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A note on signed k-matching in graphs

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Abstract

Let G be a graph of order n. For every $v \in V(G)$, let $E_G(v)$ denote the set of all edges incident with v. A signed k-matching of G is a function $f : E(G) \longrightarrow \{-1, 1\}$, satisfying $f(E_G(v)) \leq 1$ for at least k vertices, where $f(S) = \sum_{e \in S} f(e)$, for each $S \subseteq E(G)$. The maximum of the values of $f(E(G))$, taken over all signed k-matchings f of G, is called the signed k-matching number and is denoted by $\beta_S^k(G)$. In this paper, we prove that for every graph G of order n and for any positive integer $k \leq n$, $\beta_S^k(G) \geq n - k - \omega(G)$, where $w(G)$ is the number of components of G. This settles a conjecture proposed by Wang. Also, we present a formula for the computation of $\beta_S^n(G)$.

1 Introduction

Let G be a simple graph with the vertex set $V(G)$ and edge set $E(G)$. For every $v \in V(G)$, let $N(v)$ and $E_G(v)$ denote the set of all neighbors of v and the set of all edges incident with v, respectively. A *signed k-matching* of a graph G is a

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function $f : E(G) \longrightarrow \{-1, 1\}$, satisfying $f(E_G(v)) \leq 1$ for at least k vertices, where $f(S) = \sum_{e \in S} f(e)$, for each $S \subseteq E(G)$. The maximum value of $f(E(G))$, taken over all signed k-matching f, is called the *signed k-matching number* of G and is denoted by $\beta_S^k(G)$. We refer to a signed n-matching as a *signed matching*. The concept of signed matching has been studied by several authors; for instance see [1], [2], [4] and [5].

Throughout this paper, changing $f(e)$ to $-f(e)$ for an edge e is called *switching the value* of e. Let T be a trail with the edges $e_1, \ldots, e_m$ and $f : E(G) \longrightarrow \{-1, 1\}$ be a function. Call T a *good trail*, if $f(e_i) = -f(e_{i+1})$ for $i = 1, \ldots, m-1$. If $f(e_1) = a$ and $f(e_m) = b$, then we call T a *good* (a, b)*-trail*. Define $O_f(G) = \{v \in V(G) \mid d(v) \equiv 1 (\mathrm{mod}\ 2),\ f(E_G(v)) < 1\}$. A vertex is called *odd* if its degree is odd. The following conjecture was proposed in [3].

Conjecture. Let G be a graph without isolated vertices. Then for any positive integer k,

$$\beta_S^k(G) \geq n - k - \omega(G),$$

where $\omega(G)$ denotes the number of components of G.

In this note we prove this conjecture. Before stating the proof, we need the following result.

Theorem 1. *Let G be a connected graph of order n. Then for any positive integer $k \leq n$, $\beta_S^k(G) \geq n - k - 1$.*

Proof. If G is a cycle, then by Theorem 2 of [3] the assertion is obvious. Thus assume that G is not a cycle. Now, we apply induction on $|E(G)| - |V(G)|$. Since G is connected, $|E(G)| - |V(G)| \geq -1$. If $|E(G)| - |V(G)| = -1$, then G is a tree and so by Theorem 6 of [3], we are done. Now, suppose that the assertion holds for every graph H with $|E(H)| - |V(H)| \leq t\, (t \geq -1)$ and G be a connected graph such that $|E(G)| - |V(G)| = t + 1$. Since $|E(G)| - |V(G)| \geq 0$, G contains a cycle C and there exists a vertex v such that $v \in V(C)$ and $d(v) \geq 3$. Assume that $u, w \in N(v) \cap V(C)$. Let $x \in N(v) \backslash \{u, w\}$. Remove two edges vw and xv and add a new vertex v'. Join v' to both x and w. Call the new graph G'. Clearly, G' is connected and $|E(G')| - |V(G')| = t$. By the induction hypothesis, $\beta_S^{k+1}(G') \geq |V(G')| - k - 2 = n - k - 1$. We claim that $\beta_S^k(G) \geq \beta_S^{k+1}(G')$. Let f be a signed $(k+1)$-matching of G' such that $f(E(G')) = \beta_S^{k+1}(G')$. Define a function g on $E(G)$ as follows:

For every $e \in E(G) \setminus \{vx, vw\}$, let $g(e) = f(e)$. Moreover, define $g(xv) = f(xv')$ and $g(vw) = f(v'w)$. It is not hard to see that g is a k-matching of G. So $\beta_S^k(G) \geq g(E(G)) = \beta_S^{k+1}(G')$. Thus $\beta_S^k(G) \geq n - k - 1$, and the claim is proved. The proof is complete. □

Now, using the previous theorem we show that the conjecture holds.

Theorem 2. *Let G be a graph of order n without isolated vertices. Then for any positive integer $k \leq n$,*

$$\beta_S^k(G) \geq n - k - \omega(G),$$

where $\omega(G)$ denotes the number of components of G.

Proof. For the abbreviation let $\omega = \omega(G)$. If $\omega = 1$, then by Theorem 1 the assertion holds. Now, suppose that $\omega > 1$ and $G_1, \ldots, G_\omega$ are all components of G. Let $f : E(G) \longrightarrow \{-1, 1\}$, be a signed k-matching function such that $f(E(G)) = \beta_S^k(G)$. Suppose that $A \subset \{v \in V(G) \,|\, f(E_G(v)) \leq 1\}$ and $|A| = k$. Let $k_i = |\{v \in V(G_i) \cap A \,|\, f(E_G(v)) \leq 1\}|$, for $i = 1, \ldots, \omega$. Obviously, $\sum_{i=1}^{\omega} k_i = k$. By Theorem 1, $\beta_S^{k_i}(G_i) \geq |V(G_i)| - k_i - 1$, for $i = 1, \ldots, \omega$. Now, we show that $\beta_S^{k_i}(G_i) = f(E(G_i))$. By contradiction, suppose that $\beta_S^{k_i}(G_i) > f(E(G_i))$, for some i, $i = 1, \ldots, \omega$. Let $g : E(G) \longrightarrow \{-1, +1\}$ be a function such that $g(e) = f(e)$, for every $e \in E(G) \setminus E(G_i)$ and the restriction of g on $E(G_i)$ is a signed k_i-matching with $g(E(G_i)) = \beta_S^{k_i}(G_i)$. So we conclude that $g(E(G)) > \beta_S^k(G)$, a contradiction. Thus $\beta_S^k(G) = f(E(G)) = \sum_{i=1}^{\omega} f(E(G_i)) = \sum_{i=1}^{\omega} \beta_S^{k_i}(G_i) \geq \sum_{i=1}^{\omega}(|V(G_i)| - k_i - 1) = |V(G)| - k - \omega$. □

Now, suppose that G is a connected graph containing exactly $2k$ odd vertices. Let P be a partition of the edge set into m trails, say $T_1, \ldots, T_m$, for some m. Call P a *complete partition* if $m = k$. By Theorem 1.2.33 of [6], for every connected graph with $2k$ odd vertices there exists at least one complete partition. Note that for every odd vertex $v \in V(G)$, there exists i such that v is an endpoint of T_i, where $P : T_1, \ldots, T_k$ is a complete partition of G. So we obtain that the end vertices of T_i are odd and they are mutually disjoint, for $i = 1, \ldots, k$. Now, define $\tau(P) = |\{i \mid |E(T_i)| \equiv 1 (\mathrm{mod}\ 2)\}|$. Let $\eta(G) = \max \tau(P)$, taken over all complete partitions of G. In the next theorem we provide an explicit formula for the signed n-matching number of a graph.

Theorem 3. *For every non-Eulerian connected graph G of order n, $\beta_S^n(G) = \eta(G)$.*

Proof. For the simplicity, let $O_f = O_f(G)$. Let f be a signed matching such that $|O_f| = max(|O_g|)$ taken over all signed matching g with $g(E(G)) = \beta_S^n(G)$. We prove that $f(E_G(v)) \geq -1$, for every $v \in V(G)$.

By contradiction suppose that there is a vertex $v \in V(G)$ such that $f(E_G(v)) \leq -2$. Let W be a longest good $(-1, \pm 1)$-trail starting at v. Suppose that W ends at u. There are two cases:

Case 1. Assume that $u \neq v$. If W is a good $(-1, -1)$-trail, then $f(E_G(u)) \leq -1$, since otherwise there exists $e \in E_G(u) \setminus E(W)$ such that $f(e) = 1$, therefore W can be extended and it contradicts the maximality of $|E(W)|$. Now, switch the values of all edges of W to obtain a function g on $E(G)$, where $g(E_G(x)) = f(E_G(x))$ for every $x \in V(G) \setminus \{u, v\}$, and $g(E_G(x)) = f(E_G(x)) + 2$ for $x \in \{u, v\}$. Thus g is a signed matching of G such that $g(E(G)) = \beta_S^n(G) + 2$, a contradiction.
If W is a good $(-1, 1)$-trail, then $f(E_G(u)) = 1$, since otherwise there exists $e \in$

$E_G(u) \setminus E(W)$, where $f(e) = -1$, a contradiction. Now, switch the values of all the edges of W to obtain a function g on $E(G)$, where $g(E_G(x)) = f(E_G(x))$ for $x \in V(G) \setminus \{u, v\}$, $g(E_G(u)) = -1$ and $g(E_G(v)) < 1$. So g is a signed matching of G such that $g(E(G)) = \beta_S^n(G)$ and $|O_g| = |O_f| + 1$, a contradiction.

Case 2. Now, let $u = v$. Note that W is a good $(-1, -1)$-trail, since otherwise $\sum_{e \in E(W) \cap E_G(v)} f(e) = 0$ and using the inequality $f(E_G(v)) \leq -2$, we conclude that there exists $e \in E_G(v) \setminus E(W)$, such that $f(e) = -1$. Therefore W can be extended, a contradiction.

If $f(E_G(v)) \leq -3$, then switch the values of all edges of W to obtain a signed matching g such that $g(E(G)) = \beta_S^n(G) + 2$, a contradiction. Now, assume that $f(E_G(v)) = -2$. We show that $f(E_G(t)) = 0$, for every $t \in V(W) \setminus \{v\}$. By contradiction, suppose that there exists $x \in V(W) \setminus \{v\}$, such that $f(E_G(x)) \neq 0$. Let $e_1, \ldots, e_m$ be all edges of W. Assume that e_i and e_{i+1} are two consecutive edges of W which are incident with x. With no loss of generality, assume that $f(e_i) = -1$. First, suppose that $f(E_G(x)) \leq -1$. Call the sub-trail induced on the edges $e_1, e_2, \ldots, e_i$ by W_1. Clearly, W_1 is a good $(-1, -1)$-trail. Switch the values of all edges of W_1 to obtain a signed matching g such that $g(E(G)) = \beta_S^n(G) + 2$, a contradiction. Next, suppose that $f(E_G(x)) = 1$. Call the sub-trail induced on the edges $e_{i+1}, \ldots, e_m$ by W_2. Clearly, W_2 is a good $(1, -1)$-trail. Switch the values of all edges of W_2 to obtain a signed matching g such that $g(E_G(x)) = -1$, $g(E_G(v)) = 0$ and $g(E_G(z)) = f(E_G(z))$, for every $z \in E(G) \setminus \{x, v\}$. So $g(E(G)) = \beta_S^n(G)$ and $|O_g| = |O_f| + 1$, a contradiction. Thus, $f(E_G(t)) = 0$, for every $t \in V(W) \setminus \{v\}$.

Now, we show that $E_G(v) \subseteq E(W)$. By contradiction assume that there exists $e \in E_G(v) \setminus E(W)$. If $f(e) = 1$, then W can be extended, a contradiction. If $f(e) = -1$, then $f(E_G(v)) \leq -3$ which contradicts $f(E_G(v)) = -2$. Thus $E_G(v) \subseteq E(W)$. Since G is non-Eulerian, there are $x \in V(W) \setminus \{v\}$ and $y \in V(G)$ such that $xy \notin E(W)$. Let W' be a longest good trail in $G \setminus E(W)$ whose first vertex and first edge are x and xy, respectively. Suppose that W' ends at y' and the last edge of W' is e. We have two possibilities:

If $y' = x$, then we show that W' is a good $(1, -1)$ or $(-1, 1)$-trail. To see this, since $f(E_G(x)) = 0$, we obtain that $f(E_G(x) \setminus E(W)) = 0$. If $f(e) = f(xy)$, then there exists $e' \in E_G(x) \setminus (E(W) \cup E(W'))$ such that $f(e') = -f(xy)$. So W' can be extended, a contradiction. Thus $f(e) \neq f(xy)$. It is not hard to see that the trail with the edges $E(W) \cup E(W')$ is a good $(-1, -1)$-trail starting at v, a contradiction.

Now, suppose that $y' \neq x$. Assume that x is the common endpoint of e_j and e_{j+1}, for some j, $1 \leq j \leq m - 1$. With no loss of generality assume that $f(e_j) = -f(xy)$. Consider the trail $W'' : e_1, \ldots, e_j, W'$. Since $E_G(v) \subseteq E(W)$, $y' \neq v$. If $y' \in V(W)$, then $f(E_G(y')) = 0$ and $\sum_{z \in (E_G(W) \cup E_G(W')) \cap E_G(y')} f(z) = f(e)$. Hence there exists $e' \in E_G(y') \setminus (E_G(W) \cup E_G(W'))$ such that $f(e') = -f(e)$, which contradicts the maximality of $|E(W')|$. Thus $y' \notin V(W)$ and so W'' is a maximal good trail in G. So we reach to Case 1 which we discussed before (Note that in Case 1 we used just the maximality of the length of W).

So far we have proved that $f(E_G(z)) \geq -1$, for every $z \in V(G)$. In the sequel assume that G has exactly $2k$ odd vertices. We would like to partition G into k good trails.

Let $T : e_1, \ldots, e_m$ be a longest good trail in G. Suppose that T starts at u_1 and ends at u_2, where $u_1, u_2 \in V(G)$. First, we show that $u_1 \neq u_2$. By contradiction assume that $u_1 = u_2$. Suppose that $f(e_1) \neq f(e_m)$. Since G is non-Eulerian, there exists $e \in E(G) \backslash E(T)$ and e is adjacent to the common endpoint of e_i and e_{i+1} for some i, $i = 1, \ldots, m$ ($e_{m+1} = e_1$). With no loss of generality assume that $f(e) \neq f(e_i)$, so $T' : e, e_i, e_{i-1}, \ldots, e_1, e_m, \ldots, e_{i+1}$ is a good trail with $m + 1$ edges, a contradiction. Now, suppose that $f(e_1) = f(e_m)$. Since $\sum_{z \in E_G(u_1) \cap E(T)} f(z) = 2f(e_1)$ and $|f(E_G(u_1))| \leq 1$, we obtain that there exists $a \in E_G(u_1) \setminus E(T)$ such that $f(a) \neq f(e_1)$. So T can be extended, a contradiction.
Hence $u_1 \neq u_2$. Since $f(E_G(v)) = 0$, for every $v \in V(G)$ of even degree, we obtain that u_1 and u_2 have odd degrees. Indeed, if u_1 has even degree, then $f(E_G(u_1)) = 0$ and so T can be extended, a contradiction. Now, we show that $E_G(u_1) \cup E_G(u_2) \subseteq E(T)$. By contradiction, suppose that there is an edge $e \in E_G(u_1) \setminus E(T)$. Clearly, $f(e) = f(e_1)$. Since $\sum_{a \in E_G(u_1) \cap E(T)} f(a) = f(e_1)$, it is not hard to see that $|f(E_G(u_1))| \geq 2$, a contradiction. Hence $E_G(u_1) \cup E_G(u_2) \subseteq E(T)$.

Let $G' = G \backslash (E(T) \cup \{u_1, u_2\})$. First, we prove that G' has no Eulerian component. By contradiction suppose that H is an Eulerian component of G'. Since $|f(E_G(v))| \leq 1$, for every $v \in V(G)$, we have $f(E_G(v)) = 0$, for every $v \in V(H)$. It is straight forward to see that there is an Eulerian circuit $C : t_1, t_2, \ldots, t_{|E(H)|}$ of H such that $f(t_i) = -f(t_{i+1})$, for $i = 1, \ldots, |E(H)| - 1$. Clearly,

$$|E(C)| \equiv \sum_{e \in E(C)} f(e) \equiv \frac{\sum_{v \in V(C)} f(E_H(v))}{2} \equiv 0 \pmod{2}.$$

Hence, $f(t_1) = -f(t_{|E(H)|})$. Since G is connected and all of the edges of u_1 and u_2 belong to $E(T)$, there exists $v \in V(H) \cap V(T)$. It is not hard to see that we have a good trail with the edge set $E(T) \cup E(C)$ which is longer than T, a contradiction. So if $k = 2$, then $E(G') = \emptyset$, and $E(G)$ forms a good trail. Now, apply induction on k. Suppose that $k > 2$. Let $H_1, \ldots, H_r$ be all components of G', where H_i has k_i odd vertices ($k_i \geq 2$), for $i = 1, \ldots, r$. It is clear that f is a signed matching of H_i such that $f(E(H_i)) = \beta_S^{|V(H_i)|}(H_i)$ and $O_f(H_i) = \max O_g(H_i)$ taken over all signed matching g with $g(E(H_i)) = \beta_S^{|V(H_i)|}(H_i)$. So $E(H_i)$ can be decomposed into k_i good trails, for $i = 1, \ldots, r$. Hence, G has a complete partition, say P, into k good trails. Obviously, $f(E(G)) \leq \tau(P) \leq \eta(G)$. Thus, $\beta_S^n(G) \leq \eta(G)$. Now, we give a signed matching f such that $f(E(G)) = \eta(G)$.

Consider a complete partition P of the edge set of G, where $\tau(P) = \eta(G)$. For each trail T_i assign $+1$ and -1 to the edges of T_i, alternatively, to obtain a signed matching f where $f(E(G)) = \eta(G)$. So the proof is complete. □

Remark. For every Eulerian graph G of size m, $\beta_S^n(G) = 0$ if m is even and $\beta_S^n(G) = -1$ if m is odd. To see this, let f be a signed matching of G such that

$f(E(G)) = \beta_S^n(G)$. Since the degree of each vertex of G is even, $f(E_G(v)) \leq 0$, for every $v \in V(G)$. Thus $f(E(G)) = \frac{1}{2}\sum_{v\in V(G)} f(E_G(v)) \leq 0$. Therefore, $\beta_S^n(G) \leq 0$, if m is even and $\beta_S^n(G) \leq -1$, if m is odd. Now, consider an Eulerian circuit of G. Assign -1 and $+1$ to the edges of this Eulerian circuit, alternatingly to obtain a signed matching g with the desired property.

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