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Lower bounds for the minimum diameter of integral point sets

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Abstract

We improve a known lower bound for the minimum diameter of integral point sets.

1 Introduction

Let X be a finite subset in the m-dimensional Euclidean space $\mathbb{R}^m$. We define

$$A(X) = \{||x-y|| \mid x, y \in X, x \neq y\},$$

where $||(x_1,\ldots,x_m)|| = \sqrt{\sum_{i=1}^m x_i^2}$. We call $X \subset \mathbb{R}^m$ an *integral point set* if all elements in $A(X)$ are integers. The maximum element in $A(X)$ is called the *diameter* of an integral point set X. One of the basic problems is to determine the smallest possible diameter $d(m,n)$ of integral point sets for fixed dimension m and cardinality n of X. We have a lot of exact values of $d(m,n)$ for relatively small m and n, and an upper bound for $d(m,n)$ by a constructive method, see [3, 4, 7, 8, 9, 11]. For general m and n, a known lower bound is only $n^{1/m}\sqrt{3/2m} < d(m,n)$ [6]. In the present paper, we improve this lower bound. Namely, if $\binom{m+s-1}{s-1} + \binom{m+s-2}{s-2} < n \leq \binom{m+s}{s} + \binom{m+s-1}{s-1}$ for some $s \geq 3$, then

$$s \leq d(m,n). \tag{1}$$

Indeed this bound gives the bound $n^{1/m} < d(m,n)$, and it is better than the previous known bound.

We apply the theory on s-distance sets to prove our lower bound. We call X an s-distance set if $A(X)$ has size s. The author gave certain values, so called generalized Larman–Rogers–Seidel's ratios, which play an important role in the study of s-distance sets [10]. The ratio K_i of X with $A(X) = \{a_1,\ldots,a_s\}$ is

$$K_i = \prod_{j=1,\ldots,s, j\neq i} \frac{a_j^2}{a_j^2 - a_i^2}$$

for each $i = 1, 2, \dots, s$. Using the integrality of these ratios we can prove the non-existence of some s-distance sets. Under our assumption, we prove the non-existence of a distance set satisfying $A(X) \subset \{1, 2, \dots, s-1\}$ and obtain the bound (1).

2 Preliminaries

We prepare some notation and results. A lower bound for the size of an s-distance set is known.

Theorem 2.1 ([1, 2]). *Let X be an s-distance set in $\mathbb{R}^m$ with n points. Then $n \le \binom{m+s}{s}$.*

Let $\xi_1, \xi_2, \dots, \xi_m$ be independent variables, and let $\xi_0 = \xi_1^2 + \xi_2^2 + \cdots + \xi_m^2$. We define $W_l(\mathbb{R}^m)$ to be the linear space spanned by the monomials $\xi_0^{\lambda_0}\xi_1^{\lambda_1}\cdots\xi_m^{\lambda_m}$ with $\lambda_0+\lambda_1+\cdots+\lambda_m \le l$ and $\lambda_i \ge 0$. Then it is known that $\dim W_l(\mathbb{R}^m) = \binom{m+l}{l}+\binom{m+l-1}{l-1}$ [1]. The following lemmas are used later.

Lemma 2.2 ([10]). *Let $p_1, p_2, \dots, p_n \in W_l(\mathbb{R}^m)$, and $x_1, x_2, \dots, x_n \in X \subset \mathbb{R}^m$. Let M be the square matrix whose (i,j)-entry is $p_i(x_j)$. Then the rank of M is at most $\dim W_l(\mathbb{R}^m)$.*

Lemma 2.3. *Let $p_1, p_2, \dots, p_n \in W_l(\mathbb{R}^m)$, $x_1, x_2, \dots, x_n \in X \subset \mathbb{R}^m$, and M be the square matrix whose (i,j)-entry is $p_i(x_j)$. Assume $n > \dim W_l(\mathbb{R}^m)$, M is symmetric, every diagonal entry of M is $k \in \mathbb{Q}$, and non-diagonal entries are integers. Then k is an integer.*

Proof. By $n > \dim W_l(\mathbb{R}^m)$ and Lemma 2.2, M has a zero eigenvalue. We can express

$$M = kI + A,$$

where I is the identity matrix, A is a symmetric matrix whose diagonal entries are 0, and non-diagonal entries are integers. The eigenvalues of A are all algebraic integers. Since M has a zero eigenvalue, A has the eigenvalue $-k$. Therefore k is an algebraic integer, and k is a rational integer because $k \in \mathbb{Q}$. □

3 Lower bound on the minimum diameter

The following are known lower bounds for the minimum diameter of integral point sets.

Theorem 3.1 ([5, 6, 11]). (1) $\sqrt{\frac{3}{2m}}n^{1/m} < d(m,n)$.

(2) $\frac{1}{\sqrt{14}}n^{1/2} < d(3,n)$ *for* $n \ge 5$.

(3) $cn \le d(2,n)$ *for a sufficiently small constant* c.

(4) $3 \le d(m,n) \le 4$ *for* $m+2 \le n \le 2m$ *and* $d(m,2m) = 4$.

We improve the bound (1) in Theorem 3.1. Let $N(m,s) = \binom{m+s-1}{s-1} + \binom{m+s-2}{s-2}$. Note that $\dim W_l(\mathbb{R}^m) = N(m, l+1)$.

Theorem 3.2. *Let X be an integral point set in $\mathbb{R}^m$ with n points. If $N(m,s-1) < n \le N(m,s)$ holds for some $s \ge 3$, then we have a lower bound*

$$d(m,n) \ge s.$$

Proof. Since $\binom{m+s-2}{s-2} < N(m,s-1) < n$ and Theorem 2.1, the set X has at least $s-1$ distances. If X has at least s distances, then the diameter of X is greater than $s-1$. Now we assume X is an $(s-1)$-distance set with distances $1, 2, \dots, s-1$. If this assumption makes a contradiction, then this theorem follows.

For each $x \in X$, we define the polynomial

$$F_x(\xi) = \prod_{j=2}^{s-1} \frac{j^2 - ||x-\xi||^2}{j^2-1},$$

where $\xi = (\xi_1, \xi_2, \dots, \xi_m)$. Then $F_x(\xi) \in W_{s-2}(\mathbb{R}^m)$, and the following equalities are satisfied:

$$\begin{aligned} &F_x(x) = k = \prod_{j=2}^{s} \frac{j^2}{j^2-1}, \\ &F_x(y) = 1 \qquad \text{if } y \in X \text{ and } ||x-y||^2 = 1, \\ &F_x(y) = 0 \qquad \text{if } y \in X,\ x \ne y \text{ and } ||x-y||^2 \ne 1. \end{aligned}$$

Let M be the symmetric matrix $(F_x(y))_{x,y \in X}$. By $|X| > N(m,s-1)$, the matrix M satisfies the condition in Lemma 2.3. Therefore k is a rational integer. However

$$k = \prod_{j=2}^{s-1} \frac{j^2}{j^2-1} = \frac{2(s-1)!(s-1)!}{s!(s-2)!} = \frac{2(s-1)}{s}$$

is not a rational integer except for $s = 2$, because $\gcd(s-1, s) = 1$. This is a contradiction, and the theorem follows. □

Corollary 3.3. *The minimum diameter $d(m,n)$ is bounded below by $n^{1/m}$ for $n > N(m,2)$, $m \ge 4$.*

Proof. There exists $s \ge 3$ such that $N(m,s-1) < n \le N(m,s)$. Then $d(m,n) \ge s$.

We have

$$\begin{aligned} n \le N(m,s) &= \binom{m+s-1}{m} + \binom{m+s-2}{m} \\ &= (1+\frac{2(s-1)}{m})\binom{m+s-2}{m-1} \\ &= (1+\frac{2(s-1)}{m})\prod_{i=1}^{m-1}(1+\frac{s-1}{i}) \\ &< (1+\frac{s-1}{2})\prod_{i=1}^{m-1}(1+\frac{s-1}{1}) \\ &< s^m \le (d(m,n))^m \end{aligned}$$

for $m \ge 4$. Thus $n^{1/m} < d(m,s)$. □

Remark 3.4. The lower bound $d(m,n) > n^{1/m}$ is better than the known bound (1) in Theorem 3.1 for $n > N(m,2)$, $m \ge 4$. For $m = 2,3$, we have much better bounds (2), (3) in Theorem 3.1. The bound in Corollary 3.3 is a very rough evaluation of that in Theorem 3.2. For $s = 3$, if $m+2 < n \le (m+1)(m+4)/2$, then $3 \le d(m,n)$ by Theorem 3.2. This bound coincides with the bound (4) in Theorem 3.1 for $m+2 < n < 2m$.

For $s = 4$ in Theorem 3.2 we can obtain a better bound as follows.

Proposition 3.5. *Let X be an integral point sets in $\mathbb{R}^m$ with n points. If $N(m,3) < n \le N(m,4)$ holds, then we have a lower bound*

$$d(m,n) \ge 5.$$

Proof. If we show $A(X)$ is not a subset of $\{1,2,3,4\}$, then we obtain the desired bound. Assume $A(X) \subset \{1,2,3,4\}$. For each $x \in X$, we define

$$F_x(\xi) = \frac{(3^2 - ||x-\xi||^2)(4^2 - ||x-\xi||^2)}{(3^2-2^2)(4^2-2^2)}.$$

Then $F_x(\xi) \in W_2(\mathbb{R}^m)$, and the following hold:

$$\begin{aligned} &F_x(x) = \frac{12}{5}, \\ &F_x(y) = 2 \text{ if } ||x-y|| = 1, \\ &F_x(y) = 1 \text{ if } ||x-y|| = 2, \\ &F_x(y) = 0 \text{ if } ||x-y|| \ne 1,2. \end{aligned}$$

However $M = (F_x(y))_{x,y \in X}$ satisfies the condition in Lemma 2.3, and $F_x(x)$ must be an integer, a contradiction. Therefore we obtain the desired bound $5 \le d(m,n)$. □

Remark 3.6. For other s, we might obtain a better bound by a similar method to the proof in Proposition 3.5.

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