n-Extendability of Line Graphs, Power Graphs, and Total Graphs

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Abstract

A graph G that has a perfect matching is n-extendable if every matching of size n lies in a perfect matching of G. We show that when the connectivity of a line graph, power graph, or total graph is sufficiently large then it is n-extendable. Specifically: if G has even size and is $(2n+1)$-edge-connected or $(n+2)$-connected, then its line graph is n-extendable; if G has even order and is $(n+1)$-connected, then G^2 is n-extendable; if G has even order and is connected, then G^{2n+1} is n-extendable; if the total graph $T(G)$ has even order and is $(2n+1)$-connected, then $T(G)$ is n-extendable.

1 Introduction and terminology

All graphs considered in this paper are finite, undirected, connected and simple.

The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$ respectivly. The cardinalities of $V(G)$ and $E(G)$ are called respectively the *order* and *size* of G. The *line graph* $L(G)$ of a graph G is the graph whose vertex set is $E(G)$ and in which two vertices are joined if and only if they are adjacent edges in G. The *iterated line graph* $L^m(G)$ is defined recursively by $L^1(G) = L(G)$ and $L^m(G) = L(L^{m-1}(G))$ for $m > 1$. A *power graph* G^k (the kth power of a graph G) is the graph whose vertices are those of G and in which two distinct vertices are joined whenever the distance between them in G is at most k. The vertices and edges of a graph are called *elements*. Two elements of a graph are *neighbours* if they are either incident or adjacent. The *total graph* $T(G)$ has vertex set $V(G) \cup E(G)$ and two vertices of $T(G)$ are adjacent whenever they are neighbours in G. The *iterated total graph* $T^m(G)$ is defined recursively by $T^1(G) = T(G)$ and $T^m(G) = T(T^{m-1}(G))$ for $m > 1$. The *subdivision graph* $S(G)$ of a graph G is the graph obtained by replacing all edges of G with paths of length two. The inserted vertices are called the *subdivision* vertices of $S(G)$. We use P_{n+1} to denote a path of length n. The number of components of G of odd order is denoted by $o(G)$. A *matching* of G is a set edges no two of which are adjacent. The matching is *perfect* if it contains all the vertices of G. For the terminology and notation not defined in this paper, the reader is referred to [3].

We will need the following well known condition for the existence of a perfect matching.

Tutte's Theorem ([10]) *A graph G has a perfect matching if and only if for every subset S of vertices, $|S| \geq o(G - S)$.*

Let n and $2m$ be positive integers with $n \leq m - 1$ and let G be a graph with $2m$ vertices having a perfect matching (of size m). The graph G is said to be *n-extendable* if every matching of size n in G lies in a perfect matching.

The n-extendability of symmetric graphs was studied in [1], [7], and [8]. In this paper we investigate the n-extendability of some locally dense graphs, namely, line graphs, power graphs and total graphs. The following lemma is useful.

Lemma 1 ([4]) *(1) If a line graph is connected and has even order, then it has a perfect matching. (2) If G is a connected graph of even order, then G^2 has a perfect matching. (3) If a total graph is connected and has even order, then it has a perfect matching.*

We show that when the connectivity of line graphs, power graphs and total graphs is sufficiently large, then they are n-extendable.

2 Line graphs

In this section, a necessary and sufficient condition for a line graph to be n-extendable is given. The next two lemmas follow immediately from the definition of a line graph.

Lemma 2 *If $D \subseteq E(G)$ then $L(G - D) = L(G) - D$.*

Lemma 3 *If $D \subseteq E(G)$ then the number of non-trivial components of $G - D$ equals the number of components of $L(G) - D$.*

Theorem 4 *Let G be a graph of even size. Then $L(G)$ is n-extendable if and only if, for any collection $Q_1, Q_2, \ldots, Q_n$ of edge-disjoint P_3's in G, $G - E(Q_1) - E(Q_2) - \cdots - E(Q_n)$ does not have a component of odd size.*

Proof. Suppose $L(G)$ is n-extendable. Any edge disjoint P_3's $Q_1, Q_2, \ldots, Q_n$ of G correspond to n independent edges $e_i = u_i v_i$ of $L(G)$ $(i = 1, 2, \ldots, n)$. So $L(G) - \{u_1, v_1, \ldots, u_n, v_n\}$ has a perfect matching and therefore does not have any odd components. But each component of $L(G) - \{u_1, v_1, \ldots, u_n, v_n\}$ is the line graph of some component of $G - E(Q_1) - E(Q_2) - \cdots - E(Q_n)$. Hence no component of $G - E(Q_1) - E(Q_2) - \cdots - E(Q_n)$ has an odd number of edges.

For the converse, let edges $e_i = u_i v_i$ $(i = 1, 2, \ldots, n)$ form a matching of $L(G)$. These edges correspond to n edge disjoint P_3's $Q_1, Q_2, \ldots, Q_n$ of G. By Lemma 1, the line graph of each component of $G - E(Q_1) - E(Q_2) - \cdots - E(Q_n)$ has a perfect matching. Thus $L(G) - \{u_1, v_1, \ldots, u_n, v_n\}$ has a perfect matching and $L(G)$ is n-extendable. □

Corollary 5 *If a graph G has even size and is $(2n+1)$-edge-connected, then $L(G)$ is n-extendable.*

Proof. Let $Q_1, Q_2, \ldots, Q_n$ be n edge-disjoint P_3's of G. Since G is $(2n+1)$-edge-connected, $G - E(Q_1) - E(Q_2) - \cdots - E(Q_n)$ is connected and therefore has no component with an odd number of edges. The result now follows from Theorem 4. □

The connectivity in Corollary 5 is the least possible. Let F and H be two disjoint graphs both isomorphic to K_{2n+3} if K_{2n+3} has odd size or to K_{2n+3} with one edge deleted if K_{2n+3} has even size. Join F and H by n P_3's such that the middle vertices of the P_3's are n different vertices of F and the end vertices of the n P_3's are $2n$ different vertices of H. The resulting graph is 2n-edge-connected, but deleting the edges of the n P_3's gives a component of odd size. By Theorem 4, its line graph is not n-extendable.

We have another version of Corollary 5.

Corollary 6 *If $L(G)$ has even order and is $(2n+1)$-connected, then $L(G)$ is n-extendable.*

Corollary 7 *If a graph G has even size and is $(n+2)$-connected, then $L(G)$ is n-extendable.*

Proof. Suppose that $L(G)$ is not n-extendable. By Theorem 4 there are n edge disjoint P_3's $Q_1, Q_2, \ldots, Q_n$ of G such that $G' = G - E(Q_1) - E(Q_2) - \cdots - E(Q_n)$ has a component of odd size and is therefore disconnected. Let w_j be the middle vertex of Q_j for $1 \leq j \leq n$. Let $W = \{w_1, \ldots, w_n\}$. Note that the w_i's are not necessarily distinct. Let $v_1, \ldots, v_m$ be the distinct vertices of W. Suppose each v_i is repeated l_i

times in W. G is $(n+2)$-connected, so $G-W$ is connected. Also, since G' has at least two components of odd size, there is a component C of odd size that contains vertices only from W. Without loss of generality, let $V(C) = \{v_1, \ldots, v_r\}$. Note that $r \geq 2$ since C has odd size. Assume that l_1 is the least of the l_i's for $1 \leq i \leq r$ and that $v_1 = w_1 = \cdots = w_{l_1}$. The end vertices of $Q_1, Q_2, \ldots, Q_{l_1}$ and the vertices $v_2, \ldots, v_m$ form a cut set of order $2l_1+(r-1)+(m-r) \leq 2l_1+(1+l_3+\cdots+l_r)+(l_{r+1}+\cdots+l_m) \leq 1 + l_1 + l_2 + \cdots + l_m = n+1$, contradicting the fact that G is $(n+2)$-connected. □

The connectivity in Corollary 7 is also the least possible. Let F be K_n where $n = 4i+2$ for some i. Let H be K_{2n} with one edge deleted. Both F and H have an odd number of edges. Join F to H with n P_3's such that the middle vertices of the n P_3's are the n different vertices of F and the end vertices of the n P_3's are the $2n$ different vertices of H. The resulting graph is $(n+1)$-connected but deleting the edges of the n P_3's gives a component of odd size. By Theorem 4, its line graph is not n-extendable.

We turn now to the iterated line graph $L^m(G)$.

Lemma 8 ([5]) *(1) If G is k-connected, then $L(G)$ is k-connected. (2) If G is k-edge-connected, then $L(G)$ is $(2k-2)$-edge-connected.*

Corollary 9 *If G is $(n+2)$-connected and $L^m(G)$ has even order, then $L^m(G)$ is n-extendable.*

Proof. This follows from Corollary 7 and Lemma 8(1). □

If we relax the connectivity of G, then $L^m(G)$ is still n-extendable for sufficiently large m.

Corollary 10 *Let k, m, n be positive integers and $2^m \geq (4n-2)/k$. If G is $(k+2)$-edge-connected and $L^m(G)$ has even order then $L^m(G)$ is n-extendable.*

Proof. From Lemma 8(2), $L^{m-1}(G)$ is $(2^{m-1}k+2)$-edge-connected. The result now follows from Corollary 5. □

Corollary 11 *Let k, m, n be positive integers and $2^m \geq (4n-2)/k$. If G is $(k+2)$-connected and $L^m(G)$ has even order then $L^m(G)$ is n-extendable.*

Proof. This follows from Corollary 10 since G is at least $(k+2)$-edge-connected. □

3 Power graphs

In this section, we prove that when the connectivity of a graph G is sufficiently large, G^2 is n-extendable. We also show that for any connected graph G, G^r is n-extendable for sufficiently large r.

Lemma 12 *Let G be a k-connected graph. Then G^m is km-connected if km is less than the order of G.*

Proof. Suppose S is a cutset of G^m and S contains less than km vertices. Let u and v be vertices separated in G^m by S. Since G is k-connected, there are at least k internal vertex disjoint paths in G from u to v. They must all contain a vertex from S. There are fewer than m vertices from S in one of these paths. By choosing a different u and v if necessary, we can assume that all internal vertices of this path lie in S. Thus, in G^m, u and v are adjacent; a contradiction. □

The following result shows that if the connectivity of a graph G is large, the square of G is n-extendable.

Theorem 13 *If G is k-connected with even order and $k > n$, then G^r is n-extendable for $r \geq 2$.*

Proof. Suppose G^r is not n-extendable. There are n independent edges $e_i = u_iv_i$ $(i = 1, 2, \ldots, n)$ which do not lie in any perfect matching of G^r. Let $H = G^r - \{u_1, v_1, \ldots, u_n, v_n\}$. By Lemma 12, H is connected. By Tutte's Theorem, there is a cutset S of H such that $o(H - S) > |S|$. By parity, $o(H - S) = |S| + 2m$ for some positive integer m. Let $S' = S \cup \{u_1, v_1, \ldots, u_n, v_n\}$. Then $|S'| = |S| + 2n$ and $o(G^r - S') = o(H - S) = |S| + 2m$.

As G is k-connected, each component of $G^r - S'$ is adjacent in G to at least k vertices of S'. Suppose no two odd components of $G^r - S'$ in G have a common neighbour in S'. Then there are at least $(|S| + 2m)k$ vertices in S'. But S' has only $|S| + 2n < (|S| + 2m)k$ vertices. So at least two odd components C_1 and C_2 have in G a common neighbour v in S'. Then there is vertex u in C_1 and a vertex w in C_2 such that u and w are both adjacent to v. In G^r, u and w are adjacent. So u and w are in the same component of $G^r - S'$, contradicting the fact that C_1 and C_2 are different components of $G^r - S'$. □

The connectivity bound is sharp. Let $F = K_{n+1}$ if n is even or K_{n+2} if n is odd. Let H be isomorphic to F. Let $e_i = u_iv_i$ $(i = 1, 2, \ldots n)$ be n independent edges which are vertex disjoint from F and H. Join each u_i to every vertex of F and join each v_i to every vertex of H. The resulting graph G is n-connected. But $G^2 - \{u_1, v_1, \ldots, u_n, v_n\}$ has an odd component and therefore no perfect matching. Thus G^2 is not n-extendable.

If we relax the connectivity of G, then its power graph G^r is still n-extendable for sufficiently large r.

Theorem 14 *If G is k-connected with even order and $1 \leq k \leq n$, then G^r is n-extendable for $r \geq 2(n - k) + 3$.*

Proof. Proceed as in the first paragraph of the proof for Theorem 13. Let $C_1, C_2, \cdots, C_t$ be the components of $G^r - S'$. Let N_i be the set of vertices of S' that are adjacent in G to vertices of C_i. Since G is k-connected, each N_i contains at least k vertices. Also, the N_i are pairwise disjoint otherwise one of the components C_i contains a vertex u that is distance two from a vertex v in some other component C_j but then u and v would be in the same component of G^r. Since G is connected, there is a path P in G from a vertex w_i in N_i to a vertex w_j in N_j $(j \neq i)$. By assuming P has the minimum length among all such paths, P is contained in S' and the internal vertices of P have no vertex in N_l for $1 \leq l \leq t$. Since $|S'| = |S| + 2n$ and $t \geq |S| + 2m$, the order of P is at most $|S| + 2n - k(|S| + 2m) + 2 \leq |S| + 2n - k(|S| + 2) + 2 = 2(n-k) - |S|(k-1) + 2 \leq 2(n-k) + 2$. There is a vertex z_i in C_i and a vertex z_j in C_j adjacent to w_i and w_j respectively. Then $z_i P z_j$ is a path of length at most $2(n-k)+3$. So z_i and z_j are adjacent in G^r, contradicting the fact that C_i and C_j are different components of $G^r - S'$. □

The bound on r in Theorem 14 is the least possible. Let $G = u_0 u_1 \ldots u_{2n}\ u_{2n+1}$ be a path. Let $e_i = u_{2i-1}u_{2i}$ $(i = 1, 2, \ldots n)$. Since $G^{2n} - \{u_1, u_2, \ldots u_{2n}\}$ has an odd component (u_0 or u_{2n+1}) it does not have a perfect matching. We can replace u_0 or u_{2n+1} by odd components, and the resulting graph will still be a counterexample.

4 Total graphs

In this section we show that when the connectivity of a total graph $T(G)$ is sufficiently large, then $T(G)$ is n-extendable. We quote three useful lemmas.

Lemma 15 ([2]) *For any graph G, $T(G) = (S(G))^2$.*

Lemma 16 *Let G be a connected graph and let w be a vertex in a cutset R of $T(G)$. (1) If w is a subdivision vertex of $S(G)$, then w is adjacent to at most two components of $T(G) - R$. (2) If R contains no subdivision vertices of $S(G)$, then w is adjacent to exactly one component of $T(G) - R$.*

Proof. This follows immediately from Lemma 15. □

Theorem 17 *If $T(G)$ is $(2n+1)$-connected and has even order, then $T(G)$ is n-extendable.*

Proof. Suppose $T(G)$ is not n-extendable. There are n independent edges $e_i = u_i v_i$ $(i = 1, 2, \ldots n)$ which do not lie in a perfect matching of $T(G)$. Let $T' = T(G) - \{u_1, v_1, \ldots, u_n, v_n\}$. By Tutte's Theorem, there is a subset S' of vertices of T' such that $o(T' - S') > |S'|$. By parity, $o(T' - S') = |S'| + 2m$ for some positive integer m. Let $S = S' \cup \{u_1, v_1, \ldots, u_n, v_n\}$. Then $o(T(G) - S) = o(T' - S') = |S'| + 2m = |S| - 2n + 2m$. Let $C_1, C_2, \ldots$ denote the odd components of $T(G) - S$.

We now reduce S while keeping the relation $o(T(G) - S) = |S| - 2n + 2m$ $(m \geq 1)$. Let w be a vertex in S and replace S with $S'' = S \backslash \{w\}$.

If w is not adjacent to any odd component, then $o(T(G) - S'') = o(T(G) - S) + 1 = |S''| - 2n + 2(m+1)$.

Suppose every vertex of S is adjacent to an odd component. If w is a subdivision vertex of $S(G)$, then, by Lemma 16, w is adjacent to at most two odd components. If w is adjacent to two odd components C_i and C_j, then the subgraph of $T(G) - S''$ induced by $C_i \cup \{w\} \cup C_j$ is an odd component and $o(T(G) - S'') = |S''| - 2n + 2m$. If w is adjacent to only one odd component C_i, then again $o(T(G) - S'') = |S''| - 2n + 2m$.

If S does not contain any subdivision vertex of $S(G)$, then, by Lemma 16, w is adjacent to exactly one odd component and again $o(T(G) - S'') = |S''| - 2n + 2m$.

Repeat the process above until $|S| = 2n$. Then $o(T(G) - S) = |S| - 2n + 2m = 2m \geq 2$. Thus S is a cutset of $T(G)$ of order $2n$, a contradiction. □

If we relax the connectivity of G then its iterated total graph $T^r(G)$ is still n-extendable for sufficiently large r.

Lemma 18 ([6, 9]) *If G is k-connected, then $T(G)$ is $2k$-connected.*

Corollary 19 *Let G be k-connected and $2^r > 2n/k$. The iterated total graph $T^r(G)$ is n-extendable if it has even order.*

Proof. This follows immediately from Lemma 18 and Theorem 17. □

Note that if G is k-connected, then $T(G)$ may be exactly $2k$-connected. Let w be a vertex of degree k. Then w has $2k$ neighbours in $T(G)$ which form a cutset. On the other hand the connectivity of $T(G)$ may be considerably higher than $2k$. For example, let G be the graph formed by identifying a vertex from K_{4p} with a vertex of K_{4p+1}. Then G is 1-connected but $T(G)$ has even order and is $(8p-2)$-connected. Thus Theorem 17 is more powerful than Corollary 19.

The connectivity in Theorem 17 and inequality in Theorem 18 are sharp. Let G be a k-connected k-regular graph. Suppose $2^r k = 2n$. Since $T^i(G)$ is $2^i k$-regular, $T^i(G)$ is exactly $2^i k$-connected by Lemma 18. By Lemma 15 $T^r(G) = (S(T^{r-1}(G)))^2$. Let w be a vertex in $T^{r-1}(G)$, let w_i $(i = 1, 2, \ldots, 2^{r-1}k)$ be the vertices of $T^{r-1}(G)$ adjacent to w and let u_i be the subdivision vertex on ww_i in $S(T^{r-1}(G))$ $(i = 1, 2, \ldots, 2^{r-1}k)$. Then the $u_i w_i$ are $2^{r-1}k = n$ independent edges of $T^r(G)$. But $T^r(G) - \{u_i, w_i | i = 1, 2, \ldots, 2^{r-1}k\}$ does not have a perfect matching as w is an isolated vertex. So $T^r(G)$ is not n-extendable.

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