Thwart Numbers of Some Bipartite Graphs

D. G. Hoffman* and P. D. Johnson, Jr.*
Department of Discrete and Statistical Sciences
Auburn University, Alabama 36849, U.S.A.

E. B. Wantland
Department of Mathematics and Computer Science
Western Connecticut State University
Danbury, Connecticut 06810, U.S.A.

Abstract

For a simple graph G, let $c(G)$ denote the choice number of G, and for $k \geq \chi(G)$ let $c_k(G)$ be defined as $c(G)$ is defined, except that there are only k colors available to form the lists of colors available to the vertices. The *thwart number* of G, denoted $thw(G)$, is the smallest k such that $c_k(G) = c(G)$. To put it another way, $thw(G)$ is the smallest number ($\geq \chi(G)$) of colors you need in order to assign $c(G) - 1$ colors to each vertex of G in a manner so fiendishly contrived that all attempts to properly color G from these lists will be *thwarted.*

We survey what little is known about the thwart number in general, and use earlier work on choice numbers and restricted choice numbers to obtain results on thwart numbers of bipartite graphs. For instance, we show that $thw(K_{m,n}) = m^2$ for $n \geq m^m$, provided $m \geq 2$, and that $(m - \frac{3}{2})(m-1) \leq thw(K_{m,n}) \leq (m-1)^2$ for $(m-1)^{m-1} - (m-2)^{m-1} \leq n < m^m$, if $m \geq 3$. We also make a start toward characterizing bipartite graphs with thwart number 3.

1 Introduction, Open Problems and Generalities

The choice number $c(G)$ of a simple graph G is the smallest positive integer ℓ such that whenever the vertices of G are assigned lists (sets) of length (size) $\geq \ell$, there will be a proper coloring of G from these lists. (This means that each vertex gets a color from its list, and adjacent vertices have different colors.)

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Since a proper coloring is always possible with lists of length $c(G)$, it will be possible if all the lists consist of the same $c(G)$ colors; it follows that $\chi(G) \leq c(G)$. In Figure 1 we have the two smallest, simplest graphs for which this inequality is strict. They obviously have chromatic number 2, and they are supplied with lists of length 2 from which proper colorings are not possible, so they have choice number > 2. In fact, they each have choice number 3. We leave this as an exercise for now, but it also follows from some basic results about choice numbers that will be mentioned later.

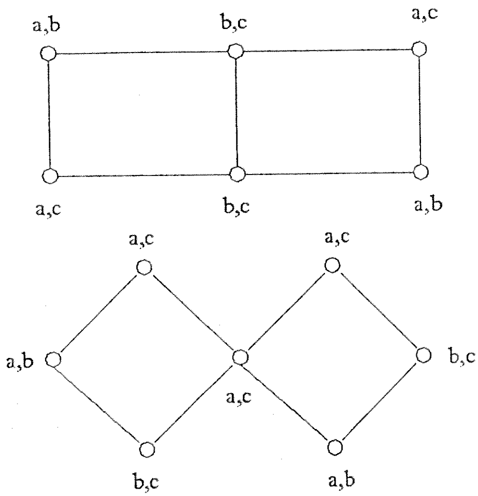

Figure 1

Notice that the list assignments to these two graphs use only three colors.

The choice number was defined independently by Erdös, Rubin, and Taylor [1] and by Vizing [11]. (Vizing's paper was in Russian, and was not known of in the non-Russian-speaking world for some years.) In both papers it is proven that Brooks' Theorem holds for the choice number: if G is connected and neither complete nor an odd cycle, then $c(G) \leq \Delta(G)$. (If G is complete, or an odd cycle, then $c(G) = \chi(G) = \Delta(G)+1$.) Notice that this implies that the first graph in Figure 1has choice number three.

In [5] we took what seemed a natural step and defined the restricted choice number $c_k(G)$ as $c(G)$ is defined, except that the lists assigned to the vertices are to be formed from a fixed set of k colors. The restricted choice number $c_k(G)$ is defined only when $k \geq \chi(G)$. It is straightforward to see that $c_{\chi(G)}(G) = \chi(G)$, that $\chi(G) \leq c_k(G) \leq \min(k, c(G))$, and that as k goes up from $\chi(G)$, $c_k(G)$ non-decreases, in leaps of lengths 0 or 1, from $\chi(G)$ to $c(G)$.

We also, in [5], introduced the thwart number of G, $thw(G)$, defined to be the smallest k such that $c_k(G) = c(G)$. To put it another way, $thw(G)$ is the smallest $k \geq \chi(G)$ such that the vertices of G can be supplied with subsets of a k-set of size

$c(G)-1$ from which no proper coloring is possible. It follows that each of the graphs in Figure 1 has thwart number 3, if you accept that they each have choice number 3.

We collect the few easy generalities that we know about the thwart number in the following theorem.

Theorem 1 *Suppose G is a simple graph. Then*

(a) $thw(G) = \chi(G)$ if and only if $c(G) = \chi(G)$;

(b) $thw(G) \geq c(G)$; and

(c) if H is a subgraph of G satisfying $c(H) = c(G)$, then $thw(H) \geq thw(G)$.

We postpone the proof of this, as of all our results, until Section 3.

Problems and Comments

1.1 When we introduced the thwart number, in [5], we posed the problem of finding "good" upper bounds on $thw(G)$. Since $thw(G)$ is the smallest number $\geq \chi(G)$ of colors from which you can assign certain lists of length $c(G)-1$ to the vertices of G, it is evident that $thw(G) \leq (c(G)-1)n$, where $n = |V(G)|$, if $c(G) > 1$, but this is not "good".

Now Paul O'Donnell has shown [8, 9] that $thw(G) \leq (c(G)-1)\sqrt{n}$, if G is non-trivial and $n \geq 4$, and he tells us that he has also shown that $thw(G) \leq n$, so our problem seems quite satisfyingly disposed of, although there may be other good upper bounds awaiting discovery. However, with regard to O'Donnell's second result, one can ask: does $thw(G) = n = |V(G)|$ imply that G is complete? Perhaps there is an answer embedded in O'Donnell's proof.

1.2 Regarding Theorem 1(c), we will see in Theorems 2 and 3, with accompanying comments, that examples abound in which H is a subgraph of G satisfying $c(H) = c(G)$ and $thw(H) > thw(G)$.

We long wondered whether or not, if you delete a vertex from a graph without changing the choice number, the thwart number can jump by more than one. It is a consequence of the proof of Theorem 2(b), below, that this is indeed the case. By that proof, for $m \geq 4$, $thw(K_{m,n}) = \lceil (m-\frac{3}{2})(m-1) \rceil$ for n ranging from approximately $\frac{3}{4}(m-1)^{m-1}$ up through $m^m - 1$. For $m \geq 5$, take $(m-1)^{m-1} \leq n < m^m$. By Theorem A below (Theorem 2 of [4]), $c(K_{m,n}) = c(K_{m-1,n}) = m$. By Theorem 2 of this paper, $thw(K_{m-1,n}) = (m-1)^2$ while, as explained above, $thw(K_{m,n}) = \lceil (m-\frac{3}{2})(m-1) \rceil$.

We still do not know if removing an edge without changing the choice number can cause the thwart number to jump by more than one.

1.3 When $c(G-v) < c(G)$, we know from Theorem 2, below, that the thwart number can drop precipitously from $thw(G)$ to $thw(G-v)$. However, we do not know of any reason why the thwart number has to non-increase as you pass from a graph to a subgraph with a smaller choice number. Nor do we know of any general estimates on how far the thwart number can drop, from $thw(G)$ to $thw(G - v)$, when it does decrease.

1.4 What can be said about the graphs G extremal for the inequality $thw(G) \geq c(G)$? In plain language, which G satisfy $thw(G) = c(G)$? By Theorem 1, the graphs G satisfying the intriguing equation $c(G) = \chi(G)$ (see [3]) are among those satisfying $thw(G) = c(G)$. It feels as though these are a special case; it might be useful to assume that $thw(G) = c(G) > \chi(G)$. We give all we know about the simplest case in Theorem 3, on bipartite graphs with thwart number 3. (These graphs have choice number 3, by Theorem 1).

1.5 We end this section by confessing to two outright oversights in [5].

1.5.1 In [5] we claim that the results and proofs in [4] "show that... $thw(K_{m,n}) = (m-1)^2$ if $m \geq 4$ and $(m-1)^{m-1} - (m-2)^{m-1} \leq n < m^m$". Theorem 2(b) here gives the true state of affairs, which surprised us.

1.5.2 In [4] it is shown that $c(K_{m,m})/\log_2 m \to 1$ as $m \to \infty$. In [5], for what reason we cannot now recall, we conjectured that $thw(K_{m,m})$ is asymptotically $2\log_2 m$. Paul O'Donnell [9] now tells us that $thw(K_{m,m})$ is more like $(\log_2 m)^2$. We await his proof, but $(\log_2 m)^2$ does seem more reasonable, for vague reasons, and O'Donnell has not been wrong yet, in our experience.

2 Main Results, and More Problems

Theorem 2 (a) *If $m \geq 2$ and $n \geq m^m$, then $thw(K_{m,n}) = m^2$.*
(b) $thw(K_{3,n}) = 3$, $3 \leq n \leq 26$.
If $m \geq 4$ and $(m-1)^{m-1} - (m-2)^{m-1} \leq n < m^m$, then $(m - \frac{3}{2})(m-1) \leq thw(K_{m,n}) \leq (m-1)^2$; furthermore, for these values of m and n, $thw(K_{m,n})$ takes on every integer value in the indicated range.

In fact, by the method of the proof of Theorem 2(b), one can easily calculate each $thw(K_{m,n})$, $m \geq 4$, $(m-1)^{m-1} - (m-2)^{m-1} \leq n < m^m$. For instance, we find that $thw(K_{4,n}) = 9$, $n = 19, 20$, and $thw(K_{4,n}) = 8$, $21 \leq n \leq 255$. However, giving the exact values of $thw(K_{m,n})$ in Theorem 2(b) would have greatly complicated the statement, so we opt for referring those interested to the proof.

The only other values of $n \geq m \geq 4$ for which we know $thw(K_{m,n})$ are given in Theorem 3.

For Theorem 3 we need some definitions. If $v \in V(G)$, the *closed neighborhood contraction of G at v* is the simple graph obtained from G by collapsing v and all its neighbors into a single vertex, adjacent to any vertex (in G) not in $N[v] = \{v\} \cup N(v)$ which was adjacent to any neighbor of v, in G. (That is, combine v and its neighbors, erase all loops resulting, and reduce all resulting multiple edges to single edges.)

Deleting a vertex of degree 1 is a special case of a closed neighborhood contraction; it has the property that it does not change the values of the choice number, nor of the restricted choice numbers, and thus not of thwart number, unless the edge that disappears is the only edge in the graph (in which case everything drops from 2 to 1). The graph resulting from G by repeatedly deleting vertices of degree 1 until none remain will be called the *torso* of G. (Erdös, Rubin, and Taylor [1] call this the *core* of G, but the term *core* sometimes means something different – see [6].)

Following [1], for r, s, t positive integers, at most one equal to 1, let $\theta(r, s, t)$ stand for the graph consisting of two vertices connected by three internally disjoint paths of lengths r, s, and t. Notice that $K_{2,3} = \theta(2, 2, 2)$.

A graph will be said to be *between* two graphs, if it contains one as a subgraph, and is a subgraph of the other.

Theorem 3 (i) *Connected graphs with any of the following torsos (in (a) - (f)) have thwart number* 3*:*

(a) $\theta(r, s, t)$, *with* r, s, t *all even or all odd, at most one equal to one, and at most one equal to two;*

(b) two even cycles connected only by a path (possibly of length zero) that intersects them only at its end-points;

(c) a graph between $K_{3,3}$ *and* $K_{3,27}$ *minus an edge;*

(d) a graph between $K_{4,4}$ *and* $K_{4,18}$*;*

(e) a graph between $K_{5,5}$ *and* $K_{5,12}$*;*

(f) a graph between $K_{6,6}$ *and* $K_{6,10}$*.*

(ii) *If* G *is bipartite with choice number* 3*, and there is a sequence of closed neighborhood contractions that transforms* G *into one of the above, then* $thw(G) = 3$.

(iii) *If* G *contains as a subgraph one of* $K_{3,27}, K_{4,19}, K_{5,13}, K_{6,11}$*, or* $K_{7,7}$*, then* $thw(G) \geq c(G) \geq 4$*; if* G*'s torso is between* $K_{2,4}$ *and* $K_{2,n}$*, for some* $n \geq 4$*, then* $thw(G) = 4$.

This theorem leaves unknown which of the graphs strictly between $K_{4,18}$ and $K_{4,19}$, or $K_{5,12}$ and $K_{5,13}$, or $K_{6,10}$ and $K_{6,11}$, or $K_{6,7}$ and $K_{7,7}$, have thwart number 3. We invite the industrious to fill these gaps.

As an example of the use of Theorem 3 (ii), it follows from Brooks' Theorem for the choice number and Theorem B, below, that if G consists of a sequence of three or more distinct, disjoint even cycles with each pair of neighbors in the sequence joined by a path that intersects the union of the cycles only at its ends, then G has choice number 3. Clearly there is a sequence of closed neighborhood contractions that takes G into a graph as described in i(b). Thus G has thwart number 3.

The usefulness of part (ii) of the theorem is greatly diminished by the difficulty of deciding when a bipartite graph has choice number 3. It has only recently been worked out ([2], [7], [8], [10]) which complete bipartite graphs have choice number 3, so we doubt very much that there is any easy characterization of bipartite graphs with choice number three. Perhaps an expert in algorithmic complexity could put us out of our misery on this matter by showing that it is very hard to decide whether or not a given bipartite graph has choice number 3.

For those who are interested in the *thwarting list assignments*, the assignments to the vertices of G of lists of length $c(G)-1$ of elements of a set with $thw(G)$ elements, from which no proper coloring is possible, here is what we know.

1. In the cases covered by Theorem 2, the thwarting assignments are essentially forced, and can be extracted from the proof.

2. In Theorem 3(i)(a), (b) and (ii), critical assignments may be obtained from those in Figures 1 or 2 (below), or from critical assignments in the cases covered by (i) (c) - (f), by a certain process that will become clear in Section 3, involving reversing the process of closed neighborhood contraction. However, it is not always true that all critical list assignments arise in this way, in these cases.

3. In Theorem 3 (i)(c) - (f), all graphs contain a copy of $K_{3,3}$; you make a thwarting list assignment by putting a thwarting list assignment on some copy of $K_{3,3}$ and then completing the list assignment any way you want, assigning 2-subsets of the 3-set of colors to the remaining vertices. Furthermore, every thwarting list assignment arises in this way, in these cases. The unique thwarting list assignment to $K_{3,3}$ assigns all three 2-subsets of the 3-set of colors to each side of the bipartition.

4. The thwarting list assignment to $K_{3,27}$, which has choice number 4 and thwart number 9, is described in the proof of Theorem 2(a), as are the thwarting list assignments to $K_{2,n}$, $n \geq 4$, which all have choice number 3 and thwart number 4. $K_{4,19}$, which has choice number 4 and thwart number 9, is dealt with in the proof of Theorem 2(b). As for the other graphs mentioned in Theorem 3(iii), namely $K_{5,13}, K_{6,11}$, and $K_{7,7}$, they all have choice number 4, and it is not known what their thwart numbers are, although it is possible that these and the thwarting list assignments can be extracted from [2], [7], [8], and [10]. It is intriguing that Erdös, Rubin, and Taylor [1] give a list assignment to $K_{7,7}$ from which no proper coloring is possible in which the lists on each side of the bipartition are the triples in a Steiner triple system of order 7. Or, if you prefer, the lists on each side are the lines in some incarnation of the Fano plane. This shows that $thw(K_{7,7}) \leq 7$ and gives powerful mystical support to the conjecture that $thw(K_{7,7}) = 7$ and that the only critical list assignments to $K_{7,7}$ are of the form described. Again, we leave these matters to those more energetic than we are.

3 Proofs and Intermediate Results

Proof of Theorem 1. (a) and (b): By definition, we always have $thw(G) \geq \chi(G)$.

If $c(G) = \chi(G)$, then a constant list assignment of $\chi(G) - 1 = c(G) - 1$ colors will thwart a proper coloring, so you certainly do not need more than $\chi(G)$ colors to make such an assignment; thus $thw(G) = \chi(G)$.

Suppose $c(G) > \chi(G)$. Since we need at least $c(G) - 1$ colors to form lists of length $c(G) - 1$, we have that $thw(G) \geq c(G) - 1$; if $thw(G) = c(G) - 1$ then the only list assignment of lists of length $c(G) - 1$ from a set of the same size would be a constant list assignment, with all $c(G) - 1$ colors available at every vertex. But $c(G) - 1 \geq \chi(G)$, so a proper coloring would be possible from such a list assignment. Therefore $thw(G) \geq c(G)$, if $c(G) > \chi(G)$. This completes the proof of (a) and (b).

As for (c), it is elementary that for each $k \geq \chi(G) \geq \chi(H)$, $c_k(H) \leq c_k(G) \leq c(G) = c(H)$. If $r = thw(H)$, then $r \geq c(H) = c(G) \geq \chi(G)$, by (b), and $c_r(H) = c(H) \leq c_r(G) \leq c(G) = c(H)$, so $c_r(G) = c(G)$. Therefore, r is no less than $thw(G)$, the smallest $k \geq \chi(G)$ such that $c_k(G) = c(G)$. □

The proof of Theorem 2 uses the following result from [4].

Theorem A *(part of Theorem 2 [4]). If $n \geq m^m$, $m \geq 1$, then $c(K_{m,n}) = m + 1$. If $m \geq 3$ and $(m-1)^{m-1} - (m-2)^{m-1} \leq n < m^m$, then $c(K_{m,n}) = m$.*

In fact, the proof of Theorem 2 below is mainly a refinement of part of the proof of Theorem A in [4]. We warn the reader who looks in [4] that "critical list assignment" there has a somewhat different meaning than "thwarting list assignment" does here. In [4] there is no restriction on the number of colors used to form the lists of length choice number minus one. A thwarting list assignment is a critical list assignment that uses as few colors as possible.

Proof of Theorem 2. (a) Suppose that $m \geq 2$ and $n \geq m^m$. From Theorem A, $c(K_{m,n}) = m + 1$. First, assign m disjoint sets of m colors each to vertices in M, the side of the bipartition of $K_{m,n}$ with m vertices, and, from the same stock of m^2 colors, assign lists of length m to the vertices in N, the other side of the bipartition, taking care to include in these lists all m^m transversals of size m of the lists on M. Clearly a proper coloring of $K_{m,n}$ is impossible from these lists. This shows that $thw(K_{m,n}) \leq m^2$.

Now consider any assignment of lists of length $m = c(K_{m,n}) - 1$ to $M \cup N$, from which no proper coloring of $K_{m,n}$ is possible. If two lists on vertices of M have a color in common, then we can color those two vertices with the same color, from their lists, and proceed to a coloring of M that uses no more than $m - 1$ colors. Since each vertex in N has m colors available, we can color N from its lists so that no color on N is among the colors we already have on M. This would constitute a proper coloring of $K_{m,n}$, from the lists. Therefore, the lists on M must be pairwise disjoint, so m^2 colors appear on them. Thus $thw(K_{m,n}) \geq m^2$, so $thw(K_{m,n}) = m^2$.

(b) As noted in [4] and earlier in this paper, the list assignment to $K_{3,3}$ consisting of the three two-subsets of $\{1, 2, 3\}$ on each side of the bipartition thwarts a proper coloring. Since $c(K_{3,n}) = 3$, $3 \leq n < 27$, by Theorem A, it follows from Theorem 1 that $thw(K_{3,n}) = 3$, $3 \leq n \leq 26$. (Indeed, by the proof of (a), the choice number of [K_{m,m^m} minus an edge] is m, for $m \geq 2$, so it follows that $thw(G) = 3$ for any graph G between $K_{3,3}$ and [$K_{3,27}$ minus an edge].)

From here on (the cases $m \geq 4$), the spirit of the proof of (b) is like that of (a). Assuming $m \geq 4$, $(m-1)^{m-1} - (m-2)^{m-1} \leq n < m^m$, the question is, how many colors do you need to put lists of length $m-1$ on $M \cup N$ so that there are no transversals (sets of representatives) of the lists on M of size $\leq m-2$, and more generally, such that each transversal of the lists on M contains some list on N? Suppose that we have such a list assignment, and let $L_1, \ldots, L_m$ be the lists on M. The requirement that the L_i have no transversal of size $\leq m-2$ implies that no color belongs to three of the L_i, and that if two of the L_i intersect, then the others are pairwise disjoint. (Otherwise, 4 of the L_i would have a transversal of size 2, and thus the whole lot would have a transversal of size $\leq m-2$.)

These considerations boil down the possibilities to the following.

I. There is one of the L_i, say L_1, which might possibly intersect each of $L_2, \ldots, L_m$, and $L_2, \ldots, L_m$ are pairwise disjoint.

II. There are three of the L_i, say L_1, L_2, L_3, which might possibly intersect each other, and $L_4, \ldots, L_m$ (if $m > 3$) are pairwise disjoint and disjoint from each of L_1, L_2, L_3. Furthermore, $S_{12} = L_1 \cap L_2$, $S_{13} = L_1 \cap L_3$, and $S_{23} = L_2 \cap L_3$ are pairwise disjoint.

Notice that in case I, at least $(m-1)^2$ colors are used, because $L_2, \ldots, L_m$ are pairwise disjoint sets of size $m-1$. If you arrange for $|L_1 \cap L_j| = 1$, $j = 2, \ldots, m$, then exactly $(m-1)^2$ colors are used, and there are exactly $(m-1)^{m-1} - (m-2)^{m-1}$ transversals of the L_i with $m-1$ elements (see [4] or do the calculation!); furthermore, every transversal of the L_i contains one of these, so you can thwart a coloring by assigning the transversals of the L_i of size $m-1$ to the vertices in N (and filling in the assignment of lists of length $m-1$ from the $(m-1)^2$ colors any way you want, in case $n > (m-1)^{m-1} - (m-2)^{m-1}$). This shows that $thw(K_{m,n}) \leq (m-1)^2$ for $(m-1)^{m-1} - (m-2)^{m-1} \leq n < m^m$.

In [4] it is shown that for $m \geq 5$, $n = (m-1)^{m-1} - (m-2)^{m-1}$, there are no list assignments to $K_{m,n}$ of type II of lists of length $m-1$ that thwart a coloring, so $thw(K_{m,n}) = (m-1)^2$ when $m \geq 5$, $n = (m-1)^{m-1} - (m-2)^{m-1}$. When $m = 4$, it is shown in [4] that there is a thwarting list assignment of type II of lists of length 3 to $K_{4,19}$, but it uses 9 colors anyway.

Therefore, for $m \geq 4$, $thw(K_{m,n})$ starts out at $(m-1)^2$ when $n = (m-1)^{m-1} - (m-2)^{m-1}$, and by Theorem 1 (c) and Theorem A, non-increases as n rises to $m^m - 1$.

To see how and when $thw(K_{m,n})$ falls as n rises we consider list assignments of type II (since all those of type I use at least $(m-1)^2$ colors). Let $x = |S_{12}|$, $y = |S_{13}|$, and $z = |S_{23}|$. Note that $x+y$, $x+z$, $y+z \leq m-1$, so $x+y+z \leq \frac{3}{2}(m-1)$. Furthermore, it is clear that arrangements can be made for $x+y+z$ to take any value between 0 and $\lfloor \frac{3}{2}(m-1) \rfloor$, inclusive. We are interested in the values of $x+y+z$ between m and $\lfloor \frac{3}{2}(m-1) \rfloor$, because the number of colors in $L_1 \cup \cdots \cup L_m$ is $m(m-1) - (x+y+z)$, which, as $x+y+z$ rises from m to $\lfloor \frac{3}{2}(m-1) \rfloor$, falls from $(m-1)^2 - 1$ to $\lceil (m - \frac{3}{2})(m-1) \rceil$. This shows that

$$(m - \frac{3}{2})(m-1) \leq thw(K_{m,n}) \leq (m-1)^2$$

for all n between $(m-1)^{m-1}-(m-2)^{m-1}$ and m^m-1, inclusive if $m \geq 4$. What remains to be shown is that all the values from $\lceil (m-\frac{3}{2})(m-1) \rceil$ to $(m-1)^2-1$ are actually taken on by $thw(K_{m,n})$, with n in that range. (We already know that $thw(K_{m,n})=(m-1)^2$ if $n=(m-1)^{m-1}-(m-2)^{m-1}$, $m \geq 4$.)

Since thwarting list assignments of length $m-1$ to $K_{m,n}$, for n in the range supposed, are either of type I or type II, either $thw(K_{m,n})=(m-1)^2$ or, if there is a type II thwarting assignment to $K_{m,n}$ with $x+y+z \geq m$, $thw(K_{m,n})=m(m-1)-(x+y+z)$ for the largest value of $x+y+z$ for which there is such an assignment. Now, $L_1,\ldots,L_m$ of type II are the lists assigned to M in a thwarting list assignment to $K_{m,n}$ (of length $m-1$) if and only if there exist n or fewer $(m-1)$-subsets of $L_1 \cup \cdots \cup L_m$, say $Z_1,\ldots,Z_k$, $k \leq n$, such that each transversal of $L_1,\ldots,L_m$ contains one of the Z_i. (These Z_i will then be among the lists assigned to N in the thwarting list assignment.) For short, let us say that the Z_i *infest the transversals* of $L_1,\ldots,L_m$. In [4] it is shown that the minimum size of a collection of $(m-1)$-sets that infest the transversals of $L_1,\ldots,L_m$, if $L_1,\ldots,L_m$ are of type II and $m \geq 4$, is

$$\begin{aligned} f(x,y,z) &= (m-1)^{m-3}[(m-1)(x+y+z)-(xy+xz+yz)] \\ &\quad +(m-1)^{m-4}(m-1-x-y)(m-1-x-z)(m-1-y-z) \end{aligned}$$

The discussion above reduces to: for $m \geq 4$ and n in the supposed range, $thw(K_{m,n})$ is the minimum of $(m-1)^2$ and the numbers $\{m(m-1)-(x+y+z);\ x,y,z$ are non-negative integers, $x+y, x+z, y+z \leq m-1$ and $n \geq f(x,y,z)\}$.

For $s=m,\ldots,\lfloor \frac{3}{2}(m-1) \rfloor$, let f_s denote the smallest number in $\{f(x,y,z);$ x,y,z are non-negative integers, $x+y$, $x+z$, $y+z \leq m-1$, and $s=x+y+z\}$. If $(m-1)^{m-1}-(m-2)^{m-1} < f_m < f_{m+1} < \cdots < f_{\lfloor \frac{3}{2}(m-1) \rfloor} < m^m$, then we are done, for then

$$thw(K_{m,f_s}) = m(m-1)-s, \quad s=m,\ldots,\lfloor \frac{3}{2}(m-1) \rfloor,$$

by the remarks above.

Verifying what needs to be verified about the f_s is a straightforward chore the gory details of which we leave to the reader. Here is one way to proceed.

1. Show that the minimum f_s is achieved when x,y,z are as nearly equal as possible by verifying that if $x \geq y \geq z$ and $x \geq z+2$, then $f(x-1,y,z+1) < f(x,y,z)$.

2. Show that $f_s < f_{s+1}$ if $m \leq s < \lfloor \frac{3}{2}(m-1) \rfloor$ by brute force: assume $x \geq y \geq z$, $m \leq x+y+z = s < \lfloor \frac{3}{2}(m-1) \rfloor$, and x,y,z are as nearly equal as possible, and verify that $f_s = f(x,y,z) < f(x,y,z+1) = f_{s+1}$. (We were not clever enough to verify this without breaking into the three cases $s=3z$, $s=3z+1$, and $s=3z+2$. It turned out to be convenient to recall that $m-1 \geq x+y = 2z+j$, $j=0,1$, or 2.)

3. Finally, it is straightforward to see that for $m \geq 4$, $f_m > (m-1)^{m-1}-(m-2)^{m-1}$ and $f_{\lfloor \frac{3}{2}(m-1) \rfloor} < m^m$. Indeed, $f_{\lfloor \frac{3}{2}(m-1) \rfloor} \simeq \frac{3}{4}(m-1)^{m-1}$. The statement about

f_m can be reviewed as a calculus exercise; alternatively, the job is done for you in [4], where it is shown that for $n = (m-1)^{m-1} - (m-2)^{m-1}$, there are no thwarting list assignments of length $m-1$, of type II, if $m \geq 5$, and if $m = 4$, there is essentially only one with $s = x + y + z = 3$.

□

Corollary 1 (a) *If $m \geq 2$ and G is a graph between K_{m,m^m} and $K_{m,n}$ for some $n > m^m$, then $thw(G) = m^2$.*

(b) *If G is a graph between $K_{3,3}$ and $K_{3,27}$ minus an edge, then $thw(G) = 3$. If $m \geq 4$ and G is a graph between $K_{m,(m-1)^{m-1}-(m-2)^{m-1}}$ and K_{m,m^m} minus an edge, then $(m - \frac{3}{2})(m-1) \leq thw(G) \leq (m-1)^2$.*

Proof: This is a corollary of Theorem 1(c) and of the proof of Theorem 2, rather than of Theorem 2 itself; at least, this is true of (b). From the proof of Theorem 2(a), or of Theorem 2 in [4], for $m \geq 3$, $c(K_{m,m^m}$ minus an edge$) = m$, and a thwarting assignment of lists of length $m-1$ using as few colors as possible to (K_{m,m^m} minus an edge) will be of type II with $x + y + z = \lfloor \frac{3(m-1)}{2} \rfloor$, whence $thw(K_{m,m^m}$ minus an edge$) = \lceil (m-1)(m - \frac{3}{2} \rceil$. □

□

To prove Theorem 3 we will need two results, Lemma 1 and Theorem B, to follow, both due to Erdös, Rubin, and Taylor [1].

Lemma 1 *If H is a closed neighborhood contraction of G, and if there is a list assignment to H of lists of length 2, from which no proper coloring of H is possible, then there is such a list assignment to G, using the same set of colors to form the lists.*

Proof: Let v be the vertex in G at which the contraction occurs, let w be the big vertex in H to which $\{v\} \cup N_G(v)$ is collapsed, and suppose that w is assigned colors a, b in the list assignment to H referred to above. Make a list assignment to G by assigning $\{a, b\}$ to v and all its neighbors, and by copying the assignment in H at every other vertex. If G were properly colorable from this assignment, with v receiving, say, the color a, then all neighbors of v would have to be colored b. Then coloring w with b and otherwise copying the coloring of G onto H, we would obtain a proper coloring of H. Therefore, there is no such proper coloring of G. □

Corollary 2 *Every closed neighborhood contraction of a bipartite graph is bipartite.*

Corollary 3 *If G is bipartite, H is obtained from G by a sequence of closed neighborhood contractions, and $c(G) = c(H) = 3$, then $thw(G) \leq thw(H)$.*

Theorem B *([1]). If G is non-trivial connected graph, then $c(G) = 2$ if and only if the torso of G is one of the following:*

(i) a single vertex;

(ii) an even cycle;

(iii) $\theta(2,2,2m)$ *for some integer* $m \geq 1$.

Corollary 4 *If* G *is a non-trivial connected graph then* $thw(G) = 2$ *if and only if the torso of* G *is one of (i), (ii), or (iii), above.*

Proof: This is straightforward from Theorem B and Theorem 1. □

Corollary 5 *The graphs in Figures 1 and 2 have choice number* 3.

Proof: We know that the choice number of each is ≥ 3. In each case, removing an edge results in a graph whose torso is an even cycle. Since removing an edge brings the choice number down by at most one, the result follows. □

Of course, for two of those three graphs, $c \leq 3$ follows from Brooks' Theorem for the choice number.

Corollary 6 *The graphs in Figures 1 and 2 have thwart number* 3.

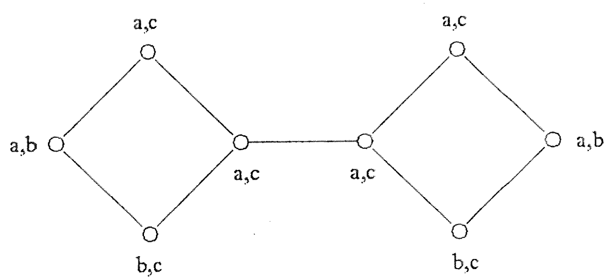

Figure 2

Proof of Theorem 3. (i)(a): By Brooks' Theorem for the choice number, $c(\theta(r,s,t)) \leq 3$ for all r,s,t positive integers, at most one equal to one. By Theorem B, $c(\theta(r,s,t)) = 3$ if, in addition, at most one of r,s,t is equal to 2, so $thw(\theta(r,s,t)) \geq c(\theta(r,s,t)) = 3$ for all r,s,t as described in (i)(a).

Each of these $\theta(r,s,t)$ is bipartite. Each can be reduced, by a sequence of closed neighborhood contractions, to one of the graphs in Figure 1. By Corollaries 3, 5, and 6, and the paragraph above, it follows that each has thwart number three.

(i)(b): Two even cycles joined by a path is certainly bipartite, and has choice number ≥ 3 by Theorem B. Removing an edge from one of the cycles results in a graph with choice number 2, by Theorem B, so the graph has choice number 3 and thwart number ≥ 3. The graph can be rendered by closed neighborhood contractions to either the second graph in Figure 1 or the graph in Figure 2. Now corollaries 3, 5, and 6 give the desired conclusion.

(i)(c)-(f): All the "endpoints" have choice number 3 by the results in [2], [4], [7], [8], and [10] (summarized in [8]), so any graph between a pair of endpoints has choice number 3. Also, any such graph contains $K_{3,3}$. Therefore, by Theorem 2(b) and Theorem 1(b) and (c), any such graph has thwart number 3.

Assertion (ii) is an easy consequence of (i) and Corollary 3, together with the observation that all the graphs mentioned in (i) have choice number 3. (See the proof of (i).)

Assertion (iii): $K_{3,27}$, $K_{4,19}$, $K_{5,13}$, $K_{6,11}$, and $K_{7,7}$ have choice number 4 by the results in [2], [4], [7], [8], and [10]. The second part of the assertion follows from Corollary 1(a), with $m = 2$. □

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