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# ON THE $n$-TH ELEMENT OF A SET OF POSITIVE INTEGERS

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**Abstract.** Given a set $A$ of positive integers and its counting function $A(x) := \#\{n \le x : n \in A\}$, we examine the size of the $n$-th element of $A$ using the size of $A(x)$.

## 1. Introduction and notation

Determining the size of the $n$-th element of a set of positive integers using the known size of the counting function of that set is a classical problem in analytic number theory. For example, letting $\pi(x)$ stand for the number of prime numbers $p \le x$, by using the Prime Number Theorem in the form $\pi(x) \sim x/\log x$ as $x \to \infty$, one can easily show that the $n$-th prime number $p_n$ satisfies

$$p_n = (1 + o(1))\, n \log n \qquad (n \to \infty)\,.$$

In fact, in 1902, by using the logarithmic integral function, Cipolla [3] improved this estimate by showing that there exists a unique sequence of polynomials

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$(Q_j)_{j\geq 1}$ with rational coefficients such that, for any given positive integer $m$,

$$p_n = n\left(\log n + \log_2 n - 1 + \sum_{j=1}^{m} \frac{(-1)^{j-1}\, Q_j\,(\log_2 n)}{\log^j n} + o\left(\frac{1}{\log^m n}\right)\right) \qquad (n\to\infty)\,. \tag{1.1}$$

Here and in what follows, we write $\log_2 x$ for $\max(1, \log\log x)$.

Another example is given by the search of an estimate for $a_n$, the $n$-th composite number. Bojarincev [2] and Shiu [12] showed that, for any given positive integer $m$,

$$a_n = n\left(1 + \frac{\beta_1}{\log n} + \frac{\beta_2}{\log^2 n} + \cdots + \frac{\beta_m}{\log^m n} + o\left(\frac{1}{\log^m n}\right)\right) \qquad (n\to\infty)\,, \tag{1.2}$$

where the $\beta_i$ are computable constants.

Finally, recall that we say that a number is a *powerful number* (or a *square-full number*) if $p \mid n$ implies that $p^2 \mid n$. Let $\wp_n$ denote the $n$-th powerful number. In 1982, Ivić and Shiu [7] showed that

$$\wp_n = \left(\frac{\zeta(3)}{\zeta(3/2)}\right)^2 n^2 + O\left(n^{5/3}\right) \qquad (n\to\infty)\,. \tag{1.3}$$

Here, we examine the problem of estimating the size of the $n$-th element of a given set $A$ of positive integers using the size of $A(x) := \#\{n \leq x : n \in A\}$, often called the counting function of $A$. We will do so in two particular cases. The first one is when $A(x) = b_1 x^{\lambda_1} + b_2 x^{\lambda_2} + R(x)$, where $R(x) = o(x^{\lambda_3})$, for some real constants $b_1 > 0$ and $b_2$, with $1 > \lambda_1 > \lambda_2 > \lambda_3 > 0$, from which we will then deduce an improvement of the estimate (1.3).

The second case is when $A(x) = \dfrac{x}{L(x)}\left(1 + O\left(\dfrac{1}{\varphi(x)}\right)\right)$ where $\varphi$ is an increasing function which tends to $+\infty$ as $x \to \infty$ and $L$ is a differentiable increasing slowly oscillating function. Recall that a function $L : [M, +\infty) \to \mathbb{R}$ continuous on $[M, +\infty)$, where $M$ is a positive real number, is said to be a *slowly oscillating function* if for each positive real number $c > 0$,

$$\lim_{x\to\infty} \frac{L(cx)}{L(x)} = 1. \tag{1.4}$$

This class of functions was introduced by Karamata [8] in 1930. His paper, along with [9] as well as the book of Seneta [11], provide some interesting properties of slowly oscillating functions. In particular, it is possible to show that a differentiable function $L$ is slowly oscillating if and only if

$$\frac{xL'(x)}{L(x)} = o(1) \qquad (x \to \infty) \tag{1.5}$$

and, in fact, that $L$ is slowly oscillating if and only if there exists $x_0 > 0$ such that

$$L(x) = C(x) \exp\left\{ \int_{x_0}^{x} \frac{\eta(t)}{t}\, dt \right\}, \tag{1.6}$$

where $\lim_{x\to\infty} C(x) = C$, for a certain constant $C \neq 0$, and $\eta(t) \to 0$ as $t \to \infty$.

We shall denote by $\mathcal{L}$ the set of increasing and differentiable slowly oscillating functions.

From here on, the letter $c$, with or without subscript, stands for an absolute positive constant, but not necessarily the same at each occurrence, while the letter $p$, with or without subscript, will always denote a prime number.

## 2. Main results

**Theorem 1.** *Given a sequence of positive integers $a_1 < a_2 < \cdots$, let $A = \{a_1, a_2, \ldots\}$ with counting function $A(x)$ satisfying*

$$A(x) = b_1 x^{\lambda_1} + b_2 x^{\lambda_2} + R(x), \tag{2.1}$$

*where*

$$R(x) = o(x^{\lambda_3}) \qquad (x \to \infty)$$

*and where $b_1 > 0$ and $b_2$ are real constants, with $1 > \lambda_1 > \lambda_2 > \lambda_3 > 0$ which satisfy*

$$\frac{3\lambda_2}{\lambda_1} - 3 < \frac{\lambda_3}{\lambda_1} - 1 \le \frac{2\lambda_2}{\lambda_1} - 2.$$

*Then*

$$a_n = \frac{n^{\frac{1}{\lambda_1}}}{b_1^{\frac{1}{\lambda_1}}} - \frac{1}{\lambda_1}\frac{b_2}{b_1^{\frac{\lambda_2+1}{\lambda_1}}} n^{\frac{\lambda_2+1}{\lambda_1}-1} + \frac{1}{2}\left(\frac{2\lambda_2+1}{\lambda_1^2} - \frac{1}{\lambda_1}\right)\frac{b_2^2}{b_1^{\frac{2\lambda_2+1}{\lambda_1}}} n^{\frac{2\lambda_2+1}{\lambda_1}-2} + \tag{2.2}$$
$$+ o\left(n^{\frac{\lambda_3+1}{\lambda_1}-1}\right).$$

**Theorem 2.** *Given a sequence of positive integers* $a_1 < a_2 < \cdots$, *let* $A = \{a_1, a_2, \ldots\}$ *with counting function* $A(x)$ *satisfying*

$$A(x) = \frac{x}{L(x)}\left(1 + O\left(\frac{1}{\varphi(x)}\right)\right) \qquad (x \to \infty), \tag{2.3}$$

*where* $\varphi$ *is an increasing function which tends to* $+\infty$ *as* $x \to \infty$ *and where* $L \in \mathcal{L}$ *with corresponding function* $\eta(t)$ *defined implicitly by* (1.6). *Moreover, assume that* $\eta(t)$ *is a decreasing function and that*

$$C(x) = C + O\left(\frac{1}{\psi(x)}\right) \qquad (x \to \infty)\,, \tag{2.4}$$

*where* $\psi(x)$ *is an increasing function which tends to* $+\infty$ *as* $x \to \infty$. *Then,*

$$a_n = n\,\frac{C\left(a_n\right)}{C(n)}\,L(n)\,\exp\left\{\int_n^{a_n} \frac{\eta(t)}{t}\,dt\right\}\left(1 + O\left(\frac{1}{\varphi(n)}\right)\right) \qquad (n \to \infty) \tag{2.5}$$

*and*

$$a_n = n\,L(n)^{1/(1-\delta(n))}\left(1 + O\left(\frac{1}{\varphi(n)} + \frac{1}{\psi(n)}\right)\right) \qquad (n \to \infty), \tag{2.6}$$

*where* $\delta$ *is some function satisfying* $\eta(a_n) < \delta(n) < \eta(n)$ *for all integers* $n \geq x_0$.

*Moreover, if there exists a positive constant* $c$ *such that*

$$\eta(n) \int_{x_0}^{n} \frac{\eta(t)}{t}\,dt \to c \qquad (n \to \infty), \tag{2.7}$$

*then*

$$a_n = \left(e^c + o(1)\right) nL(n) \qquad (n \to \infty). \tag{2.8}$$

## 3. Proof of Theorem 1

To prove Theorem 1, we use an approach already used by Copil and Panaitopol [4] to estimate the size of the $n$-th non powerful number.

First, observe that it follows from (2.1) that

$$n = A\left(a_n\right) = b_1 a_n^{\lambda_1} + b_2 a_n^{\lambda_2} + R\left(a_n\right),$$

so that

$$a_n^{\lambda_1} = \frac{n}{b_1}\left(1 - b_2\frac{a_n^{\lambda_2}}{n} - \frac{R(a_n)}{n}\right)$$

thereby implying that

$$a_n = \frac{n^{1/\lambda_1}}{b_1^{1/\lambda_1}}\left(1 - b_2\frac{a_n^{\lambda_2}}{n} - \frac{R(a_n)}{n}\right)^{1/\lambda_1}. \tag{3.1}$$

In particular, since both expressions $b_2\frac{a_n^{\lambda_2}}{n}$ and $\frac{R(a_n)}{n}$ goes to 0 as $n \to \infty$, we have $a_n = \frac{n^{1/\lambda_1}}{b_1^{1/\lambda_1}}(1 + o(1))$ and

$$a_n = \frac{n^{1/\lambda_1}}{b_1^{1/\lambda_1}} + O\left(n^{\frac{\lambda_2+1}{\lambda_1}-1}\right). \tag{3.2}$$

Moreover, for any $\alpha > 0$,

$$(1-y)^{\alpha} = 1 - \alpha y + \frac{1}{2}\left(\alpha^2 - \alpha\right)y^2 + O\left(y^3\right) \qquad \text{as } y \to 0.$$

Thus

$$\begin{aligned}&\left(1 - b_2\frac{a_n^{\lambda_2}}{n} - \frac{R(a_n)}{n}\right)^{1/\lambda_1} = \\ &= 1 - \frac{b_2}{\lambda_1}\frac{a_n^{\lambda_2}}{n} + \frac{1}{2}\left(\frac{1}{\lambda_1^2} - \frac{1}{\lambda_1}\right)b_2^2\frac{a_n^{2\lambda_2}}{n^2} + O\left(\frac{R(a_n)}{n}\right).\end{aligned} \tag{3.3}$$

Substituting this estimate in (3.1) yields

$$a_n = \frac{n^{1/\lambda_1}}{b_1^{1/\lambda_1}} - \frac{1}{\lambda_1}\frac{b_2}{b_1^{\frac{\lambda_2+1}{\lambda_1}}}n^{\frac{\lambda_2+1}{\lambda_1}-1} + O\left(n^{\frac{2\lambda_2+1}{\lambda_1}-2}\right). \tag{3.4}$$

Using this estimate and the fact that $R(a_n) = o\left(a_n^{\lambda_3}\right) = o\left(n^{\lambda_3/\lambda_1}\right)$, we can replace the RHS of (3.3) by

$$1 - \frac{1}{\lambda_1}\frac{b_2}{b_1^{\lambda_2/\lambda_1}}n^{\frac{\lambda_2}{\lambda_1}-1} + \frac{1}{2}\left(\frac{2\lambda_2+1}{\lambda_1^2} - \frac{1}{\lambda_1}\right)\frac{b_2^2}{b_1^{2\lambda_2/\lambda_1}}n^{\frac{2\lambda_2}{\lambda_1}-2} + o\left(n^{\frac{\lambda_3}{\lambda_1}-1}\right), \tag{3.5}$$

which substituted back in (3.4) yields (2.2). ∎

## 4. Proof of Theorem 2

It follows from estimate (2.3) that

$$n = A(a_n) = \frac{a_n}{L(a_n)}\left(1+O\left(\frac{1}{\varphi(a_n)}\right)\right) \qquad (n\to\infty) \tag{4.1}$$

and therefore that

$$a_n = nL(a_n)\left(1+O\left(\frac{1}{\varphi(n)}\right)\right) \qquad (n\to\infty). \tag{4.2}$$

Since $L$ is a slowly oscillating function, we have

$$L\left(a_n\right) = C\left(a_n\right)\exp\left(\int_{x_0}^{a_n}\frac{\eta(t)}{t}\mathrm{d}t\right) = \frac{C\left(a_n\right)}{C(n)}L(n)\exp\left(\int_{n}^{a_n}\frac{\eta(t)}{t}\mathrm{d}t\right). \tag{4.3}$$

Combining (4.2) and (4.3) proves (2.5).

Now, let $\alpha = \alpha(n)$ be the unique positive integer satisfying $2^{\alpha-1}n < < nL(a_n) \le 2^\alpha n$, so that $\alpha = \left\lceil \frac{\log L(a_n)}{\log 2}\right\rceil = \frac{\log L\left(a_n\right)}{\log 2} + \epsilon(n)$, where $0 \le \le \epsilon(n) < 1$. On the one hand, since $\eta(t)$ is decreasing and positive, we have

$$\begin{aligned}
\int_n^{a_n}\frac{\eta(t)}{t}\,dt &\le \int_n^{2n}\frac{\eta(t)}{t}\,dt + \int_{2n}^{2^2n}\frac{\eta(t)}{t}\,dt + \cdots + \int_{2^{\alpha-1}n}^{2^\alpha n}\frac{\eta(t)}{t}\,dt < \\
&< \eta(n)\log 2 + \eta(2n)\log 2 + \cdots + \eta(2^{\alpha-1}n)\log 2 \le \\
&\le \eta(n)\alpha\log 2 = \eta(n)\log 2\left(\frac{\log L\left(a_n\right)}{\log 2} + \epsilon(n)\right).
\end{aligned} \tag{4.4}$$

On the other hand,

$$\begin{aligned}
\int_n^{a_n}\frac{\eta(t)}{t}\,dt &\ge \int_n^{2n}\frac{\eta(t)}{t}\,dt + \int_{2n}^{2^2n}\frac{\eta(t)}{t}\,dt + \cdots + \int_{2^{\alpha-2}n}^{2^{\alpha-1}n}\frac{\eta(t)}{t}\,dt > \\
&> \eta(2n)\log 2 + \eta(4n)\log 2 + \cdots + \eta(2^{\alpha-1}n)\log 2 > \\
&> \eta(2^{\alpha-1}n)(\alpha-1)\log 2 \ge \eta\left(a_n\right)\log 2\left(\frac{\log L\left(a_n\right)}{\log 2} + \epsilon(n)\right).
\end{aligned} \tag{4.5}$$

It follows from (4.3), (4.4) and (4.5) that there exists a function $\delta$ satisfying $\eta(a_n) < \delta(n) < \eta(n)$ for all integers $n \ge x_0$ such that

$$\int_n^{a_n}\frac{\eta(t)}{t}\mathrm{d}t = \delta(n)\log L\left(a_n\right).$$

Combining this result with (2.4), we get

$$L\left(a_n\right)=L(n)^{1/(1-\delta(n))}\left(1+O\left(\frac{1}{\psi(n)}\right)\right). \tag{4.6}$$

Substituting (4.6) in (4.2) proves (2.6). Finally, (2.8) follows easily from (2.7). Indeed, by (4.6), we have

$$L\left(a_n\right)=L(n)^{1+\delta(n)+O\left(\delta^2(n)\right)}. \tag{4.7}$$

We have

$$L(n)^{\delta(n)}\leq L(n)^{\eta(n)}=C(n)^{\eta(n)}\exp\left(\eta(n)\int_{x_0}^{n}\frac{\eta(t)}{t}\mathrm{d}t\right)=e^c+o(1)$$

and

$$L(n)^{\delta(n)}\geq L(n)^{\eta(a_n)}=L\left(a_n\right)^{\eta(a_n)}\left(\frac{L(n)}{L\left(a_n\right)}\right)^{\eta(a_n)}=\left(\frac{L(n)}{L\left(a_n\right)}\right)^{\eta(a_n)}\left(e^c+o(1)\right).$$

Since

$$\log L\left(a_n\right)-\log L(n)=\int_{n}^{a_n}\frac{\eta(t)}{t}\mathrm{d}t\left(1+o(1)\right)=\delta(n)\log L\left(a_n\right)\left(1+o(1)\right),$$

it follows that

$$\left(\frac{L(n)}{L\left(a_n\right)}\right)^{\eta(a_n)}=1+o(1)$$

and thus

$$L(n)^{\delta(n)}=e^c+o(1). \tag{4.8}$$

Moreover, using (4.8),

$$L(n)^{\delta(n)^2}\leq\left(L(n)^{\eta(n)}\right)^{\delta(n)}=\left(e^c+o(1)\right)^{\delta(n)}=1+o(1).$$

Combining this last result with (4.8) and (4.7) gives

$$L\left(a_n\right)=L(n)\left(e^c+o(1)\right)\qquad(n\to\infty).$$

Substituting this estimate in (4.2) yields

$$a_n=\left(e^c+o(1)\right)nL(n).$$

■

## 5. Applications of Theorem 1

We provide two applications.

First, we shall prove that there exists a positive constant $C$ such that, as $n \to \infty$,

$$\wp_n = \left(\frac{\zeta(3)}{\zeta(3/2)}\right)^2 n^2 - 2\frac{\zeta(2/3)}{\zeta(2)}\left(\frac{\zeta(3)}{\zeta(3/2)}\right)^{\frac{8}{3}} n^{\frac{5}{3}} + \tag{5.1}$$
$$+\frac{7}{3}\left(\frac{\zeta(2/3)}{\zeta(2)}\right)^2\left(\frac{\zeta(3)}{\zeta(3/2)}\right)^{\frac{10}{3}} n^{\frac{4}{3}} + R_0(n),$$

where

$$R_0(n) \ll n^{4/3}\exp\left(-C\left(\log n\right)^{3/5}\left(\log_2 n\right)^{-1/5}\right). \tag{5.2}$$

In order to prove (5.1), we first recall the 1958 result of Bateman and Grosswald [1]

$$P_2(x) := \#\{n \le x : n \text{ powerful}\} = \frac{\zeta(3/2)}{\zeta(3)}x^{1/2} + \frac{\zeta(2/3)}{\zeta(2)}x^{1/3} + R(x), \tag{5.3}$$

where

$$R(x) \ll x^{1/6}\exp\left(-C\left(\log x\right)^{3/5}\left(\log_2 x\right)^{-1/5}\right), \tag{5.4}$$

which is the best known error term and is due to Suryanarayana and Sitaramachandra Rao [14].

Then setting $\lambda_1 = 1/2$, $\lambda_2 = 1/3$, $\lambda_3 = 1/6$, $b_1 = \frac{\zeta(3/2)}{\zeta(3)}$ and $b_2 = \frac{\zeta(2/3)}{\zeta(2)}$ in Theorem 1, keeping track of the explicit error term given by (5.2), estimate (5.1) follows.

As a second application, we consider the general case of $k$-full numbers.

Recall that, given an integer $k \ge 2$, we say that a positive integer $n$ is said to be *$k$-full* if $p \mid n$ implies that $p^k \mid n$. We denote by $P_k(x)$ the number of $k$-full integers $\le x$ and by $\wp_{n,k}$ the $n$-th $k$-full number.

Ivić and Shiu [7] obtained that

$$P_k(x) = \gamma_{0,k}x^{1/k} + \gamma_{1,k}x^{1/(k+1)} + \cdots + \gamma_{k-1,k}x^{1/(2k-1)} + \Delta_k(x), \tag{5.5}$$

where the constants $\gamma_{i,k}$ are given explicitly and $\Delta_k(x)$ is a suitable error term.

Using (5.5) in the particular case $k = 3$ and Theorem 1, we can prove that there exists a positive constant $C$ and constants $A_3$, $A_4$ and $A_5$ such that, as $n \to \infty$,

$$\wp_{n,3} = (A_3)^{-3} n^3 - 3\,(A_3)^{\frac{-15}{4}} A_4 n^{\frac{11}{4}} - 3\,(A_3)^{\frac{-18}{5}} A_5 n^{\frac{13}{5}} + \\ + \frac{21}{4} (A_3)^{\frac{-9}{2}} (A_4)^2 n^{\frac{5}{2}} + R(n),$$

where

$$R(n) \ll n^{\frac{19}{8}} \exp\left(-C\,(\log n)^{3/5} (\log_2 n)^{-1/5}\right).$$

**Remark 1.** Observe that explicit values for the constants $A_i$ were obtained by Shiu [12]. Moreover, for $k \geq 4$, as $n \to \infty$, one can prove that

$$\text{(5.6)}\quad \wp_{n,k} = \left(\frac{n}{\gamma_{0,k}}\right)^k - k\frac{\gamma_{1,k}}{(\gamma_{0,k})^{\frac{k(k+2)}{k+1}}} n^{\frac{k^2+k-1}{k+1}} - k\frac{\gamma_{2,k}}{(\gamma_{0,k})^{\frac{k(k+3)}{k+2}}} n^{\frac{k^2+2k-2}{k+2}} + R_k(n),$$

where $R_k(n) \ll n^{\frac{k^2+k-2}{k+1}}$.

**Remark 2.** Observe that explicit values for the constants $\gamma_{i,k}$ are given in Bateman and Grosswald [1] and Erdős and Szekeres [6]. Moreover, for $k > 4$, additional terms on the right hand side of (5.6) can be provided.

## 6. Applications of Theorem 2

We provide three applications.

1. Fix a positive integer $k$ and let

$$A = A_k = \{n \in \mathbb{N} : \omega(n) = k\} = \{a_n : n \in \mathbb{N}\},$$

where $\omega(n)$ stands for the number of distinct prime factors of $n$. It is well known that, as $x \to \infty$,

$$A(x) = \frac{x}{L(x)}\left(1 + O\left(\frac{1}{\log\log x}\right)\right),$$

where $L(x) = \frac{(k-1)!\log x}{(\log_2 x)^{k-1}}$ (see for instance Theorem 10.4 in the book of De Koninck and Luca [5]). It follows from Theorem 2 that

$$a_n = n\frac{(k-1)!\log n}{(\log_2 n)^{k-1}}\left(1 + O\left(\frac{1}{\log\log n}\right)\right) \qquad (n \to \infty).$$

2. Consider the set $A$ of those integers $n \geq 2$ such that $\lfloor z(n) \rfloor - \lfloor z(n-1) \rfloor = 1$, where $z(n) = n/e^{\sqrt{\log n}}$. Using a computer, we easily obtain the first elements of $A$, so that we may write

$$A = \{3, 9, 16, 24, 33, 42, 51, 61, 71, 82, 93, \ldots\} = \{a_n : n \in \mathbb{N}\}.$$

Clearly $|A(x) - z(x)| \leq 1$ for all $x \geq 2$. Then, since condition (2.7) of Theorem 2 is satisfied with $c = 1/2$, we get from (2.8) that

$$a_n = \left(\sqrt{e} + o(1)\right) n e^{\sqrt{\log n}} \qquad (n \to \infty).$$

3. Let $W = \{a_1, a_2, \ldots\}$ be the set of those positive integers which can be written as the sum of two squares. It has been known since Euler that a positive integer can be represented as a sum of two squares if and only if each of its prime factors of the form $4k+3$ occurs with an even power, so that

$$W = \{2, 5, 8, 10, 13, 17, 18, 20, 25, 26, 29, 32, 34, 37, 40, 41, 45, 50, \ldots\}.$$

In 1908, Landau [10] showed that

$$W(x) = (B + o(1)) \frac{x}{\sqrt{\log x}} \qquad (x \to \infty),$$

where $B = \dfrac{1}{\sqrt{2}} \displaystyle\prod_{p \equiv 3 \pmod 4} \left(\sqrt{1 - \frac{1}{p^2}}\right)^{-1} = 0.7642236\ldots$. In 1986, Shiu [13] showed that

$$W(x) = \frac{Bx}{\sqrt{\log x}} \left(1 + O\left(\frac{1}{\log x}\right)\right). \tag{6.1}$$

Since one can show that

$$L(x) := \frac{1}{B}\sqrt{\log x} = \frac{1}{B} \exp\left\{\int_e^x \frac{1}{2\log t} \frac{dt}{t}\right\}, \tag{6.2}$$

it follows from (6.1) and (6.2) that the corresponding functions $\varphi(x)$, $\eta(x)$, $C(x)$ and $\psi(x)$ from the statement of Theorem 2 are given by

$$\varphi(x) = \log x, \qquad \eta(x) = \frac{1}{2\log x}, \qquad C(x) = 1/B, \qquad \psi(x) = \infty.$$

Hence, it follows from (2.5) that

$$a_n = \frac{1}{B} n \sqrt{\log a_n} \left(1 + O\left(\frac{1}{\log n}\right)\right).$$

Thus, using the logarithm on this formula, one can improve it to

$$a_n = \frac{1}{B} n \sqrt{\log n} \left(1 + \frac{1}{4} \frac{\log\log n}{\log n} + O\left(\frac{1}{\log n}\right)\right) \qquad (n \to \infty).$$

## References


[1] **Bateman, P. and E. Grosswald**, On a theorem of Erdős and Szekeres, *Illinois J. Math.*, **2** (1958), 88–98.

[2] **Bojarincev, A.E.,** Asymptotic expressions for the $n$th composite number, *Ural. Gos. Univ. Mat. Zap.*, **6** (1987), 21–43.

[3] **Cipolla, M.,** La determinazione dell' $n^{imo}$ numero primo, *Rend. Acad. Sci. Fis. Mat. Napoli*, **8** (1902), 132–166.

[4] **Copil, V. and L. Panaitopol,** Properties of non powerful numbers, *J. Inequal. Pure Appl. Math.*, **9** (2008), no. 1, Article 29, 8.

[5] **De Koninck, J.-M. and F. Luca,** *Analytic Number Theory: Exploring the Anatomy of Integers*, Graduate Studies in Mathematics, Vol. 134, American Mathematical Society, Providence, Rhode Island, 2012.

[6] **Erdős, P. and G. Szekeres,** Über die Anzahl der Abelschen Gruppen gegebener Ordnung und über ein verwandtes zahlentheoretisches Problem (in German), *Acta Litt. Sci. Szeged*, (1934), 95–102.

[7] **Ivić, A. and P. Shiu,** The distribution of powerful integers, *Illinois J. Math.*, **26** (1982), no. 4, 576–590.

[8] **Karamata, J.,** Sur un mode de croissance régulière des fonctions, *Mathematica (Cluj)*, **4** (1930), 38–53.

[9] **Karamata, J.,** Sur un mode de croissance régulière. Théorèmes fondamentaux, *Bull. Soc. Math. France*, **61** (1933), 55–62.

[10] **Landau, E.,** Über die Einteilung der positiven ganzen Zahlen in vier Klassen nach der Mindestzahl der zu ihrer additiven Zusammensetzung erforderlichen Quadrate, *Arch. Math. Phys.*, **(3)13** (1908), 305-312; Collected works, Vol. 4, Thales Verlag, Essen, 1986, 59-66.

[11] **Seneta, E.,** *Regularly Varying Functions*, Springer-Verlag, Berlin, 1976.

[12] **Shiu, P.,** The distribution of cube-full numbers, *Glasgow Math. J.*, **33** (1991), no. 3, 287–295.

[13] **Shiu, P.,** Counting sums of two squares: the Meissel–Lehmer method, *Math. Comp.*, **47** (1986), no. 175, 351–360.

[14] **Suryanarayana, D. and R.R. Sitaramachandra,** The distribution of square-full integers, *Arkiv für Matematik*, **11** (1973), 195–201.

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