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On the spectrum of the forced matching number of graphs*

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Abstract

Let G be a graph that admits a perfect matching. A forcing set for a perfect matching M of G is a subset S of M, such that S is contained in no other perfect matching of G. This notion originally arose in chemistry in the study of molecular resonance structures. Similar concepts have been studied for block designs and graph colorings under the name defining set, and for Latin squares under the name critical set. Recently several papers have appeared on the study of forcing sets for other graph theoretic concepts such as dominating sets, orientations, and geodetics. Whilst there has been some study of forcing sets of matchings of hexagonal systems in the context of chemistry, only a few other classes of graphs have been considered.

Here we study the spectrum of possible forced matching numbers for the grids $P_m \times P_n$, discuss the concept of a forcing set for some other specific classes of graphs, and show that the problem of finding the smallest forcing number of graphs is NP–complete.

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1 Introduction and preliminaries

Let G be a graph that admits a perfect matching. A forcing set for a perfect matching M of G is a subset S of M, such that S is contained in no other perfect matching of G.

Example 1. *In Figure 1 a forcing set of size 6 is shown for a perfect matching in an 8×12 grid; that is $P_8 \times P_{12}$. The bold edges form a matching, and the edges in the forcing set are indicated by small circles.*

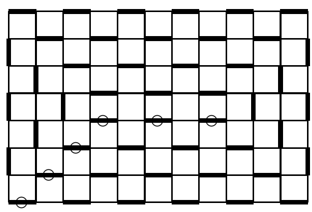

Figure 1: A forcing set for a perfect matching in $P_8 \times P_{12}$.

The matching in the Example 1 has a pattern which will be used in the next section. It is called a concentrated alternating cycles matching or a CACM of size 8×12, and is defined in general for a $P_{2m} \times P_{2n}$ as follows: a CACM of size $2m \times 2n$ is a special matching in $P_{2m} \times P_{2n}$, in which the vertices of the first row and also the last row are matched horizontally, and the remaining vertices of the first column and the last column are matched vertically, so that these matched edges form an alternating cycle. We continue this process recursively for the remaining vertices, which form a grid of size $(2m-2) \times (2n-2)$.

The smallest cardinality of a forcing set of M is called the forced matching number, and is denoted by $f(G, M)$, which we will henceforth call the forcing number of M. Also, let $f(G)$ and $F(G)$, respectively, denote the minimum and maximum of $f(G, M)$ over the set of all perfect matchings M of G. As all our matchings will be perfect, we drop the use of "perfect" after this point.

The notion of a forcing number originally arose in chemistry in 1987 in the study of molecular resonance structures [10]. Later, in [9], Harary introduced the concept of the forcing number of a perfect matching and of other concepts in graphs. Since then, papers have appeared on the forced orientation number of graphs [4, 7] , dominating sets [3], and geodetics [5].

Similar concepts have been studied under the name defining set for block designs [8, 17] for graph colorings [13], and under the name critical set for Latin squares [6, 2]. There has been some study of forcing sets of matchings of hexagonal systems (in the context of chemistry), and only a few other classes of graphs have been considered [14, 15, 11, 16]. One of the interesting problems is the study of the spectrum of forcing numbers of a given graph; to this end, the following definition is taken from [1].

Definition . *The* spectrum of forcing numbers *for a graph G is a set of natural numbers defined as:*
$\text{Spec}(G) = \{k \mid$ *there exists a matching M of G such that $f(G, M) = k$*$\}$.

The spectra of hypercubes is studied in [1]. In Section 2, we study the spectrum of $P_m \times P_n$ and show that there are no gaps in the spectra of forcing numbers of certain types of graphs which include $P_m \times P_n$ and stop signs. Recall that an (n, k) stop sign $(k \leq n-1)$ is a graph obtained from $P_{2n} \times P_{2n}$ by deleting all of the vertices along the k diagonals closest to each of the four corners [11]. In Section 3, we further discuss the concept of forcing numbers for some specific classes of graphs such as $P_m \times P_n$, $C_m \times P_n$, and $C_{2n} \times C_{2n}$. Finally in Section 4, we investigate the computational complexity of the problem of finding the forcing number of a graph.

2 Spectrum

A natural question is: which finite subsets of natural numbers are the spectra of some graph or other? In order to answer this question we need the following lemma.

Lemma 1. *If G is a graph with* $\text{Spec}(G) = A$*, then for any integer k, there exists a graph H with* $\text{Spec}(H) = \{x + k \mid x \in A\}$.

Proof. The graph H can be constructed by adding a union of k disjoint copies of C_4 (cycles of size 4) to G. Trivially $\text{Spec}(H) = \{x + k \mid x \in A\}$. ■

Next, for a given n we define a graph G_n by replacing alternate edges in C_{2n} by a cycle of size 4. Each of the edges of C_{2n} which is not replaced, is a *forcing* edge. This is illustrated for $n = 4$ in Figure 2. Each of the bold edges from C_8 is a forcing edge.

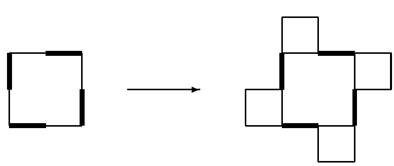

Figure 2: The graph G_4, obtained from C_8.

The following trivial lemma is to facilitate the proof of the subsequent theorem.

Lemma 2. *We have:* $\text{Spec}(G_i) = \{1, i\}$.

Theorem 1. *For any finite set $A \subset N$, there exists a graph G with* $\text{Spec}(G) = A$. *Indeed, G can be chosen to be a planar bipartite graph.*

Proof. Using Lemma 1, we can assume that $1 \in A$. Firstly, for every $i \in A$ $(i \neq 1)$, we assign a corresponding graph G_i, introduced above. We select one forcing edge from each G_i, and construct a graph G by identifying these G_i on the selected edges. Note that this common edge e is a forcing set for G. Thus $1 \in \mathrm{Spec}(G)$. We claim that $\mathrm{Spec}(G) = A$. If we have a matching M which *does not* contain e, then both ends of e must be matched with some other vertices in one of the G_i, say G_l. Then M generates l disjoint alternating cycles of size 4 in G_l, so any forcing set of M has at least l edges from G_l. Also observe that a forcing set of size l for G_l is also a forcing set for G. In fact the constructed graph G is planar and bipartite. ∎

Next, we study the spectra of some special graphs. First we give a simple proof of a theorem determining the spectrum of the grid $P_{2n} \times P_{2n}$. We then generalize that proof, to show that there are no gaps in the spectra of some specific graphs including $P_{2m} \times P_{2n}$ and stop signs.

So our result is that the spectrum of any such graph contains all the numbers between the smallest and the largest forcing number. Hence if we find the largest and the smallest forcing number for those graphs, then the spectrum is precisely determined.

Definition. *In a graph with a matching M, a* matching 2–switch *is an operation defined by the replacement of edges of M in an alternating cycle of size four with the edges not in M.*

The following lemma and its immediate corollary are instrumental to our results.

Lemma 3. *A* matching 2–switch *on a matching M does not change the forcing number by more than* 1.

Proof. Suppose that $e_1 = \{u_1, v_1\}$ and $e_2 = \{u_2, v_2\}$ are two edges of M that form an alternating cycle $(u_1 v_1 v_2 u_2)$. At least one of these two edges must be in the forcing set of M. Now consider a new matching M' which is obtained by removing the edges e_1 and e_2 from M, and adding $e'_1 = \{u_1, u_2\}$ and $e'_2 = \{v_1, v_2\}$ to it. If S is a forcing set for M, then $(S \cup \{e'_1, e'_2\}) \backslash \{e_1, e_2\}$ is a forcing set for M', so the forcing number of M' is at most one more than the forcing number of M. The same argument holds when we convert M' to M. ∎

Corollary 1. *In a graph G with a sequence of matchings $M_1, M_2, \ldots, M_s$, such that M_{i+1} is obtained from M_i by a* matching 2–switch*, all the numbers between $f(G, M_1)$ and $f(G, M_s)$ appear in the set consisting of the forcing numbers of $M_1, M_2, \ldots, M_s$.*

Now we are ready to determine the spectrum of forcing numbers of $P_{2n} \times P_{2n}$. Pachter and Kim proved the following theorem.

Theorem A. [15] *Let M be a matching of $P_{2n} \times P_{2n}$. Then $n \leq f(P_{2n} \times P_{2n}, M) \leq n^2$.*

In the following theorem we show that $f(P_{2n} \times P_{2n}, M)$ actually takes on *all* the values between n and n^2.

Theorem 2. *We have:* $\mathrm{Spec}(P_{2n} \times P_{2n}) = \{n, \ldots, n^2\}$.

Proof. By Corollary 1, it is sufficient to convert a matching with forcing number n^2 to a matching with forcing number n, by repeatedly applying matching 2–switches. We illustrate a process for this, using the example graph $P_6 \times P_6$ (that is when $n = 3$) in Figure 3. The matching M_{a_1} has forcing number n^2 (=9), and M_{a_s} which is a CACM has forcing number n (=3). It is easily seen that it is possible to convert M_{a_1} to M_{a_2} and M_{a_2} to M_{a_3} by applying matching 2–switches. By performing the same operations recursively on the inner $(2n-2) \times (2n-2)$ grid in M_{a_3}, we finally obtain the matching M_{a_s}.

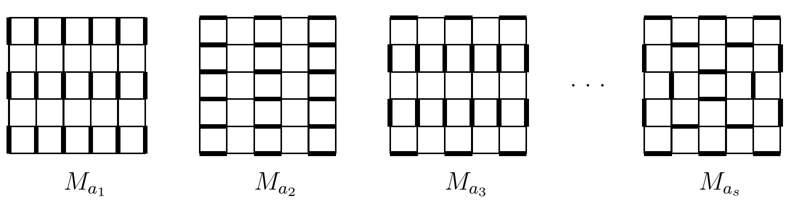

Figure 3: Applying matching 2–switches to reduce forcing numbers.

It should be easy to see that this procedure is valid for any n. Since $f(P_{2n} \times P_{2n}, M_{a_1}) = n^2$ and $f(P_{2n} \times P_{2n}, M_{a_s}) = n$, so $\mathrm{Spec}(P_{2n} \times P_{2n}) = \{n, \ldots, n^2\}$. ■

Next we generalize the method applied in the proof of Theorem 2 to more general graphs. To facilitate this, we label the vertices of $P_n \times P_n$ by ordered pairs (i, j), where $1 \leq i, j \leq n$; and i is the row number and j is the column number of that vertex.

Definition. *An induced subgraph G of a grid with vertex set $V(G)$ is called a* column continuous subgrid *if it has the following property:*

- *If $(i_1, j), (i_2, j) \in V(G)$ where $i_1 < i_2$, then for all integers i, such that $i_1 \leq i \leq i_2$, we have $(i, j) \in V(G)$.*

A row continuous subgrid also may be defined similarly. Note that a column continuous subgrid is not necessarily row continuous, but by rotating a column continuous subgrid, one obtains a row continuous subgrid. Suppose G is an induced subgraph of $P_n \times P_n$ which has a matching M. An (i, j, k)-bracket is a bracket shaped subset of the edges of M (e.g. Figure 4) as in the following:

$$\{ \ \{(i,j),(i,j+1)\}, \\ \{(i+1,j),(i+2,j)\},\{(i+3,j),(i+4,j)\},\ldots,\{(i+2k-1,j),(i+2k,j)\}, \\ \{(i+2k+1,j),(i+2k+1,j+1)\} \ \};$$

and the following set of edges is called an (i, j, k)-skew bracket (of type I) $(k > 0)$:

$\{ \ \{(i, j), (i, j+1)\},$
$\{(i+1, j), (i+2, j)\}, \{(i+3, j), (i+4, j)\}, \ldots, \{(i+2k-1, j), (i+2k, j)\},$
$\{(i+2k, j+1), (i+2k, j+2)\} \ \}.$

A skew bracket (of type II) is defined similarly as the following set of edges:

$\{ \ \{(i, j+1), (i, j+2)\},$
$\{(i, j), (i+1, j)\}, \{(i+2, j), (i+3, j)\}, \ldots, \{(i+2k-2, j), (i+2k-1, j)\},$
$\{(i+2k, j), (i+2k, j+1)\} \ \}.$

See Figure 4 for an example.

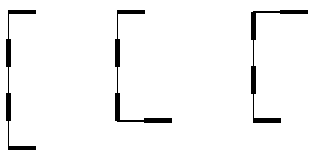

Figure 4: An $(i, j, 2)$-bracket and $(i, j, 2)$-skew brackets of type I and II.

Lemma 4. *Let G be a column continuous subgrid of $P_n \times P_n$. If M is a matching in G which contains an (i, j, k)-bracket, then we can apply matching 2-switches to M only on the edges which have both endpoints in the following set of vertices:*

$$\{(a, b) \mid i \leq a \leq i+2k+1, j \leq b \leq n\} \cap V(G),$$

so that the resulting matching contains the following edges:

$$\{\{(i, j), (i+1, j)\}, \{(i+2, j), (i+3, j)\}, \ldots, \{(i+2k, j), (i+2k+1, j)\}\}.$$

Proof. Note that we want to show that M can be changed to a matching such that all the edges in it which touch the set of vertices (a, j) in the j-th column, for $i \leq a \leq i+2k+1$, are all vertical. We apply mathematical induction on k. The case $k = 0$ is trivial. Suppose the statement is true for $k = p$. Consider an $(i, j, p+1)$-bracket. There are two cases.

The first case is where all the edges of M which touch the set of vertices $A = \{(i+1, j+1), (i+2, j+1), \ldots, (i+2p+2, j+1)\}$ are all vertical (obviously $A \subseteq V(G)$). It is easy to verify the lemma in this case.

If it is not the first case, then some of the edges which touch the set A are horizontal. The horizontal and vertical edges which touch A make some $(x, j+1, t)$-brackets, with $t \leq p$. We choose one of these brackets and apply the induction hypothesis to it, increasing the number of vertical edges which touch A by 1. By repeating this process we can convert all of the matching edges touching A to vertical matching edges, which is the first case. Note that the induction hypothesis ensures that converting an $(x, j+1, t)$-bracket does not have any effect on previously converted vertical edges. ■

Corollary 2. *Let G be a column continuous subgrid. If M is a matching in G, then by applying matching 2–switches we can convert M to a matching which contains no (i, j, k)-bracket.*

Proof. Let j $(1 \leq j \leq n)$ be the minimum value for which there exists some bracket in the j-th column. By using Lemma 4, we can destroy this bracket by matching 2–switches. If we continue this process, there will be no bracket left in this column, and so the value of j increases. Repeating this process removes all brackets. ■

Lemma 5. *Let G be a column continuous subgrid. If M is a matching in G in which there is no bracket, then there is also no skew bracket of any type in M.*

Proof. Assume to the contrary that M has no bracket, but that there does exist, for example, an (i, j, k)-skew bracket of type I in M. Since there is an odd number of vertices in the set $\{(i+1, j+1), (i+2, j+1), \ldots, (i+2k-1, j+1)\}$, the presence of matching edges in the (i, j, k)-skew bracket leads to the presence of at least one bracket in the column $j+1$, which is a contradiction. A similar argument holds if we assume that M contains a skew bracket of type II. ■

Theorem 3. *There are no gaps in the spectrum of a column continuous subgrid.*

Proof. Assume that G is a column continuous subgrid. We show that it is possible to convert a given matching of G to any other matching of G, by applying matching 2–switches.

Suppose we have two matchings in G. By Corollary 2 we remove all brackets from both of these matchings and end up with matchings say M and M'. If $M \neq M'$, then there exists a cycle which is alternating in M and M'. So if we consider the first column which is touched by this cycle, at least one of M and M' contains either a bracket or a skew bracket, and this contradicts Lemma 5 for neither M nor M' contains a bracket. ■

Note that the assumption that the graph involved is an "induced subgraph" of a grid is obviously necessary for the result of Theorem 3. Also the assumption that it be "column continuous" is necessary, as can be seen from the fact that $\text{Spec}(G_4) = \{1, 4\}$, where G_4 is shown in Figure 2. Indeed one can give infinitely many examples to show the necessity of this condition.

Since both $P_m \times P_n$ and the (n, k) stop sign are column continuous subgrids, we have the following corollary.

Corollary 3. *There are no gaps in the spectrum of forcing numbers of $P_m \times P_n$ and in the spectrum of forcing numbers of an (n, k) stop sign.*

The spectra of stop signs follow from the following theorem and Corollary 3.

Theorem B. [11] *Let G be an (n, k) stop sign and M be a matching of G. The forcing number of M is bounded by*

$$n \leq f(G, M) \leq (n - \lceil \frac{k-1}{2} \rceil)(n - \lfloor \frac{k+1}{2} \rfloor),$$

and the bounds are sharp.

3 Some special classes of graphs

In this section we study $F(G)$, where G is from some special classes of graphs: a product of two paths, a product of a cycle and a path, or a product of two cycles. We also introduce an upper bound for the smallest forcing number of a product of two paths. Pachter and Kim pointed out the following useful result.

Theorem C. [15, 12] *If G is a planar bipartite graph and M is a matching in G, then the forcing number of M is equal to the maximum number of disjoint M-alternating cycles.*

3.1 $P_m \times P_n$

Applying the same method as in [15] we see that:

$$F(P_m \times P_n) = \lfloor \frac{m}{2} \rfloor \cdot \lfloor \frac{n}{2} \rfloor.$$

In contrast, finding $f(P_m \times P_n)$ does not seem to be so easy. We introduce a pattern which gives an upper bound for it.

Theorem 4. *We have:*

- **(i)** $f(P_{2k} \times P_{(2k+1)l+r}) \leq kl + \lceil \frac{r-1}{2} \rceil$, *where* $0 \leq r \leq 2k$ *and* $l \geq 1$;
- **(ii)** $f(P_{2k+1} \times P_{(2k+2)l+2r}) \leq kl + r$, *where* $0 \leq 2r \leq 2k+1$ *and* $l \geq 1$.

Proof. We construct a matching M for which there is a forcing set of the desired size in the statement of the theorem.

(i) We choose the following l columns: $1, (2k+1)+1, 2(2k+1)+1, \ldots, (l-1)(2k+1)+1$; and also the last column if r is even. There are $2k$ vertices in each column, we take a matching in each of the chosen columns. Ignoring the chosen columns we have $l-1$ blocks of size $2k \times 2k$ and one block of height $2k$ and width varying with r. We substitute a CACM of appropriate size into each one of these blocks (see Figure 5).

This matching M has a forcing set of size $k(l-1) + \lceil \frac{2k+r-1}{2} \rceil = kl + \lceil \frac{r-1}{2} \rceil$ as shown in Figure 5.

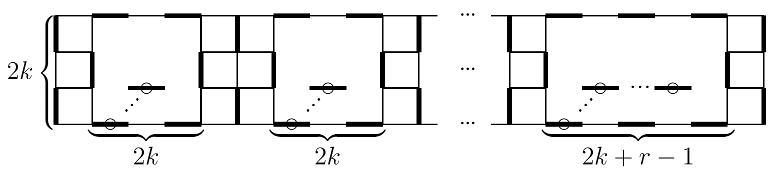

Figure 5: The pattern of M when r is odd.

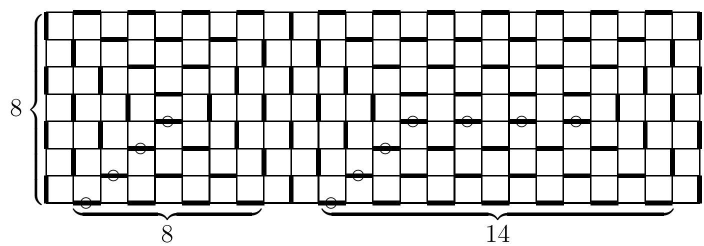

Figure 6: A forcing set of size 11 for $P_8 \times P_{25}$.

Figure 6 demonstrates M for $P_8 \times P_{25}$.

(ii) To deal with this case we construct a matching in a similar fashion to that of the previous case. To facilitate this, we introduce some notation. A UCACM and a DCACM of size $(2m-1) \times 2n$ are built from a CACM of size $2m \times 2n$ by removing the vertices of the first row, and the last row, respectively.

In this case we partition $P_{2k+1} \times P_{(2k+2)l+2r}$ to $(l-1)$ blocks of size $(2k+1)\times(2k+2)$ and one last block of size $(2k+1) \times (2k+2r+2)$, and then replace each block alternatively with a UCACM or a DCACM of appropriate size. This is illustrated in Figure 7 for the case $P_5 \times P_{28}$.

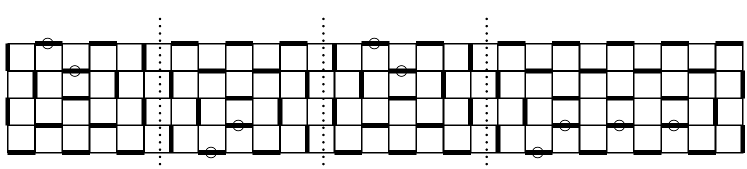

Figure 7: A forcing set of size 10 for $P_5 \times P_{28}$.

Again the resulting matching has a forcing set of the desired size. ■

Note that in the previous theorem, in each case there are appropriate number of M-alternating disjoint cycles which Theorem C implies that the size of the corresponding forcing sets are smallest. Based on observations of small cases, we conjecture that the bounds in Theorem 4 are sharp.

3.2 $P_m \times C_{2n}$

The following theorem gives the exact value for the size of a largest forcing set for $P_m \times C_{2n}$.

Theorem 5. *For every $k, n \geq 1$ we have:*

$$F(P_{2k} \times C_{2n}) = kn \text{ and } F(P_{2k+1} \times C_{2n}) = kn + 1.$$

Proof. Consider $P_m \times C_{2n}$ drawn as $2n$ "vertical" copies of P_m and m "horizontal" copies of C_{2n} on the set of vertices in the columns. The graph $P_m \times C_{2n}$ is planar and bipartite, so by Theorem C for any matching M, $f(P_m \times C_{2n}, M)$ is equal to the maximum number of disjoint M-alternating cycles.

Since the girth of $P_{2k} \times C_{2n}$ is 4, its largest forcing number is not greater than $\frac{4kn}{4} = kn$. A matching which has all edges horizontal clearly has forcing number kn.

For $P_{2k+1} \times C_{2n}$, suppose that M is a matching, and let $\mathcal{A}$ be a set of disjoint M-alternating cycles. If there is an M-alternating cycle in $\mathcal{A}$ which intersects a column exactly *once*, then it is at least of size $2n$. In this case there are at most $\frac{(2k)(2n)}{4} = kn$ other cycles in $\mathcal{A}$, and we are done.

So assume that there is no M-alternating cycle in $\mathcal{A}$ which intersects some column in exactly one vertex. In $\mathcal{A}$, each cycle has at least two vertices of intersection with each column that it intersects, so each column intersects at most k cycles in $\mathcal{A}$. Now, as there are $2n$ columns, if we count all cycles, we get $k(2n)$. But in this way each cycle is counted at least twice, as it intersects at least two *different* columns. So there are at most $\frac{k(2n)}{2} = kn$ cycles.

In this case, a matching which has all edges horizontal clearly has forcing number equal to $kn + 1$. ■

The following interesting problems remain open.

Problem 1. *Find* $F(P_{2m} \times C_{2n+1})$.

Problem 2. *Find* $f(P_m \times C_n)$.

3.3 $C_{2n} \times C_{2n}$

It is conjectured in [16] that $F(C_{2n} \times C_{2n}) = n^2$. A result in this direction is given in the following theorem.

Theorem 6. *We have:* $F(C_{2n} \times C_{2n}) \leq n^2 + \frac{n}{2}$.

Proof. Let M be a matching in $C_{2n} \times C_{2n}$ which has the largest forcing number. We show that there exists a forcing set of size less than or equal to $n^2 + \frac{n}{2}$ for M. The number of edges in M is $2n^2$, and at least n^2 of these edges are in the same direction ("horizontal" or "vertical"). Without loss of generality, suppose at least n^2 of the edges in M are horizontal. So there exists a row, say r in which at least $\frac{n}{2}$ edges of M are horizontal. Thus, there are at most $n + \frac{n}{2}$ matching edges which touch this row. We take all these matching edges in our forcing set.

Removing the vertices we chose in our forcing set, we get a planar graph, and we consider two cases. First, the case in which all the matching edges of row r are horizontal. In this case, we have already chosen n edges and the rest of the graph is a $P_{2n-1} \times C_{2n}$, which by Theorem 5 needs at most $n(n-1)+1$ edges to be forced. In the second case, we have chosen at most $n + \frac{n}{2}$ edges and the graph obtained after deleting those vertices has at most $2n-1$ vertices in each column and also has at least one column with exactly $2n-2$ vertices. Since we have a column which contains $2n-2$ vertices, by using the technique of the previous theorem, we can say that the largest forcing number of the resulting graph is at most $n(n-1)$. So the forcing number of M is at most $n(n-1) + n + \frac{n}{2} = n^2 + \frac{n}{2}$. ■

4 Computational complexity

In [1], Adams, Mahdian, and Mahmoodian studied the following problem and gave a proof for its NP-completeness.

- Smallest forcing set problem
  Instance: A graph G, a matching M in G, and an integer k.
  Question: Is there any subset S of at most k edges of M, such that S is a forcing set for M?

Theorem D. [1] Smallest forcing set *is NP-complete for bipartite graphs with maximum degree* 3.

They also left an open question which we answer in this section. The question is finding the computational complexity of the following problem:

- Smallest forcing number of graph
  Instance: A graph G and an integer k.
  Question: Is there any matching in G with the forcing number of at most k?

We use Theorem D to prove that this problem is also NP-complete even for bipartite graphs with maximum degree 4.

Theorem 7. Smallest forcing number of graph *is NP-complete for bipartite graphs with maximum degree* 4.

Proof. It is clear that the problem is in NP. We prove the NP-completeness by reducing SMALLEST FORCING SET to this probem. Let G be a bipartite graph with maximum degree 3 and M_G be a matching in G. We construct a new graph H with maximum degree 4 as follows:

- G is a subgraph of H, and
- For any edge $e = \{x, y\} \in E(G) - M_G$, we add vertices x_e and y_e to H plus three edges $\{x, y_e\}$, $\{x_e, y_e\}$, and $\{x_e, y\}$.

Note that H satisfies the conditions of the theorem and any forcing set for the matching M_G also forces a matching in H. We claim that the smallest forcing number of H is equal to the smallest forcing number of M_G. We can assume that x_e is matched to y_e, otherwise we have the following case: x_e is matched to y and y_e is matched to x. Any forcing set contains one of these two edges, and choosing one will force the choice of the other edge. So it is obvious that in this case a matching 2–switch on these edges will not change the forcing number. With this assumption, every matching in G corresponds uniquely to a matching in H and vice versa. For every matching M'_G in G, we denote the corresponding matching in H by M'_H. Now consider a matching L_G in G. For every edge $e = \{x, y\}$ in $L_G - M_G$, the four vertices x, y, x_e, and y_e constitute an alternating cycle for L_H, so at least one edge from this alternating cycle should be in the forcing set, and since choosing the edge e forces the choice of the other edge, we can assume that e is in the forcing set. Thus a forcing set F for L_H consists of $L_G - M_G$ plus some edges in $L_G \cap M_G$. It is not hard to see that $F' = (M_G - L_G) \cup (F \cap M_G)$ is a forcing set for M_G. Since $|M_G - L_G| = |L_G - M_G|$ and $F \subseteq L_G$, we have $|F'| = |F|$. ∎

For the problem of finding a smallest forcing set for a given matching in a planar graph, we have a polynomial algorithm [15], so it is interesting to ask the following question:

Question 1. *What is the computational complexity of the following problem: Given a planar graph G, find the smallest forcing number of G.*

After studying the computational complexity of the problem of finding the smallest forcing number of a graph it is natural to do the same for the largest forcing number. So we ask also the following question, and leave it as an open problem.

Question 2. *What is the computational complexity of the following problem: Given a graph G, find the largest forcing number of G.*

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