# The construction of four-weight spin models by using Hadamard matrices and M-structure

Dedicated to Professor Joji Kajiwara on the occasion of his 60th birthday

Mieko Yamada*
Department of Mathematics
Kyushu University
Fukuoka 812, Japan

**Abstract.** The concept of spin models was introduced by V.F. Jones in 1989. K. Kawagoe, A. Munemasa and Y. Watatani generalized it by removing the condition of symmetry. Recently E. Bannai and E. Bannai further generalized the concept of spin models, to give four-weight spin models or generalized spin models. Before this, F. Jaeger first pointed out the relation between spin models and association schemes.

K. Nomura constructed a family of symmetric spin models of Jones type of loop variable $4\sqrt{n}$ from Hadamard matrices of order $4n$. V. G. Kac and M. Wakimoto showed that spin models of Jones type and 4-weight spin models can be constructed by using Lie algebras.

Recently K. Nomura proved that every symmetric four-weight spin model comes from a symmetric spin model of Jones type by a twisting product construction. In this paper, we prove that a symmetric spin model of Jones type, which was introduced by Jones, can be constructed from a four-weight spin model such that two of the four functions (not necessarily all) are symmetric.

On the other hand, it is well known that the tensor product of two four-weight spin models is also a four-weight spin model. We give a construction of a four-weight spin model, which is not the tensor product construction. Namely if there exists a four-weight spin model of loop variable $D$ satisfying a certain condition, we can construct a four-weight spin model of loop variable $2D$ from it, which also satisfies the same condition. We give an example of a four-weight spin model satisfying this condition, constructed from Hadamard matrices and complex Hadamard matrices. It means that there exists an infinite family of four-weight spin models. We prove these results by using an M-structure.

*This work was supported in part by a Grant-in-Aid for General Scientific Research from the Ministry of Education, Science and Culture.

# 1 Introduction

The concept of spin models was introduced by V.F. Jones [4] in 1989 to give the link invariant. K. Kawagoe, A. Munemasa and Y. Watatani [6] generalized it by removing the condition of symmetry. Recently E. Bannai and E. Bannai [1] further generalized the concept of spin models to give four-weight spin models or generalized spin models.

**Definition 1** (E. Bannai-E. Bannai, [1]) Let $X$ be a finite set and $w_i$ $(i = 1, 2, 3, 4)$ be functions on $X \times X$ to $C$. Let $W_i = (w_i(\alpha, \beta))_{\alpha,\beta\in X}$ for $i$=1,2,3,4. Then $(X, w_1, w_2, w_3, w_4)$ is a 4-weight spin model of loop variable $D$ if the following conditions are satisfied for any $\alpha$, $\beta$ and $\gamma \in X$:

(1) $W_1^t \circ W_3 = J$, $W_2^t \circ W_4 = J$;

(2) $W_1 W_3 = nI$, $W_2 W_4 = nI$;

(3a) $\sum_{x\in X} w_1(\alpha, x) w_1(x, \beta) w_4(\gamma, x) = D w_1(\alpha, \beta) w_4(\gamma, \alpha) w_4(\gamma, \beta)$;

(3b) $\sum_{x\in X} w_1(x, \alpha) w_1(\beta, x) w_4(x, \gamma) = D w_1(\beta, \alpha) w_4(\alpha, \gamma) w_4(\beta, \gamma)$,

where $D^2 = n = |X|$ and $\circ$ means Hadamard product. The conditions (3a) and (3b) are called the star triangle relations.

Let $L$ be a diagram of an oriented link. We color the regions of $L$ in black and white so that the unbounded region is colored in white and adjacent regions have different colors. We construct an oriented graph assigning a black region to a vertex and a crossing to an edge. We get exactly four kinds of crossings according to the colors of the regions and the orientations of the links. Then we attach four weights, 1,2,3,4 to the four kinds of edges, namely to four kinds of crossings, respectively. Then we get an oriented graph with four kinds of weights.

Denote the weight $n$ for an edge $\alpha \to \beta$ by $n(\alpha \to \beta)$. Let $X$ be a finite set with $|X| = n = D^2$. Let $w_1, w_2, w_3$ and $w_4$ be complex valued functions defined on $X \times X$. Under these assumptions, the partition function $Z_L$ is defined by

$$Z_L = D^{-v(L)} \sum_{\sigma} \prod_{\alpha\to\beta} w_{n(\alpha\to\beta)}(\sigma(\alpha), \sigma(\beta))$$

where a state $\sigma$ is a map from the vertices of the graph to $X$ and $v(L)$ is the number of vertices of the graph.

If $(X, w_1, w_2, w_3, w_4)$ is a four-weight spin model with loop variable $D$, then the partition function $Z_L$ is invariant under the Reidemeister moves of types II and III. See the details in [1].

We consider the special case of 4-weight spin models. Let $\epsilon$ and $\epsilon'$ be from $\{+, -\}$. A four-weight spin model with $W_1, W_2 \in \{W_\epsilon, W_\epsilon^t\}$ and $W_3, W_4 \in \{W_{\epsilon'}, W_{\epsilon'}^t\}$ is called a generalized spin model of Jones type. A four-weight spin model with $W_1, W_4 \in \{W_\epsilon, W_\epsilon^t\}$ and $W_2, W_3 \in \{W_{\epsilon'}, W_{\epsilon'}^t\}$ is called a generalized spin model of pseudo-Jones type. Further, a four-weight spin model with $W_1, W_3 \in \{W_\epsilon, W_\epsilon^t\}$ and $W_2, W_4 \in \{W_{\epsilon'}, W_{\epsilon'}^t\}$ is called a generalized spin model of Hadamard type.

We give the precise definition of generalized spin models of Jones type, which was introduced by Kawagoe, Munemasa and Watatani.

**Definition 2** ([1], [6]) $(X, w_+, w_-)$ is a generalized spin model of Jones type if the following conditions are satisfied for any $\alpha, \beta$ and $\gamma \in X$.

(1J) $W_+^t \circ W_- = J$;

(2J) $W_+ W_- = nI$;

(3J) (Star triangle relation)

$$\sum_{x \in X} w_+(\alpha, x) w_+(x, \beta) w_-(x, \gamma) = D w_+(\alpha, \beta) w_-(\alpha\gamma) w_-(\beta, \gamma),$$

where $D^2 = n = |X|$ and $\circ$ means Hadamard product. If a generalized spin model of Jones type satisfies the following additional condition,

(0J) $w_+(\alpha, \beta) = w_+(\beta, \alpha)$, $w_-(\alpha, \beta) = w_-(\beta, \alpha)$,

it is called symmetric, that is, it is the the original spin model due to Jones [4].

There are five kinds of generalized spin models of Hadamard type. See the detailed definitions in [1].

F. Jaeger first pointed out the relation between spin models and association schemes [3]. K. Nomura [7] constructed a family of symmetric spin models of Jones type of loop variable $4\sqrt{n}$ from Hadamard matrices of order $4n$. V.G. Kac and M. Wakimoto [5] showed that spin models of Jones type and 4-weight spin models can be constructed by using Lie algebras.

Let $H$ be a complex Hadamard matrix. If we put $W_+ = H$ and $W_- = \overline{H^t}$, then $W_+$ and $W_-$ satisfy the conditions (1J) and (2J). We can verify the condition (3J). In other words, a complex Hadamard matrix which satisfies the condition (3J) is a spin model of Jones type. Similarly, if we put $W_1 = H_1$, $W_3 = \overline{H_1^t}$, $W_2 = H_2$ and $W_4 = \overline{H_2^t}$ where $H_1$ and $H_2$ are complex Hadamard matrices, the conditions (1) and (2) in Definition 1 hold. Thus we can verify the conditions (3a) and (3b).

It turns out that we need to find a complex Hadamard matrix or a pair of complex Hadamard matrices, which satisfy the star triangle relations.

# 2 Symmetric spin models of Jones type

The concept of M-structures generalizes a number of concepts in Hadamard matrices, including Williamson matrices, Goethals-Seidel matrices, Wallis-Whiteman matrices and generalized quaternion type matrices [10]. We found many symmetric Williamson matrices and many Hadamard matrices using the concept of an M-structure. The concept of an M-structure leads to the new concept of strong Kronecker product introduced by J. Seberry and X.-M. Zhang [9]. On the other hand, K. Nomura proved that every symmetric four-weight spin model comes from a symmetric spin model of Jones type by a twisted product construction [8]. Motivated by Nomura's work, we get the following result by using an M-structure.

**Theorem 1** *Let $X$ be a finite set and assume that $w_1, w_2, w_3$ and $w_4$ are complex valued functions on $X \times X$. Define the matrices $U_+$ and $U_-$ as follows:*

$$U_+ = \begin{pmatrix} W_1 & -W_1 & W_2 & W_2 \\ -W_1 & W_1 & W_2 & W_2 \\ W_2^t & W_2^t & W_1 & -W_1 \\ W_2^t & W_2^t & -W_1 & W_1 \end{pmatrix}, \quad U_- = \begin{pmatrix} W_3 & -W_3 & W_4 & W_4 \\ -W_3 & W_3 & W_4 & W_4 \\ W_4^t & W_4^t & W_3 & -W_3 \\ W_4^t & W_4^t & -W_3 & W_3 \end{pmatrix}.$$

*Then $U_+$ and $U_-$ form a generalized spin model of symmetric Jones type if and only if $(X, w_1, w_2, w_3, w_4)$ is a four-weight spin model with the conditions $W_1^t = W_1$ and $W_3^t = W_3$.*

**Proof.** First assume that $(X, w_1, w_2, w_3, w_4)$ is a four-weight spin model of loop variable $D$ such that $W_1^t = W_1$ and $W_3^t = W_3$. It is clear that $U_+$ and $U_-$ are symmetric. Since $W_1 \circ W_3^t = J$ and $W_2 \circ W_4^t = J$, the condition (1J) holds. The condition $W_2 W_4 = nI$ inplies $W_2^t W_4^t = nI$. From this and $W_1 W_3 = nI$, we can verify the condition (2J) easily. We can verify the star triangle relation (3J).

For $\alpha, \beta, \gamma$, we denote the left-hand side of the condition (3J) by $S(\alpha, \beta, \gamma)$. The star triangle relation becomes

$$(*) \quad S(\alpha+jn, \beta+kn, \gamma+ln) = 2Du_+(\alpha+jn, \beta+kn)u_-(\alpha+jn, \gamma+ln)u_-(\beta+kn, \gamma+ln)$$

for any $\alpha, \beta$ and $\gamma \in X$ and any $j, k, l$ with $0 \leq j, k, l \leq 3$.

$$\begin{aligned} S(\alpha,\beta,\gamma) &= \sum_{x\in X} w_1(\alpha,x)w_1(x,\beta)w_3(x,\gamma) + \sum_{x\in X}(-w_1(\alpha,x))(-w_1(x,\beta))(-w_3(x,\gamma)) \\ &\quad + \sum_{x\in X} w_2(\alpha,x)w_2(\beta,x)w_4(x,\gamma) + \sum_{x\in X} w_2(\alpha,x)w_2(\beta,x)w_4(x,\gamma) \\ &= 2\sum_{x\in X} w_2(\alpha,x)w_2(\beta,x)w_4(x,\gamma). \end{aligned}$$

Theorem 2 in E. Bannai and E. Bannai's paper [1] showed that four relations $(III_1)$, $(III_4)$, $(III_5)$, $(III_8)$ given in [1] are equivalent each other and four relations $(III_2)$, $(III_3)$, $(III_6)$, $(III_7)$ are also equivalent each other. The relations $(III_1)$ and $(III_6)$ are (3a) and (3b) in Definition 1 respectively. The relation $(III_8)$ gives

$$\sum_{x\in X} w_2(\alpha,x)w_2(\beta,x)w_4(x,\gamma) = Dw_1(\beta,\alpha)w_3(\alpha,\gamma)w_3(\gamma,\beta).$$

Since $W_1$ and $W_3$ are symmetric, we have

$$\begin{aligned} S(\alpha,\beta,\gamma) &= 2Dw_1(\alpha,\beta)w_3(\alpha,\gamma)w_3(\beta,\gamma) \\ &= 2Du_+(\alpha,\beta)u_-(\alpha,\gamma)u_-(\beta,\gamma). \end{aligned}$$

$$\begin{aligned} &S(\alpha,\beta+2n,\gamma+2n) \\ &\quad = \sum_{x\in X} w_1(\alpha,x)w_2(x,\beta)w_4(\gamma,x) + \sum_{x\in X}(-w_1(\alpha,x))w_2(x,\beta)w_4(\gamma,x) \\ &\quad\quad + \sum_{x\in X} w_2(\alpha,x)w_1(x,\beta)w_3(x,\gamma) + \sum_{x\in X} w_2(\alpha,x)(-w_1(x,\beta))(-w_3(x,\gamma)) \end{aligned}$$

$$
\begin{aligned}
&= 2\sum_{x\in X} w_2(\alpha,x)w_1(x,\beta)w_3(x,\gamma)\\
&= 2\sum_{x\in X} w_1(x,\beta)w_2(\alpha,x)w_3(x,\gamma).
\end{aligned}
$$

The relation $(III_2)$ in [1]

$$\sum_{x\in X} w_1(x,\beta)w_2(\alpha,x)w_3(\gamma,x) = 2Dw_2(\alpha,\beta)w_3(\gamma,\beta)w_4(\gamma,\alpha)$$

is also satisfied by Theorem 2 in [1]. Therefore we get

$$
\begin{aligned}
S(\alpha,\beta+2n,\gamma+2n) &= 2Dw_2(\alpha,\beta)w_3(\gamma,\beta)w_4(\gamma,\alpha)\\
&= 2Du_+(\alpha,\beta+2n)u_-(\gamma+2n,\beta+2n)u_-(\gamma+2n,\alpha).
\end{aligned}
$$

By a method similar to the above, we can verify the relation $(*)$ for all $\alpha,\beta,\gamma$ and for all $j,k,l$ with $0\le j,k,l\le 3$.

Next we assume $U_+$ and $U_-$ form a generalized spin model of symmetric Jones type. $W_1$ and $W_3$ are both symmetric as $U_+$ and $U_-$ are symmetric. From $U_+\circ U_-^t = J$, we obtain $W_1\circ W_3^t = J$ and $W_2\circ W_4^t = J$. Since $U_+U_- = 4nI$, we have

$$2W_1W_3 + 2W_2W_4 = 4nI$$

and

$$-2W_1W_3 + 2W_2W_4 = 0.$$

It means that $W_1W_3 = W_2W_4 = nI$. Since the star triangle relation $(*)$ is satisfied, we have the eight relations $(III_1)-(III_8)$. Proposition 1 and Theorem 2 in [1] showed that if the conditions $W_1\circ W_3^t = W_2\circ W_4^t = J$ and $W_1W_3 = W_2W_4 = nI$ are satisfied, then these eight equations reduce to the two equations (3a) and (3b). Hence $W_1,W_2,W_3,W_4$ form a four-weight spin model where $W_1$ and $W_3$ are symmetric. □

**Corollary 1** *If there exists a generalized spin model of Hadamard type such that $W_+$ or $W_-$ is symmetric, then there exists a generalized spin model of symmetric Jones type.*

# 3 An infinite family of four-weight spin models

It is known that the tensor product of two four-weight spin models is also a four-weight spin model. In this section, we give another construction for a four weight spin model.

**Theorem 2** *Let $X$ be a finite set and assume that $(X,w_1,w_2,w_3,w_4)$ is a four-weight spin model. Define the matrices $U_1,U_2,U_3$ and $U_4$ as follows:*

$$
U_1=\begin{pmatrix} W_1 & W_1 & W_1^t & -W_1^t\\ W_1 & W_1 & -W_1^t & W_1^t\\ W_1^t & -W_1^t & W_1 & W_1\\ -W_1^t & W_1^t & W_1 & W_1\end{pmatrix},\quad
U_2=\begin{pmatrix} W_2 & W_2 & W_2^t & -W_2^t\\ W_2 & W_2 & -W_2^t & W_2^t\\ W_2^t & -W_2^t & W_2 & W_2\\ -W_2^t & W_2^t & W_2 & W_2\end{pmatrix},
$$

$$U_3 = \begin{pmatrix} W_3 & W_3 & W_3^t & -W_3^t \\ W_3 & W_3 & -W_3^t & W_3^t \\ W_3^t & -W_3^t & W_3 & W_3 \\ -W_3^t & W_3^t & W_3 & W_3 \end{pmatrix}, \quad U_4 = \begin{pmatrix} W_4 & W_4 & W_4^t & -W_4^t \\ W_4 & W_4 & -W_4^t & W_4^t \\ W_4^t & -W_4^t & W_4 & W_4 \\ -W_4^t & W_4^t & W_4 & W_4 \end{pmatrix}.$$

*If $(X, w_1, w_2, w_3, w_4)$ is a four-weight spin model and the following relations are satisfied for any $\alpha, \beta$ and $\gamma$ in $X$, then $U_1, U_2, U_3$ and $U_4$ form a four-weight spin model.*

(1) $\sum_{x\in X} w_1(x,\alpha)w_1(\beta,x)w_4(\gamma,x) = Dw_1(\alpha,\beta)w_4(\alpha,\gamma)w_4(\beta,\gamma)$

(2) $\sum_{x\in X} w_1(x,\alpha)w_1(x,\beta)w_4(x,\gamma) = Dw_1(\beta,\alpha)w_4(\gamma,\alpha)w_4(\beta,\gamma)$

(3) $\sum_{x\in X} w_1(\alpha,x)w_1(\beta,x)w_4(x,\gamma) = Dw_1(\beta,\alpha)w_4(\alpha,\gamma)w_4(\gamma,\beta)$

(4) $\sum_{x\in X} w_1(\alpha,x)w_1(x,\beta)w_4(x,\gamma) = Dw_1(\beta,\alpha)w_4(\gamma,\alpha)w_4(\gamma,\beta)$

(5) $\sum_{x\in X} w_1(\alpha,x)w_1(\beta,x)w_4(\gamma,x) = Dw_1(\alpha,\beta)w_4(\alpha,\gamma)w_4(\gamma,\beta)$

(6) $\sum_{x\in X} w_1(x,\alpha)w_1(x,\beta)w_4(\gamma,x) = Dw_1(\alpha,\beta)w_4(\gamma,\alpha)w_4(\beta,\gamma)$.

**Proof.** Assume that $W_1, W_2, W_3$ and $W_4$ form a four weight spin model of loop variable $D$, $D^2 = n = |X|$ and the relations (1)-(6) are satisfied. From the condition $W_1W_3 = nI$, we can verify $U_1U_3 = 4nI$. The condition $U_2U_4 = nI$ also holds. Denote by $S_a(\alpha,\beta,\gamma)$ the left-hand side of the star triangle relation (3a).

$$\begin{aligned} & S_a(\alpha,\beta,\gamma) \\ = & \sum_{x\in X} w_1(\alpha,x)w_1(x,\beta)w_4(\gamma,x) + \sum_{x\in X} w_1(\alpha,x)w_1(x,\beta)w_4(\gamma,x) \\ & + \sum_{x\in X} w_1(x,\alpha)w_1(\beta,x)w_4(x,\gamma) + \sum_{x\in X}(-w_1(x,\alpha))(-w_1(\beta,x))(-w_4(x,\gamma)) \\ = & 2\sum_{x\in X} w_1(\alpha,x)w_1(x,\beta)w_4(\gamma,x). \end{aligned}$$

Since $W_1, W_2, W_3$ and $W_4$ form a four-weight spin model,

$$S_a(\alpha,\beta,\gamma) = 2Dw_1(\alpha,\beta)w_4(\gamma,\alpha)w_4(\gamma,\beta).$$

$$\begin{aligned} & S_a(\alpha,\beta,\gamma+2n) \\ = & \sum_{x\in X} w_1(\alpha,x)w_1(x,\beta)w_4(x,\gamma) + \sum_{x\in X} w_1(\alpha,x)w_1(x,\beta)(-w_4(x,\gamma)) \\ & + \sum_{x\in X} w_1(x,\alpha)w_1(\beta,x)w_4(\gamma,x) + \sum_{x\in X}(-w_1(x,\alpha))(-w_1(\beta,x))w_4(\gamma,x) \\ = & 2\sum_{x\in X} w_1(x,\alpha)w_1(\beta,x)w_4(\gamma,x). \end{aligned}$$

¿From the relation (1), we get

$$S_a(\alpha,\beta,\gamma+2n) = 2Dw_1(\alpha,\beta)w_4(\alpha,\gamma)w_4(\beta,\gamma).$$

Similarly we can verify the star triangle relations (3a) and (3b) for $\alpha+jn, \beta+kn, \gamma+ln$ where all $\alpha,\beta,\gamma \in X$ and all $j,k,l$ with $0 \le j,k,l \le 3$. □

**Corollary 2** *If $W_1, W_2, W_3$ and $W_4$ form a four-weight spin model with the following conditions,*

$$(**) \quad \begin{cases} w_1(\beta,\alpha) &= (-1)^{\alpha+\beta} w_1(\alpha,\beta), \\ w_4(\beta,\alpha) &= (-1)^{\alpha+\beta} w_4(\alpha,\beta), \end{cases}$$

*for any $\alpha$ and $\beta$, $0 \le \alpha, \beta \le n-1$, then $W_1, W_2, W_3$ and $W_4$ satisfy the relations (1)-(6) of Theorem 2. Hence $U_1, U_2, U_3$ and $U_4$ form a four-weight spin model. Furthermore $U_1$ and $U_4$ also satisfy the condition $(**)$. It means that there exists an infinite family of four-weight spin models if there exist a four-weight spin model satisfing the conditions $(**)$.*

**Proof.** Under the assumption, we verify the relation (1).

$$\begin{aligned} &\textstyle\sum_{x\in X} w_1(x,\alpha)w_1(\beta,x)w_4(\gamma,x) \\ &= \begin{cases} \sum_{x\in X} w_1(x,\alpha)w_1(x,\beta)w_4(\gamma,x) & \text{when } \alpha \text{ and } \beta \text{ are both even, or both odd,} \\ -\sum_{x\in X} w_1(x,\alpha)w_1(x,\beta)w_4(\gamma,x) & \text{otherwise,} \end{cases} \\ &= \begin{cases} Dw_1(\alpha,\beta)w_4(\gamma,\alpha)w_4(\gamma,\beta) & \text{when } \alpha \text{ and } \beta \text{ are both even, or both odd,} \\ -Dw_1(\alpha,\beta)w_4(\gamma,\alpha)w_4(\gamma,\beta) & \text{otherwise,} \end{cases} \\ &= Dw_1(\alpha,\beta)w_4(\alpha,\gamma)w_4(\beta,\gamma). \end{aligned}$$

Similarly, by distinguishing two cases when $\alpha$ and $\gamma$ are both even or both odd, and otherwise, we can verify the relation (2). That the relations (3)-(6) also hold is shown by a similar method.

The element $u_i(\beta,\alpha)$ is given by an element of $W_i$ according to the values of $\alpha$ and $\beta$, for $i = 1, 4$.

$$\begin{aligned} u_i(\beta,\alpha) &= \begin{cases} w_i(\beta,\alpha) \\ w_i(\alpha,\beta) \\ -w_i(\alpha,\beta) \end{cases} = \begin{cases} (-1)^{\alpha+\beta} w_i(\alpha,\beta) \\ (-1)^{\alpha+\beta} w_i(\beta,\alpha) \\ -(-1)^{\alpha+\beta} w_i(\beta,\alpha) \end{cases} = (-1)^{\alpha+\beta} \begin{cases} w_i(\alpha,\beta) \\ w_i(\beta,\alpha) \\ -w_i(\beta,\alpha) \end{cases} \\ &= (-1)^{\alpha+\beta} u_i(\alpha,\beta). \end{aligned}$$

□

We give an example satisying the condition $(**)$ of Corollary 2.

**Example.** Let $i$ be a primitive fourth root of unity.

$$W_1 = \begin{pmatrix} 1 & -i & 1 & i \\ i & 1 & -i & 1 \\ 1 & i & 1 & -i \\ -i & 1 & i & 1 \end{pmatrix}, \quad W_3 = \overline{W_1^t}, \quad W_2 = \begin{pmatrix} 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{pmatrix}, \quad W_4 = W_2^t.$$

Notice that $W_2$ and $W_4$ are both Hadamard matrices of order 4 and $W_1$ and $W_3$ are both complex Hadamrd matrices of order 4. Therefore from Corollary 2, we obtain the following remark.

**Remark.** There exists an infinite family of four weight spin models with loop variable $2^s$, where $s$ is a positive integer.

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*(Received 15/3/94)*