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Path partitions of almost regular graphs

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Abstract

The path partition number of a graph is the minimum number of paths required to partition the vertices. We consider upper bounds on the path partition number under minimum and maximum degree assumptions.

1 Introduction

Unless otherwise defined, our notation follows that of [1]. The *path partition number* of a graph G, denoted by $\mu(G)$, is the minimum number of vertex disjoint paths needed to cover the vertices of G. The path partition number, originally studied in [4], has been considered by several research groups. See [3], [5] and [6] for some examples.

In particular, Reed [5] proved the following.

Theorem 1.1 ([5])**.** *If G is a connected 3-regular graph on n vertices, then $\mu(G) \leq \lceil n/9 \rceil$.*

He also conjectured the following.

Conjecture 1.1 ([5])**.** *If G is a 2-connected 3-regular graph on n vertices, then $\mu(G) \leq \lceil n/10 \rceil$.*

This conjecture was recently settled by Yu [6].

In [2], Magnant and Martin considered the path partition number of regular graphs.

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Conjecture 1.2 ([2]). *For any positive integer k, if G is a k-regular graph on n vertices, then*

$$\mu(G) \leq \frac{n}{k+1}.$$

This conjecture was proven for $k \leq 5$ in [2] by counting edges between the ends and interior vertices of paths in a minimum path partition.

Theorem 1.2 ([2]). *For any positive integer $0 \leq k \leq 5$, if G is a k-regular graph on n vertices, then*

$$\mu(G) \leq \frac{n}{k+1}.$$

The original motivation for the work in [2] was to provide a bound on the path partition number under minimum and maximum degree assumptions but the authors soon realized that the regular case was already quite difficult.

In this work, we revisit the motivating question from [2]. We conjecture the following, and prove this conjecture when $\delta = 1, 2$ in Sections 3 and 4 respectively.

Conjecture 1.3. *Given positive integers δ and Δ with $\Delta \geq \delta$, if G is a graph on n vertices with $\delta \leq \delta(G) \leq \Delta(G) \leq \Delta$, then*

$$\mu(G) \leq \max\left\{\frac{n}{\delta+1}, \frac{(\Delta-\delta)n}{\Delta+\delta}\right\}.$$

If true, this conjecture would be sharp by the following constructions. When $\Delta \leq \delta+2$, the case where the first bound is at least as large as the second bound, we use the same construction used in [2], a collection of disjoint complete graphs $K_{\delta+1}$. Such a graph clearly satisfies the degree assumptions with path cover number equal to $\frac{n}{\delta+1}$.

When $\Delta > \delta + 2$, let G be a collection of disjoint copies of $K_{\delta,\Delta}$. This graph clearly has $\delta(G) = \delta$ and $\Delta(G) = \Delta$. For each path in our path partition, the number of vertices used from the larger side (of order Δ) of a component is at most one more than the number of vertices used from the smaller side (of order δ). Thus, each component requires at least $\Delta - \delta$ paths. Since each component has $\Delta + \delta$ vertices, we see that $\mu(G) \geq \frac{(\Delta-\delta)n}{\Delta+\delta}$.

General bounds for all values of δ and Δ are not known. In Section 5, we provide a general upper bound on $\mu(G)$ when G is almost regular in the sense that $\Delta(G) < 2\delta(G)$.

2 Preliminaries

In this section, we present general structural results that will be used in the proofs of our main results. Since every path partition of a spanning subgraph is a path partition of the entire graph, we immediately get the following fact.

Fact 2.1. *If G^* is a spanning subgraph of G, then $\mu(G) \leq \mu(G^*)$.*

Next we consider the local structure near vertices of low degree.

Lemma 2.1. *If G contains two adjacent vertices u and v each of degree 2 and G' is the graph constructed from G by contracting the edge uv to a single vertex w, then $\mu(G) = \mu(G')$.*

Note that this lemma applies even if u and v share a common neighbor x, meaning that the edge contraction merges the edges ux and vx into a single edge wx.

Proof. First consider a minimum path partition $\mathscr{P}$ of G, so using $\mu(G)$ paths. Suppose that there is a path $P \in \mathscr{P}$ containing the edge uv. Then, in G', let P' denote the path P with the edge uv contracted and let $\mathscr{P}'$ denote the set $\mathscr{P}$ with P' in place of P. Here $\mathscr{P}'$ is a path partition of G' using $\mu(G)$ paths so $\mu(G') \leq \mu(G)$ in this case. Next suppose that there is no path in $\mathscr{P}$ containing the edge uv. Since $d(u) = d(v) = 2$, this means that u and v must be the ends of paths in $\mathscr{P}$. If they are not opposite ends of the same path, then the edge uv can be used to join two paths, contradicting the minimality of $\mathscr{P}$ so u and v must be opposite ends of a single path P. Suppose $P = uu^+ \ldots v^-v$ and let $P' = u^+ \ldots v^-w$. Then we see that if $\mathscr{P}'$ is the path partition $\mathscr{P}$ with P replaced by P', then $\mathscr{P}'$ (with the edge contracted as above) is a path partition of G' using $\mu(G)$ paths. Therefore, we get

$$\mu(G') \leq \mu(G). \tag{1}$$

Next consider a minimum path partition $\mathscr{P}'$ of G', so using $\mu(G')$ paths. Let P' be the path containing the vertex w. Let w^- and w^+ denote the predecessor and successor on P' of w respectively and note that one or both of these vertices may not exist. Let P denote the path P' with the segment w^-ww^+ replaced by the segment w^-uvw^+ where the labels of the vertices u and v are chosen so that u and v are adjacent to w^- and w^+ respectively. Let $\mathscr{P}$ denote the path partition of G coming from $\mathscr{P}'$ where the path P' is replaced by P. Then $\mathscr{P}$ is a path partition of G using $\mu(G')$ paths so we get $\mu(G) \leq \mu(G')$. This along with (1) implies that $\mu(G) = \mu(G')$, as desired. $\square$

When the minimum degree and maximum degree are not too far apart, the following result, an adaptation of a claim from [2], shows that we may assume there are no singleton paths in a minimum path partition.

Lemma 2.2. *If $\Delta(G) < 2\delta(G)$, then there exists a path partition of G with $\mu(G)$ paths containing no paths of order* 1.

Proof. Consider a minimum path cover $\mathscr{P}$ of G with the smallest number of paths of order 1. Suppose there exists a path $P_1 \in \mathscr{P}$ of order 1 and let v be the vertex of P_1. Certainly v is not adjacent to an endpoint of another path in $\mathscr{P}$. First suppose v is adjacent to an internal vertex w of a path P of order at least 4. Then the two paths P_1 and P may be replaced by two paths of order at least 2 each. Therefore, every neighbor of v is the midpoint of a path of order 3 in $\mathscr{P}$.

Our assumption that $\Delta(G) < 2\delta(G)$ implies that $\delta(G) \geq 1$. Hence, v is adjacent to a vertex w_1 on a path $v_1v_1x_1$ in $\mathscr{P}$. The paths P_1 and $v_1w_1x_1$ may be replaced by the path vw_1x_1 and v_1, so by the previous argument, all the neighbors of v_1, and similarly all the neighbors of x_1, are midpoints of paths of order 3 in $\mathscr{P}$. Since $deg(q) \geq 3$, it follows that $\delta(G) \geq 2$. Hence $\mathscr{P}$ contains paths $v_1w_1x_1, v_2w_2x_2, \ldots, v_tw_tx_t$ such that all the neighbors of all the vertices in $\{v, v_1, x_1, v_2, x_2, \ldots, v_t, x_t\}$ are contained in $\{w_1, w_2, \ldots, w_t\}$. Hence $2t\delta(G) \leq t\Delta(G)$, contradicting our assumption. □

3 Minimum Degree 1

In this section, we consider graphs with minimum degree at least 1 and maximum degree at most Δ.

Theorem 3.1. *Given an integer Δ with $\Delta > 1$, if G is a graph on n vertices with $1 \leq \delta(G) \leq \Delta(G) \leq \Delta$, then*

$$\mu(G) \leq \max\left\{\frac{n}{2}, \frac{(\Delta-1)n}{\Delta+1}\right\}.$$

Proof. Let G^* denote a spanning edge-minimal subgraph of G which still satisfies $\delta(G^*) \geq 1$. Then at least one vertex of every edge in G^* has degree equal to 1, namely, each component of G^* is a star. This proof is by induction on the number of components in G^*.

If $\Delta(G^*) = 1$, then G^* is 1-regular, in other words, a matching. By Theorem 1.2, we get $\mu(G^*) = \frac{n}{2}$, as desired. Thus, we may assume $\Delta(G^*) \geq 2$.

If G^* is a single star, then G has $n-1 \geq 2$ edges, so then $\mu(G^*) = n-2 = \frac{(\Delta-1)n}{\Delta+1}$ since $\Delta = n-1$. Since we have assumed $\Delta(G^*) \geq 2$, there exists a vertex $v \in V(G^*)$ such that $deg_{G^*}(v) = k \geq 2$. Let $v_1, v_2, \cdots, v_k$ be the neighbors of v, leaves of G^*. Denote

$$G' = G^* \setminus \{v, v_1, v_2, \cdots, v_k\}.$$

Then we have $1 \leq \delta(G')$ and so, by induction on n,

$$\begin{aligned}
\mu(G^*) &\leq \mu(G') + k - 1 \\
&\leq \frac{(\Delta-1)\,(n-(k+1))}{\Delta+1} + k - 1 \\
&= \frac{(\Delta-1)n + 2(k-\Delta)}{\Delta+1} \\
&\leq \frac{(\Delta-1)n}{\Delta+1}.
\end{aligned}$$

By Fact 2.1, we have $\mu(G) \leq \mu(G^*)$, hence $\mu(G) \leq \dfrac{(\Delta-1)n}{\Delta+1}$ as desired. □

4 Minimum Degree 2

In this section, we consider graphs G with $\delta(G) \geq 2$.

Theorem 4.1. *Given an integer Δ with $\Delta > 2$, if G is a graph on n vertices with $2 \leq \delta(G) \leq \Delta(G) \leq \Delta$, then*

$$\mu(G) \leq \max\left\{\frac{n}{3}, \frac{(\Delta-2)n}{\Delta+2}\right\}.$$

Proof. Let G^* be an edge-minimal spanning subgraph of G which satisfies $\delta(G^*) \geq 2$. Then at least one vertex of each edge in G^* has degree equal to 2. The proof is by induction on n.

First suppose G^* contains two adjacent vertices u and v each of degree 2. By Lemma 2.1, we have

$$\mu(G) \leq \mu(G^*) = \mu(G'),$$

where G' is obtained by contracting the edge uv in G^* to a new vertex w. Since $|V(G')| < |V(G)|$, if $\delta(G') \geq 2$, we may apply induction on n to get $\mu(G) < \max\left\{\frac{n}{3}, \frac{(\Delta-2)n}{\Delta+2}\right\}$. If $\delta(G') = 1$, then u and v must have a common neighbor x in G^*. If $deg_{G^*}(x) = 3$, then let P be the path $uvxx_1 \cdots x_t$ (of order at least 3) in G^* where $deg_{G^*}(x_i) = 2$ for all i and $t \geq 0$. Then, if we remove $V(P)$, we may apply induction on n in the remaining graph and use P as one of the paths in the resulting path partition to find a path partition of G with the desired number of paths. Thus, we may assume $deg_{G^*}(x) \geq 4$ so $\Delta(G) \geq 4$. We may then remove the path $P = uv$ and apply induction on $G^* \setminus P$ to obtain a path partition of G with at most $\frac{(\Delta-2)n}{\Delta+2} \geq \frac{n}{2}$ paths as desired.

Thus, we may assume that there are no adjacent vertices with degree equal to 2 in G^*. We divide the vertices in G^* into two sets.

$$A = \{v \in G^* : deg_{G^*}(v) = 2\};$$
$$B = \{v \in G^* : deg_{G^*}(v) > 2\}.$$

By the observations above and the edge-minimality of G^*, we see that A and B each induce independent sets in G^*.

Claim 4.1. $\mu(G^*) \leq |A| - |B|$.

Proof. We apply a greedy algorithm to choose a path partition of G^*. Let $P^1 = w_{11}v_{11}w_{12}v_{12} \cdots w_{1k_1}$ be a longest path in G^*.

Within $G^* \setminus V(P)$, each vertex of degree 0 requires its own (singleton) path in the partition. Suppose v is a vertex of degree 0 in $G^* \setminus V(P)$. Then v is not adjacent to an end vertex of P by the maximality of P. Thus, every neighbor of v is an internal vertex of P and hence has degree at least 3 in G^*. Since B is an independent set, this implies that $v \in A$. Let $V_{0,1}$ denote the vertices of degree 0 in $G^* \setminus V(P)$ and note that $V_{0,1} \subseteq A$. Then define $G_1^* = G^* \setminus (V(P^1) \cup V_{0,1})$.

Repeating this process, we let P^i denote a longest path in G_{i-1}^*. Let $V_{0,i}$ denote the vertices of degree 0 and note that $V_{0,i} \subseteq A$ as before. Then define $G_i^* = G_{i-1}^* \setminus (P^i \cup V_{0,i})$. These paths constructed look like those pictured in Figure 1. Note that whenever a vertex $a \in N(b)$ for $b \in B$ gets used in a path, then b would also be used on the same path. Hence, the end vertices of each of these paths must be in A.

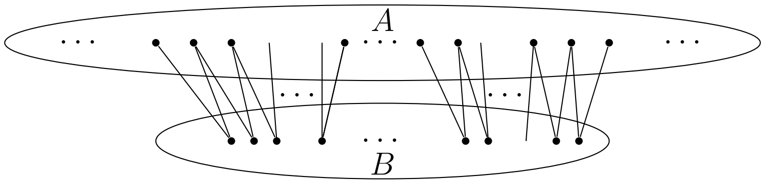

Figure 1: Counting a path partition

If we let s be the number of steps required to construct our paths and let $k_i = |P^i \cap A|$, for all $1 \leq i \leq s$, then

$$|B| = \sum_{i=1}^{s}(k_i - 1)$$

and we have

$$\mu(G^*) \leq s + |A| - \sum_{i=1}^{s} k_i = |A| - |B|,$$

completing the proof of Claim 4.1. □

By the definitions of the sets A and B, we get

$$|A| + |B| = n \tag{2}$$

and by counting the number of edges between A and B, we get

$$2|A| \leq \Delta|B|. \tag{3}$$

Combining (2) and (3) by solving each for $|B|$, we get

$$\frac{(\Delta+2)|A|}{\Delta} \leq n. \tag{4}$$

Then, by Claim 4.1, (3) and (4),

$$\begin{aligned}
\mu(G^*) &\leq |A| - |B| \\
&\leq |A| - \frac{2}{\Delta}|A| \\
&= \frac{(\Delta-2)}{\Delta+2}\frac{(\Delta+2)}{\Delta}|A| \\
&\leq \frac{(\Delta-2)n}{\Delta+2}
\end{aligned}$$

Hence,

$$\mu(G) \leq \mu(G^*) \leq \frac{(\Delta-2)n}{\Delta+2}$$

completing the proof of Theorem 4.1 □

5 Graphs with $\Delta(G) < 2\delta(G)$

In this section we provide a simple proof for a general upper bound for $\mu(G)$ for graphs G with $\Delta(G) < 2\delta(G)$.

Theorem 5.1. *Suppose G is a graph of order n with $\Delta(G) = \Delta < 2\delta = 2\delta(G)$, then $\mu(G) \leq \frac{(\Delta-2)n}{2(\Delta+\delta-4)}$.*

Proof. First note that by Lemma 2.2, there exists a minimum path partition $\mathscr{P}$ of G where every path has order at least 2. We partition $V(G)$ into the following:

- the set A of end vertices of paths in $\mathscr{P}$;
- the set I of internal vertices of paths in $\mathscr{P}$.

It is easy to see that $n = |A| + |I|$ and $\mu(G) = \frac{|A|}{2}$. Since $\mathscr{P}$ is a minimum path partition of G, there is no edge between vertices in A except possibly between two end vertices of the same path in $\mathscr{P}$. Otherwise joining two paths of $\mathscr{P}$ yields a smaller path partition.

Each vertex in A has at least $\delta - 2$ non-path neighbors (neighbors of a vertex outside its own path) in I while each vertex in I has at most $\Delta-2$ non-path neighbors, we have

$$(\Delta - 2)|I| = (\Delta - 2)(n - |A|) \geq (\delta - 2)|A|.$$

Consequently we have

$$(\Delta - 2)n \geq (\Delta + \delta - 4)|A| = 2(\Delta + \delta - 4)\mu(G),$$

implying that $\mu(G) \leq \frac{(\Delta-2)n}{2(\Delta+\delta-4)}$. □

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