AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **54** (2012), Pages 217–230

Related decompositions and new constructions of the Higman-Sims and Hall-Janko graphs

Emre Kolotoglu Nicola Pace Spyros S. Magliveras

Center for Cryptology and Information Security
and Department of Mathematical Sciences
Florida Atlantic University
Boca Raton, FL 33431
U.S.A.
ekolotog@fau.edu npace@fau.edu spyros@fau.edu

Abstract

In two recent papers [Hafner, *Elec. J. Combin.* **11** (R77) (2004), 1–33; Ilić, Pace and Magliveras, *J. Combin. Math. Combin. Comput.* **80** (2012), 267–275] it was shown that the Higman-Sims graph Γ can be decomposed into a disjoint union of five double Petersen graphs. In the second of these papers, it was further shown that all such decompositions fall into a single orbit under the action of sporadic simple group HS, which is of index two in the full automorphism group of Γ. In this article we prove that the Hall-Janko graph Θ can be decomposed into a disjoint union of double co-Petersen graphs. We find all such decompositions, and prove they fall into a single orbit under the action of the sporadic simple group $J_2 = HJ$. The stabilizer in J_2 of such a decomposition is $D_5 \times A_5$. There are striking similarities between the decompositions of Γ and Θ just described. Finally, motivated by these decompositions, we obtain new constructions of the Higman-Sims and Hall-Janko graphs from Petersen and co-Petersen graphs.

Introduction

A graph $\mathcal{G} = (X, E)$, where X is the set of vertices (or points) and E the set of edges of $\mathcal{G}$, is called a *strongly regular graph* and denoted by $\mathrm{srg}(v, k, \lambda, \mu)$, if it has v vertices where every vertex has degree k, every pair of adjacent vertices have exactly λ common neighbours, and every pair of non-adjacent vertices have exactly μ common neighbours.

The Higman-Sims and Hall-Janko groups, of orders 44,352,000 and 604,800 respectively, are two among the 26 sporadic simple groups. The Higman-Sims group HS

was discovered by Higman and Sims in 1967 [5], as a group of automorphisms of the unique $\text{srg}(100, 22, 0, 6)$, Γ, known as the Higman-Sims graph. Interestingly, the existence of the Higman-Sims graph had been known by Dale Mesner some eleven years earlier before it was independently re-discovered by Higman and Sims. Mesner had discovered the graph in his 1956 Ph.D. dissertation as graph $NL_2(10)$ of *negative Latin square type*; see [9, 13]. Unfortunately, Mesner did not consider the automorphism group of $NL_2(10)$, so sporadic HS remained unknown until 1967.

In 1965 Janko constructed a new sporadic simple group J_1, and predicted, but did not establish, the existence of three other sporadic simple groups which were later to be called J_2, J_3 and J_4. About the same time that Higman and Sims established the existence of HS, Marshal Hall Jr. and David Wales constructed a new sporadic simple group, the "Hall-Janko-Wales" group HJ, see [4], which was in fact the group J_2 earlier predicted by Janko. Thus HJ is also known as J_2. Like HS in its action on Γ, the group J_2 is a subgroup of index 2 of the full group of automorphisms of the unique $\text{srg}(100, 36, 14, 12)$, Θ, known as the Hall-Janko graph.

In his 1970 Ph.D. dissertation, Magliveras determined the maximal subgroups of HS, and in the process, discovered that the Petersen graph was contained in Γ; see [11, 12]. In particular, if P is the adjacency matrix of the Petersen graph π, and $z \in HS$ is an involution fixing points, then $|\text{fix}\,(z)| = 20$, and Γ restricted to $\text{fix}\,(z)$ is the "*double Petersen*" graph $\psi := P \otimes J$, where J is the 2×2 all-ones matrix. Similarly, if $z \in J_2$ is an involution fixing points, then $|\text{fix}\,(z)| = 20$, and Θ restricted to $\text{fix}\,(z)$ is the "*double co-Petersen*" graph $\psi' := \overline{P} \otimes J$, where $\overline{P}$ is the adjacency matrix of the complement of the Petersen graph π and J is the 2×2 all-ones matrix.

In two recent papers [3, 8], it was shown that the Higman-Sims graph can be decomposed into a disjoint union of graphs isomorphic to the double Petersen graph. In this paper, in a fashion similar to the work of the authors of [8], we prove that the Hall-Janko graph can be decomposed into a disjoint union of double co-Petersen graphs. We find all such decompositions, and prove that they fall into a single orbit under the action of J_2. The automorphism group of a decomposition is $D_5 \times A_5$. Finally, we start from these decompositions and reconstruct the Higman-Sims and Hall-Janko graphs.

1 Preliminaries

We employ standard terminology and properties of group actions as the reader may find in [1, 7, 15]. An action of group G on set X will be denoted by $G|X$. For $x \in X$, $g \in G$, x^g denotes the image of x under g. A group $G|X$ is called *transitive* if it has exactly one orbit, i.e. for all $x, y \in X$, $x^g = y$ for some $g \in G$. The orbits of the induced action $G|(X \times X)$ of a transitive group action $G|X$ are called the *orbitals* of $G|X$. The *rank* of a transitive group action $G|X$ is defined as its number of orbitals or equivalently the number of orbits of $G_x|X$, where G_x is the stabilizer in G of $x \in X$.

If $A \subseteq X$, $G_{[A]}$ denotes the *pointwise* stabilizer of A in G, and $G_{(A)}$ the *setwise* stabilizer of A. Moreover, if $P = \{A_1, A_2, \ldots, A_k\}$ is a partition of X, we write $G_{[[A_1],\ldots,[A_k]]}$ for the subgroup of G which fixes each of the blocks of the partition pointwise. Further, we write $G_{[(A_1),\ldots,(A_k)]}$ for the subgroup of G fixing each of the blocks A_i setwise, i.e. possibly permuting the elements within each A_i. Finally, we write $G_{((A_1),\ldots,(A_k))}$ for the subgroup of G fixing the partition as a whole, i.e. which (possibly) permutes the blocks A_i among themselves. Clearly, $G_{[[A_1],\ldots,[A_k]]} \leq G_{[(A_1),\ldots,(A_k)]} \leq G_{((A_1),\ldots,(A_k))}$.

If G is a group and $x \in G$, we denote by $C(x) = C_G(x)$ the centralizer of x in G, that is, $C(x) := \{y \in G \mid xy = yx\}$. Further, σ_x denotes the order of $C(x)$. If $K_1 = \{1\}$, $K_2, \ldots, K_c$ are the conjugacy classes of G, we write σ_i for the order of $C(x_i)$, where $x_i \in K_i$. We write:

$$(K_i \times K_j \to K_k) := \{(a,b) \in K_i \times K_j \mid ab \in K_k\}, \quad i,j,k \in \{1,\ldots,c\}, \tag{1}$$

and denote the cardinality of $(K_i \times K_j \to K_k)$ by $|K_i \times K_j \to K_k|$. Moreover, we write:

$$\langle K_i \times K_j \to K_k \rangle = \{\langle a,b \rangle \mid (a,b) \in (K_i \times K_j \to K_k)\} \tag{2}$$

where $\langle a,b \rangle$ denotes the subgroup of G generated by a and b.

The structure constants of the center of the group algebra $\mathbb{Z}G$ are denoted by $a_{i,j,k}$; thus

$$K_i K_j = \sum_{k=1}^{c} a_{i,j,k} K_k, \quad i,j \in \{1,\ldots,c\}. \tag{3}$$

We also have

$$a_{i,j,k} = \frac{|G|}{\sigma_i \sigma_j} \sum_{t=1}^{c} \frac{\chi_t(i)\chi_t(j)\overline{\chi_t(k)}}{\chi_t(1)} \tag{4}$$

where $\chi_t(i)$ is the value of the irreducible ordinary character χ_t of G on the elements of the class K_i. The character table of J_2 is presented below in Table 1.

Let $\mathcal{G} = (X, E)$ be an undirected graph without loops. If $x, y \in X$, we denote by $d(x,y)$ the distance in $\mathcal{G}$ between x and y. Further, if r is a non-negative integer, by the *sphere* of radius r about x we mean the set:

$$S_r(x) := \{y \in X \mid d(x,y) = r\}.$$

A t-(v,k,λ) design is a set of k-subsets, called blocks, of a set X of v elements, called points, where each subset of t points belong to exactly λ blocks.

Both the Higman-Sims graph Γ and the Hall-Janko graph Θ have 100 vertices, and the groups HS and J_2 have rank-3, primitive permutation representations on the 100 vertices of their respective graphs. We will represent the graphs Γ and Θ on the vertex set $X = \{0, 1, 2, \ldots, 99\}$. Both Γ and Θ are connected and have diameter 2.

For Γ, we have:

Table 1: Character Table of the Hall-Janko group

$\|x\|$	1	2_1	2_2	4	8	3_1	3_2	6_1	6_2	12	5_1	5_2	5_3	5_4	10_1	10_2	10_3	10_4	15_1	15_2	7
σ_x	$\|G\|$	1920	240	96	8	1080	36	24	12	12	300	300	50	50	20	20	10	10	15	15	7
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	14	-2	2	2	0	5	-1	1	-1	-1	3θ	$3\overline{\theta}$	ϵ	$\overline{\epsilon}$	$-\theta$	$-\overline{\theta}$	θ	$\overline{\theta}$	0	0	0
χ_3	14	-2	2	2	0	5	-1	1	-1	-1	$3\overline{\theta}$	3θ	$\overline{\epsilon}$	ϵ	$-\overline{\theta}$	$-\theta$	$\overline{\theta}$	θ	0	0	0
χ_4	21	5	-3	1	-1	3	0	-1	0	1	γ	$\overline{\gamma}$	2θ	$2\overline{\theta}$	$-\overline{\theta}$	$-\theta$	0	0	θ	$\overline{\theta}$	0
χ_5	21	5	-3	1	-1	3	0	-1	0	1	$\overline{\gamma}$	γ	$2\overline{\theta}$	2θ	$-\theta$	$-\overline{\theta}$	0	0	$\overline{\theta}$	θ	0
χ_6	36	4	0	4	0	9	0	1	0	1	-4	-4	1	1	0	0	-1	-1	-1	-1	1
χ_7	63	15	-1	3	1	0	3	0	-1	0	3	3	-2	-2	-1	-1	0	0	0	0	0
χ_8	70	-10	-2	2	0	7	1	-1	1	-1	5θ	$5\overline{\theta}$	0	0	θ	$\overline{\theta}$	0	0	$-\theta$	$-\overline{\theta}$	0
χ_9	70	-10	-2	2	0	7	1	-1	1	-1	$5\overline{\theta}$	5θ	0	0	$\overline{\theta}$	θ	0	0	$-\overline{\theta}$	$-\theta$	0
χ_{10}	90	10	6	-2	0	9	0	1	0	1	5	5	0	0	1	1	0	0	-1	-1	-1
χ_{11}	126	14	6	2	0	-9	0	-1	0	-1	1	1	1	1	1	1	-1	-1	1	1	0
χ_{12}	160	0	4	0	0	16	1	0	1	0	-5	-5	0	0	-1	-1	0	0	1	1	-1
χ_{13}	175	15	-5	-1	-1	-5	1	3	1	-1	0	0	0	0	0	0	0	0	0	0	0
χ_{14}	189	-3	-3	-3	1	0	0	0	0	0	3θ	$3\overline{\theta}$	ϵ	$\overline{\epsilon}$	$-\theta$	$-\overline{\theta}$	$-\theta$	$-\overline{\theta}$	0	0	0
χ_{15}	189	-3	-3	-3	1	0	0	0	0	0	$3\overline{\theta}$	3θ	$\overline{\epsilon}$	ϵ	$-\overline{\theta}$	$-\theta$	$-\overline{\theta}$	$-\theta$	0	0	0
χ_{16}	224	0	-4	0	0	8	-1	0	-1	0	δ	$\overline{\delta}$	-2θ	$-2\overline{\theta}$	1	1	0	0	θ	$\overline{\theta}$	0
χ_{17}	224	0	-4	0	0	8	-1	0	-1	0	$\overline{\delta}$	δ	$-2\overline{\theta}$	-2θ	1	1	0	0	$\overline{\theta}$	θ	0
χ_{18}	225	-15	5	-3	-1	0	3	0	-1	0	0	0	0	0	0	0	0	0	0	0	1
χ_{19}	288	0	4	0	0	0	-3	0	1	0	3	3	-2	-2	-1	-1	0	0	0	0	1
χ_{20}	300	-20	0	4	0	-15	0	1	0	1	0	0	0	0	0	0	0	0	0	0	-1
χ_{21}	336	16	0	0	0	-6	0	-2	0	0	-4	-4	1	1	0	0	1	1	-1	-1	0

$$|G| = 604800\,, \quad \theta = (1+\sqrt{5})/2\,, \quad \overline{\theta} = (1-\sqrt{5})/2\,,$$
$$\gamma = 3+\theta\,, \quad \delta = 1-4\overline{\theta}\,, \quad \epsilon = 1+\theta\,,$$
$$\overline{\gamma} = 3+\overline{\theta}\,, \quad \overline{\delta} = 1-4\theta\,, \quad \overline{\epsilon} = 1+\overline{\theta}\,.$$

1. For $x \in X$, $|S_1(x)| = 22$ and $|S_2(x)| = 77$.

2. If $y \in S_1(x)$, then $|S_1(x) \cap S_1(y)| = 0$, $|S_1(x) \cap S_2(y)| = |S_1(y) \cap S_2(x)| = 21$ and $|S_2(x) \cap S_2(y)| = 56$.

3. If $y \in S_2(x)$, then $|S_1(x) \cap S_1(y)| = 6$. $|S_1(x) \cap S_2(y)| = |S_1(y) \cap S_2(x)| = 16$, and $|S_2(x) \cap S_2(y)| = 60$.

4. For $x \in X$, the design whose points are the elements of $S_1(x)$ and blocks the 77 subsets $\{S_1(x) \cap S_1(y) \mid y \in S_2(x)\}$ of X, is a 3-$(22, 6, 1)$ design.

For Θ, we have:

1. For $x \in X$, $|S_1(x)| = 36$ and $|S_2(x)| = 63$.

2. If $y \in S_1(x)$, then $|S_1(x) \cap S_1(y)| = 14$, $|S_1(x) \cap S_2(y)| = |S_1(y) \cap S_2(x)| = 21$ and $|S_2(x) \cap S_2(y)| = 42$.

3. If $y \in S_2(x)$, then $|S_1(x) \cap S_1(y)| = 12$, $|S_1(x) \cap S_2(y)| = |S_1(y) \cap S_2(x)| = 24$, and $|S_2(x) \cap S_2(y)| = 38$.

4. For $x \in X$, the design whose points are the elements of $S_1(x)$ and blocks the 63 subsets $\{S_1(x) \cap S_1(y) \mid y \in S_2(x)\}$ of X, is a 1-$(36, 12, 21)$ design, which will be denoted by D_x.

If a and b are positive integers and $a < b$ we set $[a, b] = \{a, a+1, \ldots, b-1, b\}$. Further, 0 and 100 denote the same vertex.

2 Co-Petersen subgraphs in Θ

We begin by selecting two generators α_1 and β_1 for $G \cong J_2$ in its representation on 100 points. In particular, α_1 is chosen so that it is an involution fixing points. Then, strong generators for G are computed from α_1 and β_1 using the Schreier-Sims algorithm; see [14]. The reader can repeat the process using a computer algebra system like MAGMA or GAP. Next, the original generators are appropriately conjugated so that the resulting involution α will fix precisely $\{1, 2, \ldots, 20\}$ pointwise. The resulting generators α, β are now in canonical form, and will demonstrate directly the nature of the objects under consideration. To conserve space we have written α and β as:

$$\alpha = \left(\begin{array}{c|cccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline 0 & 32 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 \\ 2 & 20 & 97 & 84 & 94 & 83 & 88 & 91 & 81 & 99 & 89 \\ 3 & 86 & 98 & 0 & 90 & 93 & 95 & 87 & 82 & 96 & 85 \\ 4 & 92 & 69 & 77 & 64 & 74 & 67 & 62 & 66 & 79 & 76 \\ 5 & 63 & 70 & 78 & 73 & 71 & 80 & 68 & 65 & 75 & 61 \\ 6 & 72 & 59 & 46 & 50 & 43 & 57 & 47 & 45 & 56 & 41 \\ 7 & 51 & 54 & 60 & 53 & 44 & 58 & 49 & 42 & 52 & 48 \\ 8 & 55 & 27 & 37 & 24 & 22 & 39 & 30 & 36 & 25 & 29 \\ 9 & 33 & 26 & 40 & 34 & 23 & 35 & 38 & 21 & 31 & 28 \end{array}\right), \beta = \left(\begin{array}{c|cccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline 0 & 70 & 54 & 53 & 37 & 7 & 17 & 47 & 26 & 20 & 75 \\ 1 & 72 & 13 & 32 & 60 & 50 & 57 & 6 & 22 & 65 & 18 \\ 2 & 71 & 52 & 76 & 12 & 78 & 19 & 90 & 64 & 43 & 92 \\ 3 & 15 & 42 & 51 & 3 & 4 & 80 & 85 & 61 & 95 & 77 \\ 4 & 2 & 66 & 55 & 35 & 56 & 99 & 91 & 63 & 79 & 36 \\ 5 & 45 & 83 & 49 & 38 & 69 & 74 & 93 & 46 & 59 & 73 \\ 6 & 29 & 84 & 58 & 96 & 10 & 87 & 82 & 8 & 14 & 0 \\ 7 & 1 & 98 & 81 & 97 & 89 & 48 & 94 & 41 & 44 & 88 \\ 8 & 86 & 27 & 39 & 23 & 33 & 21 & 28 & 25 & 9 & 31 \\ 9 & 34 & 30 & 11 & 24 & 5 & 40 & 16 & 62 & 67 & 68 \end{array}\right)$$

Reading α and β is straightforward. For example, $\alpha(83) = 24$ and $\beta(8) = 20$. We easily recover the graph Θ by first computing the stabilizer G_x for any particular $x \in X$, then computing the point orbits of G_x on X. These orbits will have lengths 1, 36 and 63. We select the orbit of length 36 as the set of neighbors of x. In particular for $0 \in X$ and the generators α and β given,

$$\begin{aligned} S_1(0) \;=\; & \{1, 10, 13, 15, 18, 20, 23, 24, 25, 27, 39, 40, 42, 48, 52, 54, 55, 58, 63, \\ & 65, 72, 73, 74, 76, 82, 84, 87, 88, 89, 91, 92, 94, 95, 97, 98, 99\}. \end{aligned}$$

Since $S_1(x^g) = (S_1(x))^g$, for any $x \in X$, $g \in G$, by the transitivity of G on X we easily obtain the spheres of radius 1 with centers at each vertex of the graph.

The Hall-Janko group has 21 conjugacy classes of elements. In particular there are two conjugacy classes of involutions, denoted here by K_{2_1} and K_{2_2}. Further, there are two classes of elements of order 3, denoted by K_{3_1} and K_{3_2}, and four conjugacy classes of elements of order 5, which we denote by K_{5_1}, K_{5_2}, K_{5_3} and K_{5_4}. If $z \in K_{2_1}$, then z fixes exactly 20 points of X; on the other hand, elements of K_{2_2} fix no points. Let f be an element of order 5 in G. If $f \in K_{5_1}, K_{5_2}$, then $|C(f)| = 300$. If $f \in K_{5_3}, K_{5_4}$, then $|C(f)| = 50$.

The following proposition deals with the existence of co-Petersen and double co-Petersen graphs in Θ and characterizes such subgraphs.

Proposition 2.1 *(i) Let $z \in G$ be an involution that fixes points of X. Then the subgraph of Θ with vertex set* fix (z) *is isomorphic to the double co-Petersen graph ψ.*

(ii) If γ is a co-Petersen subgraph of Θ with vertex set Y, then the pointwise stabilizer of Y is of order 2; in particular, $G_{[Y]} = \langle z \rangle$, where $z \in K_{2_1}$, and γ is a subgraph of a double co-Petersen graph.

Proof. It can be observed by performing a direct computation that if $z \in G$ is an involution which fixes points of X, i.e. $z \in K_{2_1}$, then the subgraph on fix (z) constitutes a double co-Petersen graph. Here, we give a proof of this fact by examining the generator α of G. We see directly that the 20 points $\{1, 2, \ldots, 20\}$ fixed by α constitute a subgraph of Θ isomorphic to the double co-Petersen graph ψ.

Now, suppose that γ is any co-Petersen subgraph of Θ with vertex set $Y = \{y_0, \ldots, y_9\}$, and label the y_i's so that y_0 is adjacent to $\{y_1, y_2, y_3\}$. We proceed to determine $G_{[Y]}$. Since G is transitive on X we have that $|G_{y_0}| = |G|/100 = 6048$. We also know that G_{y_0} is isomorphic to the special unitary group $U_3(3)$ and is transitive on $S_1(y_0)$. Thus, $G_{[y_0,y_1]}$ is isomorphic to $GL_3(2) \cong PSL_2(7)$, and has order $168 = 6048/36$. Consider the action of $G_{[y_0,y_1]}$ on $S_1(y_0) \setminus \{y_1\}$. There is one orbit of size 21 and two orbits of size 7. If y_2 is contained in an orbit of order 21, then $H = G_{[y_0,y_1,y_2]}$ is isomorphic to the dihedral group D_4 and has order $168/21 = 8$. If y_2 is contained in an orbit of order 7, then $H = G_{[y_0,y_1,y_2]}$ is isomorphic to $\mathcal{S}_4$ and has order $168/7 = 24$.

If $H \cong D_4$, we can note that $|\text{fix}\,(H)| = 6$. In particular, we have that $\text{fix}\,(H) = \{y_0, y_1, y_2, y_3, y_4, y_9\}$, where $y_3, y_4 \in S_1(y_0)$ and $y_9 \in S_2(y_0)$. Thus, $H = G_{[y_0,y_1,y_2,y_3,y_4,y_9]}$. Let $y_5 \in S_1(y_0) \setminus \{y_1, y_2, y_3, y_4\}$. Then, y_5 is contained in an H-orbit of order 4 or 2. In the first case, we have that $G_{[Y]} = H_{y_5}$ is isomorphic to $\mathbb{Z}_2$. In the latter case, H_{y_5} is isomorphic to the Klein group V_4. Finally, let $y_6 \in Y \setminus \text{fix}\,(H_{y_5})$; we have $G_{[Y]} = H_{[y_5,y_6]} \cong \mathbb{Z}_2$.

If $H \cong \mathcal{S}_4$, then we consider the action of H on the set $S_1(y_0) \setminus \{y_1, y_2\}$. Now $S_1(y_0) \setminus \{y_1, y_2\}$ is the union of H-orbits of size $3, 4, 6, 12$. Let $y_3 \in S_1(y_0) \setminus \{y_1, y_2\}$. If y_3 belongs to an orbit of size $3, 6, 12$, then $H_{y_3} \cong D_4, V_4, \mathbb{Z}_2$, respectively. From this point, these cases can be handled as for $H \cong D_4$. Finally, if y_3 belongs to an orbit of size 4, then $H_{y_3} \cong \mathcal{S}_3$. If $y_4 \in S_1(y_0) \setminus \{y_1, y_2, y_3\}$, we have $H_{[y_3,y_4]} \cong \mathbb{Z}_2$.

Thus, we have $G_{[Y]} \cong \mathbb{Z}_2$; hence there exists an involution z such that $G_{[Y]} = \langle z \rangle$. Since all involutions fixing points are conjugate in G, we have $z \in K_{2_1}$ and $Y \subseteq \text{fix}\,(z)$. From part (i), $\text{fix}\,(z)$ is the vertex set of a double co-Petersen subgraph of Θ. □

Corollary 2.1 *The Hall-Janko graph Θ contains subgraphs isomorphic to the double co-Petersen graph ψ, and consequently Θ also contains co-Petersen subgraphs.*

Proposition 2.2 *If $z \in K_{2_1}$ and $A = \text{fix}\,(z)$, then $G_{(A)} = C_G(z)$, is of order 1920, is transitive on A and on $X \setminus A$, and is a maximal subgroup of G.*

Proof. The result follows directly from the list of maximal subgroups of G and their structure as stated in [2] and [4]. □

3 Decompositions

Let $\mathcal{B} = \{A \subset X \mid A = \text{fix}\,(z), z \in K_{2_1}\}$. Then $|\mathcal{B}| = [G : C(z)] = 315$, for $z \in K_{2_1}$. We now wish to examine whether it is possible to find a partition $\{B_i\}_{i=1}^{5}$ of X consisting of blocks $B_i \in \mathcal{B}$. If such a partition exists, the induced subgraphs on the blocks of the partition would constitute a decomposition of Θ into double co-Petersen graphs. In this section we show that such partitions exist and they all fall into a single orbit under the action of G. Moreover, we are able to count the total number of distinct such partitions.

Let A be a 315×100 $(0,1)$-matrix whose rows are the characteristic vectors of the distinct sets B_i in $\mathcal{B}$. In particular, arrange A so that its i^{th} row is the fix of the i^{th} involution in K_{2_1}. We view A as the incidence matrix of a combinatorial design $\mathcal{D} = (X, \mathcal{B})$ with points X and blocks $\mathcal{B}$. Now $\mathcal{D}$ is a 1-$(100, 20, 63)$ design and fails to be a 2-design. If $y \in S_1(x)$ then there are $\lambda_1 = 21$ blocks containing $x, y \in X$, while if $y \in S_2(x)$ there are $\lambda_2 = 7$ blocks passing through x and y.

We define a graph with vertices the blocks in $\mathcal{B}$, where two blocks are adjacent if and only if they are disjoint. We then search for 5-cliques in this graph. Our program SYNTH, which was designed to construct t-(v, k, λ) designs from Kramer-Mesner matrices [10], determines all possible solutions from A in less than 16 seconds on a desktop computer. There are in all 1008 cliques of size 5. We consider the first solution, and collect the corresponding five involutions $z_1', z_2', \ldots, z_5'$ for this solution. We subsequently relabel Θ by conjugating simultaneously the 5 involutions and initial generators of G by a sequence of transpositions of the symmetric group $\mathcal{S}_{100}$ so that, after relabeling, the five involutions become $z_1 = \alpha, z_2, \ldots, z_5$, with $\text{fix}\,(z_i) = [20i - 19, 20i]$.

Theorem 3.1 *(i) There exists a decomposition $\rho = \{B_i\}_{i=1}^{5}$ of Θ into a disjoint union of five double co-Petersen graphs.*

(ii) There are 1008 distinct partitions of X into five disjoint blocks $\{B_i\}_{i=1}^{5}$ with $B_i \in \mathcal{B}$, i.e. 1008 decompositions of Θ into double co-Petersen graphs.

(iii) If $\rho = \{B_i\}_{i=1}^{5}$ is any one of the decompositions, and z_1, z_2 are the involutions fixing the blocks B_1 and B_2, then $D_5 = \langle z_1, z_2 \rangle$ is a dihedral subgroup of order 10, with the remaining involutions $z_3, z_4, z_5 \in D_5$ fixing the remaining blocks B_3, B_4, B_5 of ρ.

(iv) The cyclic subgroup of order 5 in D_5 permutes the blocks of ρ among themselves, and the centralizer $C_G(D_5)$ is a subgroup of G isomorphic to the alternating group A_5.

Proof. (i) and (ii) We seek solutions $X \in \{0,1\}^{315}$ to the matrix system of equations $XA = J$ where J is the all 1's 1×100 row vector. There are in all 1008 solutions.

(iii) and (iv) We analyze the first solution we obtained which is described above Theorem 3.1. For that particular solution, we have $z_1 = \alpha$ and $z_2 = (\beta\alpha\beta\alpha\beta\alpha\beta^2\alpha\beta\alpha\beta^2)^6$. One can check that property (iii) holds for this solution. So, let $D_5 = \langle z_1, z_2 \rangle$. We further verify that an element of order 5 in D_5 permutes the blocks B_i among themselves; thus, $D_5 \leq G_{((B_1),\ldots,(B_5))}$. Computing $C := C_G(D_5) = C_G(z_1) \cap C_G(z_2)$ yields that C is isomorphic to the alternating group A_5. Thus, property (iv) also holds for this particular solution. Further, by direct computation, we verify that C is transitive on each of the five blocks B_i, that is $C \leq G_{[(B_1),\ldots,(B_5)]}$. It follows that there is a subgroup of G which is the direct product $D_5 \times C$ of order 600 which is transitive on X.

Now, since an element f of order 5 in D_5 commutes with an element of order 3 in C, all the elements of order 5 in D_5 come from $K_{5_1} \cup K_{5_2}$. More precisely, we see by direct computation, (also by considering the power map of HJ as described in [2]) that f and f^2 are not conugate in G; thus without loss we may assume that $f, f^{-1} \in K_{5_1}$ and $f^2, f^3 \in K_{5_2}$. Hence, the total number of cyclic subgroups F of order 5, with 3 dividing the order of $C_G(F)$, is $|K_{5_1} \cup K_{5_2}|/4 = 1008$. It also follows that $\mathcal{D} = \langle K_{2_1} \times K_{2_1} \to K_{5_1} \rangle = \langle K_{2_1} \times K_{2_1} \to K_{5_2} \rangle$.

We finally note that $D_5 \times C$ is contained in the normalizer of D_5, as well as the normalizer of C. Computing the normalizer $N = N_G(D_5)$ we find that N is of order 600, hence $N = D_5 \times C$. Moreover, checking with the list of maximal subgroups of G we see that indeed N is a maximal subgroup of G, and thus $N = G_{((B_1),\ldots,(B_5))}$. But $|\mathcal{D}| = [G : N] = 604800/600 = 1008$. Since $G_{\rho^g} = (G_\rho)^g$, the 1008 decompositions form a single orbit under the action of G, and properties (iii) and (iv) hold for each of the 1008 decompositions. □

Remark 3.1 Since there is a one-to-one correspondence between the decompositions of Θ into double co-Petersen graphs and dihedral groups of type $(2_1, 2_1, 5_1 \cup 5_2)$, a second way of counting resolutions is to count all dihedral subgroups in $\langle K_{2_1} \times K_{2_1} \to K_{5_1} \rangle$.

From the character table of G, we compute:

$$a_{2_1,2_1,5_j} = \frac{|G|}{\sigma_{2_1}\sigma_{2_1}} \sum_{t=1}^{21} \frac{\chi_t(2_1)\chi_t(2_1)\overline{\chi_t(5_j)}}{\chi_t(1)} = 5, \qquad j = 1, 2.$$

Thus,

$$|K_{2_1} \times K_{2_1} \to K_{5_1}| \;=\; |K_{2_1} \times K_{2_1} \to K_{5_2}| \;=\; 5 \cdot \frac{|G|}{300} \;=\; 10080.$$

However, in any dihedral D_5

$$|K_2 \times K_2 \to K_{5_+} \cup K_{5_-}| = 20.$$

Therefore,

$$\begin{aligned}|\langle K_{2_1} \times K_{2_1} \to K_{5_1}\rangle| &= (|K_{2_1} \times K_{2_1} \to K_{5_1}| + |K_{2_1} \times K_{2_1} \to K_{5_2}|)/20 \\ &= 2 \cdot 10080/20 \\ &= 1008.\end{aligned}$$

Remark 3.2 A third, independent way of checking the correctness of the number of decompositions is to count all subgroups isomorphic to A_5 commuting with a dihedral of order 10 of type $(2_1, 2_1, 5_1 \cup 5_2)$, as above. Computation shows that such an A_5 is in $\langle K_{2_2} \times K_{3_1} \to K_{5_3}\rangle = \langle K_{2_2} \times K_{3_1} \to K_{5_4}\rangle$.

A similar count as above yields:

$$a_{2_2,3_1,5_j} = \frac{|G|}{\sigma_{2_2}\sigma_{3_1}} \sum_{t=1}^{21} \frac{\chi_t(2_2)\chi_t(3_1)\overline{\chi_t(5_j)}}{\chi_t(1)} = 5, \qquad j = 3, 4.$$

Thus,

$$|K_{2_2} \times K_{3_1} \to K_{5_3}| = |K_{2_2} \times K_{3_1} \to K_{5_4}| = 5 \cdot \frac{|G|}{50} = 60480.$$

However, in any A_5

$$|K_2 \times K_3 \to K_{5_1} \cup K_{5_2}| = 120.$$

Therefore the total number of A_5's of type $(2_2, 3_1, 5_2 \cup 5_3)$ is $120960/120 = 1008$. The centralizer of each such an A_5 is a dihedral subgroup of order 10 and type $(2_1, 2_1, 5_1 \cup 5_2)$ and each of these dihedrals gives rise to a resolution.

4 New construction of the Higman-Sims and Hall-Janko Graphs

In this section of the paper, we show how to reconstruct the Higman-Sims graph Γ and Hall-Janko graph Θ from 5 double Petersen and 5 double co-Petersen graphs respectively. We start with a well-known labeling of the Petersen graph on the set of 2-subsets of a 5-set. Let $V_P = \{S \subset \mathbb{Z}_5 : |S| = 2\}$. Then, the set of edges in the Petersen graph can be defined on V_P as follows.

$$S \text{ is adjacent to } S' \iff S \cap S' = \emptyset.$$

An obvious way of defining the double Petersen graph is to take its vertex set as $V_P \times \mathbb{Z}_2$, but there is a better way. If we take the ordered pairs of a 5-set instead of its unordered pairs, the definition of the Petersen graph immediately extends to the double Petersen graph, i.e. if we let $V_{DP} = \{(y, z) \in \mathbb{Z}_5 \times \mathbb{Z}_5 : y \neq z\}$, then the set of edges in the double Petersen graph can be defined on V_{DP} as follows.

$$(y, z) \text{ is adjacent to } (y', z') \iff \{y, z\} \cap \{y', z'\} = \emptyset.$$

Note that, on this labeling of the double Petersen graph, the subgraphs on $V_{DP}^{+} = \{(y,z) \in V_{DP} : z - y \in \{1,2\}\}$ and $V_{DP}^{-} = \{(y,z) \in V_{DP} : z - y \in \{-1,-2\}\}$ are both isomorphic to the Petersen graph.

We can define the double co-Petersen graph on V_{DP} in a similar way.

$$(y,z) \text{ is adjacent to } (y',z') \iff |\{y,z\} \cap \{y',z'\}| = 1.$$

Similarly, the subgraphs of the double co-Petersen graph on V_{DP}^{+} and V_{DP}^{-} are both isomorphic to the co-Petersen graph.

Our goal is now to relabel the vertex set and construct Γ and Θ on the set $V = \{(x,y,z) \in \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5 : y \neq z\}$ such that the subgraph of Γ (respectively, Θ) on $V_x = \{(x,y,z) : (y,z) \in V_{DP}\}$ is the double Petersen (respectively, double co-Petersen) graph.

After making some observations on the decomposition of Γ into 5 double Petersen graphs, we got the following definition of the Higman-Sims graph on V.

Type 1: (x,y,z) is adjacent to $(x,y',z') \iff \{y,z\} \cap \{y',z'\} = \emptyset$.
Type 2: (x,y,z) is adjacent to $(x',z,y) \iff x' \neq x$.
Type 3: (x,y,z) is adjacent to $(x \pm 1, y, z') \iff z' \neq z$.
Type 4: (x,y,z) is adjacent to $(x \pm 2, y', z) \iff y' \neq y$.

It is left to the reader to verify that the graph defined as above is indeed isomorphic to the Higman-Sims graph.

Remark 4.1

(i) Note that the subgraph of Γ on V_x is the double-Petersen graph for each $x \in \mathbb{Z}_5$, and these edges are the Type 1 edges.

(ii) Let $V^{+} = \{(x,y,z) \in V : z - y \in \{1,2\}\}$ and $V^{-} = \{(x,y,z) \in V : z - y \in \{-1,-2\}\}$. One can see that the subgraphs of Γ on V^{+} and V^{-} are both isomorphic to the unique srg$(50,7,0,1)$, namely the Hoffman-Singleton graph, see [6].

(iii) Let $\sigma, \tau : \mathbb{Z}_5 \to \mathbb{Z}_5$ be defined as $\sigma(x) = x + 1$ and $\tau(x) = -x$. The group generated by σ and τ is the dihedral group D_5. It was shown in [8] that the automorphism group (inside HS) of the decomposition of Γ into 5 double Petersen graphs is of order 1200, and contains $D_5 \times A_5$ as a subgroup of index 2. Actually, the automorphism group of the decomposition (inside the full automorphism group of Γ) is of order 2400 and contains $D_5 \times \mathcal{S}_5$ as a subgroup of index 2. The action of $D_5 \times \mathcal{S}_5$ can now be easily seen from the four adjacency relations given above. We basically see that Γ is invariant under the action of D_5 on the x-coordinates and $\mathcal{S}_5$ simultaneously on the y-coordinates and z-coordinates of the vertices.

(iv) In [3], Paul Hafner has given a definition of the Higman-Sims graph on $\mathbb{Z}_4 \times \mathbb{Z}_5 \times \mathbb{Z}_5$ using 10 types of linear and quadratic equations. We will now give a similar definition of the graph on $V' = \mathbb{Z}_5{}^* \times \mathbb{Z}_5 \times \mathbb{Z}_5$ using 3 types of equations only. Define the mapping $\Phi : V \to V'$ as $\Phi((x, y, z)) = (z - y, x, 1 - y - z)$. Note that Φ is a bijection. It is left to the reader to verify that under Φ we get the following definition of the Higman-Sims graph on V'.

(x, y, z) is adjacent to $(x, y, z') \iff z' - z = \pm x$.
(x, y, z) is adjacent to $(-x, y', z') \iff z' - z = x(y' - y)^2 \pm x$.
(x, y, z) is adjacent to $(x', y', z') \Leftarrow x' \neq \pm x$ and $z' - z = -(x' - x)(y' - y)^2$.

Using a similar process, the Hall-Janko graph Θ can be constructed from 5 double co-Petersen graphs and the set of edges can be defined on the vertex set V as follows.

Type 1: (x, y, z) is adjacent to $(x, y', z') \iff |\{y, z\} \cap \{y', z'\}| = 1$.
Type 2: (x, y, z) is adjacent to $(x \pm 1, y', z') \Leftarrow (y, z, y', z') \in (0, 1, 2, 3)^{A_5}$.
Type 3: (x, y, z) is adjacent to $(x \pm 2, y', z') \Leftarrow (y, z, y', z') \in (0, 1, 3, 2)^{A_5}$.
Type 4: (x, y, z) is adjacent to $(x \pm 1, y', z) \iff y' \neq y$.
Type 5: (x, y, z) is adjacent to $(x \pm 2, y, z') \iff z' \neq z$.

Here, $(0, 1, 2, 3)^{A_5}$ denotes the orbit of $(0, 1, 2, 3)$ under the action of A_5 on the set of ordered 4-tuples of $\mathbb{Z}_5$.

Remark 4.2

(i) Note that the subgraph of Θ on V_x is the double co-Petersen graph for each $x \in \mathbb{Z}_5$, and these edges are the Type 1 edges.

(ii) It was shown in the previous section that the automorphism group of the decomposition of Θ into 5 double co-Petersen graphs is $D_5 \times A_5$. The action of $D_5 \times A_5$ can now be easily seen from the 5 adjacency relations given above. Similar to the Higman-Sims graph, we see that Θ is invariant under the action of D_5 on the x-coordinates and A_5 simultaneously on the y-coordinates and z-coordinates of the vertices.

(iii) If we consider the Higman-Sims and Hall-Janko graphs as subgraphs of the complete graph K_{100} on V, we can see from the adjacency relations that these two graphs are disjoint and so K_{100} is a disjoint union of Γ, Θ and R, where R is a 41-regular graph which can be defined on V as follows.

(x, y, z) is adjacent to (x, z, y).
(x, y, z) is adjacent to $(x', y, z) \iff x' \neq x$.
(x, y, z) is adjacent to $(x \pm 1, y', z') \Leftarrow (y, z, y', z') \in (0, 1, 3, 2)^{A_5}$.
(x, y, z) is adjacent to $(x \pm 2, y', z') \Leftarrow (y, z, y', z') \in (0, 1, 2, 3)^{A_5}$.
(x, y, z) is adjacent to $(x', z', y) \iff x' \neq x$ and $z' \neq z$.
(x, y, z) is adjacent to $(x', z, y') \iff x' \neq x$ and $y' \neq y$.

(iv) Moreover if we identify the pairs in the Higman-Sims and Hall-Janko graphs, and derive two graphs, say Γ' and Θ' on 50 vertices, we see that these two graphs are complements of each other, i.e. let $V'' = \mathbb{Z}_5 \times V_P$, and define Γ' and Θ' on V'' as follows.

$(x, \{y, z\})$ is adjacent to $(x', \{y', z'\})$ in Γ' (respectively, Θ') $\iff$
(x, y, z) is adjacent to (x, y', z') in Γ (respectively, Θ) or
(x, y, z) is adjacent to (x, z', y') in Γ (respectively, Θ) or
(x, z, y) is adjacent to (x, y', z') in Γ (respectively, Θ) or
(x, z, y) is adjacent to (x, z', y') in Γ (respectively, Θ).

Then, Γ' and Θ' are 19-regular and 30-regular graphs, respectively, and the complete graph K_{50} on V'' is the disjoint union of Γ' and Θ'.

5 Decomposition of the Higman-Sims and Hall-Janko Graphs into 10-cycles

In this section of the paper, we show that Higman-Sims and Hall-Janko graphs can both be edge-decomposed into 10-cycles.

Let Γ^* be the Higman-Sims graph with Type 1 edges removed and let Θ^* be the Hall-Janko graph with Type 1 edges removed, i.e. Γ^* is the set of edges connecting the 5 double Petersen graphs in Γ, and Θ^* is the set of edges connecting the 5 double co-Petersen graphs in Θ. We will first prove that Γ^* and Θ^* can both be edge-decomposed into disjoint 10-cycles.

For all $(y, z) \in V_{DP}$, define $V_{(y,z)} = \{(x, y, z) : x \in \mathbb{Z}_5\}$. First note that, for all $(y, z) \in V_{DP}$, the sets $V_{(y,z)}$ form independent sets of size 5 in both Γ and Θ. Hence, for all $(y, z), (y', z') \in V_{DP}$ where $(y, z) \neq (y', z')$, the subgraphs of Γ and Θ (and therefore Γ^* and Θ^*) on $V_{(y,z)} \cup V_{(y',z')}$ are bipartite graphs on 10 vertices. Let $\Gamma^*_{(y,z,y',z')}$ and $\Theta^*_{(y,z,y',z')}$ denote the subgraphs of Γ^* and Θ^* on $V_{(y,z)} \cup V_{(y',z')}$ respectively. Therefore, Γ and Θ (and hence Γ^* and Θ^*) are both disjoint unions of $\binom{20}{2}$ bipartite graphs on 10 vertices each.

Now, for all $(y, z), (y', z') \in V_{DP}$ where $(y, z) \neq (y', z')$, define:
$S_{(y,z,y',z')} = \{m \in \mathbb{Z}_5 : (x, y, z)$ is adjacent to $(x + m, y', z')$ in $\Gamma\}$, and
$T_{(y,z,y',z')} = \{m \in \mathbb{Z}_5 : (x, y, z)$ is adjacent to $(x + m, y', z')$ in $\Theta\}$.
Then, we get the following.

Lemma 5.1 *1. $y' = z$ and $z' \neq y$ or $y' \neq z$ and $z' = y \Rightarrow S_{(y,z,y',z')} = \emptyset$ and $T_{(y,z,y',z')} = \{0\}$.*

2. $(y, z, y', z') \in (0, 1, 2, 3)^{A_5} \Rightarrow S_{(y,z,y',z')} = \{0\}$ and $T_{(y,z,y',z')} = \{\pm 1\}$.

3. $(y, z, y', z') \in (0, 1, 3, 2)^{A_5} \Rightarrow S_{(y,z,y',z')} = \{0\}$ and $T_{(y,z,y',z')} = \{\pm 2\}$.

4. $y' = y$ and $z' \neq z \Rightarrow S_{(y,z,y',z')} = \{\pm 1\}$ and $T_{(y,z,y',z')} = \{0, \pm 2\}$.

5. $y' \neq y$ and $z' = z \Rightarrow S_{(y,z,y',z')} = \{\pm 2\}$ and $T_{(y,z,y',z')} = \{0, \pm 1\}$.

6. $y' = z$ and $z' = y \Rightarrow S_{(y,z,y',z')} = \{\pm 1, \pm 2\}$ and $T_{(y,z,y',z')} = \emptyset$.

The proof of the lemma is left to the reader. Note that, in all cases we get

$$|T_{(y,z,y',z')}| \equiv |S_{(y,z,y',z')}| + 1 \pmod 5.$$

The lemma also provides us a way of constructing the Hall-Janko graph from the Higman-Sims graph. Here we will make use of this lemma to decompose Γ^* and Θ^* into disjoint 10-cycles. Note that the edge between (x, y, z) and $(x + m, y', z')$ is of Type 1 in both Γ and Θ if and only if $m = 0$. Therefore, if we define $S^*_{(y,z,y',z')} = \{m \in \mathbb{Z}_5 : (x, y, z) \text{ is adjacent to } (x + m, y', z') \text{ in } \Gamma^*\}$, and $T^*_{(y,z,y',z')} = \{m \in \mathbb{Z}_5 : (x, y, z) \text{ is adjacent to } (x + m, y', z') \text{ in } \Theta^*\}$, then we have that $S^*_{(y,z,y',z')} = \emptyset$, $S^*_{(y,z,y',z')} = \{\pm 1\}$, $S^*_{(y,z,y',z')} = \{\pm 2\}$ or $S^*_{(y,z,y',z')} = \{\pm 1, \pm 2\}$, and $T^*_{(y,z,y',z')} = \emptyset$, $T^*_{(y,z,y',z')} = \{\pm 1\}$ or $T^*_{(y,z,y',z')} = \{\pm 2\}$. It can easily be seen that if $S^*_{(y,z,y',z')} = \{\pm 1\}$ or $S^*_{(y,z,y',z')} = \{\pm 2\}$ then $\Gamma^*_{(y,z,y',z')}$ is a 10-cycle, and if $S^*_{(y,z,y',z')} = \{\pm 1, \pm 2\}$ then $\Gamma^*_{(y,z,y',z')}$ is a disjoint union of two 10-cycles. Therefore, decomposition of Γ^* (and similarly Θ^*) into disjoint bipartite graphs gives rise to a decomposition into 10-cycles. Therefore, we get the following lemma.

Lemma 5.2 *Γ^* and Θ^* can both be decomposed into disjoint 10-cycles.*

Now we will give a decomposition of the double Petersen graph into disjoint 10-cycles.

Lemma 5.3 *The double Petersen graph can be decomposed into disjoint 10-cycles.*

Proof: First note that the double Petersen graph consists of 60 edges. The following six 10-cycles are disjoint and therefore decompose the double Petersen graph. $((y, y+1), (y+2, y-2), (y-1, y+1), (y, y-2), (y+2, y+1), (y, y-1), (y+1, y-2), (y, y+2), (y-2, y+1), (y-1, y+2), (y, y+1))$ for each $y \in \mathbb{Z}_5$, together with the 10-cycle $((0, 1), (3, 2), (4, 0), (2, 1), (3, 4), (1, 0), (2, 3), (0, 4), (1, 2), (4, 3), (0, 1))$. □

Theorem 5.1 *The Higman-Sims graph can be decomposed into disjoint 10-cycles.*

Proof: Edges of Type 1 can be decomposed into disjoint 10-cycles by Lemma 5.3 and the remaining edges can be decomposed by Lemma 5.2. □We get a decomposition of the Hall-Janko graph into disjoint 10-cycles in a similar way.

Lemma 5.4 *The double co-Petersen graph can be decomposed into disjoint 10-cycles.*

Proof: First note that the double co-Petersen graph consists of 120 edges. The following twelve 10-cycles are disjoint and therefore decompose the double co-Petersen graph. $((y, y+1), (y-1, y), (y, y+2), (y+1, y+2), (y+1, y-2), (y+1, y), (y, y-2), (y-2, y-1), (y-2, y+1), (y+1, y-1), (y, y+1))$ and the co-ordinatewise swap of this 10-cycle, namely $((y+1, y), (y, y-1), \ldots, (y+1, y))$ for each $y \in \mathbb{Z}_5$, together with the 10-cycles $((0, 1), (0, 4), (3, 4), (3, 2), (1, 2), (1, 0), (4, 0), (4, 3), (2, 3), (2, 1), (0, 1))$ and $((2, 4), (1, 4), (1, 3), (0, 3), (0, 2), (4, 2), (4, 1), (3, 1), (3, 0), (2, 0), (2, 4))$. □

Theorem 5.2 *The Hall-Janko graph can be decomposed into disjoint* 10*-cycles.*

Proof: This follows from Lemma 5.4 and Lemma 5.2. □

References

[1] N.L. Biggs and A.T. White, *Permutation Groups and Combinatorial Structures*, London Math Soc Lecture Notes **33**, Cambridge Univ. Press, London, N.Y., Melbourne (1979).

[2] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker and R.A. Wilson, *Atlas of finite groups—Maximal subgroups and ordinary characters for simple groups*, Clarendon press, Oxford (1985).

[3] P.R. Hafner, On the graphs of Hoffman-Singleton and Higman-Sims, *Elec. J. Combin.* **11** (R77) (2004), 1–33.

[4] M. Hall Jr. and D. Wales, The Simple Group of Order 604,800, *J. Algebra* **9** (1968), 417–450.

[5] D.G. Higman and C.C. Sims, A simple group of order 44,352,000, *Math. Zeitschrift* **105** (1968), 110–113.

[6] A.J. Hoffman and R.R. Singleton, On Moore graphs with diameters 2 and 3, *IBM J. Res. Develop.* **4** (1960), 497–504.

[7] B. Huppert, *Endliche Gruppen I*, Springer-Verlag (1967).

[8] I. Ilić, N. Pace and S. Magliveras, Decomposing the Higman-Sims graph into double Petersen graphs, *J. Combin. Math. Combin. Comput.* **80** (2012), 267–275.

[9] T.B. Jajcayova and R. Jajcay, On the contributions of Dale Mesner, *Bull. Inst. Combin. Appl.* **36** (2002), 46–52.

[10] E.S. Kramer, S.S. Magliveras and D.W. Leavitt, Construction procedures for t-designs and the existence of new simple 6-designs, *Ann. Discr. Math.* **26** (1985), 247–274.

[11] S.S. Magliveras, *The subgroup structure of the Higman-Sims simple group*, Ph.D. Dissertation, Birmingham University, England (1970), 1–141 plus appendices.

[12] S.S. Magliveras, The subgroup structure of the Higman-Sims simple group, *Bull. Amer. Math. Soc.* **77** (4) (1971), 535–539.

[13] D.M. Mesner, A new family of partially balanced incomplete block designs with some Latin square design properties, *Ann. Math. Stat.* **38** (1967), 571–581.

[14] A. Seress, *Permutation Group Algorithms*, Cambridge Univ Press (2003).

[15] H. Wielandt, *Finite Permutation Groups*, Academic Press, New York (1964).

(Received 2 Jan 2012; revised 5 May 2012)