C')| \geq 3$, C has at least three chords. By $|C| = |P_1 \cup P_2| \geq 7$, a shorter chorded cycle exists in $\langle P_1 \cup P_2 \rangle$ as in the proof of Lemma 3.2. Thus the lemma holds. □

Lemma 3.7. *Let P_1, P_2 be two vertex-disjoint paths, and let u_1, u_2 ($u_1 \neq u_2$) be in that order on P_1. Suppose $d_{P_2}(u_i) \geq 2$ for each $i \in \{1, 2\}$. Then there exists a chorded cycle in $\langle P_1[u_1, u_2] \cup P_2 \rangle$.*

Proof. Let $P_2 = v_1, \ldots, v_t$, and let $v_i, v_j \in N_{P_2}(u_1)$ with $i < j$. If u_2 has a neighbor that lies in $P_2[v_1, v_i]$ or $P_2[v_j, v_t]$, then we can easily form a chorded cycle in $\langle P_1[u_1, u_2] \cup P_2 \rangle$. Thus both of u_2's neighbors in P_2 must lie in $P_2(v_i, v_j)$, call them $v_\ell, v_{\ell'}$ with $\ell < \ell'$. Then $P_1[u_1, u_2], v_{\ell'}, P_2^-[v_{\ell'}, v_i], u_1$ is a cycle with chord $u_2 v_\ell$. □

Lemma 3.8. *Let H be a connected graph of order at least 4. Suppose H contains neither a chorded cycle nor a Hamiltonian path. Let $P_1 = u_1, \ldots, u_s$ ($s \geq 3$) be a longest path in H, and let $P_2 = v_1, \ldots, v_t$ ($t \geq 1$) be a longest path in $H - P_1$. Then the following statements hold.*
(i) $N_{H-P_1}(u_i) = \emptyset$ *for each* $i \in \{1, s\}$.
(ii) $d_H(u_i) = d_{P_1}(u_i) \leq 2$ *for each* $i \in \{1, s\}$.
(iii) $N_{H-(P_1 \cup P_2)}(v_j) = \emptyset$ *for each* $j \in \{1, t\}$.
(iv) $d_{P_2}(v_j) \leq 2$ *for each* $j \in \{1, t\}$.
(v) $u_1 u_s \notin E(H)$.
(vi) *If* $d_H(v_1) \leq d_H(v_t)$, *then* $d_H(\{u_1, u_s, v_1\}) \leq 6$.

Proof. Since P_1 is a longest path, clearly, (i) holds. By (i), $d_H(u_i) = d_{P_1}(u_i)$ for each $i \in \{1, s\}$. Since H does not contain a chorded cycle, $d_{P_1}(u_i) \leq 2$ for each $i \in \{1, s\}$. Thus (ii) holds. Since P_2 is a longest path in $H - P_1$, clearly, (iii) holds. Also, since H does not contain a chorded cycle, (iv) holds. Furthermore, since H is connected and P_1 is a longest path in H, $u_1 u_s \notin E(H)$. Thus (v) holds.

Finally, we prove (vi). Let $X = \{u_1, u_s, v_1\}$. By (ii), $d_H(u_i) \leq 2$ for each $i \in \{1, s\}$. If $d_H(v_1) \leq 2$, then $d_H(X) \leq 6$, and (vi) holds. Thus we may assume $d_H(v_1) \geq 3$. Then $d_H(v_t) \geq 3$ by the assumption. If $t = 1$, then $d_{P_1}(v_1) \geq 3$. Thus there exists a chorded cycle in $\langle v_1 \cup P_1 \rangle$, a contradiction. If $t = 2$, then $d_{P_1}(v_1) \geq 2$ and $d_{P_1}(v_2) \geq 2$ by (iii), and so by Lemma 3.7, there exists a chorded cycle in $\langle P_1 \cup P_2 \rangle$, a contradiction. Thus we may assume $t \geq 3$. By Lemma 3.7, $d_{P_1}(v_j) \leq 1$ for some $j \in \{1, t\}$. Suppose $j = 1$, that is, $d_{P_1}(v_1) \leq 1$. By (iii) and (iv), $d_{P_2}(v_1) = 2$. Since $N_{P_1}(v_\ell) \neq \emptyset$ for each $\ell \in \{1, t\}$ by (iii) and (iv), there exists a cycle with chord adjacent to v_1 in $\langle P_1 \cup P_2 \rangle$, a contradiction. If $j = t$, that is, $d_{P_1}(v_t) \leq 1$, then we get a contradiction as in the case where $j = 1$. Thus (vi) holds. □

Lemma 3.9. *Let H be a graph containing a path P. If there exist nested edges between vertices of P, then H contains a chorded cycle.*

Proof. Let v_1, v_2, v_3, v_4 be in that order on P. Suppose $v_1 v_4$ and $v_2 v_3$ are nested edges. Then $P[v_1, v_4], v_1$ is a cycle with chord $v_2 v_3$. □

Lemma 3.10. *Let H be a graph containing a path $P = v_1, v_2, \ldots, v_t$ $(t \geq 4)$. For any $2 \leq i \leq t-2$, if v_i has a right edge and v_{i+1} has a left edge, then H contains a chorded cycle.*

Proof. Let $v_iv_j \in E(H)$ with $i+2 \leq j \leq t$ and $v_{i+1}v_\ell \in E(H)$ with $1 \leq \ell \leq i-1$. Then $P[v_\ell, v_i], v_j, P^-[v_j, v_{i+1}], v_\ell$ is a cycle with chord v_iv_{i+1}. □

Lemma 3.11. *Let H be a graph containing a path $P = v_1, \ldots, v_t$ $(t \geq 3)$, and not containing a chorded cycle. If $v_1v_i \in E(H)$ for some $i \geq 3$, then $d_P(v_j) \leq 3$ for any $j \leq i-1$ and in particular, $d_P(v_{i-1}) = 2$. And if $v_tv_i \in E(H)$ for some $i \leq t-2$, then $d_P(v_j) \leq 3$ for any $j \geq i+1$ and in particular, $d_P(v_{i+1}) = 2$.*

Proof. Suppose $v_1v_i \in E(H)$ for some $i \geq 3$. No vertex v_j with $j \leq i-1$ has a left edge, otherwise the edge nests with v_1v_i, and by Lemma 3.9, H contains a chorded cycle, a contradiction. Also, no vertex v_j with $j \leq i-1$ has two or more right edges, otherwise the edges nest, and again H contains a chorded cycle, a contradiction. Thus $d_P(v_j) \leq 3$ for any $j \leq i-1$. Furthermore, v_{i-1} cannot have a right edge by Lemma 3.10. Thus $d_P(v_{i-1}) = 2$. By symmetry, the same proof shows that if $v_tv_i \in E(H)$ for some $i \leq t-2$, then $d_P(v_j) \leq 3$ for any $j \geq i+1$ and $d_P(v_{i+1}) = 2$. □

Lemma 3.12. *Let H be a graph containing a path $P = v_1, \ldots, v_t$ $(t \geq 6)$, and not containing a chorded cycle. If $d_P(v_1) = 1$, then $d_P(v_i) = 2$ for some $3 \leq i \leq 5$, or if $v_1v_3 \in E(H)$, then $d_P(v_i) = 2$ for some $4 \leq i \leq 6$.*

Proof. Suppose either $d_P(v_1) = 1$ or $v_1v_3 \in E(H)$. If $d_P(v_1) = 1$, then we let $i = 3$, and if $v_1v_3 \in E(H)$, then we let $i = 4$. Vertex v_i cannot have a left edge, otherwise in the first case, we have $d_P(v_1) = 2$, and in the second case, we get a chorded cycle by Lemmas 3.9 and 3.10. Thus we have a contradiction in either case. If $d_P(v_i) = 2$, then the lemma holds. Thus suppose $d_P(v_i) \geq 3$. Then v_i must have a right edge, say v_iv_j with $j \geq i+2$. If $j = i+2$, then $d_P(v_{i+1}) = 2$, otherwise we get a contradiction by Lemma 3.10. Thus $j > i+2$. By Lemma 3.10, v_{i+1} cannot have a left edge. If $d_P(v_{i+1}) = 2$, then the lemma holds. Thus $d_P(v_{i+1}) \geq 3$, and v_{i+1} has a right edge, say $v_{i+1}v_\ell$ for some $\ell \geq i+3$. If $\ell \leq j$, then we have nested edges and a chorded cycle by Lemma 3.9, a contradiction. Thus $\ell > j$. By the same arguments as for v_{i+1}, either $d_P(v_{i+2}) = 2$, or v_{i+2} has a right edge $v_{i+2}v_{\ell'}$ for some $\ell' > \ell$. In the later case, $P[v_i, v_{i+2}], v_{\ell'}, P^-[v_{\ell'}, v_j], v_i$ is a cycle with chord $v_{i+1}v_\ell$, a contradiction. Thus $d_P(v_{i+2}) = 2$, and the lemma holds. □

Lemma 3.13. *Let H be a graph containing a path $P = v_1, \ldots, v_t$ $(t \geq 6)$, and not containing a chorded cycle. If $d_P(v_t) = 1$, then $d_P(v_i) = 2$ for some $t-4 \leq i \leq t-2$, or if $v_tv_{t-2} \in E(H)$, then $d_P(v_i) = 2$ for some $t-5 \leq i \leq t-3$.*

Proof. The lemma follows from the proof of Lemma 3.12 by symmetry. □

Lemma 3.14. *Let H be a graph of order at least 13. Suppose H does not contain a chorded cycle. If H contains a Hamiltonian path, then there exists an independent set X of four vertices in H such that $d_H(X) \leq 8$.*

Remark 3. We consider the following graph H of order 12. (See Fig. 1.) Then H satisfies all the conditions except for the order in Lemma 3.14. However, H does not contain an independent set X of four vertices such that $d_H(X) \leq 8$. Thus $|H| \geq 13$ is necessary.

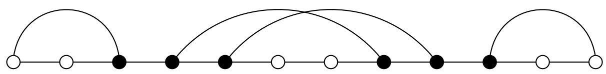

Fig. 1. The graph H of order 12. The white vertex ($\circ$) shows degree 2, and the black vertex ($\bullet$) shows degree 3.

Proof. Let $P = v_1, \dots, v_t$ ($t \geq 13$) be a Hamiltonian path in H. If $v_1v_t \in E(H)$, then $d_H(v) = 2$ for any $v \in V(H)$, otherwise, a chorded cycle exists in H, a contradiction. Then $X = \{v_1, v_3, v_5, v_7\}$ is an independent set of four vertices such that $d_H(X) = 8$. Thus we may now assume $v_1v_t \notin E(H)$. Since P is a Hamiltonian path in H, note $d_P(v) = d_H(v)$ for any $v \in V(P)$. Also, $d_H(v_1) \leq 2$ and $d_H(v_t) \leq 2$ by Lemma 3.9.

Case 1. Suppose $d_H(v_1) = 1$ and $d_H(v_t) = 1$.

By Lemmas 3.12 and 3.13, $d_H(v_i) = 2$ for some $3 \leq i \leq 5$ and $d_H(v_j) = 2$ for some $t-4 \leq j \leq t-2$. Since $t \geq 13$, $v_iv_j \notin E(H)$. Thus $X = \{v_1, v_i, v_j, v_t\}$ is the desired set.

Case 2. Suppose $d_H(v_1) = 1$ and $d_H(v_t) = 2$, or $d_H(v_1) = 2$ and $d_H(v_t) = 1$.

In this case, we may assume $d_H(v_1) = 1$ and $d_H(v_t) = 2$, otherwise, we consider the reverse orientation of P. Let $v_tv_j \in E(H)$ for some $2 \leq j \leq t-2$. Suppose $2 \leq j \leq t-5$. Since $d_H(v_t) = 2$, $v_{j+1}v_t \notin E(H)$ and $v_{j+3}v_t \notin E(H)$. By Lemma 3.11, $d_H(v_{j+1}) = 2$ and $d_H(v_{j+3}) \leq 3$. Then $X = \{v_1, v_{j+1}, v_{j+3}, v_t\}$ is the desired set. Thus $t-4 \leq j \leq t-2$. By Lemma 3.12, $d_H(v_i) = 2$ for some $3 \leq i \leq 5$. If $j \in \{t-4, t-3\}$, then v_{j+1} is still non-adjacent to v_t and $d_H(v_{j+1}) = 2$ by Lemma 3.11. Since $t \geq 13$, $v_iv_{j+1} \notin E(H)$. Then $X = \{v_1, v_i, v_{j+1}, v_t\}$ is the desired set. Thus $j = t-2$. By Lemma 3.13, $d_H(v_\ell) = 2$ for some $t-5 \leq \ell \leq t-3$. Since $t \geq 13$, $v_iv_\ell \notin E(H)$. Then $X = \{v_1, v_i, v_\ell, v_t\}$ is the desired set.

Case 3. Suppose $d_H(v_1) = 2$ and $d_H(v_t) = 2$.

Suppose $v_1v_3 \in E(H)$ or $v_tv_{t-2} \in E(H)$. Then we may assume $v_1v_3 \in E(H)$, otherwise, we consider the reverse orientation of P. By Lemma 3.12, $d_H(v_i) = 2$ for some $4 \leq i \leq 6$. If $v_tv_{t-2} \in E(H)$, then $d_H(v_j) = 2$ for some $t-5 \leq j \leq t-3$ by Lemma 3.13. As before, since $t \geq 13$, $v_iv_j \notin E(H)$. Then $X = \{v_1, v_i, v_j, v_t\}$ is the desired set. Thus $v_tv_{t-2} \notin E(H)$. Then $v_tv_s \in E(H)$ for some $s \leq t-3$. By Lemma 3.11, $d_H(v_{s+1}) = 2$. Note $s \geq 3$ since $v_1v_3 \in E(H)$. If $v_{s+1} \notin \{v_{i-1}, v_i, v_{i+1}\}$, then $X = \{v_1, v_i, v_{s+1}, v_t\}$ is the desired set. Thus $v_{s+1} \in \{v_{i-1}, v_i, v_{i+1}\}$. This implies that $v_s \in \{v_{i-2}, v_{i-1}, v_i\}$. Note $v_s \neq v_i$ since $v_tv_s \in E(H)$ and $d_H(v_i) = 2$.

Thus $v_s \in \{v_{i-2}, v_{i-1}\}$. Since $v_i \in \{v_4, v_5, v_6\}$ and $s \geq 3$, $v_s \in \{v_3, v_4, v_5\}$. If $d_H(v) = 2$ for some $v \in \{v_{s+4}, v_{s+5}\}$, then $X = \{v_1, v_i, v, v_t\}$ is the desired set. Thus $d_H(v) \geq 3$ for each $v \in \{v_{s+4}, v_{s+5}\}$. Furthermore, neither v_{s+4} nor v_{s+5} has a right edge, otherwise, this edge nests with $v_s v_t$, and H contains a chorded cycle by Lemma 3.9, a contradiction. Thus both v_{s+4} and v_{s+5} have left edges. It follows that $v_{s+4}v_\ell, v_{s+5}v_{\ell'} \in E(H)$, and then $\ell < \ell' < s$, otherwise, we have nested edges and a chorded cycle by Lemma 3.9, a contradiction. Then $P[v_\ell, v_s], v_t, P^-[v_t, v_{s+4}], v_\ell$ is a cycle with chord $v_{\ell'}v_{s+5}$, a contradiction.

Suppose $v_1v_3 \notin E(H)$ and $v_tv_{t-2} \notin E(H)$. Then $v_1v_i \in E(H)$ for some $4 \leq i \leq t-1$ and $v_tv_j \in E(H)$ for some $2 \leq j \leq t-3$. Note $i \neq j+1$, otherwise, H contains a cycle with chord v_jv_{j+1}, a contradiction. By Lemma 3.11, $d_H(v_{i-1}) = 2$ and $d_H(v_{j+1}) = 2$. If $i \notin \{j+2, j+3\}$, then $X = \{v_1, v_{i-1}, v_{j+1}, v_t\}$ is the desired set. Thus $i \in \{j+2, j+3\}$. Now we claim that $d_H(v_{\ell_1}) = 2$ for some $\ell_1 \in \{3, 4\}$. If $j \in \{2, 3\}$, then $d_H(v_{j+1}) = 2$ by Lemma 3.11. Suppose $4 \leq j \leq t-3$. If $d_H(v_3) \geq 3$, then $v_3v_{i'} \in E(H)$ for some $i' > i$ by Lemma 3.9. Then $P[v_1, v_j], v_t, P^-[v_t, v_i], v_1$ is a cycle with chord $v_3v_{i'}$, a contradiction. Thus $d_H(v_3) = 2$. In all cases, the claim holds. By symmetry, $d_H(v_{\ell_2}) = 2$ for some $\ell_2 \in \{t-3, t-2\}$. Then $X = \{v_1, v_{\ell_1}, v_{\ell_2}, v_t\}$ is the desired set. Thus Lemma 3.14 holds. □

Lemma 3.15. *Let $k \geq 2$ be an integer, and let G be a graph. Suppose G does not contain k vertex-disjoint chorded cycles. Let $\{C_1, \ldots, C_{k-1}\}$ be a minimal set of $k-1$ vertex-disjoint chorded cycles in G, $H = G - \mathscr{C}$, where $\mathscr{C} = \cup_{i=1}^{k-1} C_i$, and $X \subseteq V(H)$ with $|X| = 4$. Suppose H contains a Hamiltonian path. Then $d_{C_i}(X) \leq 12$ for each $1 \leq i \leq k-1$.*

Proof. Suppose not, then $d_{C_i}(X) \geq 13$ for some $1 \leq i \leq k-1$. Let $X = \{x_1, x_2, x_3, x_4\}$. By Lemma 3.3, $d_{C_i}(x_j) \leq 4$ for each $1 \leq j \leq 4$. Now we consider degree sequences defined in Section 1 (Introduction) from four vertices of X to C_i. Recall that when we write (d_1, d_2, d_3, d_4), we assume $d_{C_i}(x_j) = d_j$ for each $1 \leq j \leq 4$, since it is sufficient to consider the case of equality. Without loss of generality, we may assume $d_{C_i}(x_1) \geq d_{C_i}(x_2) \geq d_{C_i}(x_3) \geq d_{C_i}(x_4)$. Then the possible degree sequences from X to C_i are $(4, 4, 4, 1)$, $(4, 4, 3, 2)$ or $(4, 3, 3, 3)$. Since $d_{C_i}(x_1) = 4$, $|C_i| = 4$ by Lemma 3.3. Let $C_i = v_1, v_2, v_3, v_4, v_1$. We show the existence of two vertex-disjoint chorded cycles in $\langle H \cup C_i \rangle$, and then G contains k vertex-disjoint chorded cycles, a contradiction. Now we consider the following three cases based on the degree sequences.

Case 1. The sequence is $(4, 4, 4, 1)$.

Then $d_{C_i}(x_j) = 4$ for each $1 \leq j \leq 3$ and $d_{C_i}(x_4) = 1$. Without loss of generality, we may assume $x_4v_1 \in E(G)$. Since H is connected, there exists a path from x_4 to some other $x \in X$ not containing $X - \{x_4, x\}$. Without loss of generality, we may assume there exists a path P in H connecting x_4 and x_3. Since $d_{C_i}(x_3) = 4$, $v_1, v_2 \in N_{C_i}(x_3)$. Then $x_4, v_1, v_2, x_3, P[x_3, x_4]$ is a cycle with chord x_3v_1. For each $j \in \{1, 2\}$, since $d_{C_i}(x_j) = 4$, $v_3, v_4 \in N_{C_i}(x_j)$. Then x_1, v_3, x_2, v_4, x_1 is the other cycle with chord v_3v_4. Thus we have two vertex-disjoint chorded cycles in $\langle H \cup C_i \rangle$, a contradiction.

Case 2. The sequence is $(4, 4, 3, 2)$.

Then $d_{C_i}(x_1) = d_{C_i}(x_2) = 4$, $d_{C_i}(x_3) = 3$, and $d_{C_i}(x_4) = 2$. Since H is connected, there exists a path P from x_4 to some other $x \in X$ not containing $X - \{x_4, x\}$.

First suppose $x = x_3$, that is, the path P connects x_4 and x_3. Since $d_{C_i}(x_3) = 3$, without loss of generality, we may assume $v_j \in N_{C_i}(x_3)$ for each $1 \le j \le 3$. Assume $v_1 \in N_{C_i}(x_4)$. Then $P[x_3, x_4], v_1, v_2, x_3$ is a cycle with chord x_3v_1. For each $j \in \{1, 2\}$, since $d_{C_i}(x_j) = 4$, $v_3, v_4 \in N_{C_i}(x_j)$. Then x_1, v_3, x_2, v_4, x_1 is the other cycle with chord v_3v_4. Thus we have two vertex-disjoint chorded cycles in $\langle H \cup C_i\rangle$, a contradiction. Hence $v_1 \notin N_{C_i}(x_4)$. Similarly, $v_3 \notin N_{C_i}(x_4)$ by symmetry. Since $d_{C_i}(x_4) = 2$, $v_2 \in N_{C_i}(x_4)$. Then $P[x_3, x_4], v_2, v_1, x_3$ is a cycle with chord x_3v_2. Since $v_3, v_4 \in N_{C_i}(x_j)$ for each $j \in \{1, 2\}$, x_1, v_3, x_2, v_4, x_1 is the other cycle with chord v_3v_4. Thus we have two vertex-disjoint chorded cycles in $\langle H \cup C_i\rangle$, a contradiction.

Next suppose $x = x_1$ (or x_2), that is, the path P connects x_4 and x_1 (or x_2). Without loss of generality, we may assume P connects x_4 and x_1. Since $d_{C_i}(x_3) = 3$, without loss of generality, we may assume $v_j \in N_{C_i}(x_3)$ for each $1 \le j \le 3$. Assume $v_1 \in N_{C_i}(x_4)$. Since $d_{C_i}(x_1) = 4$, $v_1, v_4 \in N_{C_i}(x_1)$. Then $P[x_1, x_4], v_1, v_4, x_1$ is a cycle with chord x_1v_1. Since $d_{C_i}(x_2) = 4$, $v_2, v_3 \in N_{C_i}(x_2)$. Then x_2, v_2, x_3, v_3, x_2 is the other cycle with chord v_2v_3. Thus we have two vertex-disjoint chorded cycles in $\langle H \cup C_i\rangle$, a contradiction. Hence $v_1 \notin N_{C_i}(x_4)$. Similarly, $v_3 \notin N_{C_i}(x_4)$ by symmetry. Since $d_{C_i}(x_4) = 2$, $v_4 \in N_{C_i}(x_4)$, and since $d_{C_i}(x_1) = 4$, $v_3, v_4 \in N_{C_i}(x_1)$. Then $P[x_1, x_4], v_4, v_3, x_1$ is a cycle with chord x_1v_4. Since $d_{C_i}(x_2) = 4$, $v_1, v_2 \in N_{C_i}(x_2)$. Then x_2, v_1, x_3, v_2, x_2 is the other cycle with chord v_1v_2. Thus we have two vertex-disjoint chorded cycles in $\langle H \cup C_i\rangle$, a contradiction.

Case 3. The sequence is $(4, 3, 3, 3)$.

Then $d_{C_i}(x_1) = 4$ and $d_{C_i}(x_j) = 3$ for each $2 \le j \le 4$. Since H contains a Hamiltonian path by the assumption, we let P be the Hamiltonian path. We may assume the order of x_1, x_2, x_3, x_4 on P is either x_1, x_2, x_3, x_4 or x_2, x_1, x_3, x_4, otherwise we consider the reverse orientation of P. Since $d_{C_i}(x_4) = 3$, the vertex x_4 is adjacent to at least two consecutive vertices on C_i. Without loss of generality, we may assume $v_1, v_2 \in N_{C_i}(x_4)$. Since $d_{C_i}(x_3) = 3$, without loss of generality, we may assume $v_1 \in N_{C_i}(x_3)$. Then $P[x_3, x_4], v_2, v_1, x_3$ is a cycle with chord x_4v_1.

Next we prove that if x_1, x_2 (resp. x_2, x_1) are in that order on P, then there exists the other chorded cycle in $\langle P[x_1, x_2] \cup \{v_3, v_4\}\rangle$ (resp. $\langle P[x_2, x_1] \cup \{v_3, v_4\}\rangle$). Suppose that x_1, x_2 are in that order on P. (If x_2, x_1 are in that order on P, then we consider the reverse orientation of $P[x_2, x_1]$.) Since $d_{C_i}(x_1) = 4$, $v_3, v_4 \in N_{C_i}(x_1)$, and since $d_{C_i}(x_2) = 3$, $v_\ell \in N_{C_i}(x_2)$ for some $\ell \in \{3, 4\}$. If $v_3 \in N_{C_i}(x_2)$, then $P[x_1, x_2], v_3, v_4, x_1$ is the other cycle with chord x_1v_3. If $v_4 \in N_{C_i}(x_2)$, then $P[x_1, x_2], v_4, v_3, x_1$ is the other cycle with chord x_1v_4. Thus we have two vertex-disjoint chorded cycles in $\langle H \cup C_i\rangle$, a contradiction. □

4 Proof of Theorem 1.3

Suppose G does not contain a chorded cycle.

Claim 4.1. *G is connected.*

Proof. Suppose not, then $\mathrm{comp}(G) \geq 2$. Let $G_1, G_2, \ldots, G_{\mathrm{comp}(G)}$ be the components of G. First suppose $\mathrm{comp}(G) \geq 3$. By Theorem 1.1, there exists $x_i \in V(G_i)$ for each $1 \leq i \leq 3$ such that $d_{G_i}(x_i) \leq 2$. Then $X = \{x_1, x_2, x_3\}$ is an independent set and $d_G(X) \leq 6$. This contradicts the $\sigma_3(G)$ condition. Next suppose $\mathrm{comp}(G) = 2$. Without loss of generality, we may assume $|G_1| \geq |G_2|$. Since $|G| \geq 7$, $|G_1| \geq 4$. If G_1 is complete, then G_1 contains a chorded cycle. Thus G_1 is not complete. By Theorem 1.2, there exist non-adjacent $x_0, x_1 \in V(G_1)$ such that $d_{G_1}(\{x_0, x_1\}) \leq 4$. On the other hand, by Theorem 1.1, there exists $x_2 \in V(G_2)$ such that $d_{G_2}(x_2) \leq 2$. Then $X = \{x_0, x_1, x_2\}$ is an independent set and $d_G(X) \leq 6$. This contradicts the $\sigma_3(G)$ condition. Thus Claim 4.1 holds. □

Let $P_1 = u_1, \ldots, u_s$ be a longest path in G. Note $s \geq 3$ since $|G| \geq 7$ and G is connected by Claim 4.1.

Claim 4.2. *G contains a Hamiltonian path.*

Proof. Suppose not, then P_1 is not a Hamiltonian path in G. Thus $V(G - P_1) \neq \emptyset$. Let $P_2 = v_1, \ldots, v_t$ $(t \geq 1)$ be a longest path in $G - P_1$. Without loss of generality, we may assume $d_G(v_1) \leq d_G(v_t)$. Let $X = \{u_1, u_s, v_1\}$. By Lemma 3.8 (i), (v), and (vi), X is an independent set and $d_G(X) \leq 6$. This contradicts the $\sigma_3(G)$ condition. Thus Claim 4.2 holds. □

By Claim 4.2, P_1 is a Hamiltonian path in G. Note $s = |G| \geq 7$. If $u_1u_s \in E(G)$, then $d_G(u) = 2$ for any $u \in V(G)$, otherwise a chorded cycle exists in G, a contradiction. Then $X = \{u_1, u_3, u_5\}$ is an independent set and $d_G(X) = 6$. This contradicts the $\sigma_3(G)$ condition. Thus $u_1u_s \notin E(G)$. Since P_1 is a Hamiltonian path in G, note $d_{P_1}(u) = d_G(u)$ for any $u \in V(P_1)$. We also note $d_{P_1}(u_i) \leq 2$ for each $i \in \{1, s\}$. Suppose $d_{P_1}(u_1) = 1$. By Lemma 3.12, $d_G(u_i) = 2$ for some $3 \leq i \leq 5$. Since $s \geq 7$, $X = \{u_1, u_i, u_s\}$ is an independent set and $d_G(X) \leq 6$, a contradiction. Thus $d_{P_1}(u_1) = 2$. Now suppose $u_1u_3 \in E(G)$. By Lemma 3.12, $d_G(u_i) = 2$ for some $4 \leq i \leq 6$. If $s \geq 8$, then $X = \{u_1, u_i, u_s\}$ is an independent set and $d_G(X) \leq 6$, a contradiction. Thus $s = 7$. Then $d_G(u_j) \geq 3$ for each $j \in \{4, 5\}$, otherwise we get a contradiction, since $X = \{u_1, u_j, u_7\}$ for some $j \in \{4, 5\}$ would be an independent set with $d_G(X) \leq 6$. Thus $d_G(u_6) = 2$ by Lemma 3.12. Since u_4 does not have a left edge by Lemmas 3.9 and 3.10, u_4 must have a right edge. Since $d_G(u_6) = 2$, $u_4u_7 \in E(G)$. By Lemma 3.11, $d_G(u_5) = 2$, a contradiction. Thus $u_1u_3 \notin E(G)$, that is, $u_1u_i \in E(G)$ for some $4 \leq i \leq s - 1$. By Lemma 3.11, $d_G(u_{i-1}) = 2$. Then $X = \{u_1, u_{i-1}, u_s\}$ is an independent set and $d_G(X) \leq 6$, a contradiction. This completes the proof of Theorem 1.3. □

5 Proof of Theorem 1.4

By Theorem 1.3, we may assume $k \geq 2$. Suppose Theorem 1.4 does not hold. Let G be an edge-maximal counter-example. If G is complete, then G contains k vertex-disjoint chorded cycles. Thus we may assume G is not complete. Let $xy \notin E(G)$ for some $x, y \in V(G)$, and define $G' = G + xy$, the graph obtained from G by adding the edge xy. Since G' is not a counter-example by the edge-maximality of G, G' contains k vertex-disjoint chorded cycles $C_1, \ldots, C_k$. Without loss of generality, we may assume $xy \notin \cup_{i=1}^{k-1} E(C_i)$, that is, G contains $k-1$ vertex-disjoint chorded cycles. Over all sets of $k-1$ vertex-disjoint chorded cycles in G, choose $C_1, \ldots, C_{k-1}$ with $\mathscr{C} = \cup_{i=1}^{k-1} C_i$, $H = G - \mathscr{C}$, and with P_1 be a longest path in H, such that

(A1) $|\mathscr{C}|$ is as small as possible,

(A2) subject to (A1), $\text{comp}(H)$ is as small as possible, and,

(A3) subject to (A1) and (A2), $|P_1|$ is as large as possible.

We may assume H does not contain a chorded cycle, otherwise G contains k vertex-disjoint chorded cycles, a contradiction.

Claim 5.1. *H has order at least* 13.

Proof. Suppose $|H| \leq 12$. First suppose $|C_i| \leq 8$ for each $1 \leq i \leq k-1$. Since by assumption, $|G| \geq 8k+5$, it follows that $|H| \geq (8k+5) - 8(k-1) = 13$, a contradiction. Thus $|C_i| \geq 9$ for some $1 \leq i \leq k-1$. Without loss of generality, we may assume C_1 is a longest cycle in $\mathscr{C}$. Then $|C_1| \geq 9$. By Lemma 3.2, C_1 has at most two chords, and if C_1 has two chords, then these chords must be crossing. For integers t and r, let $|C_1| = 3t + r$, where $t \geq 3$ and $0 \leq r \leq 2$.

Subclaim 5.1.1. *The cycle C_1 contains t (≥ 3) vertex-disjoint sets $X_1, \ldots, X_t$ of three independent vertices each in G such that $d_{C_1}(\cup_{i=1}^{t} X_i) \leq 6t + 4$.*

Proof. For any $3t$ vertices of C_1, their degree sum in C_1 is at most $3t \times 2 + 4 = 6t + 4$, since C_1 has at most two chords. Thus it only remains to show that C_1 contains t vertex-disjoint sets of three independent vertices each. Start anywhere on C_1 and label the first $3t$ vertices of C_1 with labels 1 through t in order, starting over again with 1 after using label t. If $r \geq 1$, label the remaining r vertices of C_1 with the labels $t+1, \ldots, t+r$. (See Fig. 2.) The labeling above yields t vertex-disjoint sets of three vertices each, where all the vertices labeled with 1 are one set, all the vertices labeled with 2 are another set, and so on. Given this labeling, since $t \geq 3$, any vertex x in C_1 has a different label than x^- and x^+. Let C_0 be the cycle obtained from C_1 by removing all chords. Then the vertices in each of the t sets are independent in C_0. Thus the only way vertices in the same set are not independent in C_1 is if the endpoints of a chord of C_1 were given the same label. Note any vertex labeled i is distance at least 3 in C_0 from any other vertex labeled i. Thus even if we exchange the label of x in C_0 for the one of x^- (or x^+), the vertices in each of the resulting t sets are still independent in C_0.

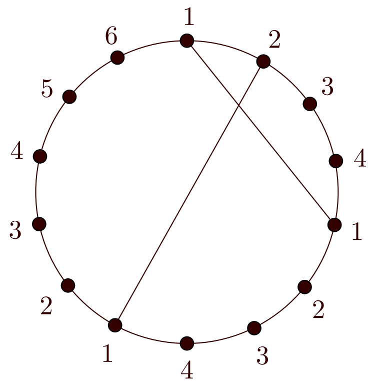

Fig. 2. An example when $t = 4$ and $r = 2$.

Case 1. No chord of C_1 has both endpoints with the same label.

Then there exist t vertex-disjoint sets of three independent vertices each in C_1.

Case 2. Exactly one chord of C_1 has both endpoints with the same label.

Recall that C_1 has at most two chords, and if C_1 has two chords, then these chords must be crossing. Since $|C_1| \geq 9$, even if C_1 has two chords, each chord has an endpoint x such that there exists some vertex $x' \in \{x^-, x^+\}$ which is equal to no endpoint of the other chord. Choose such an endpoint x of the chord whose endpoints were assigned the same label, and exchange the label of x for the one of x'. Then no chord of C_1 has endpoints with the same label, and the vertices in each of the resulting t sets are independent in C_1. Thus there exist t vertex-disjoint sets of three independent vertices each in C_1.

Case 3. Two chords of C_1 each have both endpoints with the same label.

Then the two chords are crossing. Since endpoints of a chord have the same label in this case, recall that these endpoints have distance at least 3. Suppose there is an endpoint x of one chord of C_1 which is adjacent to an endpoint $y\,(= x^+)$ of the other chord on C_1. (See Fig. 3 (a).) Now we exchange the label of x for the one of y. Then no chord of C_1 has endpoints with the same label, and the vertices in each of the resulting t sets are independent in C_1. Thus there exist t vertex-disjoint sets of three independent vertices each in C_1.

Suppose no endpoint of one chord of C_1 is adjacent to an endpoint of the other chord on C_1. (See Fig. 3 (b).) Let x_1x_2, y_1y_2 be the two distinct chords of C_1. Since the two chords are crossing, without loss of generality, we may assume x_1, y_1, x_2, y_2 are in that order on C_1. Now we exchange the labels of x_1 and x_1^+, and next the ones of y_2 and y_2^-. Then no chord of C_1 has endpoints with the same label, and the vertices in each of the resulting t sets are independent in C_1. Thus there exist t vertex-disjoint sets of three independent vertices each in C_1. □

Since $|C_1| \geq 9$, $d_{C_1}(v) \leq 2$ for any $v \in V(H)$ by (A1) and Lemma 3.3. Thus, since $|H| \leq 12$ by our assumption, it follows that $|E(H, C_1)| \leq 24$. Let $X_1, \ldots, X_t$ be as in Subclaim 5.1.1, and let $\mathscr{X} = X_1 \cup \cdots \cup X_t$. By the $\sigma_3(G)$ condition,

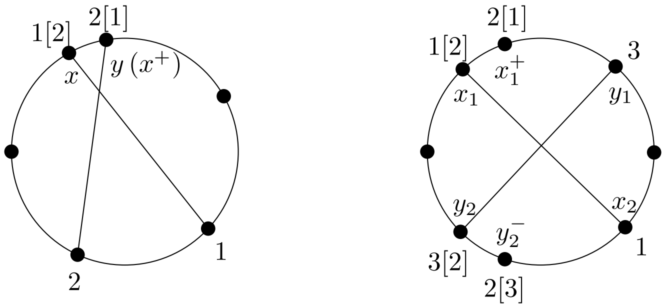

Fig. 3. Examples: (a) – the labels of x and y are 1 and 2, (b) – the labels of x_1 and y_2 are 1 and 3. ([i] means i is a new label for a vertex after the exchange.)

$d_G(\mathscr{X}) \geq t(9k-2)$. Suppose $k = 2$. Then $\mathscr{C}$ has only one cycle C_1. Since $k = 2$ and $t \geq 3$, $|E(C_1, H)| \geq d_H(\mathscr{X}) \geq t(9k-2) - (6t+4) = 10t - 4 \geq 26$, a contradiction.

Now suppose $k \geq 3$. Then we have

$$\begin{aligned} |E(\mathscr{X}, \mathscr{C} - C_1)| &= d_G(\mathscr{X}) - d_{C_1}(\mathscr{X}) - d_H(\mathscr{X}) \\ &\geq t(9k-2) - (6t+4) - 24 \\ &= 9kt - 8t - 28, \end{aligned}$$

and since $t \geq 3$,

$$\begin{aligned} 9kt - 8t - 28 &= 9t(k-1) + t - 28 \geq 9t(k-1) - 25 \\ &> 9t(k-1) - 9t \\ &= 9t(k-2). \end{aligned}$$

Thus $|E(\mathscr{X}, C')| > 9t$ for some C' in $\mathscr{C} - C_1$, since $\mathscr{C} - C_1$ contains $k-2$ vertex-disjoint chorded cycles. Let $h = \max\{d_{C'}(v) \,|\, v \in \mathscr{X}\}$. Let v^* be a vertex of $\mathscr{X}$ such that $d_{C'}(v^*) = h$. If $h \leq 3$, then $|E(\mathscr{X}, C')| \leq 3 \times 3t = 9t$, a contradiction. Thus $h \geq 4$. By the maximality of C_1, $|C'| \leq |C_1| = 3t + r$. It follows that $h = d_{C'}(v^*) \leq |C'| \leq 3t + r$. Recall $t \geq 3$ and $0 \leq r \leq 2$. Then

$$\begin{aligned} |E(\mathscr{X} - \{v^*\}, C')| &\geq (9t+1) - d_{C'}(v^*) \geq (9t+1) - (3t+r) \\ &= 6t - r + 1 \geq 17. \end{aligned} \tag{1}$$

Since $h = d_{C'}(v^*) \geq 4$, let v_1, v_2, v_3, v_4 be neighbors of v^* in that order on C'. Note v_1, v_2, v_3, v_4 partition C' into four intervals $C'[v_i, v_{i+1})$ for all $1 \leq i \leq 4$, where $v_5 = v_1$. By (1), there exist at least 17 edges from $C_1 - v^*$ to C'. Thus $C'[v_i, v_{i+1})$ for some $1 \leq i \leq 4$ contains at least five of these edges. Without loss of generality, we may assume $i = 4$, that is, $C'[v_4, v_1)$. Then by Lemma 3.6, $\langle (C_1 - v^*) \cup C'[v_4, v_1) \rangle$ contains a chorded cycle not containing at least one vertex of $\langle (C_1 - v^*) \cup C'[v_4, v_1) \rangle$. Note $v^*, C'[v_1, v_3], v^*$ is a cycle with chord v^*v_2, and it uses no vertices from $C'[v_4, v_1)$. Thus we have two shorter vertex-disjoint chorded cycles in $\langle C_1 \cup C' \rangle$, contradicting (A1). Hence Claim 5.1 holds. □

Claim 5.2. *H is connected.*

Proof. Suppose not. First we prove the following subclaim.

Subclaim 5.2.1. *Let X be an independent set of three vertices in H such that $d_H(X) \leq 6$. Then there exists some C in $\mathscr{C}$ such that the degree sequences from the vertices of X to C are $(4, 4, 2)$ or $(4, 3, 3)$. Furthermore, then $|C| = 4$.*

Proof. By the $\sigma_3(G)$ condition, $d_{\mathscr{C}}(X) \geq (9k-2) - 6 = 9k - 8 > 9(k-1)$. Thus there exists some C in $\mathscr{C}$ such that $d_C(X) \geq 10$. By Lemma 3.3, $d_C(x) \leq 4$ for any $x \in X$. It follows that the degree sequences from three vertices of X to C are $(4, 4, 2)$ or $(4, 3, 3)$. Then by Lemma 3.3, $|C| = 4$. □

Now we consider the following two cases based on comp(H).

Case 1. Suppose comp$(H) \geq 3$.

Let H_1, H_2, H_3 be three distinct components of H. For each $1 \leq i \leq 3$, let x_i be an endpoint of a longest path in H_i. Since H does not contain a chorded cycle, $d_{H_i}(x_i) \leq 2$ for each $1 \leq i \leq 3$. Note x_i for each $1 \leq i \leq 3$ is not a cutvertex of H_i, since x_i is an endpoint of a longest path. Then $X = \{x_1, x_2, x_3\}$ is an independent set and $d_H(X) \leq 6$. By Subclaim 5.2.1, the degree sequences from three vertices of X to some C in $\mathscr{C}$ are $(4, 4, 2)$ or $(4, 3, 3)$, and $|C| = 4$. Without loss of generality, we may assume $d_C(x_1) \geq d_C(x_2) \geq d_C(x_3)$. Let $C = v_1, v_2, v_3, v_4, v_1$. By the degree sequences, x_2 and x_3 have a common neighbor in C. Without loss of generality, we may assume $v_4 \in N_C(x_2) \cap N_C(x_3)$. Then $\langle H_2 \cup H_3 \cup v_4 \rangle$ is connected. Since $d_C(x_1) = 4$, $v_i \in N_C(x_1)$ for each $1 \leq i \leq 3$. Then $C' = x_1, v_1, v_2, v_3, x_1$ is a 4-cycle with chord x_1v_2. Replacing C in $\mathscr{C}$ by C', we consider the new H'. Since $H_1 - x_1$ is connected, comp$(H') \leq$ comp$(H) - 1$. This contradicts (A2).

Case 2. Suppose comp$(H) = 2$.

Let H_1, H_2 be two distinct components of H. Recall P_1 is a longest path in H. Without loss of generality, we may assume H_1 contains P_1. Let $P_1 = u_1, \ldots, u_s$. Then $|H_1| \geq |P_1| = s$. By Claim 5.1, $|H| \geq 13$. Thus $|H_i| \geq 7$ for some $i \in \{1, 2\}$. Since H_i is connected, there exists a path of order at least 3 in H_i. Thus $s \geq 3$, since P_1 is a longest path in H. Also, we let $P_2 = v_1, \ldots, v_t$ $(t \geq 1)$ be a longest path in H_2. Since P_i for each $i \in \{1, 2\}$ is a longest path in H_i, $d_{H_1}(u_j) = d_{P_1}(u_j) \leq 2$ for each $j \in \{1, s\}$ and $d_{H_2}(v_\ell) = d_{P_2}(v_\ell) \leq 2$ for each $\ell \in \{1, t\}$. Let $X = \{u_1, u_s, v_1\}$. Then $d_H(X) \leq 6$.

First suppose $u_1u_s \notin E(H_1)$. Then X is an independent set. By Subclaim 5.2.1, the degree sequences from three vertices of X to some C in $\mathscr{C}$ are $(4, 4, 2)$ or $(4, 3, 3)$, and $|C| = 4$. Without loss of generality, we may assume $d_C(u_1) \geq d_C(u_s)$. Let $C = x_1, x_2, x_3, x_4, x_1$.

Suppose the degree sequence is $(4, 4, 2)$. By the degree sequence, since u_s and v_1 have a common neighbor in C, without loss of generality, we may assume $x_4 \in N_C(u_s) \cap N_C(v_1)$. Note u_1 is not a cutvertex of H_1, since u_1 is an endpoint of a longest path. Thus $H_1 - u_1$ is connected, and $\langle (H_1 - u_1) \cup H_2 \cup x_4 \rangle$ is also connected.

Since $d_C(u_1) = 4$, $x_j \in N_C(u_1)$ for each $1 \leq j \leq 3$. Then $C' = u_1, x_1, x_2, x_3, u_1$ is a 4-cycle with chord u_1x_2. Replacing C in $\mathscr{C}$ by C', we consider the new H'. Then $\mathrm{comp}(H') \leq \mathrm{comp}(H) - 1 = 2 - 1 = 1$. This contradicts (A2).

Suppose the degree sequence is $(4, 3, 3)$. If $d_C(u_1) = 4$ and $d_C(u_s) = d_C(v_1) = 3$, then we get a contradiction similar to the case where $(4, 4, 2)$. Thus $d_C(u_1) = d_C(u_s) = 3$ and $d_C(v_1) = 4$. Without loss of generality, we may assume $x_1 \in N_C(u_1)$. Since $d_C(v_1) = 4$, $x_i \in N_C(v_1)$ for each $2 \leq i \leq 4$. Then $C' = v_1, x_2, x_3, x_4, v_1$ is a 4-cycle with chord v_1x_3. Replacing C in $\mathscr{C}$ by C', we consider the new H'. Assume $|H_2| = 1$. Then $\mathrm{comp}(H') = 1$, a contradiction. Thus $|H_2| \geq 2$. Note $H_2 - v_1$ is connected. By (A2), $\mathrm{comp}(H') = \mathrm{comp}(H)$. Then $x_1, P_1[u_1, u_s]$ is a longer path than P_1 in H'. This contradicts (A3).

Next suppose $u_1u_s \in E(H_1)$. Since H_1 is connected and P_1 is a longest path, $C_1 = P_1[u_1, u_s], u_1$ is a Hamiltonian cycle. Assume $s \geq 4$. Let $X = \{u_1, u_3, v_1\}$. Since H_1 does not contain a chorded cycle, $u_1u_3 \notin E(H_1)$ and $d_{H_1}(u_i) = 2$ for each $i \in \{1, 3\}$. Thus X is an independent set and $d_H(X) \leq 6$. Now, letting u_3 play the role of u_s in the case where $u_1u_s \notin E(H_1)$, we get a contradiction, similarly. Hence, $s = 3$. Since C_1 is a Hamiltonian cycle in H_1, $|H_1| = 3$. Note $|H_2| \geq 10$ by Claim 5.1, and H_2 does not contain a longer path than P_1. Thus $H_2 = K_{1,p}$, where $p \geq 9$. Let $V(K_{1,p}) = \{a_1\} \cup \{b_1, b_2, \ldots, b_p\}$, and let $X = \{b_1, b_2, b_3\}$. Since $d_{H_2}(b_i) = 1$ for each $1 \leq i \leq 3$, $d_{H_2}(X) = 3$. Also, X is an independent set. By Subclaim 5.2.1, the degree sequences from three vertices of X to some C in $\mathscr{C}$ are $(4, 4, 2)$ or $(4, 3, 3)$, and $|C| = 4$. Let $C = x_1, x_2, x_3, x_4, x_1$. Without loss of generality, we may assume $d_C(b_1) \geq d_C(b_2) \geq d_C(b_3)$. Since $d_C(b_2) \geq 3$ by the degree sequences, without loss of generality, we may assume $x_i \in N_C(b_2)$ for each $2 \leq i \leq 4$. Then $C' = b_2, x_2, x_3, x_4, b_2$ is a 4-cycle with chord b_2x_3. Since $d_C(b_1) = 4$, $x_1 \in N_C(b_1)$. Replacing C in $\mathscr{C}$ by C', we consider the new H'. Note $H_2 - b_2$ is connected. By (A2), $\mathrm{comp}(H') = \mathrm{comp}(H)$. Then x_1, b_1, a_1, b_3 is a longer path than P_1. This contradicts (A3). □

Claim 5.3. *H contains a Hamiltonian path.*

Proof. Suppose not, then by Claims 5.1 and 5.2, $|H| \geq 13$ and H is connected. Recall P_1 is a longest path in H. Then $V(H - P_1) \neq \emptyset$. Let $P_1 = u_1, \ldots, u_s$ $(s \geq 3)$, and let $P_2 = v_1, \ldots, v_t$ $(t \geq 1)$ be a longest path in $H - P_1$. Without loss of generality, we may assume $d_H(v_1) \leq d_H(v_t)$. Let $X = \{u_1, u_s, v_1\}$. Then by Lemma 3.8 (i), (v), and (vi), X is an independent set and $d_H(X) \leq 6$. Noting $\sigma_3(G) \geq 9k - 2$ and Lemma 3.3, as in Subclaim 5.2.1 in the proof of Theorem 1.4, there exists some C in $\mathscr{C}$ such that the degree sequences from three vertices of X to C are $(4, 4, 2)$ or $(4, 3, 3)$, and $|C| = 4$. Let $C = x_1, x_2, x_3, x_4, x_1$ be a 4-cycle with chord x_1x_3. Without loss of generality, we may assume $d_C(u_1) \geq d_C(u_s)$.

Suppose $d_C(u_1) = 4$. By the degree sequence, u_s and v_1 have a common neighbor in C, say x_ℓ for some $1 \leq \ell \leq 4$. Note u_1 is not a cutvertex of H, since u_1 is an endpoint of a longest path. Thus $H - u_1$ is connected. Since $d_C(u_1) = 4$, $\langle u_1 \cup (C - x_\ell) \rangle$ contains a chorded 4-cycle, say C'. Replacing C in $\mathscr{C}$ by C', we consider the new H'. Note H' is connected. Then $P_1[u_2, u_s], x_\ell, P_2[v_1, v_t]$ is a longer path than P_1 in

H'. This contradicts (A3). Thus $d_C(u_1) \leq 3$, that is, $d_C(u_1) = d_C(u_s) = 3$ and $d_C(v_1) = 4$. Since $d_C(u_1) = 3$, $x_1, x_3 \in N_C(u_1)$ or $x_2, x_4 \in N_C(u_1)$.

First suppose $x_1, x_3 \in N_C(u_1)$. Recall x_1x_3 is a chord of C. Since $d_C(u_s) = 3$, without loss of generality, we may assume $x_4 \in N_C(u_s)$. Then $C' = u_1, x_1, x_2, x_3, u_1$ is a 4-cycle with chord x_1x_3. Since $d_C(v_1) = 4$, $x_4 \in N_C(v_1)$. Note $H - u_1$ is connected. Replacing C in $\mathscr{C}$ by C', we consider the new H'. Then $P_1[u_2, u_s], x_4, P_2[v_1, v_t]$ is a longer path than P_1 in H'. This contradicts (A3).

Next suppose $x_2, x_4 \in N_C(u_1)$. Since $d_C(u_1) = 3$, without loss of generality, we may assume $x_3 \in N_C(u_1)$. Since $d_C(u_s) = 3$, without loss of generality, we may assume $x_4 \in N_C(u_s)$. Then $C' = u_1, x_2, x_1, x_3, u_1$ is a 4-cycle with chord x_2x_3. Since $d_C(v_1) = 4$, $x_4 \in N_C(v_1)$. Note $H - u_1$ is connected. Replacing C in $\mathscr{C}$ by C', we consider the new H'. Then $P_1[u_2, u_s], x_4, P_2[v_1, v_t]$ is a longer path than P_1 in H'. This contradicts (A3). □

By Claims 5.1, 5.3, and Lemma 3.14, there exists an independent set X of four vertices in H such that $d_H(X) \leq 8$. Let $X = \{x_1, x_2, x_3, x_4\}$, and let $X_1 = \{x_1, x_2, x_3\}, X_2 = \{x_1, x_2, x_4\}, X_3 = \{x_1, x_3, x_4\}$, and $X_4 = \{x_2, x_3, x_4\}$. Then $3|X| = \sum_{i=1}^{4} |X_i|$. Note X_i for each $1 \leq i \leq 4$ is an independent set. By the $\sigma_3(G)$ condition,

$$3 \cdot d_G(X) = \sum_{i=1}^{4} d_G(X_i) \geq 4\sigma_3(G) \geq 4(9k - 2) = 36k - 8.$$

On the other hand, by Claim 5.3 and Lemma 3.15,

$$3 \cdot d_G(X) = 3(d_{\mathscr{C}}(X) + d_H(X)) \leq 3(12(k - 1) + 8) = 36k - 12,$$

a contradiction. This completes the proof of Theorem 1.4. □

Acknowledgments

The authors would like to thank the referees for their careful reading and helpful suggestions. The first author is supported by the Heilbrun Distinguished Emeritus Fellowship from Emory University. The second author is supported by JSPS KAKENHI Grant Number JP19K03610.

References

[1] S. Chiba, S. Fujita, Y. Gao and G. Li, On a sharp degree sum condition for disjoint chorded cycles in graphs, *Graphs Combin.* 26 (2010), 173–186.

[2] S. Chiba and T. Yamashita, Degree conditions for the existence of vertex-disjoint cycles and paths: A survey, *Graphs Combin.* 34 (2018), 1–83.

[3] K. Corrádi and A. Hajnal, On the maximal number of independent circuits in a graph, *Acta Math. Acad. Sci. Hungar.* 14 (1963), 423–439.

[4] H. Enomoto, On the existence of disjoint cycles in a graph, *Combinatorica* 18 (4) (1998), 487–492.

[5] D. Finkel, On the number of independent chorded cycles in a graph, *Discrete Math.* 308 (22) (2008), 5265–5268.

[6] S. Fujita, H. Matsumura, M. Tsugaki and T. Yamashita, Degree sum conditions and vertex-disjoint cycles in a graph, *Australas. J. Combin.* 35 (2006), 237–251.

[7] Y. Gao, X. Lin and H. Wang, Vertex-disjoint double chorded cycles in bipartite graphs, *Discrete Math.* 342 (9) (2019), 2482–2492.

[8] R. J. Gould, Graph Theory, Dover Pub. Inc. Mineola, N.Y. 2012.

[9] R. J. Gould, K. Hirohata and A. Keller, On vertex-disjoint cycles and degree sum conditions, *Discrete Math.* 341 (1) (2018), 203–212.

[10] T. Molla, M. Santana and E. Yeager, Disjoint cycles and chorded cycles in a graph with given minimum degree, *Discrete Math.* 343 (6) (2020), 111837.

[11] H. Wang, On the maximum number of independent cycles in a graph, *Discrete Math.* 205 (1-3) (1999), 183–190.

(Received 11 July 2019; revised 12 Mar 2020)