On Minimal Outerplanar Graphs of Given Diameter

Ladislav Stacho *

Institute for Informatics, Slovak Academy of Sciences,

P.O.Box 56, Dúbravská cesta 9, 840 00 Bratislava 4

Slovak Republic

Abstract

We characterize outerplanar graphs with every edge in a p-cycle for odd $p \geq 3$, which are minimal of diameter $k \geq \lceil \frac{3p}{2} \rceil - 2$. For $p = 3$ it answers the problem posed by Plesník in [10]. Moreover, we show that there exists no outerplanar graph with every edge in a p-cycle for even $p \geq 4$, which is minimal of diameter $k \geq \lceil \frac{3p}{2} \rceil - 2$.

1 Introduction and known results

Given a graph G, let $V(G)$ and $E(G)$ denote its vertex-set and edge-set, respectively. The distance of two vertices u and v in the graph G is denoted by $d_G(u, v)$, or simply $d(u, v)$ if it does not cause any confusion. The diameter of G is denoted by $diam(G)$. A graph G with $diam(G) = k$ is called a *minimal graph of diameter* k (or diameter-critical graph) iff $diam(G - e) > k$ for every edge $e \in E(G)$. These graphs have been studied by several authors, see for example [1, 4, 5, 6, 7, 8, 10] and certain parts of the surveys [2] and [3]. The characterization of these graphs seems to be a difficult problem and has not been solved completely yet. However, there are some partial results. For example, those minimal graphs of diameter 2 which are planar and contain no 3-cycle are completely described in [8]. Moreover, one can analogously define minimal tournaments, which are characterized in [9].

In contrast with [8], minimal graphs of diameter k with every edge in a 3-cycle have been studied in [10]. For notational convenience, if a minimal graph of diameter k has every edge in a p-cycle, we call it (k, p)-*minimal graph*. Following this notation, Plesník [10] presents infinite classes of planar $(k, 3)$-minimal graphs for every $k \geq 2$. Moreover, he deals with outerplanar $(k, 3)$-minimal graphs, proves that no outerplanar $(2, 3)$-minimal graph exists and describes one outerplanar $(k, 3)$-minimal graph

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for each integer $k \geq 3$. His graphs are a special case ($p = 3$) of the following more general definition.

Definition 1 *For each odd integer $p \geq 3$ and $k \geq p$ the graph (k,p)-OP consists of $(2k - 4\lfloor \frac{p}{2} \rfloor + 3)$-cycle C and one path of the length $p - 1$ for each edge of C, where the path joins the ends of the edge.*

For example, the graphs $(k, 3)$-OP for $k = 3, 4, 5$ are depicted in the Fig. 1.

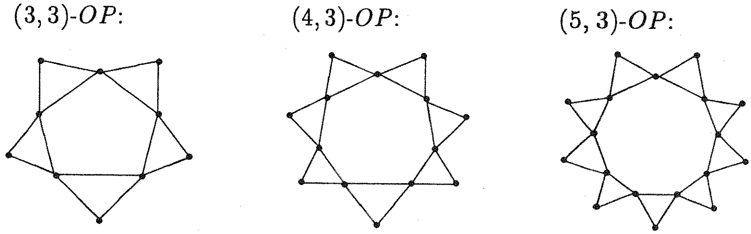

Fig. 1

Plesník asked whether other outerplanar $(k, 3)$-minimal graphs exist and posed the problem of describing all outerplanar $(k, 3)$-minimal graphs. In this paper we characterize all outerplanar (k,p)-minimal graphs for odd $p \geq 3$ and $k \geq \lceil \frac{3p}{2} \rceil - 2$. Moreover, we prove that there exists no outerplanar (k,p)-minimal graph for even $p \geq 4$ and $k \geq \lceil \frac{3p}{2} \rceil - 2$. For $p = 3$ this solves Plesník's problem.

2 Main results

First, we present some necessary definitions. A planar graph is *outerplanar* if it can be embedded in the plane so that all its vertices lie on the same face; one usually chooses this face to be exterior. The edges which determine the exterior face are called *exterior edges*; the remaining edges are *interior edges*. A walk which starts at the vertex v_1, ends at the vertex v_n and passes in order through vertices $v_2, v_3, \ldots, v_{n-1}$ is denoted by $<v_1, v_2, \ldots, v_{n-1}, v_n>$. Accordingly, an edge $e = uv$ is sometimes denoted by $<u, v>$.

We first deal with outerplanar graphs with no cutvertex. Obviously, any such outerplanar graph G on $n \geq 3$ vertices is hamiltonian and then one can label its vertices $v_1, v_2, \ldots, v_n$, such that $<v_1, v_2, \ldots, v_n, v_1>$ is the "exterior" hamiltonian cycle. Now, for each interior edge $e = v_iv_j$, $i < j$, define its *tolerance* $t(e)$ as $\min\{j - i - 1, n - j + i - 1\}$ and each vertex either from the set $A = \{v_i, v_{i+1}, \ldots, v_j\}$, if $t(e) = j - i - 1$ or from the set $B = \{v_i, v_{i-1}, \ldots, v_j\}$, if $t(e) = n - j + i - 1$ we call *tied vertex* with the edge e. Similarly, each edge $e' = u_ku_l$, where u_k and u_l are tied vertices with the edge e and e' is different from the edge e, is called *tied edge* with the edge e. If $j - i - 1 = n - j + i - 1$, then all vertices of G will be tied with e according to the above definition. In this case it will be advantageous (in proofs) to choose only one of the sets A and B to be the set of tied vertices with e. This restriction will not cause any loss of generality.

For notational convenience, if a path of the length $d_G(u,v)$ from u to v exists in G such that it does not contain an edge e, we denote the distance u and v by $d_G(u,v;e)$. We are ready to state and prove the following auxiliary results.

Lemma 1 *Let G be a minimal outerplanar graph of diameter k with no cutvertex. Let G contain an interior edge e with $t(e) \leq k-1$. Then G cannot contain any interior edge tied with the edge e.*

Proof. Suppose for a contradiction that $e = cd$ is an interior edge with $t(e) \leq k-1$ and $e' = ab$ is an interior edge which is tied with e with minimum $t(e')$. In case that each vertex of G is tied with e, we choose that set A or B (see definition of tied vertex) to be the set of tied vertices with e, which contains the vertices a and b. We shall distinguish the following three cases.

a) $t(e')$ is even; Consider the subgraph $H = G - \langle u,v \rangle$, where the vertices u and v are tied with the edge e' and $d(a,u) = d(b,v) = \frac{t(e')}{2}$ and $d(a,v) = d(b,u) = \frac{t(e')}{2} + 1$. We show that $diam(H) = k$, which contradicts the minimality of G. Since the edge e' exists, $d_H(x,y) \leq k$, where x, y are not tied vertices with the edge e'. Similarly, one can easily see that $d_H(x,y) \leq t(e') + 1 < k$, where x and y are tied vertices with the edge e'. Since $t(e')$ is even, $d_H(x,a) = d_G(x,a)$ and $d_H(x,b) = d_G(x,b)$ for all x which are tied with the edge e'. This implies that $d_H(x,y) \leq k$ for all x which are tied with e' and all y which are not tied with e', and this is the last contradiction.

b) $t(e')$ is odd and $d_G(c,a;e') = d_G(d,b;e')$ (see Fig. 2 a)); It follows from this that $d_G(c,b;e') \leq d_G(c,a;e') + 1$ and $d_G(d,a;e') \leq d_G(d,b;e') + 1$. Let $H = G - e'$. We claim that $diam(H) = k$. From the existence of the edge e, it follows that $d_H(x,y) \leq k$, where x and y are not tied with e. The fact that $t(e) \leq k-1$ implies that $d_H(x,y) \leq k$, where x and y are tied with e. Finally, it can be verified, using the above lower bounds, that each shortest $x-y$ path which contains the edge e' can be replaced by a $x-y$ path of the same length such that it does not contain the edge e', where x is tied with e and y is not tied with e. Thus, $d_H(x,y) \leq k$, which proves the claim and so this case cannot occur.

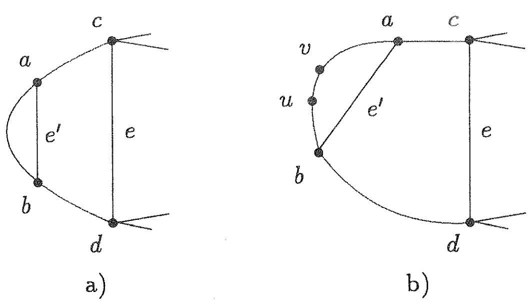

Fig. 2

c) $t(e')$ is odd and $d_G(c, a; e') \neq d_G(d, b; e')$; Suppose that $d_G(c, a; e') < d_G(d, b; e')$ (see Fig. 2 b)). This implies that $d_G(a, d; e') \leq d_G(b, d; e')$ and $d_G(a, c; e') \leq d_G(b, c; e')$. Consider the subgraph $H = G - <u, v>$, where u and v are tied with e' and $d_G(b, u) = \lfloor \frac{t(e')}{2} \rfloor$ and $d_G(a, u) = d_G(a, v) = \lceil \frac{t(e')}{2} \rceil$. Similarly, as in the previous case, one can observe that $d_H(x, y) \leq k$, where both x and y are either tied with e or are not. From the existence of the edge e', it follows that $d_H(x, y) \leq k$, where x is tied with e but it is not tied with e' and y is not tied with e. Lastly, we show that $d_H(x, y) \leq k$, where x is a tied vertex with the edge e' and y is not tied with e. Obviously, each such $x - y$ path must contain either c or d (or both these vertices). Evidently, $d_G(x, a; <u, v>) = d_G(x, a)$ for all x which are tied vertices with e'. Recall that $d_G(a, d; e') \leq d_G(b, d; e')$ and $d_G(a, c; e') \leq d_G(b, c; e')$. Now it is immediately clear that $d_H(x, y) \leq k$ for all $x, y \in H$. This contradiction proves the Lemma. □

The following Corollary can be proved using methods similar to the previous proof.

Corollary 1 *Let G be a minimal outerplanar graph of diameter k with no cutvertex. Let G contain an interior edge e with $t(e) \leq k - 1$. Then $t(e)$ must be odd.* □

Lemma 2 *Let G be an outerplanar 2-connected graph with every edge in a p-cycle, where $p \geq 3$ and is minimal of diameter $k \geq \lceil \frac{3p}{2} \rceil - 2$. Then G cannot contain any interior edge e with $t(e) > p - 2$.*

Proof. The proof consists of a number of ceses and subcases. To avoid any confusion of the reader we first outline the proof. We will consider by way of contradiction that there exists an interior edge e with minimum $t(e) > p - 2$. The main strategy in the proof is then to consider all cases of the existence of edges which are tied with the edge e and to find an edge e' for which $diam(G) = diam(G - e')$, a contradiction. Let $e = ab$ be an interior edge of G with minimum $t(e) > p - 2$. Since each edge of G must lie in a p-cycle, an interior edge $e' = uv$ which is tied with e must exist. In case that each vertex of G is tied with e, we choose that set A or B (see definition of tied vertex) to be the set of tied vertices with e, which contains the vertices u and v. From the minimality of $t(e)$, it follows that $t(e') \leq p - 2 < k - 1$ and so, from Lemma 1 it follows that no interior edge which is tied with e' exists in G. So, each tied vertex with e is tied with at most one other interior edge which is tied with e, as well. By Corollary 1, $t(e')$ is odd. We shall distinguish two cases.

a) Either some interior edge e' which is tied with e has $t(e') < p - 2$ or some exterior edge which is tied with e is not tied with any other interior edge e' which is tied with e. Since each edge must lie in a p-cycle, a shortest $a - b$ path P exists on the tied vertices with e of the length at most $p - 1$. It is immediately clear that P must contain each interior edge which is tied with e and each exterior edge which is tied with no interior edge tied with e. We can assume that $d_G(u, a; e') \leq d(v, b; e')$. We shall distinguish the following subcases.

(i) $d_G(u, a; e') < d(v, b; e')$; Let $e'' = u_1 v_1$, where u_1 and v_1 are tied with e' and $d(v_1, v) = \lfloor \frac{t(e')}{2} \rfloor$ and $d(u_1, u) = d(u_1, v) = \lceil \frac{t(e')}{2} \rceil$. Let $H = G - e''$. We claim that $diam(H) = k$. It follows from the existence of e' that $d_H(x, y) \leq k$, where both x and y are not tied with e'. Similarly, from $t(e') \le k-1$, it follows that $d_H(x,y) \le k$, where x and y are tied with e'. Now consider the case when x is tied with e' and y is not tied with e. From the assumptions of this case, it follows that $d_G(u,b;e') \le d_G(v,b;e')$ and $d_G(u,a;e') \le d_G(v,a;e')$. Now this case can be handled as a part of the case c) of the proof of Lemma 1. Finally, suppose that x is tied with e' and y is tied with e but it is not tied with e'. It is immediately clear ($t(e')$ is odd) that if $x \ne u_1$, then $d_H(x,y) = d_G(x,y) \le k$ and so we assume that $x = u_1$ and that the edge e'' lies on each $u_1 - y$ path of the length at most k (in other words it contains the vertex v, as well). In the case when y is tied with no interior edge which is tied with e, y will have to lie on the path P. It follows that $d(v,y) \le \lfloor \frac{p}{2} \rfloor$ and $d(u,y) \le \lfloor \frac{p}{2} \rfloor$. Similarly, $d_G(u_1,v) = d_G(u_1,v;e'') + 1 \le \lceil \frac{p}{2} \rceil$ and $d_G(u_1,u) = d_G(u_1,u;e'') \le \lfloor \frac{p}{2} \rfloor$. By summing these upper bounds we obtain that each such $u_1 - y$ path can be replaced by $u_1 - y$ path of the length at most k, such that it does not contain the edge e'' and so $diam(H) = k$. Now suppose that y is tied with $f = u_2v_2$, where f is tied with e. Note that $t(f) \le p-2$ and $t(f)$ is odd. Since $d_G(u_1,v) = d_G(u_1,v;e'') + 1$ and $d_G(u_1,u) = d_G(u_1,u;e'')$, the distance $d(u_1,y) = k$. We can assume that $d_G(v,u_2;e') \le d_G(v,v_2;e')$.

If $d_G(v,u_2;e') < \lfloor \frac{p-2}{2} \rfloor$, then since $d_G(u_1,v) = \lceil \frac{t(e'')}{2} \rceil \le \lfloor \frac{p-1}{2} \rfloor$ and since $k \ge \lceil \frac{3p}{2} \rceil - 2$, the distance $d_G(u_2,y)$ must be more that $\lfloor \frac{p}{2} \rfloor$, which is impossible.

If $d_G(v,u_2;e') = \lfloor \frac{p-2}{2} \rfloor$, then since the length of P is at most $p-1$, the distance $d_G(u,v_2;e') \le \lceil \frac{p-2}{2} \rceil$ and since $d_G(u_1,y) = k$, the distance $d_G(u_2,y) \ge \lfloor \frac{p}{2} \rfloor$. From this $d_G(v_2,y) \le \lfloor \frac{p}{2} \rfloor$. Recall that $d_G(u_1,u;e'') = d_G(u_1,v) \le \lfloor \frac{p-1}{2} \rfloor$. These bounds enables us to replace the $u_1 - y$ path so that it does not contain the edge e'' and has length at most k, which is a contradiction.

If $d_G(v,u_2;e') > \lfloor \frac{p-2}{2} \rfloor$, then $d_G(u,v_2;e') \le \lfloor \frac{p-2}{2} \rfloor$ and we replace the $u_1 - y$ path by using the edge $f = u_2v_2$, as well.

(ii) $d_G(u,a;e') = d(v,b;e')$ and there exists other interior edge f, which is tied with e. Note that from $t(f) \le p-2$ and from Lemma 1, it follows that e' is not tied with f. Then we choose the edge f and obtain the case (i).

(iii) $d_G(u,a;e') = d(v,b;e')$ and there exists no other interior edge which is tied with e. From $t(e) > p-2$, it follows that $t(e') = p-2$ and the length of the path P is at most $p-1$. This implies that $t(e) \le 2p-4$. Let $H = G - e'$. We claim that $diam(H) = k$. To see this, let x and y be vertices which are tied with e. Obviously, $d_H(x,y) \le p-1 < k$. If the vertices x and y are not tied with e, then $d_H(x,y) \le k$. Finally, if x is tied with e and y is not, then we use the methods of the case b) in the proof of Lemma 1.

b) Each interior edge which is tied with e has tolerance equal to $p-2$ and each exterior edge which is tied with e is tied with some interior edge which is tied with e, as well. From these facts, it follows that only certain values are possible for $t(e)$. Namely, one can observe that $t(e) = (p-1)i - 1$ for $i \ge 2$. We distinguish two subcases.

(i) We show that the assertion of this Lemma holds if $i = 2$ or 3. Firstly, let $i = 2$ and let $e_1 = ac$ and $e_2 = cb$ be two interior edges which are tied with e (see Fig. 3 a) where the case $p = 7$ is depicted). Consider the subgraph $H = G - e'$, where $e' = uv$, where u and v are tied with e_1 and $d_G(v, a) = d_G(u, a) = \lceil \frac{p-2}{2} \rceil$ and $d_G(u, c) = \lfloor \frac{p-2}{2} \rfloor$. The fact that the distance $d_H(x, y) \leq k$ for x and y which are (are not) tied with e_1 is clear. Similarly, it follows that $d_H(x, y) \leq k$, if $x \neq v$. And finally, one can easily check that $d_H(x, y) \leq k$, if $x = v$ (using the same ideas as above). The case $i = 3$ can be handled similarly if we take the edge $e' = uv$, where u and v are tied with e_1 and $d_G(u, a) = d_G(v, a) = \lceil \frac{p-2}{2} \rceil$ and $d_G(u, c) = \lfloor \frac{p-2}{2} \rfloor$ (see Fig. 3 b) where the case $p = 5$ is depicted). So, we have proved that no interior edge with the tolerance $2p - 3$ and $3p - 4$ exists in G.

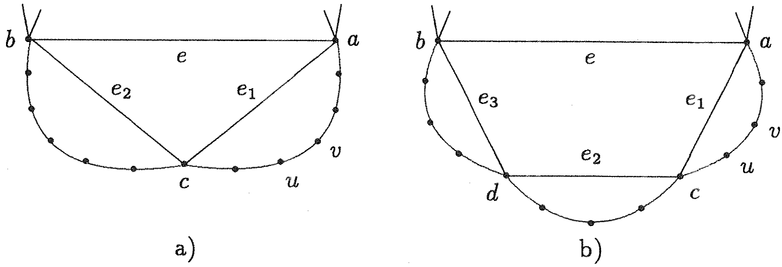

Fig. 3

(ii) In this case suppose that $i \geq 4$. We distinguish between two cases according to the value i.

aa) $i \geq 4$, even; Let $<c_k, c_{k+1}>$, $k = 0, 1, \ldots, i-1$, be interior edges which are tied with e (see Fig. 4 where the case $i = 6$ and $p = 7$ is depicted).

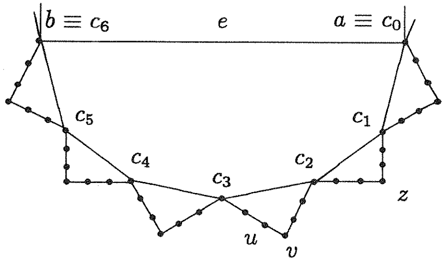

Fig. 4

Consider the edge $e' = uv$, where u and v are tied with $<c_{\frac{i}{2}}, c_{\frac{i}{2}-1}>$ and $d(c_{\frac{i}{2}}, u) = \lfloor \frac{p-2}{2} \rfloor$ and $d(c_{\frac{i}{2}-1}, v) = d(c_{\frac{i}{2}}, v) = \lceil \frac{p-2}{2} \rceil$. Let $H = G - e'$. We claim that $diam(H) = k$, which contradicts the minimality of G. Since the edge $<c_{\frac{i}{2}}, c_{\frac{i}{2}-1}>$ exists and has the tolerance $p - 2$, it follows that $d_H(x, y) \leq k$, where $x, y \neq v$. Now we show that $d_H(x, v) \leq k$ for all x from H. Suppose, by way of contradiction, that a vertex x exists here, such that $d_G(x, v) \leq k < d_H(x, v)$. We distinguish two cases.

Case 1. x is not tied with e; But $d_G(a,v) = d_G(a,v;e')$ and $d_G(b,v) = d_G(b,v;e')$ and so this case cannot occur.

Case 2. x is tied with e; If x is tied with one of the edges $<c_k, c_{k+1}>$, $k = 0, 1, \ldots, \frac{i}{2}-2$, then $d_G(x,v) = d_G(x,v;e')$. If x is tied with $<c_{\frac{i}{2}}, c_{\frac{i}{2}-1}>$, then since the tolerance of $<c_{\frac{i}{2}}, c_{\frac{i}{2}-1}>$ is $p-2 < k$, it follows that $d_G(x,v;e') < k$. Finally, suppose that x is tied with one of the edges $<c_l, c_{l+1}>$, $l = \frac{i}{2}, \frac{i}{2}+1, \ldots, i-1$; say $<c_k, c_{k+1}>$. Since $t(<c_k, c_{k+1}>)$ is odd, we can assume that $d_G(c_{k+1}, x) < d_G(c_k, x)$. From this fact a vertex y exists here, such that it is tied with $<c_k, c_{k+1}>$ and $d_G(c_{k+1}, y) \leq d_G(c_k, y)$ and $d_G(c_{k+1}, x) < d_G(c_{k+1}, y)$. Since $d_G(c_{\frac{i}{2}}, v) = d_G(c_{\frac{i}{2}}, v; e') - 1$, the length of the shortest $x-v$ path must be k and it must contain the vertices $c_{k+1}, c_k, c_{k-1}, \ldots, c_{\frac{i}{2}}$. Let z be a vertex tied with $<c_{\frac{i}{2}-1}, c_{\frac{i}{2}-2}>$, such that $d_G(c_{\frac{i}{2}-1}, z) = d_G(c_{\frac{i}{2}-2}, z) = \lceil\frac{p-2}{2}\rceil$. Since $diam(G) = k$, shortest $x-z$ and $y-z$ paths P_1, P_2 must exist in G, respectively. It can be checked that P_1 and P_2 cannot contain the vertices $c_k, c_{k-1}, \ldots, c_{\frac{i}{2}}$. On the other hand, they must contain the vertices $c_{k+1}, c_{k+2}, \ldots, c_{i-1}, b, a, c_1, c_2, \ldots, c_{\frac{i}{2}-2}$. From this, it follows that $d_G(x,z) < d_G(y,z) \leq k$. But $d_G(c_{\frac{i}{2}-2}, v; e') = d_G(c_{\frac{i}{2}-2}, z) + 1$ and so $d_G(x,v;e') \leq k$, which contradicts our assumption.

bb) $i \geq 5$, odd; This case can be handled similarly to the previous case if we take the edge $e' = uv$, which is tied with the edge $<c_{\lfloor\frac{i}{2}\rfloor}, c_{\lfloor\frac{i}{2}\rfloor-1}>$.
The proof is completed. □

Now we consider outerplanar (k,p)-minimal graphs with a cutvertex. For any subset H of $V(G)$ an induced subgraph by the set H is denoted by $\langle H\rangle$. If $x-c$ and $y-c$ are two paths denoted by A, B, respectively, then the *connecting* of the paths A and B, denoted by $A \oplus B$, is a $x-y$ path $(A \cup B) \setminus ((A \cap B) \cup \{\cup C_i\})$, where C_i is a cycle in $A \cup B$. We prove a more general result.

Theorem 1 *There exists no minimal graph of diameter k which is 2-edge-connected and has a cutvertex.*

Proof. Suppose for a contradiction that G is a minimal graph of diameter k which is 2-edge-connected and contains a cutvertex w.

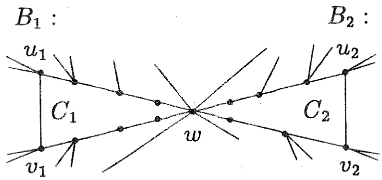

Fig. 5

Let B_1' be a connected component of $G-w$ and $B_2' = \langle(V(G-w)) - V(B_1')\rangle$. Now let $B_1 = \langle V(G) - V(B_2')\rangle$ and $B_2 = \langle V(G) - V(B_1')\rangle$. Since G does not contain any bridge, each of its edges lies in a cycle. And since w is a cutvertex, a cycle C_1 which

contains w exists in B_1 and a cycle C_2 which contains w exists in B_2. Suppose that C_1 and C_2 are minimal such cycles (see Fig. 5).
From the minimality of C_i, $i = 1, 2$, it follows that no other edge exists between the vertices of C_1 and C_2, respectively. Consider the edges $<u_1, v_1> \in C_1$ and $<u_2, v_2> \in C_2$, where $d(u_i, w) = d(v_i, w) = \lfloor \frac{|C_i|}{2} \rfloor$ if $|C_i|$ is odd or $d(u_i, w) - 1 = d(v_i, w) = \frac{|C_i|}{2}$ if $|C_i|$ is even, for $i = 1, 2$. We distinguish two cases.
a) An $x - y$ path of the length at most k which does not contain the edge $<u_1, v_1>$ exists in B_1 for all $x, y \in B_1$. Obviously, deleting $<u_1, v_1>$ does not affect the lengths of the shortest $x - y$ paths, where $x \in B_2$ and $y \in B_2$. From the minimality of C_1, it follows that $d(u_1, w) = d(u_1, w; <u_1, v_1>) = d(v_1, w) = d(v_1, w; <u_1, v_1>)$ if $|C_1|$ is odd, and $d(u_1, w) - 1 = d(u_1, w; <u_1, v_1>) - 1 = d(v_1, w) = d(v_1, w; <u_1, v_1>)$ if $|C_1|$ is even. From these facts, for $x \in B_1$ and $y \in B_2$ the shortest $x - y$ path can be chosen such that it contains either one of the vertices u_1, v_1 or neither of them and so, $d(x, y) = d(x, y; <u_1, v_1>)$. But this contradicts the minimality of G.

b) Vertices x, y exist in B_1, such that each $x - y$ path of the length at most k contains the edge $<u_1, v_1>$. Consider the shortest $x - w$ and $y - w$ paths P_1, P_2, respectively. By the methods similar to those used in case a), it can be observed that P_1 and P_2 can be chosen such that they contain at most one of the vertices u_1 and u_2. Thus, $P = P_1 \oplus P_2$ does not contain the edge $<u_1, v_1>$ and so the length of P must be more than k. From this, the length of P_1 or P_2 must be more than $\frac{k}{2}$. Since $diam(G) = k$, the length of an $x - w$ shortest path must be less than $\frac{k}{2}$ for all $x \in B_2$. Recall that each of these shortest paths can be chosen such that it does not contain the edge $<u_2, v_2>$. So, for B_2, a) holds. □

Note that no assertion similar to Theorem 1 holds for minimal graphs with a bridge, because one can see that any graph obtained from the graph (k, p)-OP, where $p \geq 3$ and $k \geq p$, by adding a new vertex for each vertex of (k, p)-OP and the edge connecting these two vertices is a minimal graph, as well. On the other hand, it can be proved (using methods similar to those used in the previous proof) that any minimal graph of given diameter cannot contain two blocks which contain at least three vertices. Now we improve Lemma 2.

Lemma 3 *Let G be an outerplanar graph with every edge in a p-cycle, where $p \geq 3$ and is minimal of diameter $k \geq \lceil \frac{3p}{2} \rceil - 2$. Then the tolerance each of its interior edges is equal to $p - 2$.*

Proof. Suppose that G has an interior edge e, with tolerance different from $p - 2$. From Theorem 1, it follows that G cannot contain any cutvertex. Further, it follows from the previous Lemmas that each interior edge of G has tolerance at most $p - 2$, and each exterior edge is tied with at most one interior edge. Thus, the tolerance $t(e) < p - 2$. In the case that each vertex of G is tied with e we obtain $diam(G) < k$, which is impossible. Since each exterior edge tied with e must lie in a p-cycle and since $t(e) < p - 2$, a cycle C, which consists of all interior edges and all exterior edges which are tied with no interior edge has length at most $p - 1$. Estimate the upper bound of $diam(G)$. Let x, y be vertices of the maximum distance in G. If

x and y lie on C, then $d(x,y) \le \lfloor \frac{p-1}{2} \rfloor < k$. If x lies on C and y does not, then $d(x,y) \le \lfloor \frac{p-1}{2} \rfloor + \lfloor \frac{p}{2} \rfloor < k$. Finally, if neither x nor y lies on C, then we have:

a) if p is odd:

$$d(x,y) \le \lfloor \frac{p-3}{2} \rfloor + 2\lfloor \frac{p}{2} \rfloor = \frac{p-3}{2} + \frac{2p-2}{2} < \lceil \frac{3p}{2} \rceil - 2$$

b) if p is even; then by Corollary 1, the tolerance of each interior edge is less than $p-2$ and so

$$d(x,y) \le \lfloor \frac{p-3}{2} \rfloor + 2\lfloor \frac{p-1}{2} \rfloor = \frac{p-4}{2} + \frac{2p-4}{2} < \lceil \frac{3p}{2} \rceil - 2.$$

In all cases $diam(G) < k$, which is a contradiction. □

We are ready to prove the main results of the paper. The following Theorem follows immediately from the previous Lemma.

Theorem 2 *There exists no (k,p)-minimal outerplanar graph for even $p \ge 4$ and $k \ge \lceil \frac{3p}{2} \rceil - 2$.*

Proof. Suppose that G is such a graph. By Theorem 1 since G is 2-edge-connected, G cannot contain any cutvertex. If G has no interior edge, then G is a p-cycle, but p is even and so G is not minimal graph. So, G must contain at least one interior edge e. From Lemma 3, it follows that $t(e) = p-2 < k$. But this contradicts Corollary 1. □

Theorem 3 *For odd $p \ge 3$ and $k \ge \lceil \frac{3p}{2} \rceil - 2$, each outerplanar (k,p)-minimal graph is isomorphic to the graph (k,p)-OP.*

Proof. Similarly to the proof of the above Theorem, each such graph has no cutvertex and the tolerance of each interior edge is $p-2$. We first prove that each exterior edge is tied with an interior edge. For this purpose, suppose that G is a (k,p)-minimal outerplanar graph, where $p \ge 3$, odd, and $k \ge \lceil \frac{3p}{2} \rceil - 2$. If G has no interior edge, then it is a p-cycle and $diam(G) = \lfloor \frac{p}{2} \rfloor < k$. So, G has at least one interior edge. If an exterior edge e is tied with no interior edge, then the length of the cycle, which consists of all interior edges and all exterior edges which are tied with no interior edge is p. From this, $diam(G) < \lceil \frac{3p}{2} \rceil - 2$, which is impossible. Now we have proved that G consists of a cycle C created by all interior edges and one path of the length $p-1$ for each edge of C, where the path joins the ends of the edge. Consequently, if C has more than $2k - 4\lfloor \frac{p}{2} \rfloor + 3$ vertices, then either G is not minimal or $diam(G) > k$; if C has fewer than $2k - 4\lfloor \frac{p}{2} \rfloor + 3$ vertices, then either G is not minimal or $diam(G) < k$. Finally, it can be easily checked that the cycle C (of all interior edges) of the graph (k,p)-OP has $2k - 4\lfloor \frac{p}{2} \rfloor + 3$ vertices, the graph (k,p)-OP is minimal of diameter k and each its edge lies in a p-cycle. So, G is isomorphic to (k,p)-OP. The Theorem is proved. □

The following problem arises.

Problem. Describe all outerplanar (k,p)-minimal graphs, for $p \geq 3$ and $k < \lceil \frac{3p}{2} \rceil - 2$.

This problem seems to be more difficult. For example, if $k \geq 4$, then it can be verified that all graphs (k,l)-OP, where $l = 3, 5, \ldots, t$, where $k - 1 \leq t \leq k$, t is odd, have each edge in a $2k+1$-cycle and are minimal of diameter k. Moreover, a $2k+1$-cycle is such graph, as well.

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