Percolation and Disorder-Resistance in Cellular Automata

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Abstract. We rigorously prove a form of disorder-resistance for a class of one-dimensional cellular automaton rules, including some that arise as boundary dynamics of two-dimensional solidification rules. Specifically, when started from a random initial seed on an interval of length L, with probability tending to one as $L \to \infty$, the evolution is a *replicator*. That is, a region of space-time of density one is filled with a spatially and temporally periodic pattern, punctuated by a finite set of other finite patterns repeated at a fractal set of locations. On the other hand, the same rules exhibit provably more complex evolution from some seeds, while from other seeds their behavior is apparently chaotic. A principal tool is a new variant of percolation theory, in the context of additive cellular automata from random initial states.

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1 Introduction

Cellular automata (CA) started from *seeds*, i.e., finite perturbations of a quiescent state, have been the subject of much empirical analysis, starting with [Wol1]. The observed behavior falls roughly into four categories: (a) the perturbation remains *localized* in the sense that it never affects sites outside a bounded interval; (b) a *periodic* structure develops and spreads; (c) a *replicating* (also called *nested* or *fractal*) structure develops, with a recursive (but sometimes complicated) description; (d) unpredictable *chaotic* (or *complex*) growth generates a space-time configuration with apparent characteristics of random fields. Many CA are capable of behavior in multiple categories depending on the choice of seed, and this is true even for some of the very simplest one-dimensional CA. An example is the *Exactly 1* rule, in which a cell is alive whenever exactly one of itself and its two neighbors were alive at the previous generation. *Exactly 1* is capable of periodic, replicating, and chaotic behavior for different seeds; see [GG3].

If a particular CA is capable of chaotic behavior from some initial seed, it appears natural to conclude, by analogy with the second law of thermodynamics, that such behavior should be generic for that CA, in the sense that almost all sufficiently long seeds yield chaotic evolution. Shadowing results from dynamical systems [Pil], with their general message of stability of chaotic trajectories, would also tend to support such a conclusion. Indeed, strong empirical evidence confirms that chaotic behavior is prevalent for many CA including *Exactly 1*; see [GG3].

In this article we exhibit a class of one-dimensional CA rules for which we rigorously prove that the *opposite* conclusion holds. Typical (random) long seeds self-organize into replicating structures, while exceptional seeds yield more complex behavior, including apparently chaotic evolution.

We focus on one-dimensional range-2 CA rules with 3 states (although our techniques in principle apply to more general one-dimensional rules). Thus, the configuration of the CA at time $t \in \{0, 1, 2, \dots\}$ is an element $\xi_t = (\xi_t(x))_{x\in\mathbb{Z}}$ of $\{0, 1, 2\}^{\mathbb{Z}}$, and for a given initial configuration ξ_0, the evolution is given by

$$\xi_{t+1}(x) = f\Big(\xi_t(x-2), \xi_t(x-1), \xi_t(x), \xi_t(x+1), \xi_t(x+2)\Big)$$

for all x, t and a fixed function f (the CA rule). (In many cases the dependence on ξ_t will actually be restricted to the range-1 neighborhood $x-1, x, x+1$). We sometimes write $\xi(x,t) = \xi_t(x)$ for the state of ξ at the space-time point $(x,t) \in \mathbb{Z} \times [0, \infty)$. In keeping with standard convention, diagrams of space-time evolution are drawn with the space coordinate x increasing from left to right, and the time coordinate t increasing from top to bottom.

A key supporting role will be played by the *1 Or 3* CA, a simple 2-state rule denoted by λ_t, and defined as follows. The states are 0 and 1, and the evolution is

$$\lambda_{t+1}(x) = \lambda_t(x-1) + \lambda_t(x) + \lambda_t(x+1) \mod 2.$$

As is well known, the additive structure of this rule enables many of its characteristics to be fully understood. (See Figure 1.1 below for an illustration.)

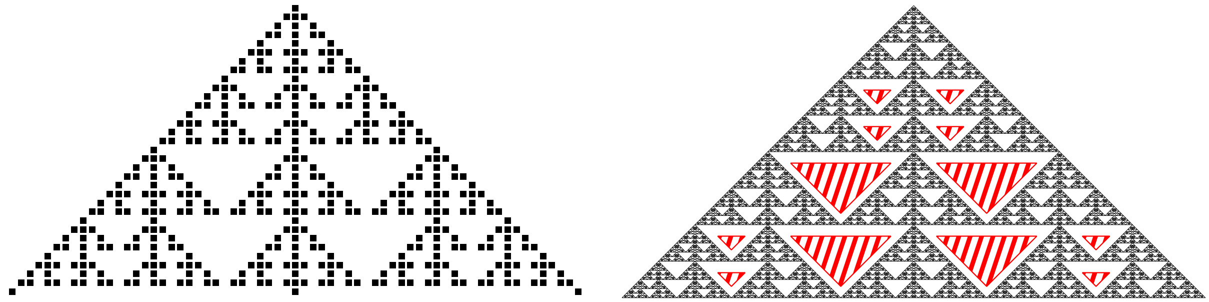

Figure 1.1: Left: the configuration $\lambda^\bullet$ of *1 Or 3*, started from a single occupied cell, up to time $t = 32$. Right: schematic depiction of a replicator. The striped regions are filled with a doubly periodic ether. The thickness of the white "buffer zones" remains constant for all time.

We consider 3-state CA rules with the following special property. For any configuration ξ, if we define $\lambda_t(x) = \mathbf{1}[\xi_t(x) = 1]$ for all x, t, then λ evolves precisely according to the *1 Or 3* CA. We also assume that state 0 is quiescent – that is, if $\xi_0 \equiv 0$ then $\xi_1 \equiv 0$. We call any CA rule satisfying these two conditions a **web CA**. The idea is that the 1s form an additive "web" which is not influenced by the distinction between 0s and 2s, while the web may affect the 2s. As we will see later, web CA also arise in analysis of two-dimensional solidification CA. We will usually be interested in evolution from a **seed**, i.e. an initial configuration ξ_0 with finite support.

One of the simplest web CA, which we call *Web-Xor*, is defined by setting $\xi_t(x) = 2$ if and only if $\lambda_t(x) = 0$ and there is a exactly one 2 among $\xi_{t-1}(x-1), \xi_{t-1}(x+1)$. (Together with the web CA condition, this is sufficient to specify the rule). Thus, 2s perform a 2-neighbor exclusive-or rule on the points that are not occupied by 1s. Figure 1.2 illustrates the evolution of *Web-Xor* from four different seeds. (States are always colored as: 0 white; 1 black or grey; 2 another color depending on the rule.) Our results imply that *typical* seeds result in behavior similar to the first picture. More specifically, we will prove that for certain classes of web CA, evolution from long random seeds yields with high probability a space-time configuration that is periodic except within some finite distance of an additive web. To state this conclusion precisely we need some more notation.

An **ether** is an element η of $\{0,2\}^{\mathbb{Z}^2}$ that is periodic in both coordinates. Two ethers are **equivalent** if one can be obtained from the other via some translation of $\mathbb{Z}^2$. In a CA configuration ξ we say that a set $K \subseteq \mathbb{Z} \times [0, \infty)$ is **filled with** η if ξ agrees with some ether equivalent to η on K. Let $\lambda^\bullet$ be the *1 Or 3* CA started from the seed consisting of a single 1 at the origin, and let $\Lambda = \{(x,t) : \lambda^\bullet(x,t) = 1\}$ be its support. See Figure 1.1. Let $\Lambda(r) \subset \mathbb{Z}^2$ be the set of space-time points at ℓ^1-distance at most r from Λ.

For a given CA, we say that a seed ξ_0 (or equivalently the resulting configuration ξ) is a **replicator** of **thickness** r and ether η if each bounded component of $\mathbb{Z}^2 \setminus \Lambda(r)$ is filled with η. See Figure 1.1. It is a straightforward fact that $\Lambda(r)$ has density 0 as a subset of $\mathbb{Z}^2$ for any r. Therefore, in a replicator, the density of 2s within the cone $\{(x,t) : |x| \le t\}$ equals the density of the ether. Furthermore, it may be shown that for any replicator (of any CA), the configuration ξ can be fully described in terms of a finite set of local patterns that are repeated at infinitely many locations prescribed by Λ. (This is the reason for the name replicator.) For more details we refer the reader to [GGP], where the concept was introduced.

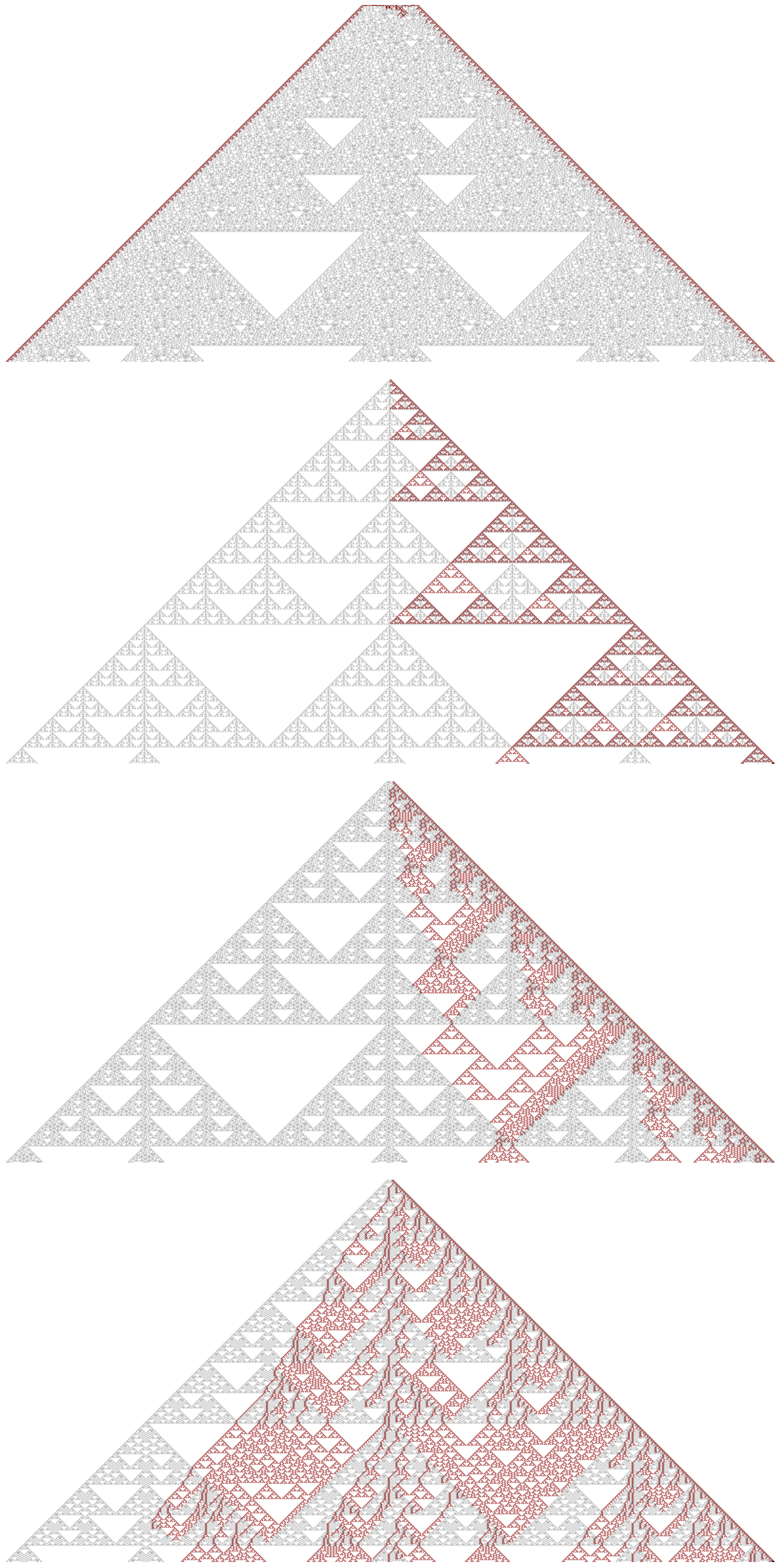

Figure 1.2: Four configurations of *Web-Xor*. The first (top) example, a replicator with zero ether, starts from a random string of 64 1s and 2s. The second and third examples, with respective seeds 12 and 11111012, are quasireplicators. The bottom example, with seed 1100112, is apparently chaotic.

Our results will apply to web CA rules satisfying two conditions which we call *diagonal-compliance* and *wide-compliance*. The conditions state that flow of information concerning the distinction between 0s and 2s is blocked by certain local patterns of 1s. The formal statements of the conditions are straightforward but somewhat technical, and we therefore postpone them to the next section. For now we note that *Web-Xor* is diagonal-compliant.

A **uniformly random binary seed** on $[0, L]$ is an initial configuration ξ_0 in which $\xi_0(x)$ takes values $0, 1$ with equal probabilities independently for all $x \in [0, L]$, and 0 outside $[0, L]$.

Theorem 1.1 (Replication from random seeds)**.** *Consider a web CA that is either diagonal-compliant or wide-compliant, started from a uniformly random binary seed on* $[0, L]$*. There exist a random variable* R_L *taking values in* $[0, \infty]$*, and a random ether* η_L *(both deterministic functions of the seed), with the following properties. We have* $\mathbf{P}(R_L = \infty) \to 0$ *as* $L \to \infty$*, and indeed the sequence* $(R_L)_{L \geq 0}$ *is tight. On the event* $R_L < \infty$*, the configuration* ξ *is a replicator of thickness* $R_L + L$ *and ether* η_L*. Furthermore, if any finite set of 0s in* ξ_0 *are changed into 2s, the same statement holds with the same* R_L *and* η_L*.*

Web CA rules may be further classified in the following way, which has implications for their production of ethers. A CA has **no spontaneous birth** (of 2s) if whenever ξ_0 contains no 2s, ξ_1 also contains no 2s. *Web-Xor* has no spontaneous birth. Figure 1.3 shows four possible evolutions of a CA rule called *Piggyback* (to be defined in the next section) that is wide-compliant and has spontaneous birth.

Theorem 1.2 (Trivial and non-trivial ethers)**.** *Assuming the conditions of Theorem 1.1,* R_L *can be chosen to have the following additional properties.*

(i) *If the CA rule has no spontaneous birth, then* $\eta_L \equiv 0$ *whenever* $R_L < \infty$*.*

(ii) *Suppose that the CA rule has spontaneous birth. If for some deterministic ether* η *we have* $R_L < \infty$ *and* $\eta_L = \eta$ *for* some *binary seed, then for uniformly random binary seeds we have*

$$\liminf_{L \to \infty} \mathbf{P}(R_L < \infty \text{ and } \eta_L = \eta) > 0.$$

Given any particular seed, there is a simple procedure to compute the random variable R_L appearing in Theorems 1.1 and 1.2, and in particular to determine whether it is finite. (See Sections 8 and 9 for details). For many CA of interest, including *Piggyback*, there are multiple non-equivalent ethers η for which the condition of Theorem 1.2(ii) indeed holds, and which hence have asymptotically non-trivial probabilities. The first two pictures in Figure 1.3 show two examples. Our methods allow the computation of explicit rigorous lower bounds on asymptotic probabilities of particular ethers. For example, in *Piggyback*, for the ether that results from the periodic initial state $(00022222)^\infty$, the $\liminf$ in the theorem is at least 0.1297, while $(0)^\infty$, $(2)^\infty$ and $(00002022)^\infty$ have lower bounds 0.5, 0.0398 and 0.0151 respectively. (In fact, more than 100 ethers have positive $\liminf$, and we believe that there are infinitely many.)

As remarked earlier, many CA rules of interest provably exhibit more complex behavior for certain exceptional seeds. One important class of behavior is formalized by the following concept introduced in [GGP]. We call a seed ξ_0 (or a configuration ξ) a **quasireplicator** with ether η if the following holds. For some **exceptional set** of space-time points $Q \supseteq \Lambda$, each bounded

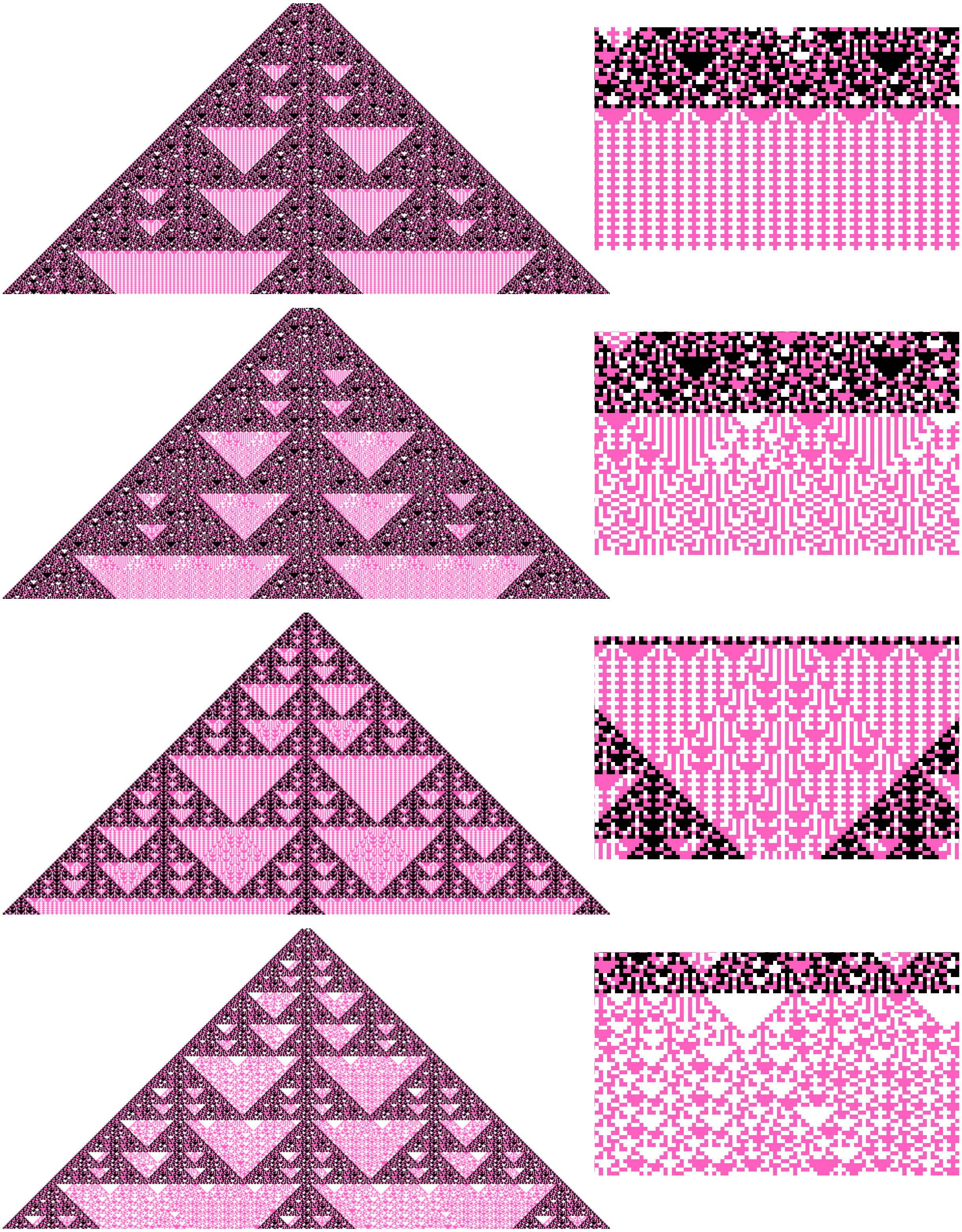

Figure 1.3: Four examples of *Piggyback* evolution: two replicators (with enlarged regions showing different ethers) from random seeds of length 30; a quasireplicator with seed 11111; and an apparently chaotic example with seed 100011011.

component of $\mathbb{Z}^2 \setminus Q$ is filled with η, while for some $a > 1$, the set $a^{-n}Q$ converges as $n \to \infty$ in Hausdorff metric to a set of Hausdorff dimension strictly less than 2.

Theorem 1.3 (Quasireplicators)**.** *For some diagonal-compliant and wide-compliant web CA rules, including Web-Xor and Piggyback, there exist seeds that are quasireplicators but not replicators.*

Examples of (provable) quasireplicators include the second and third seeds in Figure 1.2, and the third seed in Figure 1.3. Certain other seeds appear to be neither replicators nor quasireplicators, but exhibit apparently chaotic behavior, although proving this seems very challenging. The fourth examples in each figure are in this category. In some very special cases we can prove chaotic behavior in a certain conditional sense, even for an infinite family of seeds whose number grows exponentially with their length. We discuss these issues further in the next section.

Theorem 1.1 describes the space-time configuration away from Λ, and moreover states that this description is insensitive to 2s in the initial configuration. However, the result provides no information about the configuration close to Λ. The next result addresses this. The **forward cone** of a space-time point (x,t) is the set $\{(y,s) : |y-x| \le s-t\}$, and the forward cone of a set is the union of the forward cones of its points.

Theorem 1.4 (Stability)**.** *Consider a diagonal-compliant or wide-compliant web CA, started from a uniformly random binary seed on* $[0,L]$*. With probability converging to 1 as* $L \to \infty$*, the configuration of* ξ *in the forward cone of* $[0,L] \times \{\lfloor C \log L \rfloor\}$ *is unchanged if any set of 0s in* ξ_0 *are changed to 2s. Here* C *is an absolute constant. If the CA has no spontaneous birth then with probability converging to* 1 *the same cone contains no* 2*s.*

We next discuss some ideas behind our proofs. Since in a web CA the web of 1s evolves according to *1 Or 3*, it easily follows that all 1s lie in $\Lambda(L)$. In the situation of Theorem 1.1 we will prove that immediately above each bounded component of $\mathbb{Z}^2 \setminus \Lambda(L)$ there is a strip which blocks information flow. Furthermore, each such strip contains a spatially periodic configuration of 1s, with the repeating unit being identical for all strips up to translation. This is a probabilistic statement, not a deterministic one, and the height of the strip is random. It will be proved using techniques of percolation theory. In contrast with classical percolation, the space-time configuration λ of *1 Or 3* is not i.i.d., but has long-range dependence. We will make use of the key percolation result below, which we believe is interesting in its own right.

A **path** is a finite or infinite sequence π of space-time points $(x_0,t_0),(x_1,t_1),\dots,(x_n,t_n)\,(,\dots)$ with $t_{i+1} = t_i + 1$ and $|x_{i+1} - x_i| \le 1$ for all i. A path is **diagonal** if it satisfies $|x_{i+1} - x_i| = 1$ for all i. Suppose λ_0 is given, and let λ be the resulting configuration of *1 Or 3*. We say that a path π is **empty** if $\lambda(x,t) = 0$ for every (x,t) on π. A path is **wide** if it is empty and it makes no diagonal step between two 1s, i.e. it has no two consecutive points $(x,t),(y,t+1)$ with $|x-y| = 1$ but $\lambda(y,t) = \lambda(x,t+1) = 1$. (As suggested by the terminology, diagonal-compliance and wide-compliance of web CA refer to information flow being restricted to paths of the appropriate type). We now assume that the initial configuration λ_0 is uniformly random on $\mathbb{Z}$, that is, $\lambda_0(x)$ takes values $0,1$ with equal probabilities independently for all $x \in \mathbb{Z}$.

Theorem 1.5 (Subcriticality)**.** *Consider the 1 Or 3 CA from a uniformly random initial configuration on $\mathbb{Z}$. We have*

$$\mathbf{P}\big(\exists \text{ an empty diagonal path from } \mathbb{Z}\times\{0\} \text{ to } (0,t)\big) < e^{-ct}, \qquad t>0, \tag{1.1}$$

for some absolute constant $c>0$. The same conclusion holds for the existence of a wide path.

In contrast, we prove that empty paths do percolate.

Theorem 1.6 (Supercriticality)**.** *For the 1 Or 3 CA from a uniformly random initial configuration on $\mathbb{Z}$,*

$$\mathbf{P}\big(\exists \text{ an infinite empty path from } (0,0)\big) > 0.$$

We now briefly discuss background to our results. As remarked earlier, CA that exhibit chaotic behavior for typical seeds but regular behavior for some seeds are apparently very common. Empirical evidence strongly suggests that the one-dimensional rules *Exactly 1* [GG3], *Perturbed Exactly 1* [GGP], and *EEED* [GG4] are all in this category. It is natural to postulate a mechanism for this phenomenon, whereby chaos nucleates from certain local patterns, and, once started, invades all non-chaotic regions. It is tempting to conclude that this robustness of chaos might be universal law, akin to the second law of thermodynamics.

To our knowledge the first compelling evidence to the contrary was presented in [GG2], where a CA later called *Extended 1 Or 3* was introduced. This rule arises naturally as the "2-layer extremal boundary dynamics" of a classical two-dimensional CA rule, *Box 13*. *Piggyback* is also the 2-layer extremal boundary dynamics of a two-dimensional rule. See Section 2 for more information. Extremal dynamics have been utilized very effectively in the analysis of Packard snowflake CA in [GG1, GG2].

Extended 1 Or 3 was proved in [GGP] to admit both replicators and quasireplicators, and observed to generate apparent chaos from some seeds. Empirical evidence was presented that long random seeds are replicators with high probability, and thus that it is the ordered phase that is resistant to disorder. In this article we provide the first rigorous demonstration of this phenomenon. The classes of CA that we consider are strongly inspired by *Extended 1 Or 3*. We have not succeeded in proving that the conclusion of Theorem 1.1 holds in the case of *Extended 1 Or 3*, although this would follow if a certain natural conjecture (Conjecture 5.2) were established.

We note that the disorder-resistance phenomenon under consideration is somewhat reminiscent of insensitivity of CA rules to random noise in the update rule, as in [Gac1, Gra].

Much CA research has focussed on evolution from carefully chosen initial configurations — a notable rigorous example is [Coo]. In contrast, rigorous results for CA from a random initial configurations are scarce, despite their potential importance in understanding self-organization. Most such research has been focused on *nucleation*, that is, random formation of centers that orchestrate a takeover of the available space. Notable examples include bootstrap percolation [Hol, BBDM] and excitable media models [FGG]. We also mention two previous works on additive dynamics started from a product measure, [Lin] and [EN]; the latter finds an embedded random walk by an argument somewhat related to the methods in Section 5.

In many cases, percolation with long range dependence is extremely challenging to analyze rigorously (see [Win, BBS, Gac2, BS], and references therein). Nevertheless, in our setting it turns out that the additivity of *1 Or 3* allows certain judiciously chosen percolation arguments to be carried through with relative ease. Translating results from an infinite random initial configuration to finite seeds also appears daunting, since the number of random bits is now *finite*. However, additivity introduces extensive periodicity and repetition into the configuration. With care, these properties can be used to advantage. This extreme form of long-range dependence provides the link between lack of percolation and evolution from random seeds, and is also the reason for formation of ethers.

While our results provide a reasonably comprehensive picture of *subcritical* percolation behavior for certain path types (diagonal and wide), it should be emphasized that the behavior for paths of *supercitical* type (empty paths) in the evolution from finite random seeds is not well understood. We discuss open questions and prove some preliminary results in this direction in Section 6.

The article is organized as follows. In Section 2 we establish terminology, including the formal definitions of diagonal-compliance and wide-compliance, we introduce and discuss some further examples of CA having these properties, and we discuss how Theorem 1.3 is proved. Sections 3–7 are concerned entirely with properties of the additive rule *1 Or 3*, from which properties of web CA are deduced later. In Section 3 we review properties (most of them well known) of *1 Or 3* started from a single occupied site, and in Section 4 we use additivity to deduce basic properties of the evolution from random configurations. In Sections 5 and 6 we prove the percolation results, Theorems 1.5 and 1.6 respectively, and discuss other facts and open problems concerning percolation. In Section 7 we deduce key results about evolution of *1 Or 3* from random seeds. Finally we return to web CA. In Section 8 we deduce Theorems 1.1 and 1.4, and in Section 9 we prove Theorem 1.2 and show how to compute lower bounds on ether probabilities.

2 Definitions, examples, and preliminary results

2.1 Basic conventions

Throughout the paper, λ denotes the *1 Or 3* CA, while ξ denotes a web CA. All our intervals will be subsets of $\mathbb{Z}$ or of $\mathbb{Z} \times \{t\}$ for some $t \geq 0$. We adopt the convention that $[a, b] = \emptyset$ and $[a, b] \times \{t\} = \emptyset$ whenever $b < a$.

Throughout, a *site* or a *cell* will refer to an element of $\mathbb{Z}$; a *point* will be an element of space-time $\mathbb{Z} \times [0, \infty) \subset \mathbb{Z}^2$. The state of a CA ξ at cell x and time t is denoted $\xi_t(x)$ or $\xi(x, t)$, depending on whether our focus is on time evolution or the space-time configuration. When specifying a seed, we always assume that all states left unspecified are 0. In diagrams of space-time evolution, state 0 is colored white, state 1 is black or grey, and state 2 is a different non-greyscale color for each CA rule.

We say that a collection of $\{0, 1\}$-valued random variables is **uniformly random** if they are independent and take values 0 and 1 each with probability 1/2.

2.2 Compliance

In this section we formally introduce various families of web CA. As mentioned already, these will have 3 states and range 2. Thus the state of a site is $\xi_t(x) \in \{0,1,2\}$ for $x \in \mathbb{Z}$ and $t \in [0,\infty)$, and the evolution is given by

$$\xi_{t+1}(x) = f\Big(\xi_t(x-2), \xi_t(x-1), \xi_t(x), \xi_t(x+1), \xi_t(x+2)\Big)$$

for some function f.

We reiterate our standing assumption that the 1s of ξ behave as the *1 Or 3* CA. More precisely, writing

$$\delta(a) := \mathbf{1}[a=1] = a \bmod 2, \qquad a = 0,1,2,$$

we assume that

$$\delta(f(a,b,c,d,e)) = \delta(b) + \delta(c) + \delta(d) \bmod 2 \tag{2.1}$$

for all a,b,c,d,e. Thus, if we define

$$\lambda_t(x) := \delta(\xi_t(x)), \tag{2.2}$$

then (2.1) implies that λ satisfies the *1 Or 3* CA rule. We sometimes call λ the **first level** of the process. We call a CA rule that satisfies (2.1) and $f(0,0,0,0,0) = 0$ a **web** rule.

We now consider various further conditions that may be imposed on f. The idea will be that the flow of information concerning the distinction between states 0 and 2 is blocked by 1s (in various locations). Throughout the following, we take a,b,c,d,e and a',b',c',d',e' to be arbitrary satisfying $\delta(a) = \delta(a')$, $\delta(b) = \delta(b')$, etc.

We say that the rule f is **empty-compliant** if

$$f(a,b,c,d,e) = f(a',b,c,d,e');$$

that is, a cell's next state $\xi_{t+1}(x)$ depends on non-adjacent cells $\xi_t(x \pm 2)$ only through their first level. (Recall that by (2.1), the *first level* of the next state cannot depend on the non-adjacent cells at all). Similarly, we say that the rule is **diagonal-compliant** if

$$f(a,b,c,d,e) = f(a',b,c',d,e').$$

It will be convenient to express the next conditions in terms of the *new* first-level states of the neighboring cells. Thus we denote

$$\ell := \delta(a) + \delta(b) + \delta(c) \bmod 2;$$
$$r := \delta(c) + \delta(d) + \delta(e) \bmod 2,$$

so that if $(a,b,c,d,e) = (\xi_t(x-2), \dots, \xi_t(x+2))$ then $(\ell, r) = (\lambda_{t+1}(x-1), \lambda_{t+1}(x+1))$. We say that f is **wide-compliant** if it is empty-compliant and

$$\begin{aligned} c = r = 1 \quad &\text{implies} \quad f(a,b,c,d,e) = f(a',b,c,d',e'), \\ \text{and } c = \ell = 1 \quad &\text{implies} \quad f(a,b,c,d,e) = f(a',b',c,d,e'). \end{aligned}$$

In a configuration λ of *1 Or 3*, a path is said to be θ-**free** if it is empty and it contains no point (x,t) whose 5-point neighborhood $\{(x\pm 1,t),(x\pm 1,t-1),(x,t-1)\}$ contains θ or more 1s. Finally, we say a CA rule f is θ-**free-compliant** if it is empty-compliant and

$$\begin{gathered}\delta(b)+\delta(c)+\delta(d)+\delta(\ell)+\delta(r)\geq\theta\\ \text{implies}\quad f(a,b,c,d,e)=f(a',b',c',d',e').\end{gathered}$$

Recall the definition of *no spontaneous birth* from the introduction; this is equivalent to the condition that $f(a,b,c,d,e)\neq 2$ whenever $a,b,c,d,e\in\{0,1\}$.

As suggested by the terminology, the behavior of cellular automata satisfying the above conditions is constrained by paths of the appropriate types.

Lemma 2.1 (Compliance)**.** *Consider a web CA that is empty-compliant (respectively: diagonal-compliant, wide-compliant, or θ-free-compliant). Consider two initial configurations ξ_0,ξ_0' whose first levels agree (i.e. $\delta(\xi_0(x))=\delta(\xi_0'(x))$ for all x), and define the first-level dynamics λ via* (2.2)*. Fix a point (y,t). If λ has no empty path (respectively: empty diagonal, wide, or θ-free path) from any $(x,0)$ at which $\xi_0(x)\neq\xi_0'(x)$ to (y,t), then $\xi_t(y)=\xi_t'(y)$. Moreover, if the CA has no spontaneous birth, then $\xi_t(y)\neq 2$.*

Proof. Suppose, to the contrary, that $\xi_t(y)\neq\xi_t(y')$. We need to show that there exists a path of the appropriate type from $\mathbb{Z}\times\{0\}$ to (y,t). By induction, it suffices to exhibit the final step on this path. This is a straightforward verification.

To prove the final claim in the no spontaneous birth case, consider the initial state ξ_0' in which every 2 of ξ_0 is changed to 0. Then $\xi_t(y)=\xi_t'(y)=0$. ☐

Lemma 2.2 (3-free paths)**.** *In any configuration λ of 1 Or 3, any 3-free path is wide. Any 3-free-compliant web CA rule is wide-compliant.*

Proof. Assume that a 3-free path makes a leftward diagonal move on two space-time points in state 0. Denote the states a,b,c at nearby points thus:

$$\begin{matrix} a & b & 0\\ & 0 & c \end{matrix}$$

We need to show that b and c cannot be both 1. However, if $b=1$, then also $a=1$, but then $c=0$ as the path is 3-free. This establishes the first claim. A similar argument gives the second claim. ☐

We now state a simple but important lemma that says that, although the web rules have range 2, empty-compliance ensures that the "light speed" is essentially 1.

Lemma 2.3 (Light speed)**.** *Assume an empty-compliant web CA. The state $\xi(x,t)$ depends on the initial configuration ξ_0 only through the states*

$$\lambda_0(x-t-1),\quad \xi_0(x-t),\ldots,\xi_0(x+t),\quad \lambda_0(x+t+1),$$

where λ is defined by (2.2)*.*

Proof. The given states determine the following states at time 1:

$$\lambda_1(x-t), \quad \xi_1(x-t+1), \dots, \xi_1(x+t-1), \quad \lambda_1(x+t).$$

Then we use induction. $\square$

2.3 Examples of rules

We will introduce several examples of web CA, chosen to represent various behaviors. Finding such rules is not particularly difficult, and we know of many others with similar characteristics. Let the function N_1 (resp. N_2) count the number of 1s (resp. 2s) among its arguments, and $N_{12} = N_1 + N_2$.

Our first example is *Web-Xor*, whose update rule is given by

$$f(a,b,c,d,e) = \begin{cases} 1 & (b+c+d) \bmod 2 = 1 \\ 2 & (b+c+d) \bmod 2 = 0 \text{ and } N_2(b,d) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check that *Web-Xor* is diagonal-compliant and has no spontaneous birth. Examples of its evolution are given in Figure 1.2. The top example represents typical behavior: replication with zero ether from a long random seed. The middle two examples are quasireplicators, one very simple and one similar to the one in Theorem 8 of [GGP]. For many seeds including these two, quasireplication can be rigorously proved via inductive schemes that completely characterize the configuration at certain specified times. In more complicated cases, such schemes can be very laborious to construct, while in other cases it may be difficult even to determine whether the seed is a quasireplicator. We will not give proofs of quasireplication; instead we refer the reader to [GGP] for two typical examples of inductive schemes that feature in such arguments. We believe that the final example in Figure 1.2 is chaotic.

Even this simplest of rules displays a remarkable variety of behavior from "exceptional" seeds. Other interesting seeds that we have found include: 110010012 (a replicator with non-trivial pattern of 2s in the web), 110011112 (a quasireplicator with scale factor $a = 4$), 111001112 (perhaps chaotic or a very complicated quasireplicator), 10110112 (apparent chaos restricted to one side).

Modified Web-Xor also has no spontaneous birth, but the 2s obey a symmetric two-point *Or* rule in the presence of 1s:

$$f(a,b,c,d,e) = \begin{cases} 1 & (b+c+d) \bmod 2 = 1 \\ 2 & (b+c+d) \bmod 2 = 0, \text{ and} \\ & \quad \text{either } N_2(b,d) = 1 \text{ or } [N_2(b,d) > 1 \text{ and } N_1(\ell,b,c,d,r) \geq 1] \\ 0 & \text{otherwise.} \end{cases}$$

As seen in Figure 2.1, this rule is capable of "mixed replication" with two different ethers (top). Note that Theorem 1.1 implies that with high probability this does not happen for long random

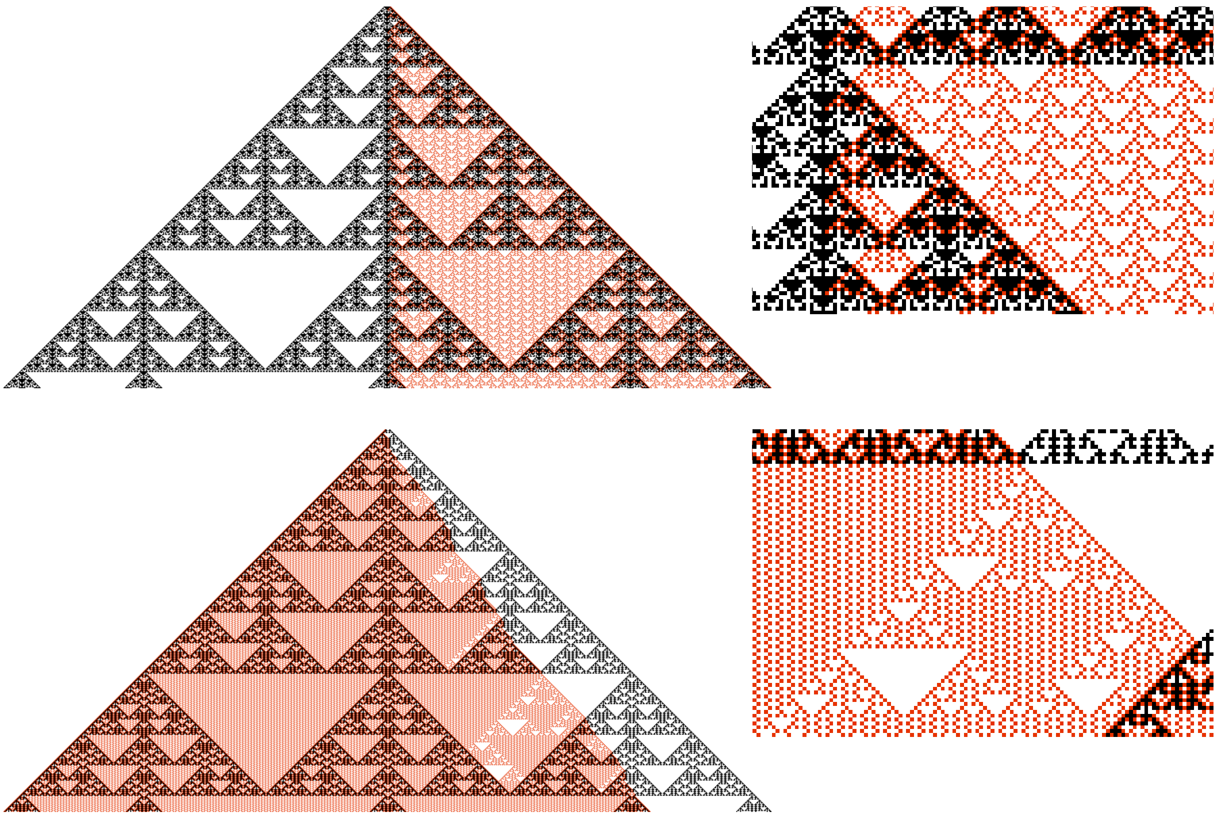

Figure 2.1: *Modified Web-Xor* with seeds 11111112 and 210001.

seeds. The bottom example is apparently a quasireplicator, although we have no proof, and it seems that the inductive methods of [GGP] do not apply. Here and in the last example of Figure 1.2, it is plausible that the evolution is driven by the advance of a front that lags behind the edge of the light cone by a power law. We will discuss this phenomenon in Section 6.

In *Web-adapted Rule 30*, 2s evolve according to *Rule 30* [Wol2], except that 2s perform the three-point *Or* rule in the presence of 1s when a neighborhood occupation number is small enough:

$$f(a,b,c,d,e) = \begin{cases} 1 & (b+c+d) \bmod 2 = 1 \\ 2 & (b+c+d) \bmod 2 = 0 \text{ and } N_{12}(\ell,b,c,d,r) \leq 2, \text{ and} \\ & \quad \text{either } w_{30}[\delta_2(b), \delta_2(c), \delta_2(d)] = 1 \\ & \quad \text{or } [N_2(b,c,d) \geq 1 \text{ and } N_1(\ell,b,c,d,r) \geq 1] \\ 0 & \text{otherwise.} \end{cases}$$

Here, w_{30} is the update rule for *Rule 30*, given by $w_{30}(a_1,a_2,a_3) = (a_1+a_2+a_3+a_2a_3) \bmod 2$, and $\delta_2(a) := \mathbf{1}[a=2]$. *Web-adapted Rule 30* is 3-free-compliant (and therefore wide-compliant) and has no spontaneous birth. See Figure 2.2 for an example. One can prove that this instance is not a replicator, but is it chaotic? There are no known methods to prove chaotic evolution, or

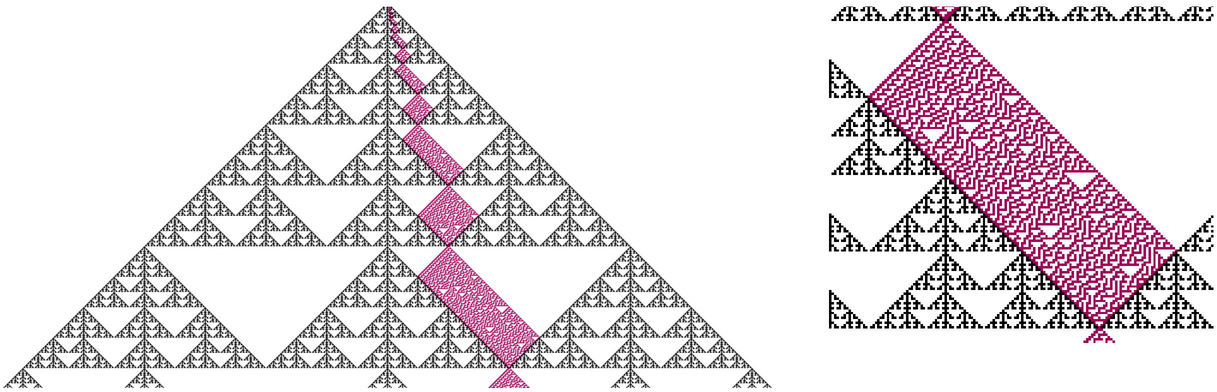

Figure 2.2: Chaotic behavior of *Web-adapted Rule 30* with seed 100010201.

even universally agreed definitions of the concept; however, suppose one accepts the reasonable premise that *Rule 30* generates a chaotic configuration ρ started from a single 1 [Wol2]. Then the example in Figure 2.2 is equally chaotic, in the sense that its evolution provably features larger and larger regions of ρ, at specific locations that are easily characterized. We will also show in Section 5 that an exponentially growing family of seeds exhibit conditional chaos in the same sense.

The above rule may be modified in various ways so as to include spontaneous birth, resulting in further rules where Theorem 1.1 applies, yet in which many provable replicators have ethers with very long temporal period, perhaps too long to be seen experimentally. In the interests of brevity we omit the details. We briefly discuss bounds on the period in Section 8.

A number of web rules arise naturally in analysis of two-dimensional CA, as we now explain. Consider a binary CA $\zeta_t \in \{0,1\}^{\mathbb{Z}^2}$, in which the new state of cell z is given by a rule defined on the Moore neighborhood $\mathcal{N}(z) := \{z' \in \mathbb{Z}^2 : ||z' - z||_\infty \leq 1\}$. We assume that state 0 is quiescent, and that the CA **solidifies**, that is, $\zeta_t(z) = 1$ implies $\zeta_{t+1}(z) = 1$; the CA rule then only needs to specify when a $z \in \mathbb{Z}^2$ becomes occupied, i.e., changes its state from 0 at time t to 1 at time $t+1$. To each such CA we associate **extremal boundary dynamics (EBB)**: assume that ζ_0 vanishes on $\mathbb{Z} \times [1,\infty)$ and let λ_t be given by ζ_t on $\mathbb{Z} \times \{t\}$. Observe that λ_t is a one-dimensional CA whose space-time configuration is a lower bound on the **final configuration** $\zeta_\infty = \cup_{t\geq 0}\zeta_t$. Now assume that we extend the boundary layer to width 2, which leads to the CA $\xi_t \in \{0,1,2\}^{\mathbb{Z}}$ with the following rule: $\xi_t(x) = 1$ if $\zeta_t(x,t) = 1$ (so that $\lambda_t = \xi_t \bmod 2$), $\xi_t(x) = 2$ if $\zeta_t(x,t) = 0$ but $\zeta_{t+1}(x,t) = 1$, and $\xi_t(x) = 0$ otherwise. Again, ξ_t is a one dimensional CA. As $\zeta_t(x,t-1) = 1$ exactly when either $\xi_{t-1}(x) = 1$ or $\xi_t(x) = 2$, ξ_t indeed determines *two* extremal layers of ζ_t and is thus called **two-level EBD**. The evolution of ξ_t also provides a lower bound on ζ_∞ and is often useful when the bound provided by λ_t "leaks" [GG2]. To conform with the rest of the paper, we assume throughout that the EBD is the *1 Or 3* CA.

The natural setting for study of the issues addressed in this paper are general web CA, a much larger class than the two-level EBD rules. The latter, however, provide many interesting examples. In fact, the different ethers, quasireplicators and (apparent) chaotic behavior were first observed in the two-level EBD generated by the *Box 13* solidification CA [GG2], in which

z becomes occupied at time $t+1$ when the the number of occupied cells in $\mathcal{N}(z)$ at time t is 1 or 3. The corresponding two-level EBD is called the *Extended 1 Or 3* CA, and is given by

$$f(a,b,c,d,e) = \begin{cases} 1 & (b+c+d) \bmod 2 = 1 \\ 2 & (b+c+d) \bmod 2 = 0 \text{ and } N_{12}(\ell, r, b, c, d) \in \{1,3\} \\ 0 & \text{otherwise,} \end{cases}$$

as is easy to check; therefore, this rule is equivalent to the one with the same name introduced in [GGP]. This rule is 4-free-compliant, and is not covered by our main theorems. However, we establish some rigorous results in Section 9.

For simplicity, assume that the two-dimensional CA ζ is **isotropic**, that is, that its rule respects all isometries of the lattice $\mathbb{Z}^2$. Then there is a convenient sufficient condition that assures wide-compliance for its two-level EBD: when the neighborhood configuration is

$$\begin{matrix} a & 1 & c \\ b & 0 & 1 \\ 0 & 0 & 0 \end{matrix}$$

the next state at the center cell is independent of c (i.e., depends only on a and b). This holds, for example, for the following solidification rule, which we call *Perturbed Box 13*. Given ζ_t, let $\mathtt{occ}_1(z)$ (resp. $\mathtt{occ}_\infty(z)$) count the number of occupied cells among the four nearest neighbors of z (resp. in $\mathcal{N}(z)$); then z becomes occupied if either

- $\mathtt{occ}_1(z) = 2$, or
- $\mathtt{occ}_1(z) \le 1$ and $\mathtt{occ}_\infty(z) \in \{1,3\}$.

See Figure 2.3 for an example.

The resulting two-level EBD has the update rule

$$f(a,b,c,d,e) = \begin{cases} 1 & (b+c+d) \bmod 2 = 1 \\ 2 & (b+c+d) \bmod 2 = 0, \text{ and} \\ & \quad \text{either } N_{12}(\ell,c,r) = 2 \\ & \quad \text{or } [N_{12}(\ell,c,r) \le 1 \text{ and } N_{12}(\ell,b,c,d,r) \in \{1,3\}] \\ 0 & \text{otherwise.} \end{cases}$$

We call this web CA *Piggyback*. It is easy to see that it is wide-compliant, and has spontaneous birth. The top two examples in Figure 1.3 start from long random seeds and are replicators with different ethers. (We will have more to say about ethers for *Piggyback* in Section 9.) The third example is provably non-replicating, as it is a quasireplicator. The bottom example appears to be chaotic. Like the bottom picture in Figure 1.2, the evolution displays a tantalizing mixture of order and disorder.

Our results on *Piggyback* have rigorous implications for the two-dimensional *Perturbed Box 13* rule (and similarly in other cases where 2-level EBD satisfies the conditions of Theorem 1.1). Here we summarize some initial observations, noting that further investigation is warranted. As

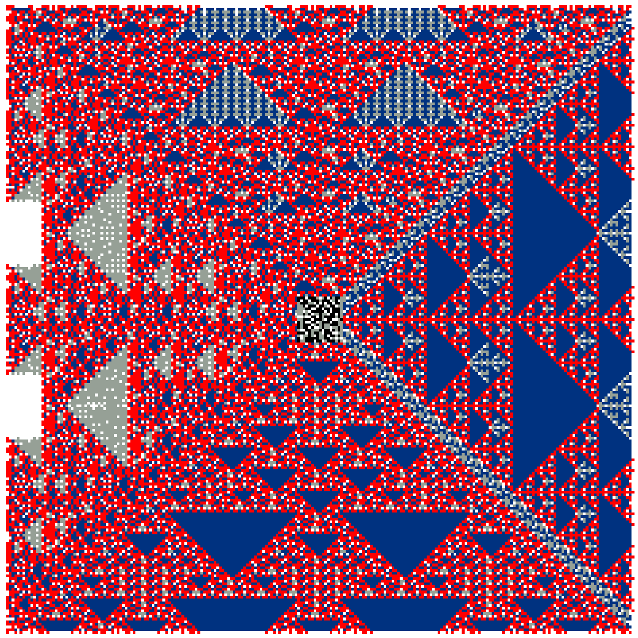

Figure 2.3: *Perturbed Box 13* started from a seed in the square $[0, 16]^2$. Initially occupied cells are black, and subsequently occupied cells are red or blue if they have state 1 or 2 respectively in the 2-level EBD, and otherwise grey. Unoccupied cells are white.

suggested by Figure 2.3, the evolution of *Perturbed Box 13* from a seed in $[0, L]^2$ is governed by four space-time configurations of *Piggyback* in four quadrants with boundaries at 45° to the axes. Depending on the behavior of each, we may make deductions about the final configuration ζ_∞. In the case of a replicator with the "solid" ether $(2)^\infty$, as in the bottom quadrant in this example, clearly no further filling of the ether is possible after the second level of the EBD. By Theorem 1.2, it follows that that *Perturbed Box 13* started from a uniformly random seed in $[0, L]^2$ results in a final configuration ζ_∞ of density 1 in $\mathbb{Z}^2$ with probability bounded away from 0 as $L \to \infty$. Certain other ethers of *Piggyback* can also be shown to fill in in a predictable manner, resulting in a corresponding ether for *Perturbed Box 13*, as in the top quadrant. A similar analysis can likely be carried through for certain simple quasireplicators such as the one in the right quadrant. When *Piggyback* is a replicator with zero ether, as in the left quadrant, it appears plausible that the subsequent filling-in by *Perturbed Box 13* results in a chaotic final configuration. See [GG1, GG2] for detailed analysis of the filling-in process for some other EBD.

We conclude by mentioning a natural rule that seems intractable by our current methods. *Web 1 Or 3* is the web CA in which 2s perform *1 Or 3* on the points not occupied by 1s:

$$f(a, b, c, d, e) = \begin{cases} 1 & (b + c + d) \bmod 2 = 1 \\ 2 & (b + c + d) \bmod 2 = 0 \text{ and } N_2(b, c, d) \bmod 2 = 1 \\ 0 & \text{otherwise.} \end{cases}$$

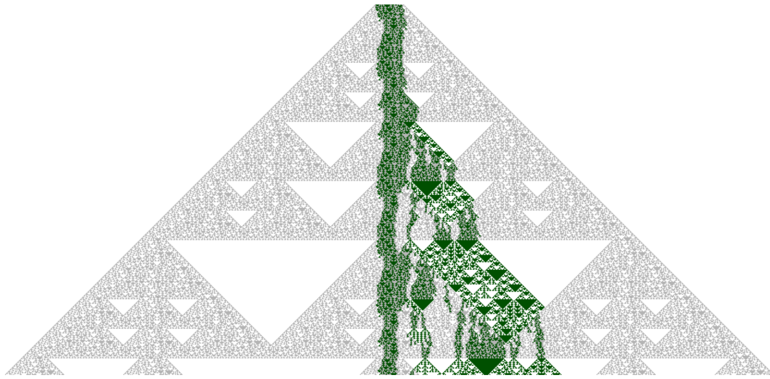

Figure 2.4: Chaotic behavior of *Web 1 Or 3* from a random seed of 32 sites.

Figure 2.4 gives an example of an evolution from a random seed of 1s and 2s, with an apparent message of near-criticality and chaos.

2.4 Generalizations

The simplest additive rule, *Xor* CA μ_t, is defined on the state space $\{0,1\}^{\mathbb{Z}}$ by

$$\mu_t(x) = \mu_{t-1}(x) + \mu_{t+1}(x) \bmod 2.$$

One might consider μ, and not λ, to be the most natural candidate for the web dynamics. However, while μ does have some points of interest (see, for example Proposition 6.5), many of the main issues we consider become trivial in this setting. For example, μ either only occupies points satisfying a parity constraint or generates an impenetrable web even for empty paths [BDR, GG1].

In the other direction one might ask whether similar results hold if λ is replaced by an arbitrary additive rule. It is indeed likely that a more general theory could be developed in this setting. One complication is that predecessors of the all-0 state will no longer necessarily be unique (as they are for *1 Or 3* — see Lemma 3.4) and as a result "mixed replicators" similar to the top example of Figure 2.1 may be the norm.

On the other hand, all our results generalize with appropriate minor changes in the definitions to CA with a quiescent state 0, first-level state 1 and other states $2, \ldots, s$.

3 Additive dynamics from a single occupied site

Recall that $\lambda^\bullet$ denotes *1 Or 3* started from a single 1. In this section we collect properties that we will need. All these results are elementary and many are well known. First is a rescaling property, illustrated in Figure 3.1.

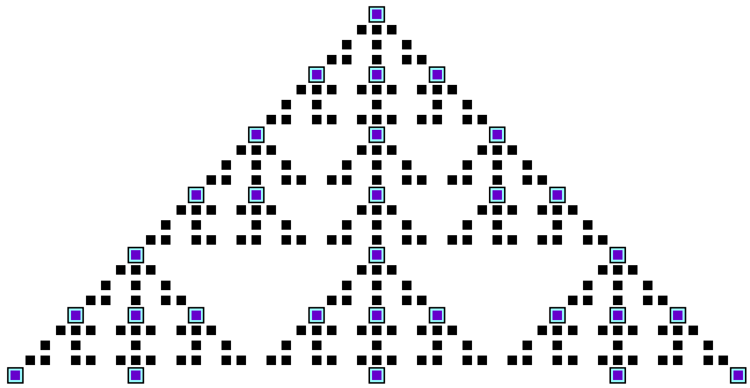

Figure 3.1: An illustration of Lemma 3.1, with $m = 2$. Highlighted points comprise a "separated out" copy of $\lambda^\bullet$.

Lemma 3.1 (Rescaling)**.** *For any nonnegative integers a and m,*

$$\lambda^\bullet_{a2^m} = \begin{cases} \lambda^\bullet_a(y) & \text{if } x = 2^m y \text{ for } y \in \mathbb{Z} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The case $m = 1$ follows from additivity on observing that $\lambda^\bullet_2$ is 10101. For $m > 1$ we apply the $m = 1$ case iteratively. □

Lemma 3.2 (Periodicity properties)**.**

(i) *For $t \geq 0$, $\lambda^\bullet_t(0) = \lambda^\bullet_t(\pm t) = 1$ while $\lambda^\bullet_t(\pm(t-1)) = t \bmod 2$.*

(ii) *For all $n \geq 0$, $\lambda^\bullet_{2^n}(x) = 1$ exactly at $x = 0, \pm 2^n$.*

(iii) *For all $n \geq 0$, $\lambda^\bullet_{2^n+2^{n-1}}(x) = 1$ exactly at $x = 0, \pm 2^n, \pm(2^n + 2^{n-1})$.*

(iv) *For any $k \geq 1$, the sequence of edge configurations of $\lambda^\bullet$ on $[t-k+1, t] \times \{t\}$ is periodic (from $t = 0$ on) with period equal to 2^p where $2^{p-1} < k \leq 2^p$.*

Proof. Parts (ii) and (iii) follow from Lemma 3.1, and (iv) follows from (ii), with (i) as a special case. □

For some purposes, the following recursive description of $\lambda^\bullet$ is useful, a variant of the one given [Wil]. See Figure 3.3 for an illustration. Given a space-time configuration A on $S_n = [0, 2^n] \times [0, 2^n - 1]$, we say that A is **placed** at a space-time point s if the configuration in $s + S_n$ is the corresponding translate of A. Let B_n be the space-time configuration of $\lambda^\bullet$ on S_n. Reflect B_n around its vertical bisector and denote the resulting configuration on S_n by $\overline{B}_n$.

Lemma 3.3 (Recursion)**.** *We have $B_0 = 10$ and $B_1 = \begin{smallmatrix} 1\,0\,0 \\ 1\,1\,0 \end{smallmatrix}$. Moreover, for $n \geq 2$, B_n is obtained by placing B_{n-1} at $(0,0)$ and at $(2^{n-1}, 2^{n-1})$; B_{n-2} at $(0, 2^{n-1})$ and at $(0, 2^{n-1} + 2^{n-2})$; and $\overline{B}_{n-2}$ at $(2^{n-2}, 2^{n-1})$ and at $(2^{n-2}, 2^{n-1} + 2^{n-2})$. All placements result in consistent state assignments at overlaps.*

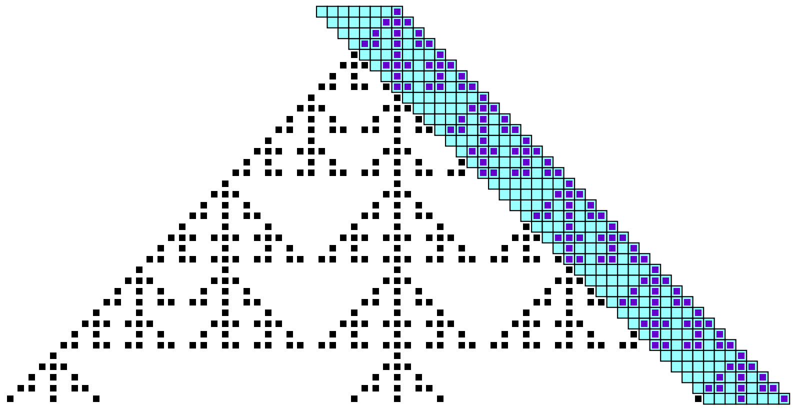

Figure 3.2: Evolution of $\lambda^\bullet$ with highlighted boundary strip of width 8 and temporal period 8.

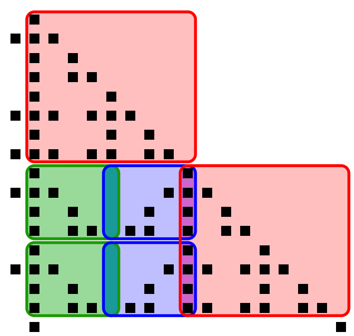

Figure 3.3: Recursive description of $\lambda^\bullet$: B_4 is composed of two copies of B_3 (red), 2 copies of B_2 (green) and two copies of $\overline{B}_2$ (blue).

Proof. This follows easily from (i), (ii) and (iii) of Lemma 3.2. □

Our results for seeds depend on the fact that $\lambda^\bullet$ has certain a unique periodic configuration above every region of 0s. This property does not hold for general additive rules.

Lemma 3.4 (Predecessors of 0)**.** *For an arbitrary initial state λ_0, suppose that $\lambda_t \equiv 0$ on $[a,b]$, but $\lambda_{t-1} \not\equiv 0$ on $[a-1,b+1]$. Then λ_{t-1} is a subword of the periodic word $(110)^\infty$ on $[a-1,b+1]$.*

Proof. Consider the 4 possible values for the pair $\lambda_{t-1}(a-1)$ and $\lambda_{t-1}(a)$. Once these states are fixed, the rest of λ_{t-1} on $[a-1,b+1]$ can be determined sequentially. □

Fix an initial state for λ. A **void** is a finite inverted triangle of the form $\cup_{i\geq 0}([a+i,b-i]\times\{t+i\})$, on which the configuration is identically 0, and that is maximal with these properties with respect to inclusion. Its **width** is $b-a+1$, and its **start time** is t.

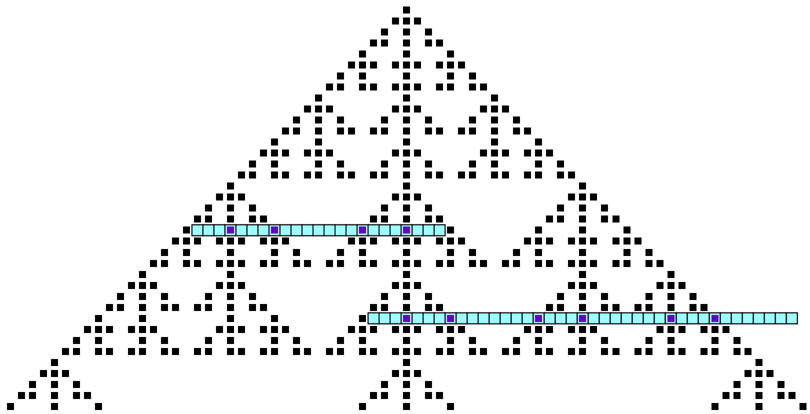

Figure 3.4: Illustration of Proposition 3.6 with $m = 2$. The highlighted intervals at distance 2^2 above two selected voids have the claimed periodic configuration.

Lemma 3.5 (Voids)**.** *In $\lambda^\bullet$, each void has width $2^k - 1$ and start time divisible by 2^{k-1} for some integer $k \geq 1$. Furthermore, for every fixed k, the union of all voids of width at least $2^k - 1$ has density 1 within the forward cone of $(0, 0)$.*

Proof. This is a straightforward application of Lemma 3.3. $\square$

Finally, we deduce the following fact, which will be crucial for our results on percolation and ethers. See Figure 3.4 for an illustration.

Proposition 3.6 (Periodic interval above a void)**.** *In $\lambda^\bullet$, assume that $[a, b] \times \{t\}$ is the top row of a void of width $2^k - 1$. For $m < k$, the state of interval $[a - 2^m, b + 2^m] \times \{t - 2^m\}$ is a segment of the following infinite periodic string of period $3 \cdot 2^m$:*

$$(1 \; \square \; 1 \; \square \; 0 \; \square)^\infty. \tag{3.1}$$

Here, $\square$ represents a string of $2^m - 1$ consecutive 0s, and the segment begins and ends with a full $\square$.

Proof. As t is divisible by 2^{k-1}, and therefore by 2^m, $\lambda_t^\bullet$ on $[a, b] \times \{t\}$ is of the form

$$\square \; 0 \; \square \; \ldots \; 0 \; \square,$$

by Lemma 3.1. Then, by the same lemma, and Lemma 3.4 applied to $\lambda_{t/2^m}$, the configuration on $[a - 2^m, b + 2^m] \times \{t - 2^m\}$ is either of the claimed type started and ended with $\square$, or all 0s. The latter possibility contradicts maximality of the original void. $\square$

4 Duality and randomness

When the initial configuration of *1 Or 3* is uniformly random (on some set), the resulting space-time configuration is of course not uniformly random but has a high degree of dependence. Nevertheless, in this section we show how to identify space-time *sets* on which the randomness is uniform. The additive structure of the CA rule ensures that the space-time configuration is a linear function (modulo 2) of the initial states, and the idea is to find cases where the associated matrix is upper triangular.

Recall that $\lambda_t^\bullet$ is the *1 Or 3* rule started with only the origin occupied. Let λ_t^A denote the rule started with the set of initially occupied sites exactly equal to $A \subseteq \mathbb{Z}$. We will extensively use the following version of cancellative duality.

Lemma 4.1 (Duality). *We have* $\lambda_t^A(x) = \sum_{y \in A} \lambda_t^\bullet(x-y) \bmod 2$.

Proof. This follows easily by additivity and induction on t. $\square$

Observe that by symmetry and translation-invariance, $\lambda_t^\bullet(x-y) = \lambda_t^\bullet(y-x) = \lambda_t^{\{x\}}(y)$.

Suppose we have an ordered set $S = \{(x_i, t_i) : i = 1, 2, \ldots, n\}$, of space-time points. A function $F : S \to \mathbb{Z}$ is a **dual assignment** for S if for all $i, j \in \{1, \ldots, n\}$,

$$\lambda^\bullet\big(x_j - F(x_i, t_i), t_j\big) = \begin{cases} 1 & \text{if } j = i, \\ 0 & \text{if } j < i. \end{cases}$$

(There is no restriction when $j > i$). We think of $F(\cdot, \cdot)$ as sites in the initial configuration. The idea is that in order to determine $\lambda^A(x_i, t_i)$, we need new information about A at each successive i.

Proposition 4.2 (Randomness via dual assignment). *Suppose that the initial configuration λ_0 of 1 Or 3 is uniformly random on some fixed set $K \subseteq \mathbb{Z}$ and deterministic on K^C. Let S be a fixed set of space-time points. If S has a dual assignment whose image is contained in K, then λ is uniformly random on S.*

Proof. Writing

$$\begin{aligned} K_i &= \{y \in K : \lambda_{t_i}(x_i - y) = 1\}, \\ K_i' &= \{y \in K^C : \lambda_{t_i}(x_i - y) = 1\}. \end{aligned}$$

and

$$c_i = \sum_{y \in K_i'} \lambda_0(y) \bmod 2,$$

we have by Lemma 4.1,

$$\lambda(x_i, t_i) = \sum_{y \in K_i} \lambda_0(y) + c_i \bmod 2.$$

But K_i contains an element, $F(x_i, t_i)$, that is not in $\bigcup_{j<i} K_j$, therefore $\lambda(x_i, t_i)$ is uniformly random conditional on $(\lambda(x_j, t_j) : j < i)$. $\square$

A particularly useful special case is that a 1 adjacent to a string of 0s in $\lambda^\bullet$ heralds uniformly random intervals in the evolution from a random seed.

Corollary 4.3 (Random intervals)**.** *Fix integers a and $L, k > 0$. Let the initial configuration λ_0 be a uniformly random binary seed on $[0, L]$, and suppose that $\lambda_t^\bullet$ on $[a, a+k]$ is 1 followed by k 0s. Then for any $x \in [-a, L-a-k]$, the configuration λ_t is uniformly random on $[x, x+k]$.*

Proof. To find a dual assignment of $[x, x+k] \times \{t\}$, order the set from left to right, and let $F(y,t) = y + a$. Clearly, the image of this assignment is contained in $[0, L]$. Now apply Proposition 4.2. □

5 Subcritical percolation

In this section we prove Theorem 1.5, which states that when *1 Or 3* is started from a uniformly random initial configuration on $\mathbb{Z}$, the probability of an empty diagonal or wide path from the initial interval $\mathbb{Z} \times \{0\}$ to the point $(0, t)$ decays exponentially in t. See Figures 5.1 and 5.3 for the diagonal and wide cases respectively. Note the contrast with Figure 6.1 in the next section for empty paths.

Our approach is to use dual assignments to control the probabilities of paths, but the details of the argument are very different for the two types of path. A diagonal path has 2 choices at each step, and any given point has state 0 with probability 1/2, suggesting a critical bound. To improve this to a subcritical bound we consider a *leftmost* path, and use special properties of λ. On the other hand, we control wide paths via a random process of space-time intervals that terminates when an interval has even length.

Later in the section we also discuss θ-free paths, and show that notwithstanding Theorem 1.5, there is an exponential family of initial configurations for which percolation by wide paths does occur.

5.1 Empty diagonal paths

Proof of Theorem 1.5, case of empty diagonal paths. We may assume without loss of generality that t is even, since the configuration at time 1 is also uniformly random, and the probability in question is strictly less than 1 for $t = 1$.

Fix a diagonal path π from $(x, 0)$ to $(0, t)$. We will find an upper bound for the probability that π is the *leftmost* empty diagonal path from $\mathbb{Z} \times \{0\}$ to $(0, t)$. To this end, partition the steps of π into segments of length 2. During each such segment, the path has one of the following forms: left-left, right-right, left-right, or right-left. When π makes a right-left move, that is $(x, s) \to (x+1, s+1) \to (x, s+2)$, the leftmost property requires a 1 at $(x-1, s+1)$; we call these points (which are not on the path) the **corner points** of the path, and let $N(\pi)$ be their number, i.e., the number of right-left segments that start at even times.

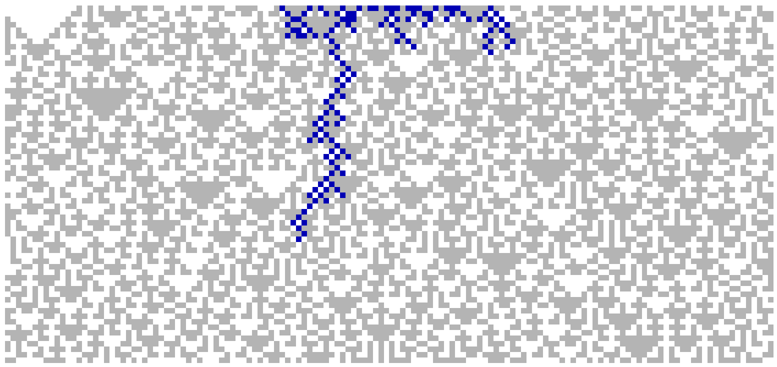

Figure 5.1: All empty diagonal paths from an interval at time 0 are highlighted in blue. The initial configuration is uniformly random.

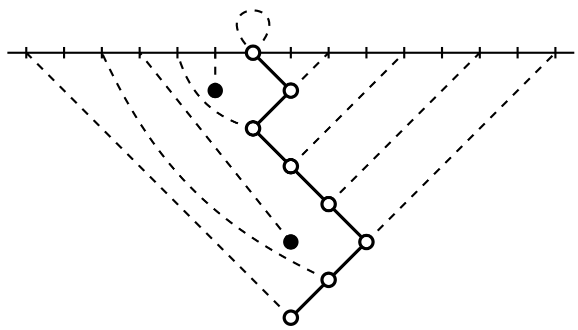

Figure 5.2: A leftmost diagonal path together with a dual assignment for the points of the path (white discs) and the corner points (black discs). The dashed lines connect each point to its assigned position in the initial configuration.

We will give a dual assignment F of the path together with the set of its corner points (see Figure 5.2 for an illustration). Order points on the path with increasing time, and place a corner point (x, s) in the ordering immediately *after* the point on the path at the time $s+1$. For every corner point (x, s), let $F(x, s) = x - s + 1$. For every point (x, s) on the path, let $F(x, s)$ be either $x - s$ or $x + s$, according to whether the path arrives to (x, s) from the right (i.e., from $(x + 1, s - 1)$) or from the left (i.e., from $(x - 1, s - 1)$), respectively. We let $F(x, 0) = x$.

To check that F is a dual assignment, we will use Lemma 3.2(i). Fix a point (x, s) on the path. All positions assigned by F to points earlier in the order lie in $[x - s + 2, x + s]$ or $[x - s, x + s - 2]$ according to whether the path arrives to (x, s) from the right of left, so the required condition is satisfied for this point. Now suppose $(x - 1, s + 1)$ is a corner point arising from the moves $(x, s) \to (x + 1, s + 1) \to (x, s + 2)$ in the path. This corner point is assigned to $F(x-1, s+1) = x-s-1$. We have $F(x, s+2) = x-s-2$, and all points earlier than $(x, s+2)$ were assigned integers at least $x-s$. Since $s+1$ is odd, $\lambda^\bullet(x-1-(x-s-1), s+1) = \lambda^\bullet(s, s+1) = 1$. Finally, since $s + 2$ is even, $\lambda^\bullet(x - (x - s - 1), s + 2) = \lambda^\bullet(s + 1, s + 2) = 0$, as required.

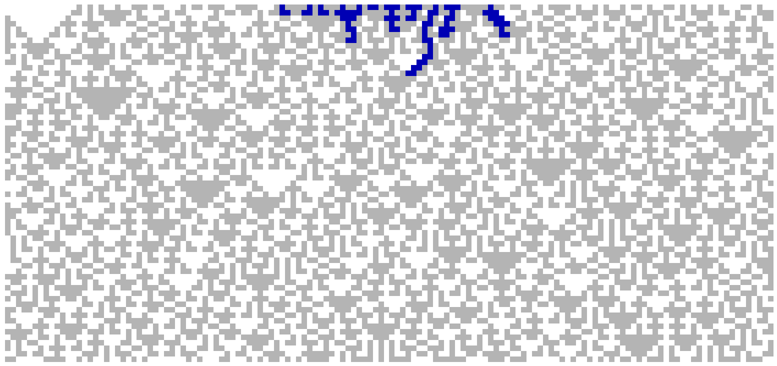

Figure 5.3: All wide paths from an interval at time 0 are highlighted in blue. The initial configuration is uniformly random.

Now, using Proposition 4.2,

$$\begin{aligned}
&\mathbf{P}(\text{an empty diagonal path from } \mathbb{Z}\times\{0\} \text{ to } (0,t) \text{ exists})\\
&\leq \sum_{\pi} \mathbf{P}(\pi \text{ is the leftmost empty diagonal path from } \mathbb{Z}\times\{0\} \text{ to } (0,t))\\
&\leq \sum_{\pi} (\tfrac{1}{2})^{t+1+N(\pi)},
\end{aligned}$$

where both sums are over all diagonal paths from $\mathbb{Z}\times\{0\}$ to $(0,t)$. Let P_t be the last sum above. Then, by considering the last two steps of the path,

$$P_{t+2} = \Big((\tfrac{1}{2})^2 + (\tfrac{1}{2})^2 + (\tfrac{1}{2})^2 + (\tfrac{1}{2})^3\Big) P_t,$$

so, recalling that t is even, $P_t = (1/2)\cdot(7/8)^{t/2}$. $\square$

As an aside, we mention that the assertion of Theorem 1.5 for diagonal paths also holds when λ is replaced by the *Xor* CA μ, with a much simpler proof, since the set of all space-time points that the origin is connected to by diagonal paths is a rectangle.

5.2 Wide paths

Proof of Theorem 1.5, case of wide paths. We will prove that

$$\mathbf{P}\big(\exists \text{ a wide path from } (0,0) \text{ to } \mathbb{Z}\times\{t\} \bigm| \lambda_0(0)=0\big) < e^{-ct} \tag{5.1}$$

for some $c>0$. This clearly suffices by translation-invariance, since there are only $2t+1$ points at time 0 from which a path can reach $(0,t)$. Therefore, we will henceforth assume that $\lambda_0(0)=0$ and that λ_0 is uniformly random elsewhere.

We recursively define intervals $I_t = [L_t, R_t]$ for $t=-1,0,\dots,T$, where $T\leq\infty$, as follows. Start with $L_{-1}=R_{-1}=0$; then let I_0 be the maximal subinterval of $\mathbb{Z}$ containing 0 on which

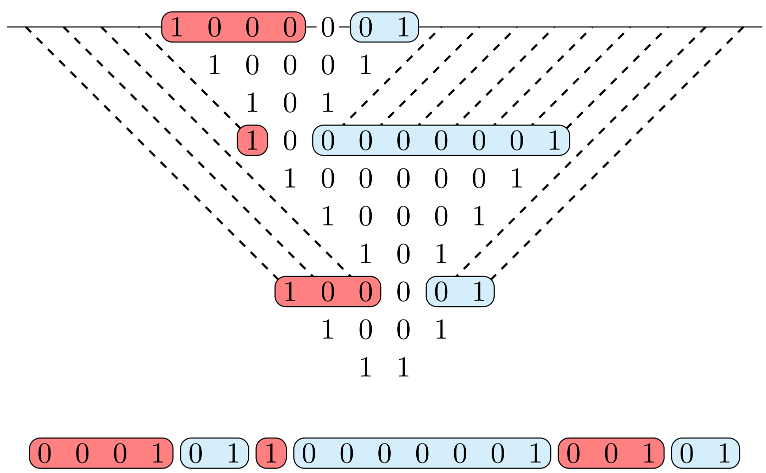

Figure 5.4: The process of intervals of zeros used to prove non-percolation by wide paths. Here the interval lengths $(|I_0|, |I_1|, \ldots, |I_T|)$ are $(5, 3, 1,\ \ 7, 5, 3, 1,\ \ 4, 2, 0)$. The witness points are highlighted. The corresponding binary sequence $(X_1, \ldots, X_{2T+1})$ is shown below; it is obtained by reading the states of the witness points in conventional text order on the page, except with the left (red) intervals reversed. The dual assignment of witness points to initial positions is shown via dashed lines (witness points in the top row are assigned to themselves).

$\lambda_0 \equiv 0$. If $I_t = \emptyset$, then we set $T = t$ and there is no I_{t+1}. Otherwise, if $|I_t| \geq 2$, then I_{t+1} is the interval $[L_t + 1, R_t - 1]$ (which is $\emptyset$ when $|I_t| = 2$). If $|I_t| = 1$, then I_{t+1} is the maximal subinterval of $\mathbb{Z}$ containing $R_t = L_t$ on which $\lambda_{t+1} \equiv 0$. Observe that for each $t < T$ we have $\lambda_t \equiv 0$ on I_t, while $\lambda_t(L_t - 1) = \lambda_t(R_t + 1) = 1$. This follows from the CA rule for λ_t by induction on t; the key observation is that if $|I_t| = 1$ then $\lambda_{t+1}(L_t) = 0$ (see Figure 5.4). Furthermore, any wide path started at $(0, 0)$ is within $\cup_{t<T}(I_t \times \{t\})$.

We now define an ordered sequence of $2T+1$ space-time points, which we call **witness** points, associated with the above sequence of intervals. If $|I_t| = 1$, we call $t + 1$ a **refresh time**; we also declare 0 a refresh time. Let $0 = \tau_0 < \tau_1 < \ldots$ be the refresh times. We build the sequence of witness points by appending certain points at each refresh time τ_i, in order. Specifically, for every i, we append all points in $I_{\tau_i} \times \{\tau_i\}$, with the exception of $z_i = (L_{\tau_i - 1}, \tau_i)$, in the following order: points to the left of z_i in the right-to-left order, followed by points to the right of z_i in the left-to-right order. Let $X_i = \lambda(s_i)$, where $s_1, \ldots, s_{2T+1}$ are the witness points in the order described. Write X for the random finite or infinite sequence given by $X = (X_1, \ldots, X_{2T+1})$ if $T < \infty$ and $X = (X_1, X_2, \ldots)$ if $T = \infty$. Our goal is to show that X is equal in distribution to a sequence of independent fair coin flips stopped at a certain a.s. finite stopping time.

Let $Y_1, Y_2, \ldots$ be independent random variables taking values 0 and 1 with equal probability. Partition this sequence into blocks of the form $0^a 1 0^b 1$, where $a, b \geq 0$, and let $S \geq 1$ be the location of the endpoint of the first such block of odd length. Then S is a.s. finite and

$$\mathbf{P}(S \geq t) < e^{-ct}, \tag{5.2}$$

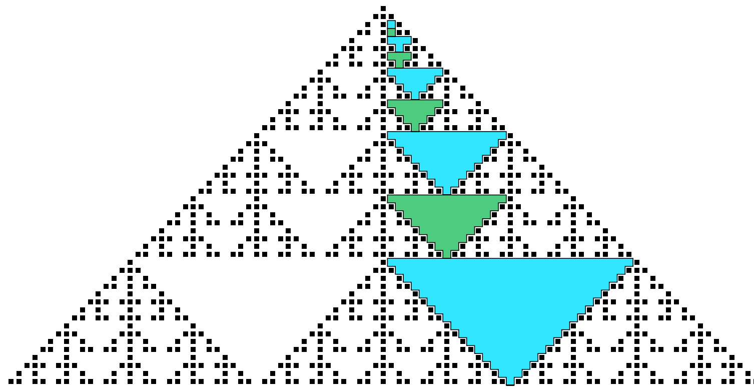

Figure 5.5: The sequence of principal voids $V_1, V_2, \ldots, V_8$, numbered from top to bottom; V_1 and V_2 each consist of a single point.

since S is at most the waiting time for the pattern 11011. Write $Y' := (Y_1, \ldots, Y_S)$. We claim

$$X \stackrel{d}{=} Y'. \tag{5.3}$$

Once (5.3) is proved, the exponential bound (5.2) implies (5.1), since $2T + 1 \stackrel{d}{=} S$.

We now proceed to prove (5.3). Since $\mathbf{P}(S < \infty) = 1$, the random variable Y' has countable support, thus it suffices to show that $\mathbf{P}(X = y) = \mathbf{P}(Y' = y)$ for any y with $\mathbf{P}(Y' = y) > 0$. Choose such a y. The event $\{X = y\}$ determines T and $I_0, \ldots, I_T$, and therefore the locations of the witness points. It follows that the event $\{X = y\}$ is precisely the event that λ takes the specified values y on these (deterministic) witness points. Now we use a dual assignment to show via Proposition 4.2

$$\mathbf{P}(X = y) = \left(\frac{1}{2}\right)^{\text{length of } y},$$

as required. Considering the witness points in their order, we assign to each (x, t) either $x + t$ or $x - t$, depending on whether it is to the right or left of z_i in its interval. (See Figure 5.4.) This is a dual assignment simply because $\lambda^\bullet_t(\pm t) = 1$ and $\lambda^\bullet_t(x) = 0$ for $x \in [-t, t]^C$. $\square$

5.3 θ-free paths

Since 3-free paths are wide (Lemma 2.2), the exponential bound (1.1) holds for 3-free paths, and there are no infinite 3-free paths when λ_0 is uniformly random on $\mathbb{Z}$. Do such paths exist started from special initial conditions? Indeed they do, as shown by our next result.

Define the sequence of **principal voids** $V_1, V_2, \ldots$ of $\lambda^\bullet$ according to Figure 5.5, and for $L > 0$ let $W_i = V_i \cap (V_i + (L, 0))$ (the notation means that V_i is translated by the vector $(L, 0)$).

Proposition 5.1 (Exceptional percolation)**.** *Assume λ_0 is a seed on $[0, L]$ which vanishes outside $4\mathbb{Z}$, where L is a multiple of 4. Define the sets W_i as above. Let i be such that $2^i > 16L^2$. From every point in W_i there is a 3-free path to some point in W_{i+1}. In particular, from any such point there is an infinite 3-free path.*

Proof. For a maximal interval $I_t \subset \mathbb{Z}$ on which λ_t vanishes, define its **successor** I_{t+1} to be the maximal interval on which λ_{t+1} vanishes and $I_t \cap I_{t+1} \neq \emptyset$; if such an interval does not exit, let $I_{t+1} = \emptyset$.

Observe that λ_t vanishes outside $2\mathbb{Z}$ at all even times t. Using this, it is easy to verify by case-checking that every maximal interval of 0s has odd length at every time. It follows that any nonempty maximal interval of 0s has a nonempty successor. Similarly, since λ_t vanishes outside $4\mathbb{Z}$ when t a multiple of 4, it is easily verified that for any t, from every point in $I_t \times \{t\}$ there is a 3-free path to some point in $I_{t+1} \times \{t+1\}$.

Let the **apex** w_i be the (unique) bottommost point of W_i. Clearly, from every point in W_i there is a 3-free path to w_i. Moreover, by the above, there is a 3-free path from w_i to $\mathbb{Z} \times \{t_{W_{i+1}}\}$, where $t_{W_{i+1}}$ is the time of the top interval of W_{i+1}. The condition $2^i > 16L^2$ ensures that this top interval contains the intersection of the forward cone of w_i with $\mathbb{Z} \times \{t_{W_{i+1}}\}$. ☐

The above result implies furthermore that there are, for some $c > 1$, at least c^L seeds on $[0, L]$ for which there is a 3-free path from $\mathbb{Z} \times \{0\}$ going through all W_i for i sufficiently large. This follows by replacing a seed λ_0 satisfying the condition of the theorem with its successor λ_t for t a sufficiently large multiple of L. If all 0s within such a seed are replaced by 2s, this also yields an exponential family of seeds for the *Web-adapted Rule 30* CA with the same chaotic properties as discussed in Section 2.

We conclude this section with the following conjecture supported by computer experiments.

Conjecture 5.2. *The exponential bound* (1.1) *in Theorem 1.5 holds when diagonal path is replaced by 4-free path.*

6 Supercritical percolation

In this section we consider empty paths from random initial conditions, and in particular we prove the percolation result Theorem 1.6. The results of this section are not needed for the proofs of Theorems 1.1–1.4, but they are of independent interest and complement those of the previous section. We also consider initial conditions where the randomness is restricted to the half-line or a finite seed. Here many questions are open, but we establish some preliminary results. The questions we consider are relevant to further understanding certain web CA behavior.

6.1 Percolation of empty paths

As Figure 6.1 suggests, the set of points reachable by empty paths emanating from an interval at time 0 form an interval at each subsequent time. With random initial conditions, this interval spreads linearly provided it survives. Proving this is the key to Theorem 1.6.

Suppose λ_0 is given. The **rightward** Z**-path** from a space-time point (x, t) is an infinite sequence of points (r_s, s), $s \geq t$ defined as follows. Start with $r_t = x$. Inductively, let r_{s+1} be the largest integer y in $(-\infty, r_s + 1]$ for which $\lambda_{s+1}(y)$ is 0; or if there is no such y we take $r_u = -\infty$

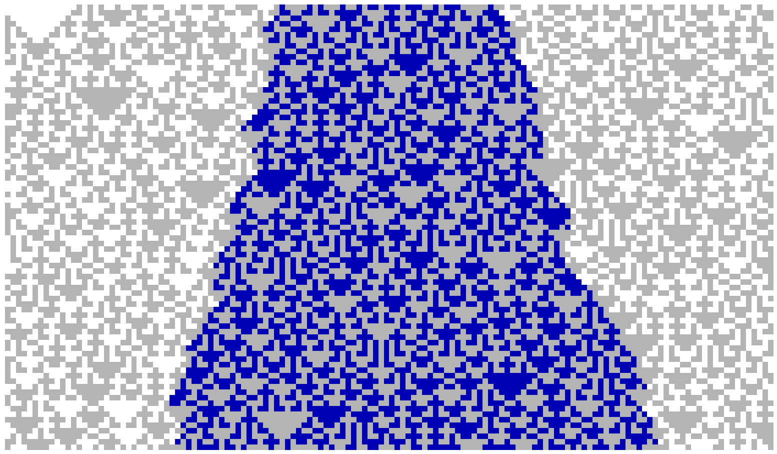

Figure 6.1: All empty paths from an interval at time 0 are highlighted in blue.

for all $u > s$. Note that $\lambda(r_s, s) = 0$ for all $s > t$ for which r_s is finite, but not necessarily for $s = t$. Analogously, we define the **leftward Z-path** (ℓ_s, s), $s \geq t$ by reversing the space coordinate in the definition.

Lemma 6.1 (Properties of Z-paths)**.** *Suppose λ is 1 Or 3 from any initial configuration.*

(i) *Suppose $\lambda(0,0) = 0$ and let (r_t, t), $t \geq 0$ be the rightward Z-path from $(0,0)$. If $x \leq r_t$ and $\lambda(x,t) = 0$ then there is a empty path from $(-\infty, 0] \times \{0\}$ to (x,t).*

(ii) *Fix an interval $[a,b]$ with $a \leq b$. Let (ℓ_t, t) be the leftward Z-path from $(a,0)$, and (r_t, t) the rightward Z-path from $(b,0)$. Suppose that $\ell_s \leq r_s$ for every $s \leq t$. Then for any $y \in [\ell_t, r_t]$ with $\lambda_t(y) = 0$, there is an empty path from $[a-2, b+2] \times \{0\}$ to (y,t).*

(iii) *Under the assumptions of* (ii)*, suppose also that $\lambda_0(a) = \lambda_0(b) = 0$. Then for any $y \in [\ell_t, r_t]$ with $\lambda_t(y) = 0$, there is an empty path from $[a,b] \times \{0\}$ to (y,t).*

(iv) *Conversely, if there is an empty path from $[a,b] \times \{0\}$ to some (y,t), then $\ell_s \leq r_s$ for all $s \leq t$, and $\ell_t \leq y \leq r_t$.*

Proof. We omit the proof of (i), as it is similar to the proof of (iii), which proceeds by induction as follows. The argument reduces to verifying (iii) at time $t = 1$. Assume $\ell_1 \leq r_1$ and take $y \in [\ell_1, r_1] \subseteq [a-1, b+1]$ with $\lambda_1(y) = 0$. Then there exists an $x \in \{y-1, y, y+1\}$ with $\lambda_0(x) = 0$. It remains to verify that x can be chosen to be in $[a,b]$. If $y \in [a+1, b-1]$ this is clear; if $y \in \{b, b+1\}$ we may take $x = b$ and if $y \in \{a, a+1\}$ we may take $x = a$.

The above argument also proves (ii): we verify the claim at time $t = 1$ and then use (iii). The last claim (iv) is an easy consequence of definitions of empty and Z-paths. $\square$

The key fact in establishing percolation of empty paths is that r_t has drift $1/4$. The proof is somewhat similar to that of non-percolation for wide paths, Theorem 1.5.

Lemma 6.2 (Drift)**.** *Suppose that the initial configuration λ_0 is uniformly random on $\mathbb{Z} \setminus \{0\}$ and $\lambda_0(0) = 0$. Let (r_t, t), $t \geq 0$ be the rightward Z-path from $(0,0)$. For every $\epsilon > 0$ there exists a constant $c = c(\epsilon) > 0$ so that $\mathbf{P}(|r_t - t/4| > \epsilon t) < e^{-ct}$.*

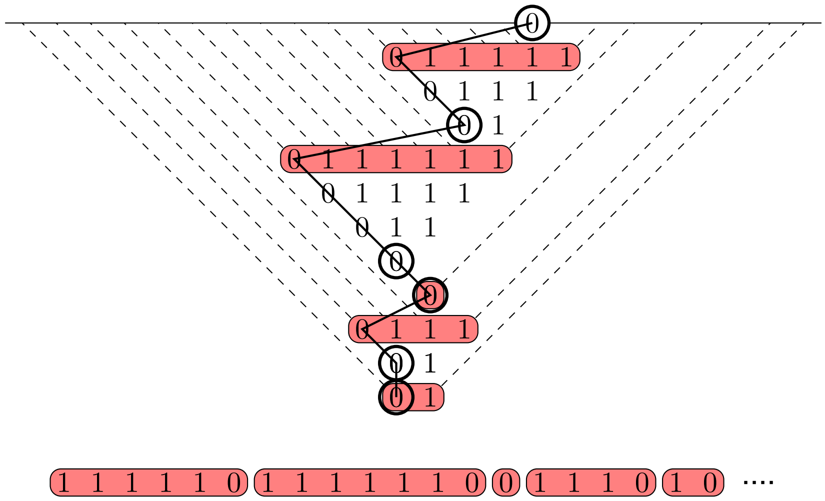

Figure 6.2: The rightward Z-path (solid lines) from the origin, together with its refresh points (circled), and witness points (highlighted in red). A dual assignment of the witness points to initial positions is indicated by the dashed lines. The states of the witness points in the order they are examined are shown below.

Proof. We first describe an exploration process that determines the rightward Z-path from the origin $(0,0)$. We designate $(0,0)$ to be the first **refresh point**. Now we examine the states of the points $(1,1),(0,1),(-1,1),(-2,1),\dots,$ in this order, until we find the first point with state 0. Let G be the number of points examined, and call them **witness points**. Since the states of these witness points are $01\cdots1$ (from left to right), certain states at the immediately following time steps are determined. Specifically, the pattern $01\cdots1$ is immediately followed by patterns of the same form, but with the length decreasing by 2 at each step and centered at the same location, ending with either 01 or 0 according to whether G was even or odd. (See Figure 6.2). We designate the location of the 0 in this last pattern to be the next refresh point. It is $(1-\lfloor G/2\rfloor,\lceil G/2\rceil)$. Now iterate the process starting at the new refresh point. Note that the rightward Z-path from $(0,0)$ consists precisely of the 0s at the left ends of the $01\cdots1$ patterns, including the refresh points. Observe also that the Z-path is determined by the locations of the refresh points, and that these are determined by examination of the witness points.

Now consider the above exploration process for the initial configuration that is uniformly random on $\mathbb{Z}\setminus\{0\}$ and 0 at 0. Let $(X_i)_{i\geq1}$ be the sequence of states of the witness points, in the order that they are examined by the exploration process. We claim that $(X_i)_{i\geq1}$ is uniformly random. It suffices to check that $(X_1,\dots,X_n)$ is uniformly random. This follows from Proposition 4.2, by the dual assignment in which a witness point (x,t) is assigned to $x+t$ if it is the rightmost witness point in its $01\cdots1$ pattern, and otherwise to $x-t$. See Figure 6.2.

Let G_i be the number of witness points examined in the row immediately immediately below the ith refresh point. Then (G_i) are i.i.d. Geometric(1/2) random variables. Furthermore, the sequence of refresh points is a random walk on $\mathbb{Z}^2$ with steps $(1-\lfloor G_i/2\rfloor,\lceil G_i/2\rceil)$. As

$\mathbf{E}\lfloor G_i/2 \rfloor = 2/3$ and $\mathbf{E}\lceil G_i/2 \rceil = 4/3$, each step has expectation vector $(1/3, 4/3)$. The proof is concluded by standard large deviation estimates. $\square$

Proof of Theorem 1.6. For L to be chosen later, consider the leftward path (ℓ_t, t) started at $(-L, 0)$ and the rightward path (r_t, t) started at $(0, L)$. Then, by a union bound and symmetry,

$$\mathbf{P}\big(\ell_t < r_t \ \forall t\big) \geq \mathbf{P}\big(\ell_t \leq -1 \text{ and } r_t \geq 1 \ \forall t\big) \geq 1 - 2\,\mathbf{P}\big(r_t \leq 0 \text{ for some } t\big).$$

By Lemma 6.2, for L large enough we have $\mathbf{P}(r_t \leq 0 \text{ for some } t) \leq 1/3$. Call a site $x \in \mathbb{Z}$ **good** if an infinite empty path starts at $(x, 0)$. Thus, by Lemma 6.1(ii),

$$\mathbf{P}\big([-L-2, L+2] \text{ contains some good site}\big) \geq \mathbf{P}\big(\ell_t < r_t \ \forall t\big) \geq 1/3.$$

Consequently, by translation-invariance, $\mathbf{P}(0 \text{ is good}) \geq 1/[3(2L+5)]$. $\square$

6.2 Empty paths for half-lines and seeds

How do empty paths behave when the initial configuration is a random seed? This question is largely unresolved. (In contrast, the next section will provide detailed answers for diagonal and wide paths). A first step would be to understand the case of a uniformly random half-line, for which the following conjecture is natural given Lemma 6.2.

Conjecture 6.3. *Suppose the initial condition λ_0 is uniformly random on $[1, \infty)$ and 0 elsewhere. Let (r_t, t), $t \geq 0$ be the rightward Z-path from $(0, 0)$. Then $r_t/t \to 1/4$ as $t \to \infty$.*

We prove that a much weaker statement holds deterministically: for an initial configuration supported in a half-line, empty paths penetrate arbitrarily far into its forward cone.

Lemma 6.4 (Unbounded penetration)**.** *Assume that the initial condition λ_0 has no 1s outside $[1, \infty)$. Let (r_t, t) be the rightward Z-path from $(0, 0)$. Then $\sup_t (r_t + t) = \infty$.*

Proof. We first observe that for any initial configuration λ_0 of *1 Or 3*, if (x, t) has state 0 and $t \geq 1$ then at least one of the three points $(x, t-1), (x \pm 1, t-1)$ has state 0 also. Iterating this we see that there must be an empty path from $\mathbb{Z} \times \{0\}$ to (x, t). We call any such path an **ancestral path** of (x, t).

Now, under the conditions or the lemma, note that for any $m \geq 0$, the sequence of configurations on the intervals $[-t, -t+m+1] \times \{t\}$ is periodic in t starting from some time t_p depending on m and λ_0. For $a \geq 0$, define the leftward diagonal $D_a := \{(a-t, t) : t \geq 0\}$. Then λ cannot be identically 1 on two consecutive diagonals D_a and D_{a+1}, and also cannot be identically 1 on D_a and identically 0 on D_{a+1}. (Indeed, in either case we deduce that λ is also 1 on D_{a-1}, leading to a contradiction by induction.)

Fix $m \geq 0$. We will show that for some t there an empty path from $(-\infty, 0] \times \{0\}$ to $\{(-t+m+1, t), (-t+m, t)\}$, which suffices by Lemma 6.1. To verify this claim, we may assume that the periodic orbit commences initially, i.e., that $t_p = 0$. There must be a time t with either $\lambda_t(-t+m+1) = 0$ or $\lambda_t(-t+m) = 0$; by periodicity there must be infinitely many such times.

Now take the *leftmost* ancestral path of one of these two points. Suppose this path does not start on $(-\infty, 0] \times \{0\}$. Then, if t is large enough, the path has a diagonal segment longer than the period of the orbit; additionally, all states immediately to the left of such a segment must be 1. By periodicity, we have, for some $a \in [0, m+1]$, infinite diagonals D_a and D_{a+1} on which λ is identically 1 and 0, respectively. This is in contradiction with our observations above. $\square$

We remark that the supremum in the above lemma cannot be replaced with a limit; a counterexample is $\lambda_0 \equiv 1$ on $[2, \infty)$ and $\lambda_0(1) = 0$.

Returning to our earlier question on seeds, Figure 6.3 (top) shows the set of all points on empty paths from $(-1, 0)$, when λ_0 is a random seed on $[0, 25]$. We believe that for typical long seeds, the right frontier of this set lags behind the right edge of the forward cone of the seed by a non-trivial power of t in the limit $t \to \infty$. This is a natural guess, since the frontier has speed 1 in the voids, but presumably speed 1/4 on the fractal set occupied by randomness. It appears plausible that such a process is a driving force behind the evolution of some exceptional seeds for web CA, including the examples in Figures 1.2 (bottom), 2.1 (bottom), and possibly 2.4.

We believe that similar power law behavior holds for some specific small seeds. One example is shown in Figure 6.3 (middle): the seed is $1\,000\,\widehat{0}\,000\,1$, and empty paths from the middle $\widehat{0}$ are highlighted. However, some seeds exhibit entirely different behavior. The bottom picture shows the empty paths from the two $\widehat{0}$s in the seed $1\,000\,\widehat{0}\,000\,1\,000\,\widehat{0}\,000\,1$. Despite apparent initial similarity to the previous case, here the rightmost point (r_t, t) reachable at time t has r_t/t bounded strictly between 0 and 1 at $t \to \infty$. Indeed, the rescaled path $2^{-n}\{(r_t, t) : t \geq 0\}$ converges as $n \to \infty$ to a variant of the Cantor function or "devil's staircase". This may be proved by an inductive scheme.

As a preliminary step towards the power law behavior postulated above, we prove a version in a simplified setting. Recall from the Section 2 that μ denotes the *Xor* additive cellular automaton rule. Given a configuration $\mu \in \{0,1\}^{\mathbb{Z}\times[0,\infty)}$, we define the χ**-path** starting from a point $(x, 0)$ to be the sequence of points $((x_t, t) : t \geq 0)$ given by $x_0 = x$ and

$$x_{t+1} = \begin{cases} x_t, & \mu(x_t, t) = 1; \\ x_t + 1, & \mu(x_t, t) = 0. \end{cases}$$

In other words, the path makes a down step from a 1, and a diagonal step from a 0. This is intended as a simplified model for a rightward Z-path, which moves with speed 1 in 0s, but with a slower speed in a random configuration.

Proposition 6.5 (Power law for *Xor*)**.** *Let μ be the Xor CA with initial configuration μ_0 equal to 1 on the two-point set $\{-1, 0\}$ and 0 elsewhere. The χ-path $((x_t, t))_{t\geq 0}$ starting from $(0, 0)$ satisfies*

$$x_t = t - \Theta\big(t^{\log 2/\log 3}\big) \quad \textit{as } t \to \infty.$$

Proof. We first note some easy facts about μ. Denote the interval of points $R(k, t) := ((i, t) : t - 2^k < i \leq t)$ on the right side of the forward cone of the origin. For any $k \geq 1$, the state-vectors $(\mu(z) : z \in R(k, t))$ on these intervals form a periodic sequence in t with period 2^{k-1}.

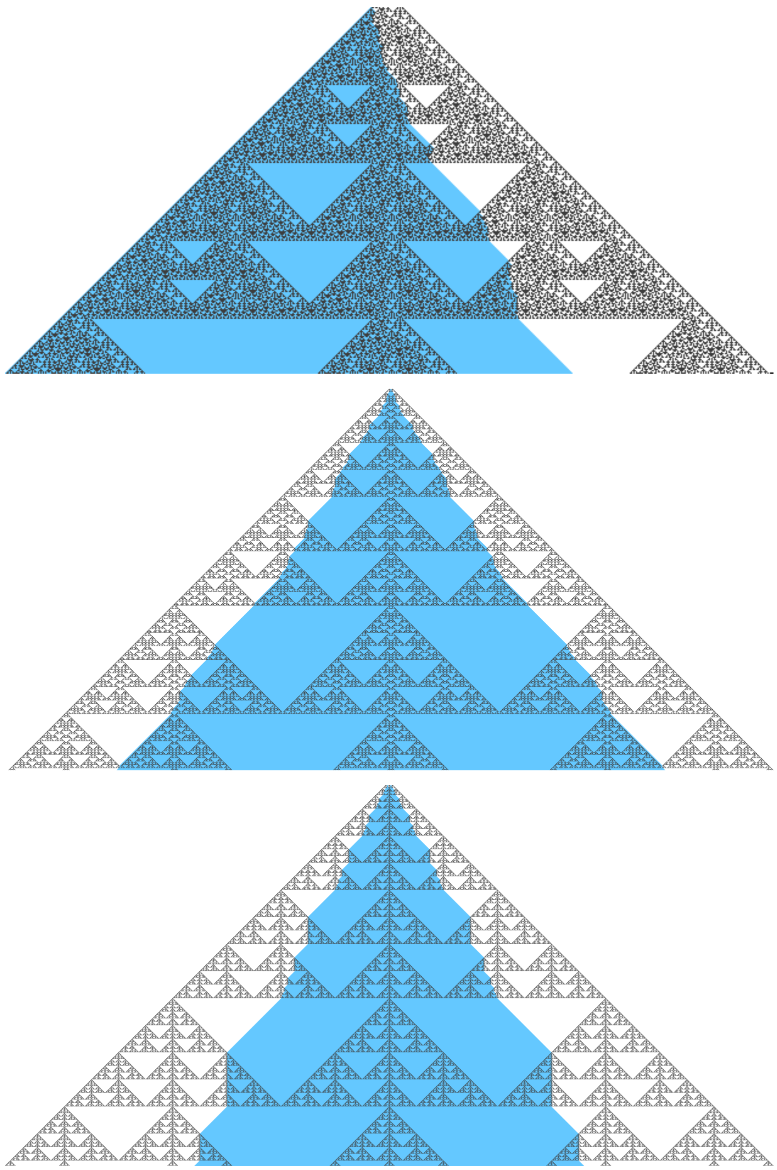

Figure 6.3: The set of all points (blue) on empty paths starting from certain initial points, in *1 Or 3* started from three different seeds: two apparent power-law cases, and a devil's staircase.

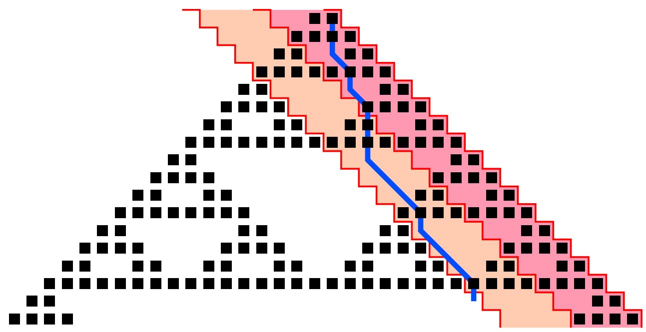

Figure 6.4: The χ-path from the origin in the Xor cellular automaton, together with the construction used in its analysis. The strips $S(2)$ (pink) and $S(3) \setminus S(2)$ (orange) are shaded.

Furthermore, the sequence of state-vectors on the intervals $R(k+1,t) \setminus R(k,t)$ consists precisely of the all-0 vector repeated 2^{k-1} times followed by the first 2^{k-1} state-vectors for $R(k,t)$ (all repeated with period 2^k). See Figure 6.4 for an illustration of the case $k = 2$.

Let $E_k := \min\{t \geq 0 : (x_t, t) \notin R(k,t)\}$; this is the time at which the χ-path leaves the diagonal strip $S(k) := \cup_t R(k,t)$. This can only happen at a down step, which can occur only at a 1 of μ in the leftmost diagonal of $S(k)$. It follows that E_k is divisible by 2^{k-1}; write $E_k = 2^{k-1} e_k$. For example (referring to Figure 6.4), we have $e_2 = 3$ and $e_3 = 4$.

In order to leave the strip $S(k+1)$, the path must first leave $S(k)$, and then leave $S(k+1) \setminus S(k)$. By the above observations on periodicity, and the fact that the path moves diagonally on 0s, we deduce that

$$e_{k+1} = \left\lfloor \frac{e_k}{2} \right\rfloor + e_k.$$

The proof is concluded using induction and obvious monotonicity properties of the χ-path. $\square$

Among many unresolved questions, we do not know whether an analogue of Proposition 6.5 holds when the χ-path is defined similarly in terms of the *1 Or 3* CA λ rather than μ.

7 Additive dynamics from random seeds

Our goal in this section is to transfer the non-percolation results for infinite random initial configurations to random seeds. The proofs exploit an intriguing interplay between randomness and periodicity in the configuration started from a random seed.

Lemma 7.1 (Random edge-intervals)**.** *Assume λ_0 is a uniformly random binary seed on $[0, L]$. For a fixed t, the state on the interval $[t, t+L] \times \{t\}$ is uniformly random.*

Proof. This is an immediate consequence of Lemma 3.2(i) and Corollary 4.3. $\square$

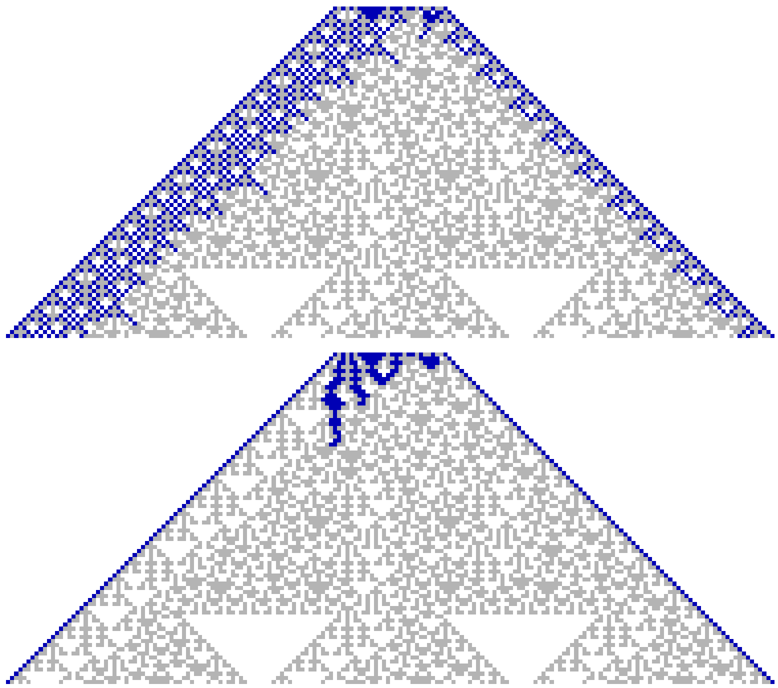

Figure 7.1: Illustrations of the set of all points reached by paths starting within an initial random seed. Top: empty diagonal paths; bottom: wide paths.

Lemma 7.2 (Edge-periodicity)**.** *For any λ_0 which is 0 on $[L+1, \infty)$, and any $k \geq 1$, the sequence of edge configurations $(\lambda(i,t) : i = t+L-k+1, \ldots, t+L)$ is periodic in t, with period at most $2k$.*

Proof. This follows from Lemma 3.2(iv) and Lemma 4.1. □

Our first result establishes that, in subcritical cases, paths from the initial state do not reach far into the forward cone of $[0, L] \times \{0\}$. This is illustrated in Figure 7.1, in which $L = 25$ and all points on paths from $\mathbb{Z} \times \{0\}$ are again depicted in blue (only one layer of points outside the forward cone is colored blue, as all such are trivially reachable from $\mathbb{Z} \times \{0\}$).

Proposition 7.3 (Percolation into the cone)**.** *Suppose λ_0 is a uniformly random binary seed on $[0, L]$. The probability that there is an empty diagonal path from $\mathbb{Z} \times \{0\}$ to the forward cone of $[0, L] \times \{\lfloor C \log L \rfloor\}$ goes to 0 as $L \to \infty$. The same is true for wide paths. Here C is an absolute constant.*

Proof. Let k be a positive integer to be chosen later satisfying $2k+1 < L$. Call a space-time point (x,t) **bad** if there exists an empty diagonal path from $\mathbb{Z} \times \{t-k\}$ to (x,t). If the state on the interval $I(x,t) := [x-k, x+k] \times \{t-k\}$ is uniformly random, then Theorem 1.5 implies that $\mathbf{P}((x,t) \text{ is bad}) \leq \exp(-ck)$ for an absolute constant $c > 0$.

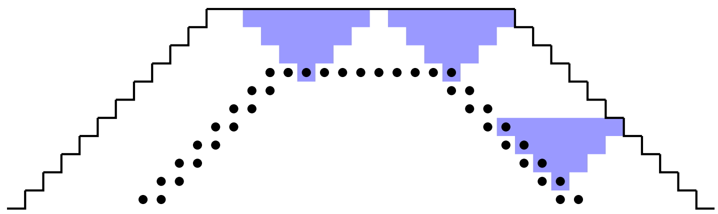

Figure 7.2: An illustration of the proof of Proposition 7.3. The outline of the forward cone of $[0, L] \times \{0\}$ is shown by the solid line (here $L = 16$). Points in the set S are shown as black discs. Any path from the top row to the region below S must pass through S. For each point in S, the top row of the associated triangle (of size $k = 3$) has a uniformly random state (three such triangles are shaded).

We define an infinite set of points S via Figure 7.2. This set has the following properties: (i) any path from $\mathbb{Z} \times \{0\}$ to the forward cone of $[0, L] \times \{2k\}$ must pass through a point in S; and (ii) for every $(x, t) \in S$, the state on the interval $I(x, t)$ defined above is uniformly random, either trivially or by Lemma 7.1.

We wish to bound the probability that S contains a bad point by a union bound. The set S is infinite, but Lemma 7.2 implies that the states of the relevant intervals $I(x, t)$ for (x, t) in the diagonal "arms" of S repeat with period at most $2(2k + 2)$. Thus, besides the at most L points on the top section of S, there are only $8(2k + 2)$ distinct cases to consider. Hence

$$\mathbf{P}(S \text{ contains a bad point}) \leq (L + 16k + 16)\, e^{-ck}.$$

The proof is completed by taking $k = \lfloor C' \log L \rfloor$ for a suitably large C' (the argument for wide paths is identical). $\square$

Similarly, we next show that to each void of $\lambda^\bullet$ there corresponds a periodic strip that blocks diagonal and wide paths. Fix a void V of $\lambda^\bullet$, and an integer $L \geq 0$. Define the **perturbed void** V to be the triangular region

$$W_L(V) = V \cap (V + (L, 0)).$$

See Figure 7.3 for an example. Note that $W_L(V) = \emptyset$ unless the width of V exceeds L. Further, fix an integer $m \geq 1$, assume the top of $W_L(V) = [a, b] \times \{t\}$, and define the following interval above $W_L(V)$:

$$J_{L,m}(V) = [a - 2^m, b + 2^m] \times t - 2^m.$$

(We set $J_{L,m}(V) = \emptyset$ when $W_L(V) = \emptyset$.)

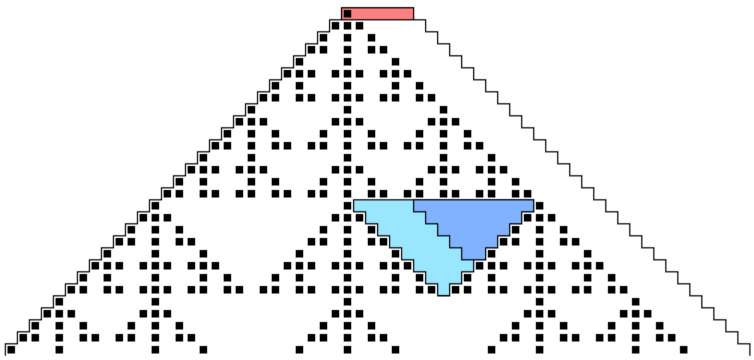

Figure 7.3: Perturbed void (dark blue), with $L = 5$, of the void with top interval $[1, 15] \times \{16\}$. The perturbed void is filled with 0s for any seed included in the interval of six red points. The forward cone of this interval is outlined.

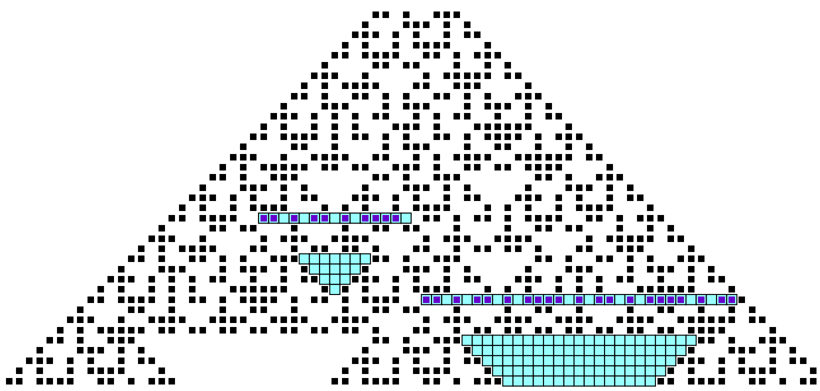

Figure 7.4: Illustration of Theorem 7.4(ii) with the seed 110100111 on $[0, 8]$ and $m = 2$ and same two voids as in Figure 3.4; the repeating string is $A = 010110101111$.

Lemma 7.4 (Periodic and random intervals above voids)**.** *Suppose the initial configuration λ_0 vanishes outside $[0, L]$. Let m be a nonnegative integer. Let V be a void of $\lambda^\bullet$ of width at least L and at least 2^m.*

(i) *λ vanishes on $W_L(V)$.*

(ii) *There exists a string A of length $3 \cdot 2^m$, depending on m and λ_0 but not on V, such that the configuration of λ on $J_{L,m}(V)$ is a subword of A^∞.*

(iii) *Now suppose that $2^{m+1} \le L$ and that λ_0 is uniformly random on $[0, L]$. Then every interval of length 2^m in $J_{L,m}(V)$ has uniformly random state.*

Proof. Claim (i) is a simple consequence of Lemma 4.1, (ii) follows from Proposition 3.6 and Lemma 4.1, and (iii) from Proposition 3.6 and Corollary 4.3. $\square$

Proposition 7.5 (Percolation into voids)**.** *Assume the conditions in Lemma 7.4(iii). Let* $\texttt{cross}(m)$ *be the event that there exists a void* V *for which there is either an empty diagonal or a wide path from* $J_{L,m}(V)$ *to* $W_L(V)$*. Then*

$$\mathbf{P}(\texttt{cross}(m)) \leq \exp(-c2^m)$$

for some universal constant c*.*

Proof. Using Theorem 1.5, and Lemma 7.4 (ii) and (iii), this follows by a similar argument to the proof of Theorem 7.3. The key point is that by Lemma 7.4(ii), only $3 \cdot 2^m$ distinct cases need to be considered in the union bound. □

8 Replication and ethers in web cellular automata

We can now prove Theorems 1.1 and 1.4 from the introduction.

Proof of Theorem 1.4. This is an immediate consequence of Proposition 7.3 and Lemma 2.1. □

Proof of Theorem 1.1. Consider a uniformly random binary seed on $[0, L]$. In the context of Proposition 7.5, let M be the smallest m with $2^{m+1} \leq L$, for which $\texttt{cross}(m)$ does not occur. If such an m does not exist, let $M = \infty$. By Proposition 7.5, M is tight as $L \to \infty$.

If $M = \infty$, take $R_L = \infty$. Assume now that $M < \infty$. Then there exists a string A' of 0s and 2s of length of $3 \cdot 2^m$ so that the top row of W_L is a segment of $(A')^\infty$ for every void. This holds because the first level configuration λ is periodic with the required period on a strip above W_L, by Lemma 7.4, while the absence of relevant paths makes the top row also periodic by Lemma 2.1. Moreover, by the same results, the periodic pattern is the same for all voids.

Consider the CA ξ started with a periodic configuration B^∞, for some string B of length σ. The evolution is periodic in time after some initial burn-in period T_B. Let $\texttt{burnin}(\sigma) = \max_B T_B$. Our random distance R_L is $\texttt{burnin}(3 \cdot 2^M)$ plus a universal additive constant. The proof is finished by Lemma 2.3. □

As remarked earlier, Theorem 1.1 implies that the union of the regions that are filled by a translate of the ether has density 1 within the forward cone of the seed. Therefore, on the event that $R_L < \infty$ the set of non-zero points has a rational density within the same forward cone. We do not know whether the same holds for *every* seed.

For an arbitrary web CA satisfying the conditions of Theorem 1.1, $\limsup_{L\to\infty} \mathbf{P}(R_L \geq r)$ decays at least as fast as a power law in r. This is easily seen from the above proof, using the fact that $\texttt{burnin}(\sigma) \leq 2^\sigma$. In cases when the dynamics restricted to 0s and 2s is additive, including *Web-Xor*, *Modified Web-Xor*, and *Piggyback*, one can easily show that the decay is exponential. Identical remarks apply to the temporal period of the ether η_L.

9 Bounds on ether probabilities

In this section we prove Theorem 1.2, and explain how explicit lower bounds on ether probabilities are proved. We also indicate how some ethers can be ruled out for certain rules.

For $m \geq 0$, we call the string A in Lemma 7.4(ii) the **level-2^m link** of the seed λ_0. (Note that the choice of A is unique up to periodic shifts.) Fix an integer $k \geq 1$ and a binary string A. Consider *1 Or 3* with initial periodic configuration $\lambda_0 = A^\infty$. If there is no empty diagonal (resp. wide, θ-free) path from $\mathbb{Z} \times \{0\}$ to $\mathbb{Z} \times \{k-1\}$ in the resulting configuration λ, then we say that that A is a **blocker** to **depth** k for diagonal (resp. wide, θ-free) paths. If $\lambda_{k-1} \not\equiv 0$, then we say that that A is **non-degenerate** to depth k.

Fix an $m \geq 1$, and let λ_0 be a uniformly random seed on $[0, L]$, with $L \geq 2^{m+1}$. For diagonal and wide paths, Proposition 7.5 implies that

$$\mathbf{P}(\text{the level-}2^m \text{ link of } \lambda_0 \text{ is a blocker to depth } 2^m) \geq 1 - \exp(-c2^m),$$

for some universal constant c.

Further, consider a web CA ξ_t with an ether $\eta \in \{0,2\}^{\mathbb{Z}^2}$. The **signature** of η is a string B such that, for some t, $\eta(\cdot, t)$ equals a (spatial) translation of B^∞, and is the first in the lexicographic order among shortest such strings. Observe that two ethers are equivalent if and only if they have the same signature. We say that a binary string A **produces** η with signature B if the initial state $\xi_0 = A^\infty$ makes ξ_t equal to a translation of B^∞ at some time t.

Lemma 9.1 (Blockers)**.** *Let ξ_t be a diagonal-compliant (respectively: wide-compliant, or θ-free-compliant) web CA. Further, let A be a string that is a blocker to depth 2^m for diagonal (respectively: wide, or θ-free) paths and produces an ether η. If a seed ξ_0 results in the level-2^m link A, then ξ is a replicator with ether η. If, in addition, the CA ξ_t has no spontaneous birth, then $\eta \equiv 0$.*

Proof. The first claim follows by the same arguments as in the proof of Theorem 1.1. The last claim follows by Lemma 2.1. □

Denote by $\texttt{Seeds}_{[a,b]}$ the set of binary seeds that vanish outside $[a, b]$, and by $\texttt{Seeds} = \cup_{a \leq b} \texttt{Seeds}_{[a,b]}$ the set of all binary seeds. Let $g : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$ be the map determined by one step of the *1 Or 3* rule (i.e., the map $\lambda_0 \mapsto \lambda_1$). It is well-known (and easy to prove) that, for $a \leq b$, the map g is injective from $\texttt{Seeds}_{[a,b]}$ to $\texttt{Seeds}_{[a-1,b+1]}$, and therefore the restriction $g|_{\texttt{Seeds}}$ is injective. We say that a binary seed λ_0 **has a predecessor** if it is in the image of $g|_{\texttt{Seeds}}$. More generally, λ_0 **has** k **predecessors**, for $k \geq 1$, if it is in the image of the k-th iteration $(g|_{\texttt{Seeds}})^k$; in that case, $(g|_{\texttt{Seeds}})^{-k}(\lambda_0)$ contains a unique seed called the the **k-th predecessor** of λ_0. We denote by $\texttt{Pred}_k$ the set of all seeds that have k predecessors. The following lemma follows immediately from the properties of $g|_{\texttt{Seeds}}$.

Lemma 9.2 (Predecessors of random seeds)**.** *Assume λ_0 is a uniformly random binary seed on $[0, L]$ and that $1 \leq k \leq L/2$. Then $\mathbf{P}(\texttt{Pred}_k) = 1/4^k$. Moreover, conditioned on $\texttt{Pred}_k$, the k-th predecessor of λ_0 is a uniform binary seed on $[k, L-k]$.*

Lemma 9.3 (Predecessors and links). *For $m \geq 0$, a seed λ_0 has 2^m predecessors if and only if its level-2^m link is 0.*

Proof. If λ_0 has k predecessors, then its level-2^m link is 0 by Lemma 3.2(ii) and Lemma 4.1.

Conversely, assume that $\lambda_0 \in \mathtt{Seeds}_{[0,L]}$ is given by a string S of length $L+1$. If n is large enough so that $2^n > 2L$ and $2^n > 2^{m+1}$, the configuration of λ_{2^n} on $[0, L]$ is S, again by Lemma 3.2(ii) and additivity. If the level-2^m link is 0, the configuration on $[0, L]$ at time $2^n - 2^m$ provides a seed λ'_0, such that $g^{2^m}(\lambda'_0) = \lambda_0$. ☐

Lemma 9.4 (Ether probabilities). *Suppose some seed $S_0 \in \mathtt{Seeds}_{[0,s-1]}$ is a replicator with some ether η, and is such that for some $m \geq 1$, the level-2^m link is a blocker to depth 2^m. Let ξ_0 be a uniformly random binary seed on $[0, L]$. Then*

$$\liminf_{L\to\infty} \mathbf{P}(\xi_0 \textit{ is a replicator with ether } \eta) \geq 2^{-s-2^{m+1}}.$$

Proof. Assume a seed S_1 with support in $[s, \infty)$ has 2^m predecessors. Form a seed S by adding the configurations of S_0 and S_1. Then, by Lemma 9.3 and Lemma 4.1, S has the same level-2^m link as S_0, and therefore, by Lemma 9.1, is a replicator with the same ether η. In the rest of the proof we apply this fact to random seeds.

Suppose now ξ_0 is a uniformly random seed in $\mathtt{Seeds}_{[0,L]}$, with L large enough so that $L - s \geq 2^{m+1}$. Let ξ'_0 (resp. ξ''_0) be the random seed that agrees with ξ_0 on $[0, s-1]$ (resp. $[s, L]$) and vanishes elsewhere. Then,

$$\begin{aligned}
\mathbf{P}(S \text{ is a replicator with ether } \eta) &\geq \mathbf{P}(\xi'_0 = S_0, \text{ and } \xi''_0 \text{ has } 2^m \text{ predecessors}) \\
&= \mathbf{P}(\xi'_0 = S_0) \cdot \mathbf{P}(\xi''_0 \text{ has } 2^m \text{ predecessors}) \\
&= 2^{-s} \cdot 4^{-2^m},
\end{aligned}$$

where the last equality follows from Lemma 9.2. ☐

Proof of Theorem 1.2. This is immediate from Lemmas 9.1 and 9.4 and the proof of Theorem 1.1. ☐

Recall that *Extended 1 Or 3* is not diagonal- or wide-compliant. However, it is 4-free compliant, and this allows us to prove the following lower bounds.

Theorem 9.5 (Replication and ether probabilities for *Extended 1 Or 3*). *Let ξ be the Extended 1 Or 3 web CA, started from a uniformly random binary seed on $[0, L]$. Then*

$$\liminf_{L\to\infty} \mathbf{P}(\xi \textit{ is a replicator}) \geq 0.826.$$

Moreover, lower bounds on $\liminf_{L\to\infty} \mathbf{P}(\xi$ is a replicator with ether $\eta)$ for certain ethers η are as in Table 9.1.

ether signature	temporal period	spatial period	density of 2s	lower bound
0	1	1	0	0.6061
02	1	2	1/2	0.0471
0002	2	4	1/2	0.0333
[7]2	4	8	3/8	0.0664
[5]202	4	8	3/8	0.0189
[15]2	8	16	5/16	0.0193
[13]202	8	16	11/32	0.0079
[11]20002	8	16	5/16	0.0024
[9]2000202	8	16	3/8	0.0085
[9]2020202	8	16	3/8	0.0006
[7]200020202	8	16	13/32	0.0045
[7]202020202	8	16	7/16	0.0105
[5]2[5]20202	8	16	7/16	0.0006

Table 9.1: Some non-equivalent ethers that provably emerge for *Extended 1 or 3* from a long random seed with positive asymptotic probability. Each ether is generated from the initial condition obtained by repeating its signature indefinitely. Here $[k]$ stands for an interval of k 0s. The last column is a rigorous lower bound for the lim inf of the probability in Theorem 1.2. The lower bounds sum to just over 0.826.

For an ether η, its **reflection** around the time axis is denoted by $\bar{\eta}$. Then η is **symmetric** if $\bar{\eta}$ is equivalent to η. Assume that B is the signature of η and $\bar{B}$ its reflection. A sufficient condition for symmetry of η is that the reflection $\bar{B}$ is a periodic shift of B. However, this is not a necessary condition: the ether with signature $B =$ [7]200020202 is symmetric as the fourth iteration of the *1 Or 3* rule applied on B^∞ yields a translation of $\bar{B}^\infty$. Thus the only non-symmetric ether in Table 9.1 is the one with signature [9]2000202. In non-symmetric cases, our tables combine the frequencies of an ether and its reflection.

Proof of Theorem 9.5. Throughout the proof, fix a positive integer $m \geq 1$ and assume that $L \geq 3 \cdot 2^m$. Using the same notation as in Proposition 3.6, assume that for some $a \geq 0$, the configuration of $\lambda^\bullet$ at some time t on $I = [a + L, a + L - 1 + 3 \cdot 2^m] \times \{t\}$ is exactly the string $A_0 = 1 \,\square\, 1 \,\square\, 0 \,\square$. For ease of reference, we will assume A_0^∞ is positioned on $\mathbb{Z}$ so that A_0 is the configuration in $[0, 3 \cdot 2^n - 1]$.

Our main tool is the map $\Phi : \mathbb{Z}_2^{L+1} \to \mathbb{Z}_2^{3 \cdot 2^m}$ that takes as argument an initial binary seed λ_0 supported on $[0, L]$ and outputs the configuration of λ on I. This is a linear map that assigns to every seed with support in $[0, L]$ its level-2^m link. The matrix of Φ (in the standard basis) has row i given by the segment $[i, L + i]$ of A_0^∞, $i = 0, \dots, 3 \cdot 2^m - 1$. (All matrix and vector coordinate indices start at 0.) It is easy to see that the matrix has rank 2^{m+1}, and therefore its image has cardinality $2^{2^{m+1}}$. The kernel of Φ^* has basis vectors y^k, $k = 0, \dots, 2^m - 1$, given by

$y_i^k = \mathbf{1}[i \bmod 2^m = k]$, $i = 0, \dots, 3 \cdot 2^m - 1$. Therefore, the image of Φ is the set

$$\Phi(\mathbb{Z}_2^{L+1}) = \left\{ b \in \mathbb{Z}_2^{3 \cdot 2^m} : b_i + b_{2^m+i} + b_{2^{m+1}+i} = 0,\ \forall i = 0, \dots, 2^m - 1 \right\}.$$

A vector in $\mathbb{Z}_2^{3\cdot 2^m}$ is naturally identified with a binary string of length $3 \cdot 2^m$ and we will do so for the rest of the proof. Let N_n be the number of strings in $\Phi(\mathbb{Z}_2^{L+1})$ that are non-degenerate to depth 2^m. Further, let N_b the number of strings in $\Phi(\mathbb{Z}_2^{L+1})$ that are non-degenerate and blockers, for 4-free paths, to the same depth 2^m.

Now suppose that ξ_0 be a uniform random binary seed on $[0, L]$ and let p_L be the probability that ξ is a replicator. We claim that

$$\liminf_{L\to\infty} p_L \geq \frac{N_b}{N_n}. \tag{9.1}$$

Recall that $\texttt{Pred}_1$ is the event that ξ_0 has a predecessor; by Lemma 9.3, $\texttt{Pred}_1^C$ is exactly the event that $\Phi(\xi_0)$ is non-degenerate to depth 2^m. Furthermore, conditioned on $\texttt{Pred}_1$, the first predecessor of ξ_0 is a uniformly random binary seed on $[1, L-1]$. Therefore,

$$\begin{aligned} p_L &\geq \mathbf{P}(\Phi(\xi_0) \text{ is a blocker to depth } 2^m \mid \texttt{Pred}_1^C)\, \mathbf{P}(\texttt{Pred}_1^C) \\ &\quad + \mathbf{P}(\xi \text{ is a replicator} \mid \texttt{Pred}_1)\, \mathbf{P}(\texttt{Pred}_1) \\ &= \frac{N_b}{N_n} \cdot \frac{3}{4} + p_{L-2}\frac{1}{4}. \end{aligned} \tag{9.2}$$

Now (9.1) follows by taking lim inf as $L \to \infty$ of the first and last expressions of (9.2). The particular bound was obtained by a computer for $m = 4$: all 2^{32} vectors in the range of Φ were checked for blocking and non-degeneracy, and the resulting tallies were $N_n = 3,221,225,472$ and $N_b = 2,663,229,504$. This concludes the proof for replication probability.

The proof for a lower bound for a particular ether η is identical except in the definition of N_b, which is now the number of strings in $\Phi(\mathbb{Z}_2^{L+1})$ that are blockers and non-degenerate to the level 2^m, *and* produce η. For example, the result for the zero ether was $N_b = 1,952,489,232$. □

Table 9.1 suggests that spatial and temporal periods of *Extended 1 Or 3* ethers are powers of 2, and that the ether $(2)^\infty$ never appears. This is addressed in our next two results.

Lemma 9.6 (Periodic configurations)**.** *Assume that λ_0 is a spatially periodic configuration whose period σ divides $3 \cdot 2^n$, and that $\lambda_t = \lambda_0$ for some t. Then σ divides 2^n. Moreover, for $\sigma \geq 1$ the temporal period equals $\sigma/2$.*

Proof. By Lemma 3.1 and Lemma 4.1, we may assume $n = 1$, and then we check that any λ_0 of period 3 leads to a constant configuration in a single time step. The last assertion follows from Lemma 3.1 and the following two easily checked facts: (1) if λ_0 is periodic with period at most 2, then $\lambda_0 = \lambda_1$; and (2) if λ_0 is periodic with period exactly 4, then $\lambda_0 \neq \lambda_1$. □

Proposition 9.7 (Possible ethers)**.** *Assume ξ_t is the Extended 1 or 3 CA. Suppose that that ξ_0 is a replicator with ether η, and that its level-2^m link is a blocker to depth 2^m for 4-free paths. Then η has spatial period that is a power of 2. Also, the signature of η is either* 0 *or it is of the form* $[a_1]2[a_2]2\dots[a_k]2$, *where $k \geq 1$ and each $[a_i]$ is a string of* 0*s of odd length a_i. In particular, $\eta \not\equiv 2$.*

ether signature	temporal period	spatial period	density of 2s	lower bound
0	1	1	0	0.5
2	1	1	1	0.0398
02	1	2	1/2	0.0142
0002	2	4	1/2	0.0258
[7]2	4	8	3/8	0.0099
[4]2022	4	8	1/2	0.0303
00020222	4	8	1/2	0.0209
00022222	4	8	5/8	0.1297
0002000200022222	8	16	11/16	0.0362
0002000200202002	8	16	9/16	0.0216

Table 9.2: Ten ethers that emerge from long random seeds for *Piggyback* with positive asymptotic probability. The conventions of Table 9.1 apply.

Proof. The first claim follows from the previous lemma and Theorem 7.4, so we proceed to prove the second claim. If $\lambda_t \equiv 0$, but $\lambda_{t-1} \not\equiv 0$, then there are, up to translation, exactly two possibilities for λ_{t-1} and λ_{t-2}:

$$\begin{array}{ccc} \cdots 1\,1\,1\,1\,0\,0\,1\,1\,1\,1\,0\,0 \cdots & \qquad & \cdots 0\,1\,0\,0\,0\,1\,0\,1\,0\,0\,0\,1 \cdots \\ \cdots 0\,1\,1\,0\,1\,1\,0\,1\,1\,0\,1\,1 \cdots & \qquad & \cdots 0\,1\,1\,0\,1\,1\,0\,1\,1\,0\,1\,1 \cdots \end{array}$$

Assume that the seed ξ_0 is such that $\delta(\xi_0)$ has exactly k predecessors. (Recall that $\delta(a) := \mathbf{1}[a = 1]$.) Then, for any n, the state of $\delta(\xi)$ on $[C-1, 2^n - C + 1] \times \{2^n - k - 2, 2^n - k - 1\}$ is a segment of one of the two configurations above. (Here, C is a constant that depends only on L.) By considering 4-free paths, we see that the left configuration implies $\xi_{2^n-k} \equiv 0$ on $[C, 2^n - C]$, while for right one implies that one ξ_{2^n-k} vanishes outside $6\mathbb{Z} \cap [C, 2^n - C]$ (after a suitable translation). As 2s evolve according to the *1 Or 3* rule in the absence of 1s, the positions of 2s started from a subset of $2\mathbb{Z}$ are a subset of $2\mathbb{Z}$ at all even times (by Lemma 3.1). The claimed form of the signature follows. $\square$

Lower bounds for ether probabilities can also be obtained for *Piggyback*, with the same proof as for Theorem 9.5.

Theorem 9.8 (Ether probabilities for *Piggyback*)**.** *Let ξ_t be the Piggyback web CA, started from a uniformly random seed of* 0*s and* 1*s on* $[0, L]$*. Lower bounds on*

$$\liminf_{L\to\infty} \mathbf{P}(\xi \text{ is a replicator with ether } \eta)$$

are as in Table 9.2.

The computer search with $m = 4$ yielded 117 different ethers for *Piggyback* with provably positive asymptotic probability, with their combined probabilities at least 0.914. The ethers listed in Table 9.2 are the ten with largest lower bounds. The only non-symmetric ether among these ten has signature [4]2022. (The initial state $(00020222)^\infty$ generates its translated reflection in 2 steps.) We do not know whether the asymptotic probability for the zero ether is exactly 1/2.

Open problems

As the earlier discussions indicate, this topic offers a rich supply of open questions. We highlight a small selection.

(i) In the *1 Or 3* cellular automaton started from a uniformly random binary string on the half-line $[0, \infty)$, what is the growth rate of the maximum integer r_t for which there is an empty path from $(-\infty, 0) \times \{0\}$ to (r_t, t)? Is it the case that $r_t/t \to 1/4$ as $t \to \infty$?

(ii) What can be said about percolation in the space-time configuration of other one-dimensional cellular automata started in an invariant measure? For example, the uniformly random binary string on $\mathbb{Z}$ is invariant for *permutative* rules (see [BL] for a definition), including *Rule 30*. Do there exist infinite diagonal, wide or empty paths?

(iii) Are there infinitely many different ethers for replicators in the *Piggyback* cellular automaton? Is there an algorithm that decides whether a given ether occurs in some replicator?

(iv) For two-dimensional *Box 13* solidification CA (see Section 2 and [GG2]) started from a uniform random seed in $[0, L]^2$, does the final configuration have rational density with probability converging to 1 as $L \to \infty$?

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