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RESULTS ON CLASSES OF FUNCTIONAL EQUATIONS
TRIBUTE TO ANTAL JÁRAI

by János Aczél and Che Tat Ng

While *individual* noncomposite functional equations in several variables had been solved at least since d'Alembert 1747 [9] and Cauchy 1821 [8], results on broad *classes* of such equations began appearing in the 1950's and 1960's. On general *methods of solution* see e.g. Aczél [1] and for *uniqueness of solutions* Aczél [2, 3], Aczél and Hosszú [6], Miller [20], Ng [21, 22], followed by several others. – Opening up and cultivating the field of *regularization* is *mainly Járai's achievement.* By regularization we mean assuming weaker regularity conditions, say measurability, of the unknown function and proving differentiability of several orders, for whole classes of functional equations. Differentiability of the unknown function(s) in the functional equation often leads to differential equations that are easier to solve.

For example, in Aczél and Chung [5] it was shown that locally Lebesgue integrable solutions of the functional equation

$$\sum_{i=1}^{n} f_i(x+\lambda_i y) = \sum_{k=1}^{m} p_k(x)q_k(y)$$

holding for x, y on open real intervals, with appropriate independence between the functions, are in fact differentiable infinitely many times. The differentiable solutions are then extracted using induced differential equations. Járai [11] showed that Lebesgue measurability and ordinary linear independence are sufficient to lead to the same solutions.

Aczél [4] called attention to some unsolved problems in the area of functional equations. One concerned Hilbert's fifth problem. Járai [15] formulated a problem that falls within that general call for non-composite functional equations in multiple variables. Here we exhibit the intricate problem he formulated and the sequence of results that led to its solution, and make references to his comprehensive book Járai [16].

Problem. *Let T and Z be open subsets of $\mathbb{R}^s$ and $\mathbb{R}^m$, respectively, and let D be an open subset of $T \times T$. Let $f : T \to Z$, $g_i : D \to T$ $(i = 1, ..., n)$, and $h : D \times Z^n \to Z$ be functions. Suppose that*

$$f(t) = h(t, y, f(g_1(t, y)), ..., f(g_n(t, y))) \quad \textit{for all} \quad (t, y) \in D\,;$$

$$h \quad \textit{is analytic}\,;$$

$g_1, ..., g_n$ *are analytic and for each* $t \in T$ *there exists a* y *for which*

$$(t,y) \in D \quad \text{and} \quad \frac{\partial g_i}{\partial y} \quad \text{has rank} \quad s \quad \text{for each} \quad i = 1, ..., n.$$

Is it true that every solution f *which is measurable, or has the Baire property, is also analytic?*

He proposed some incremental steps which may be taken to address the problem:

(I) Measurability implies continuity.

(II) Almost open solutions are continuous.

(III) Continuous solutions are locally Lipschitz.

(IV) Locally Lipschitz solutions are continuously differentiable.

(V) All p-times continuously differentiable solutions are $(p+1)$-times differentiable.

(VI) Infinitely many times differentiable solutions are analytic.

In [19] Járai and Székelyhidi outlined the above steps and gave a survey on the advances made. Many historic attributions were made to contributors in the field. Ng [23] contains results concerning the functional equation

$$f(x) + g(y) = h(T(x,y))$$

with given T. It is shown that under suitable assumptions, local boundedness of f implies the continuity of g.

Járai published a sequence of papers obtaining impressive results about that problem. [12] contains results regarding (I), (II), (IV), (V), and partially about (III). Step (III) is obtained for one variable in [13] and is generalized in [14]. In [15] Járai obtained the following result on the problem formulated above.

Theorem. *Suppose that the conditions of the Problem are satisfied and suppose that* f *has locally essentially bounded variation. Then* f *is infinitely many times differentiable.*

[16] contains, in Section 1, a summary account about the problem. We include some of it (abbreviated).

Theorem. (i) *If* h *is continuous and the functions* g_i *are continuously differentiable then every solution* f*, which is Lebesgue measurable or has Baire property, is continuous.*

(ii) *If* h *and* g_i *are* p *times continuously differentiable, then every almost everywhere differentiable solution* f *is* p *times continuously differentiable.*

(iii) *If* h *and* g_i *are* $\max\{2,p\}$ *times differentiable and there exists a compact subset* C *of* T *such that for each* $t \in T$ *there exists a* $y \in T$ *satisfying* $g_i(t,y) \in C$*, besides the other stated rank condition on* g_i*, then every solution* f*, which is Lebesgue measurable or has the Baire property, is* p *times continuously differentiable* $(1 \le p \le \infty;\ i = 1, ..., n)$.

Járai has deep insights and knowledge in the field of real analysis. He used the theorems reported in Giusti [10] swiftly, made fine and technical adaptations when necessary to get the above strong results.

In his book [16] many regularization theorems by him and others are assembled in a well organized way. For the convenience of the readers he has given several examples to illustrate how his general results can be applied to known functional equations. He devised and proved a general transfer principle which makes it possible to apply theorems concerning problems having only one unknown function also for cases with several unknown functions. A good example amongst many is the following

Theorem. *Let* $\alpha \neq \beta$ *be fixed real numbers,* $f, g_1, g_2 :]0,1[\to \mathbb{R}$*. Suppose that the functional equation*

$$\begin{aligned} &f(x) + (1-x)^{\alpha} g_1(u/(1-x)) + (1-x)^{\beta} g_2(u/(1-x)) \\ &= f(u) + (1-u)^{\alpha} g_1(x/(1-u)) + (1-u)^{\beta} g_2(x/(1-u)) \end{aligned}$$

is satisfied for all $x, u \in]0,1[$ *with* $x + u \in]0,1[$*. If the functions* f, g_1, g_2 *are Lebesgue measurable then they are* $\mathcal{C}^{\infty}$.

He offered readers some details which precede the applications of his regularization theorems. The functional equation has its source in the study of symmetric divergences and distance measures and the differentiable solutions have been reported by Sander [25]. A more elaborate example is their joint work in [18] connected to the Weierstrass sigma function (as in [7]). They extended the results of M. Bonk [7] on the functional equation

$$\chi(u+v)\phi(u-v) = \sum_{\nu=1}^{k} f_{\nu}(u) g_{\nu}(v)$$

and treated it under weaker regularity assumptions.

Section 16 of the book contains results on (VI), analyticity. Járai's results as well as those of Páles [24] are covered. In Járai, Ng and Zhang [17] a composite type functional equation is solved under different regularity assumptions. The uniqueness theorem of Ng [22] is applied to obtain continuous solutions in one case, and the differentiation steps are used to extract the differentiable solutions in another case.

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ANTAL JÁRAI HAS TURNED 60

by Zoltán Daróczy

Antal Járai was born on 25th August, 1950 in Biharkeresztes, Hungary. He attended secondary school in Debrecen. Then he studied mathematics at Kossuth Lajos University (Debrecen) between 1969–74. After graduation he started his professional career at the Department of Analysis in the Institute of Mathematics at Kossuth University. In 1976 he wrote his thesis "On Measurable Solutions of Functional Equations" and received doctoral degree. Then he held various positions as a researcher at the University in Debrecen. In the period 1992–1997 he had a research position at the University of Paderborn (Germany). Since 1997, he has been a professor of Eötvös Loránd University (Budapest). He earned candidate degree in 1990 and the doctor of the Hungarian Academy of Sciences degree in 2001.

Antal Járai's scientific activities cover a wide range of various fields. He himself considers the following areas as his fields of interest:

- functional equations,
- measure theory,
- system programming,
- computational number theory and computer algebra,
- generalized number systems.

The list above demonstrates that Antal Járai is both modern mathematician and computer scientist at the same time.

The writer of this laudation, having been his teacher and scientific supervisor in the past and being his friend and colleague now, is biased in his appreciation. I remember that student Járai was characterized by the say "his brain like a piece sponge, as it absorbs everything; on the other hand, it is sharp like a knife as he is fast and creative in addressing any problem". Antal Járai is considered to be a valuable member of the Debrecen school of functional equations, whose scientific results cannot be missed by experts in this field. Furthermore, the years spent in Paderborn play a significant role in his scientific contribution to computer science, which has ripened by now and so earned worldwide reputation. Besides these scientific achievements, his work as an educator is admirably colourful and successful. His textbooks and course-books are widely recognized in Hungary.

Most of his scientific research work concerns the theory of functional equations. In the paper "Tribute to Antal Járai", János Aczél and Che Tat Ng give a due appreciation of his scientific achievements in this field of mathematics. In measure theory he has outstanding results concerning the invariant extension of the Haar-measure and generalizations of the Steinhaus theorem. He did pioneer work in the study of interval filling sequences and in complex and higher dimensional number systems.

He started to do research in computer science as early as 1982. It is appropriate to say that besides his extensive theoretical knowledge of mathematics, he has also demonstrated his talent in solving practical problems. He wrote more than twenty system programs as an entrepreneur. The included translation programs, database management systems, floating point arithmetic algorithms and time sharing systems. His programs, some of which proved to be the fastest on the given hardware all over the world, have been installed at about hundred sites.

During the years he spent at the Universität GH Paderborn (1992–1997) as a member of Karl-Heinz Indlekofer's team, they achieved more than ten world records. Elaborating on and continuing these researches, his team in Hungary has succeeded in gaining five more world records. Working with highly efficient computational methods and elliptic curves for prime testing, he has reached outstanding results in computer algebra as well.

Antal Járai is a renaissance figure of our age. He is interested in physics, chemistry and electronics as well as in certain field of geology and biology. Most of all he is a prominent developer of mathematics and computer science.

His sons, Antal and Zoltán, born in his first marriage, are stepping in their father's footsteps. He has a daughter, Mariann, born in his second marriage. In difficult times, his wife, Ilona assisted him in his enterprise as a skilled software developer. It is a pleasant personal memory from the summer of 1985, when two couples (them and us) were travelling together to the 23^{rd} International Symposium on Functional Equations (ISFE) in Gargnano, Italy (June $2^{\text{nd}} - 11^{\text{th}}$) in a Trabant car. On the way there and back we stayed in tents at camping sites.

Antal Járai has been granted the following awards: Pro Universitate (Kossuth Lajos University, Debrecen, 1974), "Grünwald Géza Award" (Bolyai Mathematical Society, 1979), Ministry award of Ministry of Culture (1990), "For outstanding contribution to the conference" (ISFE, 1994), Award of Hungarian Academy of Science (2000), "Kalmár Award" (2008), Knight Cross, the Order of Merit of the Hungarian Republic (2010).

My dear student, friend and colleague, happy 60th birthday to you. I also wish you and your family good health and spirits.

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ON THE THEOREM OF H. DABOUSSI OVER THE GAUSSIAN INTEGERS

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Dedicated to Professor Antal Járai on his 60th birthday

Abstract. Some analogues of the theorem of Daboussi over the set of Gaussian integers are investigated.

1. Introduction

Let $c, c_1, c_2, \ldots, K, K_1, K_2, \ldots$ be positive constants, not necessarily the same at every occurrence. Let $\mathcal{M}$ be the set of complex valued multiplicative functions and $\mathcal{M}_1$ be the set of those $g \in \mathcal{M}$ for which additionally $|g(n)| \leq 1$ $(n \in \mathbb{N})$ holds as well. Let $e(\alpha) := e^{2\pi i\alpha}$.

A famous theorem of H. Daboussi published in the paper written jointly with H. Delange in [2] asserts that

$$\sup_{f \in \mathcal{M}_1} \left| \frac{1}{x} \sum_{n \leq x} f(n) e(n\alpha) \right| = \varrho_x \to 0 \qquad (x \to \infty), \tag{1.1}$$

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whenever α is an irrational number. This famous theorem has been generalized in different aspects in [1], [3]–[20]. In [2] the following assertion was proved:

Let S be an arithmetical function satisfying the following conditions:

(i) *S is almost-periodic B^1,*

(ii) *the Fourier series of S is $\lambda + \sum \lambda_\nu e(\alpha_\nu n)$, where all the α_ν are irrational.*

Then, as x tends to infinity, we have

$$\sup_{f\in\mathcal{M}_1}\left|\frac{1}{x}\sum_{n\le x} f(n)S(n) - \frac{1}{\lambda}\sum f(n)\right| \le \varrho_x(S),$$

$\varrho_x(S) \to 0$ *as* $(x\to\infty)$.

In [20] the following theorem is proved.

Let $k \ge 1$ be fixed, $J_1,\dots,J_k \subseteq [0,1)$ be such sets which are the union of finitely many intervals. Let $P_1(x),\dots,P_k(x)$ be non-constant real valued polynomials,

$$Q_{m_1,\dots,m_k}(x) = m_1P_1(x) + \dots + m_kP_k(x)$$

for $m_1,\dots,m_k \in \mathbb{Z}$.

Assume that $Q_{m_1,\dots,m_k}(x) - Q_{m_1,\dots,m_k}(0)$ has at least one irrational coefficient for every $m_1,\dots,m_k \in \mathbb{Z}$, except when $m_1 = \ldots = m_k = 0$.

Let

$$S := \{n \mid n\in\mathbb{N},\quad \{P_l(n)\}\in J_l,\quad l=1,\dots,k\}.$$

Let λ be the Lebesgue measure.

Theorem A. *Under the conditions stated for $P_1,\dots,P_k,J_1,\dots,J_k$ we have*

$$\sup_{g\in\mathcal{M}_1}\left|\frac{1}{x}\sum_{\substack{n\le x\\ n\in S}} g(n) - \frac{\lambda(J_1)\ldots\lambda(J_k)}{x}\sum_{n\le x} g(n)\right| = \tau_x, \tag{1.2}$$

$\tau_x \to 0$ *as* $x\to\infty$.

By using the same method and Theorem B we can prove

Theorem 1. *Let $J_1,\dots,J_k,P_1,\dots,P_k,S$ be as above. Let P be a non-constant real valued polynomial.*

Let $R_{m_0,m_1,\dots,m_k}(x) = m_0P(x) + Q_{m_1,\dots,m_k}(x)$. Assume that

$$R_{m_0,m_1,\dots,m_k}(x) - R_{m_0,m_1,\dots,m_k}(0)$$

has at least one irrational coefficient for every $m_0, m_1, \ldots, m_k$ *except the case when* $m_0 = m_1 = \ldots = m_k = 0$.

Then

$$\sup_{g\in\mathcal{M}_1} \frac{1}{x} \left| \sum_{\substack{n\le x\\ n\in S}} g(n)e(P(n)) \right| = \varrho_x \to 0, \qquad as \quad x\to\infty. \tag{1.3}$$

ϱ_x *may depend on* S *and on* P.

Theorem B. (See [7].) (1.3) *is true, if* $S = \mathbb{N}$.

Applying Theorem A for $g(n) = 1$ we obtain that

$$\frac{1}{x}\#\{n \le x \mid n \in S\} \to \lambda(J_1)\ldots\lambda(J_k).$$

From Theorem 1, by using Weyl's criterion for uniformly distributed sequences we get

Theorem 2. *Let* $J_1, \ldots, J_k, P, P_1, \ldots, P_k, S$ *as in Theorem* 1. *Let* $\mathcal{A}$ *be the set of additive arithmetical functions,* $S = \{t_1, t_2, \ldots\}$, $t_j < t_{j+1}$ $(j = = 1, 2, \ldots)$, $\xi_n(f) := f(t_n) + P(t_n)$ $(n = 1, \ldots)$,

$$\Delta_N(f \mid S) :=$$

$$:= \sup_{[\alpha,\beta)\subseteq[0,1)} \left| \frac{1}{N}\#\{\xi_n(f) \mod 1 \in [\alpha,\beta], n \in N\} - (\beta-\alpha) \right|. \tag{1.4}$$

Then

$$\sup_{f\in\mathcal{A}} \Delta_N(f|S) = \varrho_N \to 0 \qquad as \quad N\to\infty. \tag{1.5}$$

ϱ_N *may depend on* S.

Let $\mathcal{N}_k$ be the set of the integers the number of the prime power factors of which is k. Let $N_k(x)$ be the size of $n \le x$, $n \in \mathcal{N}_k$. In our paper [10] we proved

Theorem C. *Let* $0 < \delta(< 1)$ *be an arbitrary constant, and* α *be an irrational number. Then*

$$\lim_{x\to\infty} \sup_{\delta\le\frac{k}{\log\log x}\le 2-\delta} \sup_{f\in\mathcal{M}_1} \frac{1}{N_k(x)} \left| \sum_{\substack{m\le x\\ m\in\mathcal{N}_k}} f(m)e(m\alpha) \right| = 0. \tag{1.6}$$

The proof depends on an important assertion due to Dupain, Hall, Tenenbaum [4], namely that

$$(1.7)\qquad \sup_{\frac{k}{\log\log x}\le 2-\delta} \frac{1}{N_k(x)}\left|\sum_{\substack{m\le x\\ m\in\mathcal{N}_k}} e(m\alpha)\right| \to 0 \qquad \text{as} \quad x\to\infty.$$

Theorem 3.

1.) *Let* $P(n)=\alpha n$, $P_j(n)=\alpha_j n$, $(j=1,\dots,k)$, $J_1,\dots,J_k$ *and* S *as earlier. Assume that* $m\alpha+m_1\alpha_1+\dots+m_k\alpha_k$ *is irrational for every nontrivial choice of* $m,m_1,\dots,m_k$. *Let* $S_k(x)=\#\{n\le x\mid n\in\mathcal{N}_k,\ n\in S\}$.

Then

$$(1.8)\qquad \lim_{x\to\infty}\sup_{\delta\le\frac{k}{\log\log x}\le 2-\delta}\ \sup_{f\in\mathcal{M}_1}\frac{1}{S_k(x)}\left|\sum_{\substack{n\le x\\ n\in\mathcal{N}_k\cap S}} f(n)e(n\alpha)\right| = 0.$$

2.) *Let* $P_1,\dots,P_k,J_1,\dots,J_k$ *and* S *as earlier. Assume that* $m_1\alpha_1+\dots+$ $+m_k\alpha_k$ *is irrational for every nontrivial choice of* $m_1,\dots,m_k$. *Then*

$$(1.9)\quad \lim_{x\to\infty}\sup_{\delta\le\frac{k}{\log\log x}\le 2-\delta}\ \sup_{f\in\mathcal{M}_1}\left|\frac{1}{S_k(x)}\sum_{\substack{n\le x\\ n\in\mathcal{N}_k\cap S}} f(n) - \frac{1}{N_k(x)}\sum_{\substack{n\le x\\ n\in\mathcal{N}_k}} f(n)\right| = 0.$$

Since the Theorems 1, 2, 3 can be deduced from already published papers by the method used in [20], we omit the proofs of them. In the next section we formulate and prove Theorem 4.

2.

Let $\mathbb{Z}[i]$ be the ring of Gaussian integers, $\mathbb{Z}^*[i]=\mathbb{Z}[i]\setminus\{0\}$ be the multiplicative group of nonzero Gaussian integers.

Let χ be such an additive character on $\mathbb{Z}[i]$, for which $\chi(1)=e(A)$, $\chi(i)=$ $=e(B)$. Let $\mathcal{K}_1$ be the set of multiplicative functions $g:\mathbb{Z}^*[i]\to\mathbb{C}$ satisfying $|g(\alpha)|\le 1\quad(\alpha\in\mathbb{Z}^*[i])$. Let W be the union of finitely many convex bounded domain in $\mathbb{C}$. In our paper [11] written jointly with N.L. Bassily and J.-M. De Koninck we proved

Theorem D. *Assume that at least one of A or B is irrational. Then*

$$\lim_{x\to\infty}\sup_{g\in\mathcal{K}_1}\frac{1}{|xW|}\left|\sum_{\beta\in xW} g(\beta)\chi(\beta)\right| = 0. \tag{2.1}$$

Let $I=[0,1)\times[0,1)$, $S=S_1\cup\ldots\cup S_r\subseteq I$, where S_j are domains the boundary of which is a rectifiable continuous curve for every j. For some small $\Delta>0$ let

$$S^{(-\Delta)} = \{(u,v)\mid [u-\Delta,u+\Delta]\times[v-\Delta,v+\Delta]\subseteq S\},$$
$$S^{(+\Delta)} = \{(u,v)\mid [u-\Delta,u+\Delta]\times[v-\Delta,v+\Delta]\cap S\neq 0\}.$$

Let

$$f(x,y)=\begin{cases}1, & \text{if } (x,y)\in S\\ 0, & \text{if } (x,y)\in I\setminus S,\end{cases} \tag{2.2}$$

and let us extend the definition of f over $\mathbb{R}^2$ by

$$f(x+k,y+l)=f(x,y) \qquad (k,l\in\mathbb{Z}).$$

Let $\sum_{m,n\in\mathbb{Z}} a_{m,n}e(mx+ny)$ be the Fourier-series of $f(x,y)$. Let $\Delta>0$ be so small that $S^{(+\Delta)}\subseteq I$, and

$$f_\Delta(x,y) := \frac{1}{(2\Delta)^2}\int_{-\Delta}^{\Delta}\int_{-\Delta}^{\Delta} f(x+u)f(y+v)\,\mathrm{d}u\,\mathrm{d}v. \tag{2.3}$$

Since

$$\kappa(n) := \frac{1}{2\Delta}\int_{-\Delta}^{\Delta} e(nu)\,\mathrm{d}u = \frac{1}{4\pi i n\Delta}(e(n\Delta)-e(-n\Delta))$$

if $n\neq 0$, and $\kappa(0)=1$, therefore the Fourier coefficients $b_{m,n}$ of f_Δ are

$$b_{m,n} = a_{m,n}\kappa(m)\cdot\kappa(n).$$

Assume that for some $\delta>0$,

$$|a_{m,n}|\le c\left(\frac{1}{1+|m|^\delta}\right)\left(\frac{1}{1+|n|^\delta}\right), \tag{2.4}$$

c is a constant. Thus

$$|b_{m,n}|\le |a_{m,n}|\min\left(1,\frac{2}{|m|\Delta}\right)\min\left(1,\frac{2}{|n|\Delta}\right). \tag{2.5}$$

It is clear that $f_\Delta(u,v) = 1$ if $(u,v) \in S^{(-\Delta)}$, and $f_\Delta(u,v) = 0$ if $(u,v) \in$ $\in I \setminus S^{(+\Delta)}$.

Let $z = u + iv \in \mathbb{C}$. The fractional part of z is defined as $\{z\} = \{u\} + i\{v\}$.

Theorem 4. *Let $\gamma_j = \xi_j + i\eta_j$ $(j = 1, \dots, k)$ be distinct nonzero numbers, $\mathcal{T} = \{\beta \mid \beta \in \mathbb{Z}[i], \{\gamma_j\beta\} \in S,\ j = 1, \dots, k\}$. Assume that S satisfies the conditions stated above. Assume that $\xi_1, \dots, \xi_k, \eta_1, \dots, \eta_k$ are linearly independent over $\mathbb{Q}$. Then*

$$\lim_{x\to\infty} \sup_{g\in\mathcal{K}_1} \left| \frac{1}{|xW|} \sum_{\substack{\beta\in xW\\ \beta\in\mathcal{T}}} g(\beta) - \frac{a_{0,0}^k}{|xW|} \sum_{\beta\in xW} g(\beta) \right| = 0. \tag{2.6}$$

Here $a_{0,0,} = \lambda(S)$ =Lebesgue measure of S.

Theorem 5. *Let $S, \gamma_j, \mathcal{T}$ be as above, $\chi(u+iv) = e(Au+Bv)$. Let $\mathcal{L}$ be the lattice $\{m_1\xi_1 + \dots + m_k\xi_k + n_1\eta_1 + \dots n_k\eta_k\}$. Assume that either $nA \notin \mathcal{L}$ for $n \in \mathbb{Z}\setminus\{0\}$ or $nB \notin \mathcal{L}$ for $n \in \mathbb{Z}\setminus\{0\}$. Then*

$$\lim_{x\to\infty} \sup_{g\in\mathcal{K}_1} \left| \frac{1}{|xW|} \sum_{\substack{\beta\in xW\\ \beta\in\mathcal{T}}} g(\beta)\chi(\beta) \right| = 0. \tag{2.7}$$

Proof of Theorem 4. First we observe that

$$\begin{aligned} \#\{\beta \in xW \mid \{\gamma_j\beta\} \in S^{(+\Delta)} \setminus S^{(-\Delta)}\} &\le \\ \le c_1\lambda(S^{(+\Delta)} \setminus S^{(-\Delta)})\lambda(xW),& \end{aligned} \tag{2.8}$$

and that $\lambda(S^{(+\Delta)} \setminus S^{(-\Delta)}) \le c_2\Delta$. c_2 may depend on S. Let $F(u+iv) =$ $= f(u,v)$, $F_\Delta(u+iv) = f_\Delta(u,v)$. In this notation

$$\begin{aligned} \sum_{\substack{\beta\in xW\\ \beta\in\mathcal{T}}} g(\beta) &= \sum_{\beta\in xW} g(\beta)F(\beta\gamma_1)\dots F(\beta\gamma_k) = \\ &= \sum_{\beta\in xW} g(\beta)F_\Delta(\beta\gamma_1)\dots F_\Delta(\beta\gamma_k) + \mathcal{O}(\Delta\lambda(xW)). \end{aligned} \tag{2.9}$$

Let K be so large that

$$\sum_{n\in\mathbb{Z}} \sum_{|m|\ge K} |b_{m,n}| + \sum_{|n|\ge K} \sum_{m} |b_{m,n}| \le \Delta. \tag{2.10}$$

Since $\sum b_{m,n}$ is absolutely convergent, therefore such a K exists. (See (2.5).)

Let

$$F_\Delta^{(K)}(u+iv) = \sum_{\substack{|m|\le K\\ |n|\le K}} b_{m,n} e(mu+nv). \tag{2.11}$$

Since

$$|F_\Delta(u+iv) - F_\Delta^{(K)}(u+iv)| \le \Delta,$$

from (2.9) we have

$$\sum_{\substack{\beta\in xW\\ \beta\in\mathcal{T}}} g(\beta) = \sum_{\substack{m_1,\dots,m_k\\ n_1,\dots,n_k}}{}^{*} b_{m_1,n_1}\dots b_{m_k,n_k} \sum_{\beta\in xW} g(\beta)\chi_{m_1,\dots,n_k}(\beta).$$

The star indicates that we sum over those m_j, n_j for which $|m_j|\le K$, $|n_j|\le$ $\le K$ $(j=1,\dots,k)$, where $\chi_{m_1,\dots,n_k}(\beta) = e(\lambda \mathrm{Re}\,\beta + \mu \mathrm{Im}\,\beta)$,

$$\lambda = \sum_{j=1}^{k}(m_j\xi_j + n_j\eta_j), \quad \mu = \sum_{j=1}^{k}(n_j\xi_j - m_j\eta_j).$$

From the assumption of the theorem we have that either λ or μ is irrational, consequently, by Theorem D we have that

$$\sum_{\substack{\beta\in xW\\ \beta\in\mathcal{T}}} g(\beta) = a_{0,0}^k \sum_{\beta\in xW} g(\beta) + o_x(|xW|) + \mathcal{O}(\Delta|xW|).$$

Hence we obtain that

$$\lim_{x\to\infty} \sup_{g\in\mathcal{K}_1} \left| \frac{1}{|xW|} \sum_{\substack{\beta\in xW\\ \beta\in\mathcal{T}}} g(\beta) - \frac{a_{0,0}^k}{|xW|} \sum_{\beta\in xW} g(\beta) \right| \le c\Delta.$$

Since Δ is arbitrary, therefore our theorem is true. ■

The proof of Theorem 5 is similar. We omit it.

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ON MULTIPLICATIVE FUNCTIONS WITH SHIFTED ARGUMENTS

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Dedicated to Professor Antal Járai on his 60th anniversary

Abstract. It is proved that for given integers $a > 0$, $c > 0$, b, d with $ad - cb \neq 0$ there exists a constant $\eta > 0$ with the following property: If unimodular multiplicative functions g_1, g_2 satisfy $|g_1(p) - 1| < \eta$ and $|g_2(p) - 1| < \eta$ for all $p \in \mathcal{P}$, then

$$\liminf_{x\to\infty} \frac{1}{x} \sum_{n\le x} |g_1(an+b) - \Gamma\, g_2(cn+d)| = 0$$

may hold with some $\Gamma \in \mathbb{C} \setminus \{0\}$ if $g_1(n) = g_2(n) = 1$ for all positive integers $n \in \mathbb{N}$, $(n, ac(ad - cb)) = 1$.

1. Introduction

An arithmetic function $g(n) \not\equiv 0$ is said to be multiplicative if $(n, m) = 1$ implies that

$$g(nm) = g(n)g(m)$$

and it is completely multiplicative if this relation holds for all positive integers n and m. Let $\mathcal{M}$ and $\mathcal{M}^*$ denote the class of all complex-valued multiplicative and completely multiplicative functions, respectively. A function g is said to be

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unimodular if g satisfies the condition $|g(n)| = 1$ for all positive integers n. In the following we shall denote by $\mathcal{M}(1)$ and $\mathcal{M}^*(1)$ the class of all unimodular functions $g \in \mathcal{M}$ and $g \in \mathcal{M}^*$, respectively.

Let $\mathcal{A}, \mathcal{A}^*$ be the set of real valued additive and completely additive functions, respectively. As usual, let $\mathcal{P}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ be the set of primes, positive integers, integers, real and complex numbers, respectively. For each real number z we define $\| z \|$ as follows:

$$\| z \| = \min_{k \in \mathbb{Z}} | z - k | .$$

A. Hildebrand [1] proved the following

Theorem A. *There exists a positive constant δ with the following property. If $g \in \mathcal{M}^*(1)$ and $|g(p) - 1| \le \delta$ holds for every $p \in \mathcal{P}$, then either $g(n) = 1$ for all $n \in \mathbb{N}$ identically, or*

$$\liminf_{x \to \infty} \frac{1}{x} \sum_{n \le x} |g(n+1) - g(n)| > 0.$$

By using the ideas of Hildebrand [1] and himself, I. Kátai [2] proved the following generalization:

Theorem B. *Let $g \in \mathcal{M}^*(1)$. There exist positive constants δ and $\beta < 1$ with the property: If*

$$\limsup_{x \to \infty} \sum_{x^\beta < p < x} \frac{|g(p) - 1|}{p} < \delta$$

and

$$\liminf_{x \to \infty} \frac{1}{x} \sum_{\frac{x}{2} \le n \le x} |g(n+1) - g(n)| = 0,$$

then $g(n) = 1$ for all $n \in \mathbb{N}$ identically.

Our purpose in this paper is to prove the following

Theorem. *Let $a, c \in \mathbb{N}$, $b, d \in \mathbb{Z}$ with $ad - cb \ne 0$. There exists a constant $\eta > 0$ with the following property:*

If $g_1, g_2 \in \mathcal{M}(1)$, $|g_1(p) - 1| < \eta$ and $|g_2(p) - 1| < \eta$ for all $p \in \mathcal{P}$, then

$$\liminf_{x \to \infty} \frac{1}{x} \sum_{n \le x} |g_1(an + b) - \Gamma g_2(cn + d)| = 0$$

may hold with some $\Gamma \in \mathbb{C} \setminus \{0\}$ *if*

$$g_1(n) = g_2(n) = 1 \quad \textit{for all} \quad n \in \mathbb{N}, \; (n, ac(ad - cb)) = 1.$$

As a direct consequence we can formulate the next

Corollary. *Let* $a, c \in \mathbb{N}$, $b, d \in \mathbb{Z}$ *with* $ad - cb \neq 0$. *There exists a constant* $\eta > 0$ *with the following property:*

If $f_1, f_2 \in \mathcal{A}$, $\|f_1(p)\| < \eta$ *and* $\|f_2(p)\| < \eta$ *for all* $p \in \mathcal{P}$, *then*

$$\liminf_{x\to\infty} \frac{1}{x} \sum_{n\le x} \|f_1(an+b) - f_2(cn+d) - \Delta\| = 0$$

may hold with some $\Delta \in \mathbb{R}$ *if*

$$\|f_1(n)\| = \|f_2(n)\| = 0 \quad \textit{for all} \quad n \in \mathbb{N}, \; (n, ac(ad - cb)) = 1.$$

We note that I. Kátai [2] has conjectured that if

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} \|f(n+1) - f(n)\| = 0,$$

then there is a real number $\lambda \in \mathbb{R}$ such that

$$\|f(n) - \lambda \log n\| = 0 \quad \text{for all} \quad n \in \mathbb{N}.$$

This conjecture remains open.

2. Lemmata

N. M. Timofeev [3] proved the following assertion (see [3], Lemma 1):

Lemma 1. *Suppose that* $f_1(n)$ *and* $f_2(n)$ *are multiplicative with* $|f_1(n)| \le 1$ *and* $|f_2(n)| \le 1$ *that satisfy the condition*

$$\sum_{p\le x} \big(|f_1(p) - 1| + |f_2(p) - 1|\big) \frac{\log p}{p} \le \varepsilon(x) \log x, \tag{2.1}$$

where $\varepsilon(x)$ is a decreasing function that approaches zero as $x \to \infty$, but $\varepsilon(x)\sqrt{\log x}$ approaches infinity as $x \to \infty$, and let $a > 0,\ b,\ c > 0,\ d,\ a_j,\ b_j,\ \delta_j$ $(j = 1, 2)$ be integers with

$$a = \delta_1 a_1, \quad b = \delta_1 b_1, \quad c = \delta_2 a_2, \quad d = \delta_2 b_2,$$

$$(a_1, b_1) = 1, \quad (a_2, b_2) = 1, \quad \Delta = a_1 b_2 - a_2 b_1 \neq 0.$$

Then

$$\frac{1}{x} \sum_{n \le x} f_1(an + b) f_2(cn + d) = \prod_{p \le x} \omega_p(f_1, f_2) + O\left(\sqrt{\varepsilon(x)}\right), \tag{2.2}$$

where for $p \nmid a_1 a_2 \Delta$

$$\omega_p(f_1, f_2) = \left(1 - \frac{2}{p}\right) f_1\left(p^{\alpha_p(\delta_1)}\right) f_2\left(p^{\alpha_p(\delta_2)}\right) +$$
$$+ \sum_{r=1}^{\infty} \frac{1}{p^r}\left(1 - \frac{1}{p}\right)\left[f_1\left(p^{r+\alpha_p(\delta_1)}\right) f_2\left(p^{\alpha_p(\delta_2)}\right) + f_1\left(p^{\alpha_p(\delta_1)}\right) f_2\left(p^{r+\alpha_p(\delta_2)}\right)\right];$$

if $p \mid a_1$, but $p \nmid (a_1, a_2)$, then

$$\omega_p(f_1, f_2) = \left[f_2\left(p^{\alpha_p(\delta_2)}\right) + \sum_{r=1}^{\infty} f_2\left(p^{r+\alpha_p(\delta_2)}\right) \frac{1}{p^r}\right]\left(1 - \frac{1}{p}\right) f_1\left(p^{\alpha_p(\delta_1)}\right);$$

if $p \mid a_2$, but $p \nmid (a_1, a_2)$, then

$$\omega_p(f_1, f_2) = \left[f_1\left(p^{\alpha_p(\delta_1)}\right) + \sum_{r=1}^{\infty} f_1\left(p^{r+\alpha_p(\delta_1)}\right) \frac{1}{p^r}\right]\left(1 - \frac{1}{p}\right) f_2\left(p^{\alpha_p(\delta_2)}\right);$$

if $p \mid \Delta$, but $p \nmid a_1 a_2$, then

$$\omega_p(f_1, f_2) = \left(1 - \frac{1}{p}\right)\left[\sum_{0 \le r \le \alpha_p(\Delta) - 1} f_1\left(p^{r+\alpha_p(\delta_1)}\right) f_2\left(p^{r+\alpha_p(\delta_2)}\right) \frac{1}{p^r} + \right.$$
$$+ f_1\left(p^{\alpha_p(\Delta)+\alpha_p(\delta_1)}\right) f_2\left(p^{\alpha_p(\Delta)+\alpha_p(\delta_2)}\right)\left(1 - \frac{1}{p}\right)^{-1}\left(1 - \frac{2}{p}\right) +$$
$$+ \sum_{r \ge 1} \frac{1}{p^{r+\alpha_p(\Delta)}}\left(f_1\left(p^{r+\alpha_p(\delta_1)}\right) f_2\left(p^{\alpha_p(\delta_2)+\alpha_p(\Delta)}\right) + \right.$$
$$\left.\left. + f_1\left(p^{\alpha_p(\delta_1)+\alpha_p(\Delta)}\right) f_2\left(p^{r+\alpha_p(\delta_2)}\right)\right)\right];$$

if $p \mid (a_1, a_2)$, then

$$\omega_p(f_1, f_2) = f_1\left(p^{\alpha_p(\delta_1)}\right) f_2\left(p^{\alpha_p(\delta_2)}\right).$$

Here $\alpha_p(n)$ is the largest integer α such that p^α divides n.

Analyzing the proof of Lemma 1, one can see that it remains true in the following form:

Lemma 1′. *Assume that in the notations of Lemma 1, instead of* (2.1)

$$\sum_{p\le x}\Big(|f_1(p)-1|+|f_2(p)-1|\Big)\frac{\log p}{p}\le \delta\log x \tag{2.3}$$

if $x>x_0(\delta)$*. Then*

$$\limsup_{x\to\infty}\left|\frac{1}{x}\sum_{n\le x} f_1(an+b)f_2(cn+d)-\prod_{p\le x}\omega_p(f_1,f_2)\right|\le C\sqrt{\delta}, \tag{2.4}$$

where C *is a constant that may depend only on* a, b, c, d.

3. Proof of the theorem

Assume that the conditions of Theorem hold and

$$\sum_{n\le x_\nu}|g_1(an+b)-\Gamma g_2(cn+d)|<\varepsilon_\nu x_\nu, \tag{3.1}$$

where $\varepsilon_\nu\searrow 0$, $x_\nu\nearrow\infty$. From (3.1) it is clear that $|\Gamma|=1$ and

$$\sum_{n\le x_\nu}|\overline{\Gamma}g_1(an+b)\overline{g}_2(cn+d)-1|<\varepsilon_\nu x_\nu.$$

Since

$$|1-z|^2=2(1-\mathrm{Re}\ z)\le 2|1-z|\quad\text{when}\quad |z|=1,$$

we have

$$\sum_{n\le x_\nu}|\,\overline{\Gamma}g_1(an+b)\overline{g}_2(cn+d)-1|^2\le 2\sum_{n\le x_\nu}|\overline{\Gamma}g_1(an+b)\overline{g}_2(cn+d)-1|<2\varepsilon_\nu x_\nu,$$

which implies

$$\mathrm{Re}\ 2\overline{\Gamma}\sum_{n\le x_\nu}g_1(an+b)\overline{g}_2(cn+d)\ge 2(1-\varepsilon_\nu)x_\nu. \tag{3.1$'$}$$

Let us apply Lemma 1′ with $f_1=g_1$, $f_2=\overline{g}_2$ and $\delta=2\eta$. We obtain that

$$\prod_{p\le x}|\omega_p(g_1,\overline{g}_2)|\ge 1-C\sqrt{\delta}. \tag{3.2}$$

Assume that δ is small, $C\sqrt{\delta} < 1$. Then, from (3.2), we have

$$\sum_{p\in\mathcal{P}} \left(1 - |\omega_p(g_1, \overline{g}_2)|^2\right) < \infty.$$

If $(p, ac\Delta) = 1$, then $\alpha_p(\delta_1) = \alpha_p(\delta_2) = 0$ and

$$\omega_p(g_1, \overline{g}_2) = \left(1 - \frac{2}{p}\right) + \left(1 - \frac{1}{p}\right)\frac{1}{p}\big(g_1(p) + g_2(p)\big) + O\left(\frac{1}{p^2}\right) = 1 + \xi_p,$$

where

$$\xi_p = \frac{1}{p}\big[(g_1(p) - 1) + (g_2(p) - 1)\big] + O\left(\frac{1}{p^2}\right).$$

Therefore

$$|\omega_p(g_1, \overline{g}_2)|^2 = 1 + \xi_p + \overline{\xi}_p + |\xi_p|^2,$$

and so

$$\sum_{p\in\mathcal{P}} \left(1 - |\omega_p(g_1, \overline{g}_2)|^2\right) = 2\mathrm{Re}\,\left\{\sum_{p\in\mathcal{P}} \frac{1 - g_1(p)}{p} + \sum_{p\in\mathcal{P}} \frac{1 - g_2(p)}{p}\right\} + O(1).$$

Since

$\mathrm{Re}\,(1 - g_1(p)) \geq 0$, $\mathrm{Re}\,(1 - g_2(p)) \geq 0$ and $|1 - z|^2 = 2(1 - \mathrm{Re}\, z)$ when $|z| = 1$,

therefore

$$\sum_{p\in\mathcal{P}} \frac{|1 - g_j(p)|^2}{p} < \infty, \quad j = 1, 2. \tag{3.3}$$

Let

$$\sigma_j(x) = \sum_{\sqrt{x}\leq p\leq x} \frac{|1 - g_j(p)|^2}{p}.$$

From (3.3) we have

$$\sum_{l=0,1,\ldots} \sigma_j(x^{1/2^l}) < c,$$

where c is a constant. Since

$$\sum_{p\leq x} \frac{1}{p} = \log\log(x) + C + O\left(\frac{1}{\log x}\right) \quad \text{where} \quad C = 0.2615...,$$

by applying Cauchy's inequality, we have

$$\sum_{\sqrt{x}\leq p\leq x} \frac{|1 - g_j(p)|\log p}{p} \leq \log x \sum_{\sqrt{x}\leq p\leq x} \frac{1}{\sqrt{p}} \frac{|1 - g_j(p)|}{\sqrt{p}} \leq$$

$$\le \log x \left(\sum_{\sqrt{x} \le p \le x} \frac{1}{p} \right)^{1/2} \left(\sum_{\sqrt{x} \le p \le x} \frac{|1 - g_j(p)|^2}{p} \right)^{1/2} \le c_1 \log x \sqrt{\sigma_j(x)}.$$

Therefore

$$\sum_{2 \le p \le x} \frac{|1 - g_j(p)| \log p}{p} \le c_1 \sum_{2^l \le \log x} \left(\log x^{1/2^l} \right) \sqrt{\sigma_j(x/2^l)} = c_1 \log x \Theta_j(x),$$

where

$$\Theta_j(x) = \sum_{2^l \le \log x} \frac{\sqrt{\sigma_j(x/2^l)}}{2^l}.$$

It is clear that $\Theta_j(x) \to 0$ $(x \to \infty)$. Let

$$\varepsilon_j(y) = \max_{x \ge y} \Theta_j(x) \quad \text{and} \quad \epsilon(y) = \epsilon_1(y) + \epsilon_2(y).$$

Thus (2.1) holds with this $\epsilon(x)$.

From $(3.1)'$ and (2.2) with $f_1 = g_1$ and $f_2 = \overline{g}_2$, we obtain that

$$\operatorname{Re} \overline{\Gamma} \prod_{p \in \mathcal{P}} \omega_p(g_1, \overline{g}_2) = 1,$$

which implies that

$$|\omega_p(g_1, \overline{g}_2)| = 1 \quad \text{for all} \quad p \in \mathcal{P}$$

and

$$\prod_{p \in \mathcal{P}} \omega_p(g_1, \overline{g}_2) = \Gamma.$$

It is clear that if $(p, ac\Delta) = 1$, then $\alpha_p(\delta_1) = \alpha_p(\delta_2) = 0$ (in the notations of Lemma 1), and so

$$\omega_p(g_1, \overline{g}_2) = \left(1 - \frac{2}{p}\right) + \left(1 - \frac{1}{p}\right) \sum_{r=1}^{\infty} \frac{1}{p^r} \Big(g_1(p^r) + \overline{g}_2(p^r) \Big). \tag{3.4}$$

Let

$$\lambda_p = \sum_{r=1}^{\infty} \frac{1}{p^r} \left(g_1(p^r) + \overline{g}_2(p^r) \right).$$

It is clear that $|\lambda_p| \le \frac{2}{p-1}$, and one can check from (3.4) that $|\omega_p(g_1, \overline{g}_2)| < 1$, if $g_1(p^r) + \overline{g}_2(p^r) \ne 2$ for at least one r.

Thus we have $g_1(p^r) = g_2(p^r) = 1$ if $p \not| a_1 a_2 \Delta$, $p > \max(\delta_1, \delta_2)$.

The proof of our theorem is completed. ■

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COMPUTATIONAL INVESTIGATION OF LEHMER'S TOTIENT PROBLEM

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Dedicated to Professor Antal Járai on his 60th birthday

Abstract. Let N be a composite number for which $k \cdot \varphi(N) = N - 1$. We show that if $3 \mid N$ then $\omega(N) \geq 40\ 000\ 000$ and $N > 10^{360\ 000\ 000}$.

1. Introduction

In this paper we study a famous unanswered question, the so-called "Lehmer's Totient Problem", which was first studied by Lehmer in 1932 [1]. Lehmer asked whether there is such a composite integer N for which the equation

$$k \cdot \varphi(N) = N - 1 \tag{1}$$

holds, where φ is the Euler totient function. Then we say that N is a Lehmer number and k is the Lehmer index of N. Let us denote the set of Lehmer numbers by L. Lehmer conjectured that L is empty.

Let us consider the equation (1) in the form

$$1 = N - k \cdot \varphi(N), \tag{2}$$

from which some interesting facts follow immediately. We know that $\varphi(N)$ is always even, if $N > 1$. Thus if N is even, then $N - k \cdot \varphi(N)$ cannot be 1. Also we can observe easily that if N is not squarefree then N has a prime factor p_i for which $p_i \mid \varphi(N)$. In this case if N is a Lehmer number, then $p_i \mid 1$ would be valid which is impossible, so we get the following assertion.

Remark 1. If N is a Lehmer number, then $2 \nmid N$ and N is square-free.

Hereafter we write a Lehmer number N in the form

$$N = p_1 p_2 \dots p_n, \text{ where } 3 \le p_1 < p_2 < \dots < p_n \tag{3}$$

and $p_1, p_2, \dots p_n$ are different prime numbers.

A composite number N is called *Carmichael number* if

$$a^{N-1} \equiv 1 \pmod{N}$$

is valid for all $a \in \mathbb{Z}$, where $(a, N) = 1$. The *Carmichael function* for N is defined as the smallest positive integer $\lambda(N)$ such that

$$a^{\lambda(N)} \equiv 1 \pmod{N}$$

for every integer a that is both coprime to and smaller than N. As a matter of fact $\lambda(N)$ is the exponent of $\mathbb{Z}_N^*$, the multiplicative group of residues modulo N, i. e. $\lambda(N)$ is the least common multiple of the orders of the elements of $\mathbb{Z}_N^*$. Since the order of $\mathbb{Z}_N^*$ is $\varphi(N)$ we have $\lambda(N) \mid \varphi(N)$. Thus if $\varphi(N) \mid N-1$, then $\lambda(N) \mid N-1$. Finally we get that $a^{N-1} \equiv 1 \pmod{N}$ for all elements of $\mathbb{Z}_N^*$, which implies the next assertion.

Remark 2. Every Lehmer number is a Carmichael number.

The next observation is important for the computational investigation of the Lehmer conjecture.

Remark 3. Let $3 \le p_1 < p_2 < \dots < p_n$ are different prime numbers. If $N = p_1 p_2 \dots p_n p_{n+1}$ is a Lehmer number, then

$$p_i \nmid p_{n+1} - 1, \text{ where } 1 \le i \le n.$$

This assertion follows directly from (2). Subbarao and Siva Rama Prasad proved the following statement in [2].

Remark 4. If N is a Lehmer number and $3 \mid N$, then

$$k \equiv 1 \pmod{3}.$$

2. Previous achievements

Although the Lehmer totient problem has not yet solved, a lot of results are published concerning it. Let us denote the number of distinct prime factors of N by $\omega(N)$. Lehmer showed that if $N \in L$, than $\omega(N) \geq 7$. Improving this result Lieuwens [3] proved in 1970 that $\omega(N) \geq 11$. In 1977 Kishore [4] showed that $\omega(N) \geq 13$, and his result was increased to 14 by Cohen and Hagis [5] in 1980 using a computational method. Nowadays the best lower bound of $\omega(N)$ is 15 reached by John Renze [6] in 2004, and R. Pinch gave a computational proof of the assertion:

$$N > 10^{30}.$$

Let us suppose that $p_1 = 3$. In this case Lieuwens shoved in [3] that

$$\omega(N) \geq 212 \text{ and } N > 5.5 \cdot 10^{570}.$$

This result was improved by Subbarao and Siva Rama Prasad in [2]:

$$\omega(N) \geq 1850.$$

In 1988 Hagis [7] proved by computer the following inequalities:

$$\omega(N) \geq 298\ 848 \text{ and } n > 10^{1\ 937\ 042}. \tag{4}$$

We also mention two interesting pure mathematical results: Banks and Luca proved in [8] that the number of composite integers $N < x$ for which $\varphi(N) \mid N - 1$ is at most

$$O\left(x^{1/2}(\log\log x)^{1/2}\right).$$

Subbarao and Siva Rama Prasad showed in [2] that

$$N < (\omega(N) - 1)^{2^{(\omega(N)-1)}}.$$

3. Results

We focus on the case where $p_1 = 3$. With computational methods, we improve the results in (4) on $\omega(N)$ and N mentioned above.

We need some notations. Let $p_1 < p_2 < \ldots < p_m$ be a sequence of prime numbers. Hereafter we call this sequence a *G-sequence* if the numbers fulfill the conditions in (3). Now let r be a positive real number and $\underline{p} = p_1, \ldots, p_m$ be a G-sequence. We define the following value:

$$\begin{aligned} \min \omega(\underline{p}, r) \quad = \quad & \inf\{\omega(N) | N = p_1 p_2 \cdots p_m p_{m+1} \cdots p_n, \text{where} \\ & \qquad p_1 < \ldots < p_n \text{ is a G-sequence} \\ & \qquad \text{and the Lehmer index of N is at least } r\}. \end{aligned}$$

We define $\min N(\underline{p}, r)$ similarly, but for the infinum of N rather than $\omega(N)$. Clearly, if we set $r = 4$, these values give lower bounds for $\omega(N)$ and N if N is a Lehmer number with $3 \mid N$, since it follows from (4) that the Lehmer index of such a number is at least 4.

Unfortunately, it seems infeasible to calculate these values exactly. The greedy algorithm of choosing $p_{m+1}, \ldots, p_n$ such that we always select the smallest prime that keeps the G-sequence property might fail if r is large enough. We illustrate the intuition behind this with an example: Let $m = 1$ and $p_1 = 3$. The smallest possible value for p_2 is 5. Now if we want to extend the sequence, we will have to look for primes that are incongruent to 1 modulo 3 and 5, giving a set of 3 possible residue classes modulo 15, loosely speaking, a 3/8 fraction of all subsequent primes. If we choose $p_2 = 11$ instead, we get 9 possible residues modulo 33, a 9/20 fraction of primes, which is larger. So choosing 5 increases the Lehmer index faster, but this advantage might turn over when n becomes large, since there are more primes to choose from.

However, it is possible to give *lower bounds* with the simple greedy algorithm of choosing the minimal possible value for $p_m, \ldots, p_n$, if we only require $p_i \nmid p_j - -1$ to hold for $i < j$ with $i \leq m$. Such a sequence will be called a G_m-sequence. The estimates obtained this way are denoted by $\operatorname{est} \omega(\underline{p}, r)$ and $\operatorname{est} N(\underline{p}, r)$. We have

$$\min \omega(\underline{p}, r) \geq \operatorname{est} \omega(\underline{p}, r)$$

and also

$$\min \omega(\underline{p}, r) \geq \min \operatorname{est} \omega(\left[\underline{p}, p_{m+1}\right], r) \ , \tag{5}$$

where the minimum is taken over all p_{m+1} such that $\underline{p}, p_{m+1}$ is a G_{m+1}-sequence. The same is true for the estimates of N. Unfortunately, there are infinitely many possible p_{m+1} values, so in this form the estimate is still ineffective. Therefore we investigate the special case of G_m sequences when we add the extra condition that p_{m+1} is at least q. This will be written as $\operatorname{est} \omega([\underline{p}, q+], r)$. Note that we denote the extension of a sequence by brackets.

The algorithm is relatively simple to implement. The main idea was to transform the problem to an additive setting: instead of calculating the Lehmer index directly, we calculate the sum of the logarithms of the $\frac{p_i}{p_i-1}$, and then account for the -1 in the numerator of the Lehmer index. The logarithms of the mentioned fractions were pre-stored in a table using fixed point representation. The rounding errors and the slight imprecision caused by the -1 in the numerator of the Lehmer-index are also considered, so we found that the 64-bit fixed point representation never caused problems.

We summarize the results in the Table 1 where the estimates correspond to nodes in a rooted tree. The root is 3, and each node of the tree represents a G-sequence $p_1, \ldots, p_m$ or a sequence $p_1, \ldots, p_m, q+$. Part of this infinite tree is shown in Figure 1. The table shows the values of $\mathrm{est}\,\omega(\underline{p}, 4)$, $\mathrm{est}N(\underline{p}, 4)$, and the lower bounds coming from inequality (5), where the minimum was taken over the descendants shown in the tree.

Sequence $\underline{p}$	$\mathrm{est}\,\omega$	$\log_{10}(\mathrm{est}N)$	bound for $\min\omega$	bound for $\min N$
[3]	1540	6082	$4.0 \cdot 10^7$	$10^{3.6\cdot 10^8}$
[3, 5]	$4.9 \cdot 10^6$	$3.9 \cdot 10^7$	$4.0 \cdot 10^7$	$10^{3.6\cdot 10^8}$
[3, 11]	$1.6 \cdot 10^7$	$1.3 \cdot 10^8$	$8.1 \cdot 10^7$	$10^{7.4\cdot 10^8}$
[3, 17]	$4.8 \cdot 10^7$	$4.3 \cdot 10^8$	$8.4 \cdot 10^7$	$10^{7.6\cdot 10^8}$
[3, 23]	$> 8.7 \cdot 10^7$	$> 7.9 \cdot 10^8$	$8.7 \cdot 10^7$	$10^{7.9\cdot 10^8}$
[3, 29+]	$> 8.9 \cdot 10^7$	$> 8.1 \cdot 10^8$		
[3, 5, 17]	$4.0 \cdot 10^7$	$3.6 \cdot 10^8$		
[3, 5, 23]	$> 7.5 \cdot 10^7$	$> 6.8 \cdot 10^8$		
[3, 5, 29+]	$> 7.6 \cdot 10^7$	$> 7.0 \cdot 10^8$		
[3, 11, 17]	$> 8.1 \cdot 10^7$	$> 7.4 \cdot 10^8$		
[3, 11, 29]	$> 8.3 \cdot 10^7$	$> 7.5 \cdot 10^8$		
[3, 11, 41+]	$> 8.4 \cdot 10^7$	$> 7.7 \cdot 10^8$		
[3, 17, 23]	$> 8.4 \cdot 10^7$	$> 7.6 \cdot 10^8$		
[3, 17, 29+]	$> 8.6 \cdot 10^7$	$> 7.8 \cdot 10^8$		
[3, 23, 29]	$> 8.7 \cdot 10^7$	$> 7.9 \cdot 10^8$		
[3, 23, 41+]	$> 8.7 \cdot 10^7$	$> 7.9 \cdot 10^8$		

Table 1. This table shows our main results. For each sequence we show the estimates that were output by the program, and the estimates obtained by looking at the sequence's displayed descendants - only shown for nodes with children.

4. Further work

The efficiency of the programs can be further enhanced by parallel processing several G-sequences at a time. This can be achieved by "batch sieving"

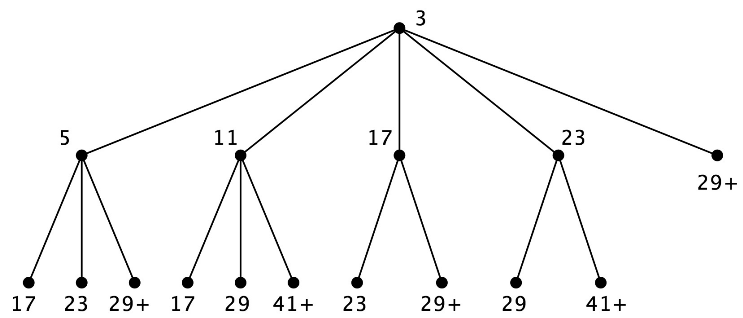

Figure 1. This figure shows part of the infinite tree of G-sequences.

that is calculating the logarithms of primes in an interval and registering which of the examined G-sequences can be extended by the sieved prime. This method will probably further improve the above results. New bounds will be published on the project's home page:

`http://compalg.inf.elte.hu/tanszek/projects.php`

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ON THE WEIGHTED LEBESGUE FUNCTION OF FOURIER–JACOBI SERIES

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Dedicated to Professor Antal Járai on his 60th birthday

Abstract. S.A. Agahanov and G.I. Natanson [1] established lower and upper bounds for the Lebesgue functions $L_n^{(\alpha,\beta)}(x)$ of Fourier–Jacobi series on the interval $[-1,1]$. The bounds differ from each other only in a constant factor depending on Jacobi parameters α and β, so their result is of final character. The aim of this paper is to extend their estimation for the weighted Lebesgue functions $L_n^{(\alpha,\beta),(\gamma,\delta)}(x)$ using Jacobi weights with parameters γ and δ. We shall also give sufficient conditions with respect to α,β,γ and δ for which the order of the weighted Lebesgue functions is $\log(n+1)$ on the whole interval $[-1,1]$.

1. Introduction

It is known that the Lebesgue functions of an approximation process play an important role in the convergence of that process. The Lebesgue functions $L_n^{(\alpha,\beta)}(x)$ (see (2.1)) of Fourier–Jacobi series have been studied by many authors.

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G. Szegő [10, 9.3.] showed that for every fixed number $\varepsilon \in (0,1)$

$$\max_{x\in[-1+\varepsilon,1-\varepsilon]} L_n^{(\alpha,\beta)}(x) \sim \log(n+1)$$
$$(n \in \mathbb{N} := \{1,2,\dots\}).$$

Here and in what follows for the positive functions $a_n, b_n : I \to \mathbb{R}$ (I is an interval of $\mathbb{R}$) the notation

$$a_n(x) \sim b_n(x) \qquad (x \in I,\ n \in \mathbb{N})$$

means that there exist positive constants c_1, c_2 independent of x and n such that

$$c_1 \le \frac{a_n(x)}{b_n(x)} \le c_2 \qquad (x \in I,\ n \in \mathbb{N}).$$

H. Rau [7] showed that the order of the Lebesgue functions at the points -1 and 1 is $n^{\sigma+\frac{1}{2}}$, where $\sigma = \max\{\alpha,\beta\}$.

S. A. Agahanov and G. I. Natanson [1] proved the following result: if $\alpha, \beta > > -\frac{1}{2}$ then

$$L_n^{(\alpha,\beta)}(x) \sim \log\left(n(1-x)^{\varepsilon(\alpha)}(1+x)^{\varepsilon(\beta)}+1\right) + \sqrt{n}\left(|P_n^{(\alpha,\beta)}(x)| + |P_{n+1}^{(\alpha,\beta)}(x)|\right)$$
$$(x \in [-1,1],\quad n \in \mathbb{N}),$$

where

$$\varepsilon(t) = \begin{cases} \frac{1}{2}, & \text{if } t \in \mathbb{R} \setminus \{\frac{1}{2}\} \\ 0, & \text{if } t = \frac{1}{2} \end{cases}$$

and $P_n^{(\alpha,\beta)}(x)$ is the nth Jacobi polynomial.

The aim of this paper is to extend this estimation by using suitable Jacobi weights. We will give conditions for the weight parameters γ and δ such that the order of the weighted Lebesgue functions $L_n^{(\alpha,\beta),(\gamma,\delta)}(x)$ is $\log(n+1)$ on the whole interval $[-1,1]$.

2. Pointwise estimate of the weighted Lebesgue function

For parameters $\alpha, \beta > -1$ we shall denote by $P_n^{(\alpha,\beta)}$ the nth Jacobi polynomial with the normalization

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n} \qquad (n \in \mathbb{N}_0 := \{0,1,2,\dots\}).$$

They are orthogonal with respect to the Jacobi weight function

$$w^{(\alpha,\beta)}(x) := (1-x)^{\alpha}(1+x)^{\beta} \qquad \big(x \in (-1,1)\big).$$

The nth *Lebesgue function of Fourier–Jacobi series* is defined by

$$L_n^{(\alpha,\beta)}(x) := \int_{-1}^{1} |K_n^{(\alpha,\beta)}(x,y)| w^{(\alpha,\beta)}(y)\,\mathrm{d}y \qquad \big(n \in \mathbb{N},\ x \in [-1,1]\big), \tag{2.1}$$

where the kernel function $K_n^{(\alpha,\beta)}(x,y)$ can be expressed as

$$\begin{aligned} K_n^{(\alpha,\beta)}(x,y) &= \sum_{k=0}^{n} \left\{h_k^{(\alpha,\beta)}\right\}^{-1} P_k^{(\alpha,\beta)}(x) P_k^{(\alpha,\beta)}(y) = \\ &= \lambda_n^{(\alpha,\beta)} \frac{P_{n+1}^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(y) - P_n^{(\alpha,\beta)}(x)P_{n+1}^{(\alpha,\beta)}(y)}{x-y}. \end{aligned} \tag{2.2}$$

Here

$$h_k^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}}{2k+\alpha+\beta+1}\,\frac{\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{\Gamma(k+1)\Gamma(k+\alpha+\beta+1)}\,, \tag{2.3}$$

and

$$\lambda_n^{(\alpha,\beta)} = \frac{2^{-\alpha-\beta}}{2n+\alpha+\beta+2}\,\frac{\Gamma(n+2)\Gamma(n+\alpha+\beta+2)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)} \tag{2.4}$$

(see [10, (4.3.3) and (4.5.2)]), where $\Gamma(p)$ $(p>0)$ is the Gamma function.

For $\gamma, \delta \geq 0$ we define the nth *weighted Lebesgue function of Fourier–Jacobi series* by

$$L_n^{(\alpha,\beta),(\gamma,\delta)}(x) := w^{(\gamma,\delta)}(x) \int_{-1}^{1} |K_n^{(\alpha,\beta)}(x,y)| w^{(\alpha-\gamma,\beta-\delta)}(y)\,\mathrm{d}y \qquad \big(n \in \mathbb{N},\ x \in [-1,1]\big). \tag{2.5}$$

For the existence of this integral, we shall assume that the parameters γ, δ satisfy the inequalities

$$\gamma < \alpha + 1, \quad \delta < \beta + 1. \tag{2.6}$$

Theorem. *Suppose that $\alpha, \beta > -\frac{1}{2}$ and $\gamma, \delta \geq 0$ satisfy the inequalities*

$$\frac{\alpha}{2} + \frac{1}{4} < \gamma < \frac{\alpha}{2} + \frac{3}{4} \quad \textit{and} \quad \frac{\beta}{2} + \frac{1}{4} < \delta < \frac{\beta}{2} + \frac{3}{4}. \tag{2.7}$$

Then we have for all $n \in \mathbb{N}$ and $x \in [-1,1]$ that

$$c_1\, w^{(\gamma,\delta)}(x)\, \phi_n^{(\alpha,\beta)}(x) \le L_n^{(\alpha,\beta),(\gamma,\delta)}(x) \le c_2\, \widetilde{w}_n^{(\gamma,\delta)}(x)\, \phi_n^{(\alpha,\beta)}(x) \tag{2.8}$$

with the constants $c_1, c_2 > 0$ independent of x and n, where

$$\phi_n^{(\alpha,\beta)}(x) := \log\left(n\sqrt{1-x^2}+1\right)+$$

$$+\sqrt{n}\left(\sqrt{1-x}+\tfrac{1}{n}\right)^{\alpha+\frac{1}{2}}\left(\sqrt{1+x}+\tfrac{1}{n}\right)^{\beta+\frac{1}{2}}\left(|P_n^{(\alpha,\beta)}(x)|+|P_{n+1}^{(\alpha,\beta)}(x)|\right),$$

and

$$\widetilde{w}_n^{(\gamma,\delta)}(x) := \left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma}\left(\frac{\sqrt{1+x}}{\sqrt{1+x}+\frac{1}{n}}\right)^{2\delta}.$$

We note that the conditions for the parameters $\alpha, \beta, \gamma, \delta$ in Theorem imply the inequalities in (2.6).

Corollary. *Suppose that $\alpha, \beta > -\frac{1}{2}$ and $\gamma, \delta \ge 0$ satisfy the inequalities* (2.7)*. Then we have*

$$\max_{x\in[-1,1]} L_n^{(\alpha,\beta),(\gamma,\delta)}(x) \;\sim\; \log(n+1) \qquad (n \in \mathbb{N}).$$

Remark. A result similar to this Corollary proved by U. Luther and G. Mastrioianni [5]. This paper does not contain a pointwise estimation (cf. (2.8)).

3. Preliminaries

In what follows for the functions $a_n, b_n : I \to \mathbb{R}$ (I is an interval of $\mathbb{R}$) the notation

$$a_n(x) = O(b_n(x)) \qquad (x \in I,\ n \in \mathbb{N})$$

means that there exists a positive constant c independent of x and n such that

$$|a_n(x)| \le c\, b_n(x) \qquad (x \in I,\ n \in \mathbb{N}).$$

3.1. Formulas for Jacobi polynomials. Here we list those well known formulas which we shall use throughout the paper.

If $\alpha, \beta > -1$ then for every $x \in [-1, 1]$ and $n \in \mathbb{N}$ we have

$$P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x) \tag{3.1}$$

(see [10, (4.1.3)]) and

$$\frac{d}{dx}\left\{P_n^{(\alpha,\beta)}(x)\right\} = \tfrac{1}{2}(n+\alpha+\beta+1)P_{n-1}^{(\alpha+1,\beta+1)}(x) \tag{3.2}$$

(see [10, (4.21.7)]).

An important bound for Jacobi polynomials can be given in this form: if $\alpha, \beta > -1$ then

$$\left|P_n^{(\alpha,\beta)}(x)\right| = O\left(n^{-\frac{1}{2}}\right)\left(\sqrt{1-x}+\frac{1}{n}\right)^{-\alpha-\frac{1}{2}} \qquad (0 \le x \le 1,\ n \in \mathbb{N}) \tag{3.3}$$

(see [6, 2.3.22]).

A more precise formula is the following. Let $\alpha, \beta > -1$. Then we have

$$P_n^{(\alpha,\beta)}(\cos s) = n^{-\frac{1}{2}}k(s)\Big(\cos(Ns+\nu) + \frac{O(1)}{n\sin s}\Big), \tag{3.4}$$

where

$$\frac{c}{n} \le s \le \pi - \frac{c}{n},\ k(s) = k^{(\alpha,\beta)}(s) = \pi^{-\frac{1}{2}}\Big(\sin\frac{s}{2}\Big)^{-\alpha-\frac{1}{2}}\Big(\cos\frac{s}{2}\Big)^{-\beta-\frac{1}{2}},$$
$$N = n + \tfrac{1}{2}(\alpha+\beta+1),\ \nu = -\left(\alpha+\tfrac{1}{2}\right)\tfrac{\pi}{2}.$$

Here c is a fixed positive number and the bound for the error term holds uniformly in the interval $\left[\frac{c}{n}, \pi - \frac{c}{n}\right]$ (see [10, (8.21.18)]).

If $\alpha, \beta, \mu > -1$ then we have uniformly in $n \in \mathbb{N}$ that

$$\int_0^1 |P_n^{(\alpha,\beta)}(y)|(1-y)^\mu\,dy \sim \begin{cases} n^{\alpha-2\mu-2}, & \text{if } 2\mu < \alpha - \frac{3}{2} \\ n^{-\frac{1}{2}}\log n, & \text{if } 2\mu = \alpha - \frac{3}{2} \\ n^{-\frac{1}{2}}, & \text{if } 2\mu > \alpha - \frac{3}{2} \end{cases} \tag{3.5}$$

(see [10, (7.34.1)]).

Let $p > 0$ be a fixed real number. Then

$$\frac{\Gamma(n+p)}{\Gamma(n)} \sim n^p \qquad (n \in \mathbb{N})$$

(see [8, p. 166]). Thus for the numbers (2.3) and (2.4) we have

$$h_n^{(\alpha,\beta)} \sim \frac{1}{n} \qquad (n \in \mathbb{N}),$$
$$\lambda_n^{(\alpha,\beta)} \sim n \qquad (n \in \mathbb{N}). \tag{3.6}$$

We introduce the notations

$$\overline{P}_n(x) := P_n^{(\alpha+1,\beta)}(x),$$
$$\widetilde{P}_n(x) := P_n^{(\alpha+1,\beta+1)}(x).$$

Using the formulas [10, (4.5.7)] we obtain that

$$\frac{1}{2}(1-x^2)\widetilde{P}_{n-1}(x) = \Big(x + \frac{\alpha-\beta}{2n+\alpha+\beta+2}\Big)P_n^{(\alpha,\beta)}(x) - \\ - \frac{2n+2}{2n+\alpha+\beta+2}P_{n+1}^{(\alpha,\beta)}(x). \tag{3.7}$$

Moreover, by [10, (4.5.4)] we have

$$\Big(1+\frac{\alpha+\beta}{2n+2}\Big)(1-x)\overline{P}_n(x) = \frac{n+\alpha+1}{n+1}P_n^{(\alpha,\beta)}(x) - P_{n+1}^{(\alpha,\beta)}(x). \tag{3.8}$$

3.2. Auxiliary results.

Lemma 1. *Suppose that $R \geq 1$ and $A < 0$ are fixed real numbers. Then with a suitable index $N \in \mathbb{N}$ we have*

$$\int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \frac{t^A}{t-s}\,\mathrm{d}t \sim \Big(s+\frac{R}{n}\Big)^A \Big[\log\Big(\frac{ns}{R}+1\Big)+1\Big] \tag{3.9}$$

uniformly in $s \in [0, \frac{\pi}{2}]$ and $n \in \mathbb{N}$, $n > N$.

Proof. Let us introduce the following notation

$$I := I(n,s,A,R) := \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \frac{t^A}{t-s}\,\mathrm{d}t$$
$$(n \in \mathbb{N},\ s \in [0, \tfrac{\pi}{2}],\ A < 0,\ R \geq 1).$$

In order to prove the statement, we split the interval $[0, \frac{\pi}{2}]$ into three parts:

$$\big[0, \tfrac{\pi}{2}\big] = \big[0, \tfrac{R}{n}\big] \cup \big(\tfrac{R}{n}, \tfrac{2\pi}{9}\big) \cup \big[\tfrac{2\pi}{9}, \tfrac{\pi}{2}\big].$$

Case 1. Let $0 \leq s \leq \frac{R}{n}$ and $t \in [s+\frac{R}{n}, \frac{2\pi}{3}]$. From $2s \leq s + \frac{R}{n} \leq t$ it follows that

$$\tfrac{1}{2}\,t \leq t - s \leq t.$$

Therefore we have

$$\int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} t^{A-1}\,\mathrm{d}t \le \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \frac{t^A}{t-s}\,\mathrm{d}t \le 2\int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} t^{A-1}\,\mathrm{d}t. \tag{3.10}$$

Since

$$\int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} t^{A-1}\,\mathrm{d}t = \frac{1}{|A|}\left[\left(s+\frac{R}{n}\right)^A - \left(\frac{2\pi}{3}\right)^A\right], \tag{3.11}$$

we obtain the following upper estimation of I:

$$I \le \frac{2}{|A|}\left(s+\frac{R}{n}\right)^A \left[\log\left(\frac{ns}{R}+1\right)+1\right]. \tag{3.12}$$

Now, let us consider the lower estimation. If $n \ge \frac{6R}{\pi}$ and $A < 0$, then $\left(\frac{n\pi}{3R}\right)^A \le 2^A$. Therefore using (3.10) and (3.11) we get

$$\begin{aligned} I &\ge \frac{1}{|A|}\left[\left(s+\frac{R}{n}\right)^A - \left(\frac{2\pi}{3}\right)^A\right] = \frac{1}{|A|}\left(s+\frac{R}{n}\right)^A\left[1-\left(\frac{\frac{2\pi}{3}}{s+\frac{R}{n}}\right)^A\right] \ge \\ &\ge \frac{1}{|A|}\left(s+\frac{R}{n}\right)^A\left[1-\left(\frac{\frac{2\pi}{3}}{\frac{2R}{n}}\right)^A\right] = \frac{1}{|A|}\left(s+\frac{R}{n}\right)^A\left[1-\left(\frac{n\pi}{3R}\right)^A\right] \ge \\ &\ge \frac{1-2^A}{|A|}\left(s+\frac{R}{n}\right)^A = \frac{1-2^A}{|A|}\left(s+\frac{R}{n}\right)^A \frac{1+\log 2}{1+\log 2} \ge \\ &\ge \frac{1-2^A}{|A|(1+\log 2)}\left(s+\frac{R}{n}\right)^A\left[1+\log\left(\frac{ns}{R}+1\right)\right], \end{aligned}$$

where we used the fact that from $\frac{ns}{R} \le 1$ it follows that $\log 2 \ge \log\left(\frac{ns}{R}+1\right)$.

Consequently,

$$I \ge c\left(s+\frac{R}{n}\right)^A\left[\log\left(\frac{ns}{R}+1\right)+1\right]$$
$$\left(s \in [0, \tfrac{R}{n}],\ A < 0,\ R \ge 1,\ n \ge \tfrac{6R}{\pi}\right),$$

with a constant $c > 0$ independent of s and n.

This inequality together with (3.12) prove (3.9), if $0 \le s \le \frac{R}{n}$.

Case 2. Let $\frac{R}{n} < s < \frac{2\pi}{9}$. Then $s + \frac{R}{n} < 2s < 3s < \frac{2\pi}{3}$. Now we split the integral I into two parts:

$$I = \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \frac{t^A}{t-s}\,\mathrm{d}t = \int_{s+\frac{R}{n}}^{3s} \frac{t^A}{t-s}\,\mathrm{d}t + \int_{3s}^{\frac{2\pi}{3}} \frac{t^A}{t-s}\,\mathrm{d}t =: I_1 + I_2.$$

For I_1 we have

$$\begin{aligned} I_1 &= \int_{s+\frac{R}{n}}^{3s} \frac{t^A}{t-s}\,\mathrm{d}t \le \left(s+\frac{R}{n}\right)^A \int_{s+\frac{R}{n}}^{3s} \frac{1}{t-s}\,\mathrm{d}t = \\ &= \left(s+\frac{R}{n}\right)^A \left[\log(2s) - \log\frac{R}{n}\right] = \left(s+\frac{R}{n}\right)^A \log\left(\frac{2ns}{R}\right) = \\ &= \left(s+\frac{R}{n}\right)^A \left[\log 2 + \log\frac{ns}{R}\right] \le \left(s+\frac{R}{n}\right)^A \left[\log\left(\frac{ns}{R}+1\right)+1\right]. \end{aligned}$$

If $3s \le t$ then $s \le \frac{1}{3}\,t$, i.e. $s + \frac{2}{3}\,t \le t$. Thus

$$\tfrac{2}{3}\,t \le t - s \le t.$$

Therefore for I_2 we get

$$\begin{aligned} I_2 &= \int_{3s}^{\frac{2\pi}{3}} \frac{t^A}{t-s}\,\mathrm{d}t \le \frac{3}{2}\int_{3s}^{\frac{2\pi}{3}} t^{A-1}\,\mathrm{d}t = \frac{3}{2|A|}\left[(3s)^A - \left(\frac{2\pi}{3}\right)^A\right] \le \\ &\le \frac{3}{2|A|}(2s)^A \le \frac{3}{2|A|}\left(s+\frac{R}{n}\right)^A. \end{aligned}$$

Summarizing the above formulas we obtain that there exists a constant $c > 0$ independent of n and s such that

$$I \le c\left(s+\frac{R}{n}\right)^A \left[\log\left(\frac{ns}{R}+1\right)+1\right] \tag{3.13}$$
$$\left(s \in \left(\tfrac{R}{n}, \tfrac{2\pi}{9}\right),\ A < 0,\ R \ge 1,\ n \ge \tfrac{6R}{\pi}\right).$$

For the lower estimation of I it is enough to consider the integral I_1. Since

$s+\frac{R}{n} \le t \le 3s \le 3\left(s+\frac{R}{n}\right)$, thus by $A<0$ we get that

$$\begin{aligned} I_1 = \int_{s+\frac{R}{n}}^{3s} \frac{t^A}{t-s}\,\mathrm{d}t \ge 3^A\left(s+\frac{R}{n}\right)^A \int_{s+\frac{R}{n}}^{3s} \frac{1}{t-s}\,\mathrm{d}t = \\ = 3^A\left(s+\frac{R}{n}\right)^A\left(\log(2s)-\log\frac{R}{n}\right) = \\ = 3^A\left(s+\frac{R}{n}\right)^A \log\left(2\frac{ns}{R}\right). \end{aligned} \tag{3.14}$$

The following inequality holds:

$$\frac{\log(2x)}{\log(x+1)+1} > \frac{\log 2}{1+\log 2} \qquad (x \ge 1). \tag{3.15}$$

Indeed, if $x \ge 1$ then

$$\begin{aligned} \frac{\log(2x)}{\log(x+1)+1} \ge \frac{\log(2x)}{\log(2x)+1} = 1-\frac{1}{\log(2x)+1} \ge \\ \ge 1-\frac{1}{1+\log 2} = \frac{\log 2}{1+\log 2}. \end{aligned}$$

Since $\frac{ns}{R} \ge 1$ we obtain from (3.14) and (3.15) that

$$I \ge I_1 \ge \frac{3^A \log 2}{1+\log 2}\left(s+\frac{R}{n}\right)^A\left[\log\left(\frac{ns}{R}+1\right)+1\right],$$

which together with (3.13) prove (3.9), if $\frac{R}{n} < s < \frac{2\pi}{9}$.

CASE 3. Let $\frac{2\pi}{9} \le s \le \frac{\pi}{2}$ and $t \in \left[s+\frac{R}{n}, \frac{2\pi}{3}\right]$. Then

$$s+\frac{R}{n} \le t \le \frac{2\pi}{3} \le 3s \le 3\left(s+\frac{R}{n}\right), \tag{3.16}$$

so we have the following upper estimation of I:

$$\begin{aligned} I = \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \frac{t^A}{t-s}\,\mathrm{d}t \le \left(s+\frac{R}{n}\right)^A \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \frac{1}{t-s}\,\mathrm{d}t = \\ = \left(s+\frac{R}{n}\right)^A\left[\log\left(\frac{2\pi}{3}-s\right)-\log\frac{R}{n}\right] = \\ = \left(s+\frac{R}{n}\right)^A \log\left[\frac{n}{R}\left(\frac{2\pi}{3}-s\right)\right] \le \left(s+\frac{R}{n}\right)^A \log\left(\frac{2ns}{R}\right) = \\ = \left(s+\frac{R}{n}\right)^A\left[\log\frac{ns}{R}+\log 2\right] \le \left(s+\frac{R}{n}\right)^A\left[\log\left(\frac{ns}{R}+1\right)+1\right]. \end{aligned} \tag{3.17}$$

For the lower estimation of I we use the condition $A<0$ and (3.16). Then we have

$$I=\int\limits_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \frac{t^A}{t-s}\,\mathrm{d}t \geq 3^A\left(s+\frac{R}{n}\right)^A \int\limits_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \frac{1}{t-s}\,\mathrm{d}t =$$

$$(3.18)\qquad = 3^A\left(s+\frac{R}{n}\right)^A \log\left[\frac{n}{R}\left(\frac{2\pi}{3}-s\right)\right] =$$

$$= 3^A\left(s+\frac{R}{n}\right)^A \log\left(\frac{\pi}{2}\cdot\frac{4}{3}\frac{n}{R}-\frac{ns}{R}\right) \geq$$

$$\geq 3^A\left(s+\frac{R}{n}\right)^A \log\left(\frac{1}{3}\frac{ns}{R}\right).$$

The following inequality is true:

$$(3.19)\qquad \frac{\log\left(\frac{1}{3}\,x\right)}{\log(x+1)+1} > \frac{\log\frac{4}{3}}{\log(8e)} \qquad (x\geq 4).$$

Indeed, if $x\geq 4$, then

$$\frac{\log\left(\frac{1}{3}\,x\right)}{\log(x+1)+1} > \frac{\log\left(\frac{1}{3}\,x\right)}{\log(2x)+1} = \frac{\log\left(\frac{1}{3}\,x\right)}{\log\left(\frac{1}{3}\,x\right)+\log 6+1} =$$

$$= 1-\frac{\log(6e)}{\log\left(\frac{1}{3}\,x\right)+\log(6e)} \geq 1-\frac{\log(6e)}{\log\frac{4}{3}+\log(6e)} = \frac{\log\frac{4}{3}}{\log(8e)}.$$

Let $n\geq\frac{18R}{\pi}$. Then $\frac{ns}{R}\geq\frac{n}{R}\frac{2\pi}{9}\geq 4$. Thus using (3.18) and (3.19) we obtain

$$I\geq 3^A\frac{\log\frac{4}{3}}{\log(8e)}\left(s+\frac{R}{n}\right)^A\left[\log\left(\frac{ns}{R}+1\right)+1\right],$$

which together with (3.17) prove (3.9), if $\frac{2\pi}{9}\leq s\leq\frac{\pi}{2}$.

Lemma 1 is proved. ■

Lemma 2. *If $A>-1$, $n\in\mathbb{N}$ and $s\in\left(\frac{1}{n},\frac{\pi}{2}\right]$, then there exists a constant $c>0$ independent from s and n such that*

$$\int\limits_0^{s-\frac{1}{n}} \frac{t^A}{s-t}\,\mathrm{d}t \leq c\left(s+\frac{1}{n}\right)^A \log(ns+1).$$

Proof. Consider the following identity:

$$\int_0^{s-\frac{1}{n}} \frac{t^A}{s-t}\,\mathrm{d}t = \frac{1}{s}\int_0^{s-\frac{1}{n}} \frac{t^A[(s-t)+t]}{s-t}\,\mathrm{d}t =$$

$$= \frac{1}{s}\int_0^{s-\frac{1}{n}} t^A\,\mathrm{d}t + \frac{1}{s}\int_0^{s-\frac{1}{n}} \frac{t^{A+1}}{s-t}\,\mathrm{d}t =: I_1 + I_2.$$

For I_1 we have

$$I_1 = \frac{1}{s}\int_0^{s-\frac{1}{n}} t^A\,\mathrm{d}t = \frac{1}{s}\,\frac{\left(s-\frac{1}{n}\right)^{A+1}}{A+1} \le c\,s^A,$$

where $c>0$ is independent of s and n. From $A+1>0$ it follows that

$$I_2 = \frac{1}{s}\int_0^{s-\frac{1}{n}} \frac{t^{A+1}}{s-t}\,\mathrm{d}t \le s^A \int_0^{s-\frac{1}{n}} \frac{1}{s-t}\,\mathrm{d}t = s^A \log(ns),$$

therefore

$$I_1 + I_2 \le c\,s^A\big(1+\log(ns)\big) \le c\,s^A \log(ns+1).$$

Since

$$\frac{1}{2} \le \frac{s}{s+\frac{1}{n}} = 1 - \frac{1}{ns+1} \le 1,$$

we have that there exists a $c>0$ independent of s and n such that

$$s^A \le c\left(s+\frac{1}{n}\right)^A,$$

which proves our statement. ■

4. Proof of Theorem

In this section we shall use the following notations:

$$P_n(x) := P_n^{(\alpha,\beta)}(x), \quad \lambda_n := \lambda_n^{(\alpha,\beta)}.$$

By (3.1) we have the following symmetry property of the kernel function (2.2)

$$K_n^{(\alpha,\beta)}(x,y)=K_n^{(\beta,\alpha)}(-x,-y)$$
$$\big(x,y\in[-1,1],\quad n\in\mathbb{N},\quad \alpha,\beta>-1\big).$$

Using this we obtain the symmetry property of the weighted Lebesgue function:

$$L_n^{(\alpha,\beta),(\gamma,\delta)}(-x)=L_n^{(\beta,\alpha),(\delta,\gamma)}(x)$$
$$\big(x,y\in[-1,1],\quad n\in\mathbb{N},\quad \alpha,\beta>-1,\quad \gamma,\delta\ge 0\big), \tag{4.1}$$

which means that it is enough to prove (2.8) for $x\in[0,1]$ only.

From now on we will assume that $x\in[0,1]$.

In what follows, C or c (or $C_1, C_2, \dots, c_1, c_2, \dots$) will always denote a positive constant (not necessarily the same at different occurrences) independent of n and x. Also, N will always denote a fixed natural number, not necessarily the same at different occurrences.

4.1. Upper estimation of $L_n^{(\alpha,\beta),(\gamma,\delta)}(x)$. In order to estimate (2.5) we split the integral into two parts:

$$\int_{-1}^{1}|K_n^{(\alpha,\beta)}(x,y)|w^{(\alpha-\gamma,\beta-\delta)}(y)\,\mathrm{d}y=\int_{-1}^{-\frac12}\dots\,\mathrm{d}y\;+\int_{-\frac12}^{1}\dots\,\mathrm{d}y.$$

In the second integral we use the substitutions

$$y=\cos t\quad \left(0\le t\le \tfrac{2\pi}{3}\right)\quad\text{and}\quad x=\cos s\quad \left(0\le s\le \tfrac{\pi}{2}\right),$$

and consider the following two cases:

$$\text{(i)}\quad \tfrac1n\le s\le\tfrac{\pi}{2}\quad\text{and}\quad\text{(ii)}\quad 0\le s\le\tfrac1n.$$

In the first case we split the second integral into three parts:

$$\int_{-\frac12}^{1}\dots\,\mathrm{d}y=\int_{0}^{\frac{2\pi}{3}}\dots\,\mathrm{d}t=\int_{0}^{s-\frac1n}\dots\,\mathrm{d}t+\int_{s-\frac1n}^{s+\frac1n}\dots\,\mathrm{d}t+\int_{s+\frac1n}^{\frac{2\pi}{3}}\dots\,\mathrm{d}t.$$

Thus we have

$$L_n^{(\alpha,\beta),(\gamma,\delta)}(x)=:\sum_{k=1}^{4}J_k,$$

where

$$J_1 = w^{(\gamma,\delta)}(x) \int_{-1}^{-\frac{1}{2}} |K_n^{(\alpha,\beta)}(x,y)| w^{(\alpha-\gamma,\beta-\delta)}(y)\,\mathrm{d}y,$$

$$J_2 = w^{(\gamma,\delta)}(x) \int_{s+\frac{1}{n}}^{\frac{2\pi}{3}} |K_n^{(\alpha,\beta)}(x,\cos t)| w^{(\alpha-\gamma,\beta-\delta)}(\cos t)\sin t\,\mathrm{d}t,$$

$$J_3 = w^{(\gamma,\delta)}(x) \int_{s-\frac{1}{n}}^{s+\frac{1}{n}} |K_n^{(\alpha,\beta)}(x,\cos t)| w^{(\alpha-\gamma,\beta-\delta)}(\cos t)\sin t\,\mathrm{d}t,$$

$$J_4 = w^{(\gamma,\delta)}(x) \int_{0}^{s-\frac{1}{n}} |K_n^{(\alpha,\beta)}(x,\cos t)| w^{(\alpha-\gamma,\beta-\delta)}(\cos t)\sin t\,\mathrm{d}t.$$

In the second case the lower bound in J_3 is 0 and $J_4 := 0$.

4.1.1. Estimation of J_1. Here we use the formula (2.2). Since $x \geq 0$ we have $|x-y| \geq \frac{1}{2}$ $(-1 \leq y \leq -\frac{1}{2})$. Consequently,

$$J_1 = w^{(\gamma,\delta)}(x) \int_{-1}^{-\frac{1}{2}} \lambda_n \frac{|P_{n+1}(x)P_n(y) - P_n(x)P_{n+1}(y)|}{|x-y|} w^{(\alpha-\gamma,\beta-\delta)}(y)\,\mathrm{d}y \leq$$

$$\leq 2\lambda_n w^{(\gamma,\delta)}(x)|P_n(x)| \int_{-1}^{-\frac{1}{2}} |P_{n+1}(y)| w^{(\alpha-\gamma,\beta-\delta)}(y)\,\mathrm{d}y +$$

$$+2\lambda_n w^{(\gamma,\delta)}(x)|P_{n+1}(x)| \int_{-1}^{-\frac{1}{2}} |P_n(y)| w^{(\alpha-\gamma,\beta-\delta)}(y)\,\mathrm{d}y.$$

By (3.1) we have

$$\int_{-1}^{-\frac{1}{2}} |P_n(y)| w^{(\alpha-\gamma,\beta-\delta)}(y)\,\mathrm{d}y = \int_{-1}^{-\frac{1}{2}} |P_n^{(\alpha,\beta)}(y)|(1-y)^{\alpha-\gamma}(1+y)^{\beta-\delta}\,\mathrm{d}y \leq$$

$$\leq c \int_{-1}^{-\frac{1}{2}} |P_n^{(\alpha,\beta)}(y)|(1+y)^{\beta-\delta}\,\mathrm{d}y = c\int_{\frac{1}{2}}^{1} |P_n^{(\beta,\alpha)}(y)|(1-y)^{\beta-\delta}\,\mathrm{d}y \leq$$

$$\leq c \int_0^1 |P_n^{(\beta,\alpha)}(y)|(1-y)^{\beta-\delta}\,\mathrm{d}y.$$

Since $\delta < \frac{\beta}{2} + \frac{3}{4}$, i.e. $2(\beta-\delta) > \beta - \frac{3}{2}$ it follows by (3.5) that the last integral has the upper bound $cn^{-\frac{1}{2}}$. Consequently,

$$\int_{-1}^{-\frac{1}{2}} |P_n(y)| w^{(\alpha-\gamma,\beta-\delta)}(y)\,\mathrm{d}y = O\big(n^{-\frac{1}{2}}\big) \qquad (n \in \mathbb{N}).$$

Collecting the above formulas and using (3.6) we obtain

$$J_1 = O(\sqrt{n}) w^{(\gamma,\delta)}(x) \Big(|P_n^{(\alpha,\beta)}(x)| + |P_{n+1}^{(\alpha,\beta)}(x)| \Big) \qquad \big(x \in [0,1],\ n \in \mathbb{N}\big). \tag{4.2}$$

4.1.2. Estimation of J_2. The expression

$$J_2 = w^{(\gamma,\delta)}(x) \int_{s+\frac{1}{n}}^{\frac{2\pi}{3}} |K_n^{(\alpha,\beta)}(x,\cos t)| w^{(\alpha-\gamma,\beta-\delta)}(\cos t) \sin t\,\mathrm{d}t$$

may be simplified by using the following formulas:

$$w^{(\gamma,\delta)}(x) = (1-x)^\gamma (1+x)^\delta \sim (1-x)^\gamma \qquad \big(x \in [0,1]\big),$$

$$w^{(\alpha-\gamma,\beta-\delta)}(\cos t)\sin t = (1-\cos t)^{\alpha-\gamma}(1+\cos t)^{\beta-\delta}\sin t \sim t^{2(\alpha-\gamma)+1} \qquad \big(t \in [0, \tfrac{2\pi}{3}]\big),$$

$$x - y = \cos s - \cos t = 2 \sin\frac{t+s}{2} \sin\frac{t-s}{2} \sim t^2 - s^2 \sim t(t-s) \qquad \big(s \in [0, \tfrac{\pi}{2}],\quad t \in [s, \tfrac{2\pi}{3}]\big).$$

Thus by (2.2) and (3.6) we have uniformly in $x \in [0,1]$ and $n \in \mathbb{N}$ that

$$J_2 \sim (1-x)^\gamma \int_{s+\frac{1}{n}}^{\frac{2\pi}{3}} |K_n^{(\alpha,\beta)}(x,\cos t)|\, t^{2(\alpha-\gamma)+1}\,\mathrm{d}t \sim$$

$$\sim n(1-x)^\gamma \int_{s+\frac{1}{n}}^{\frac{2\pi}{3}} \Big| P_{n+1}(x)P_n(\cos t) - P_n(x)P_{n+1}(\cos t) \Big| \frac{t^{2(\alpha-\gamma)}}{t-s}\,\mathrm{d}t.$$

Following the idea of [1, p. 15] we use the identity

$$P_{n+1}(y)P_n(x) - P_n(y)P_{n+1}(x) =$$

(4.3)
$$= \left(1 + \frac{\alpha+\beta}{2n+2}\right)\left[(1-x)\overline{P}_n(x)P_n(y) - (1-y)\overline{P}_n(y)P_n(x)\right],$$

which may be verified by using (3.8).

Thus we have uniformly in $x \in [0,1]$ and $n \in \mathbb{N}$ that

$$J_2 = O(n)(1-x)^{\gamma+1}|\overline{P}_n(x)| \int_{s+\frac{1}{n}}^{\frac{2\pi}{3}} |P_n(\cos t)| \frac{t^{2(\alpha-\gamma)}}{t-s}\, \mathrm{d}t +$$

$$+ O(n)(1-x)^{\gamma}|P_n(x)| \int_{s+\frac{1}{n}}^{\frac{2\pi}{3}} |\overline{P}_n(\cos t)| \frac{t^{2(\alpha-\gamma)+2}}{t-s}\, \mathrm{d}t =$$

$$= O(\sqrt{n})(1-x)^{\gamma+1}|\overline{P}_n(x)| \int_{s+\frac{1}{n}}^{\frac{2\pi}{3}} \frac{t^{\alpha-2\gamma-\frac{1}{2}}}{t-s}\, \mathrm{d}t +$$

$$+ O(\sqrt{n})(1-x)^{\gamma}|P_n(x)| \int_{s+\frac{1}{n}}^{\frac{2\pi}{3}} \frac{t^{\alpha-2\gamma+\frac{1}{2}}}{t-s}\, \mathrm{d}t =: J_{21} + J_{22},$$

where we used (3.3) and $\sqrt{1-\cos t} \sim t$ ($t \in [0, \frac{2\pi}{3}]$).

From the condition $\frac{\alpha}{2} + \frac{1}{4} < \gamma$ it follows that $\alpha - 2\gamma - \frac{1}{2} < -1$, so by Lemma 1, $s \sim \sqrt{1-x}$ ($\cos s = x \in [0,1]$) and (3.3) we obtain

$$J_{21} = O(\sqrt{n})(1-x)^{\gamma+1}|\overline{P}_n(x)| \int_{s+\frac{1}{n}}^{\frac{2\pi}{3}} \frac{t^{\alpha-2\gamma-\frac{1}{2}}}{t-s}\, \mathrm{d}t =$$

$$= O(1)\left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma+2} (\log(n\sqrt{1-x}+1)+1).$$

Similarly, for J_{22} we have (since $\alpha - 2\gamma + \frac{1}{2} \in (-1,0)$)

$$J_{22} = O(\sqrt{n})(1-x)^{\gamma}|P_n(x)| \int_{s+\frac{1}{n}}^{\frac{2\pi}{3}} \frac{t^{\alpha-2\gamma+\frac{1}{2}}}{t-s}\, \mathrm{d}t =$$

$$= O(1)\left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma}\Big(\log\left(n\sqrt{1-x}+1\right)+\sqrt{n}\left(\sqrt{1-x}+\tfrac{1}{n}\right)^{\alpha+\frac{1}{2}}|P_n(x)|\Big).$$

Finally we obtain the estimate

$$\begin{aligned} J_2 = O(1)\left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma}\Big(\log\left(n\sqrt{1-x}+1\right)+ \\ +\sqrt{n}\left(\sqrt{1-x}+\tfrac{1}{n}\right)^{\alpha+\frac{1}{2}}\left(|P_n(x)|+|P_{n+1}(x)|\right)+1\Big), \end{aligned} \tag{4.4}$$

which holds uniformly in $x \in [0,1]$ and $n \in \mathbb{N}$, $n > N$.

4.1.3. Estimation of J_3*.* The expression J_3 may be simplified (see the estimate of J_2):

$$J_3 \sim (1-x)^{\gamma} \int_{s-\frac{1}{n}}^{s+\frac{1}{n}} |K_n^{(\alpha,\beta)}(x,\cos t)|\, t^{2(\alpha-\gamma)+1}\,\mathrm{d}t$$
$$(x \in [0,1],\ s \in [0, \tfrac{\pi}{2}]),$$

if $s \geq \frac{1}{n}$ (the lower bound of the integral is 0 if $0 \leq s \leq \frac{1}{n}$). For the kernel function we shall use the following estimates (see (3.3) and (3.6))

$$\begin{aligned} \left|K_n^{(\alpha,\beta)}(x,\cos t)\right| &= \Big|\sum_{k=0}^{n}\frac{1}{h_k}P_k(x)P_k(\cos t)\Big| = \Big|\frac{1}{h_0}+\sum_{k=1}^{n}\frac{1}{h_k}P_k(x)P_k(\cos t)\Big| = \\ &= O(1)\Big(1+\sum_{k=1}^{n}k|P_k(x)|\,|P_k(\cos t)|\Big) = \\ &= O(1)\Big(1+\sum_{k=1}^{n}k\,k^{-\frac{1}{2}}\left(\sqrt{1-x}+\tfrac{1}{k}\right)^{-\alpha-\frac{1}{2}}k^{-\frac{1}{2}}\left(t+\tfrac{1}{k}\right)^{-\alpha-\frac{1}{2}}\Big) = \\ &= O(1)\Big(1+n\left(\sqrt{1-x}+\tfrac{1}{n}\right)^{-\alpha-\frac{1}{2}}t^{-\alpha-\frac{1}{2}}\Big) \\ &\qquad (x \in [0,1],\, t \in [0, \tfrac{2\pi}{3}]). \end{aligned}$$

If $\frac{1}{n} < s \leq \frac{\pi}{2}$ then we have uniformly in $x = \cos s$ that

$$J_3 = O(1)\,(1-x)^{\gamma}\left\{\int_{s-\frac{1}{n}}^{s+\frac{1}{n}} t^{2(\alpha-\gamma)+1}\,\mathrm{d}t + \frac{n}{\left(\sqrt{1-x}+\frac{1}{n}\right)^{\alpha+\frac{1}{2}}}\int_{s-\frac{1}{n}}^{s+\frac{1}{n}} t^{\alpha-2\gamma+\frac{1}{2}}\,\mathrm{d}t\right\}.$$

Since

$$\int_{s-\frac{1}{n}}^{s+\frac{1}{n}} t^A \sim \frac{s^A}{n} \qquad \left(\tfrac{1}{n} \le s \le \pi,\ n \in \mathbb{N},\ A > -1\right),$$

we obtain by $s \sim \sqrt{1-x}$ that

$$J_3 = O(1)(1-x)^\gamma \left\{ \frac{s^{2(\alpha-\gamma)+1}}{n} + \frac{s^{\alpha-2\gamma+\frac{1}{2}}}{\left(\sqrt{1-x}+\frac{1}{n}\right)^{\alpha+\frac{1}{2}}} \right\} =$$

$$= O(1)(1-x)^\gamma \left\{ s^{2(\alpha-\gamma+1)} + \frac{1}{\left(\sqrt{1-x}+\frac{1}{n}\right)^{2\gamma}} \right\} =$$

$$= O(1)(1-x)^\gamma \frac{1}{\left(\sqrt{1-x}+\frac{1}{n}\right)^{2\gamma}} = O(1)\left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}} \right)^{2\gamma}.$$

If $0 \le s \le \frac{1}{n}$ then (see the definition of J_3 in Section 4.1) we get

$$J_3 = O(1)\,(1-x)^\gamma \left\{ \int_0^{s+\frac{1}{n}} t^{2(\alpha-\gamma)+1}\,\mathrm{d}t + \frac{n}{\left(\sqrt{1-x}+\frac{1}{n}\right)^{\alpha+\frac{1}{2}}} \int_0^{s+\frac{1}{n}} t^{\alpha-2\gamma+\frac{1}{2}}\,\mathrm{d}t \right\}.$$

Since $\gamma < \alpha+1$ and $\gamma < \frac{\alpha}{2}+\frac{3}{4}$ we have $2(\alpha-\gamma)+1 > -1$ and $\alpha-2\gamma+\frac{1}{2} > -1$. So by

$$\int_0^{s+\frac{1}{n}} t^A\,\mathrm{d}t \sim \left(s+\frac{1}{n}\right)^{A+1} \qquad (s \ge 0,\ A > -1)$$

we obtain

$$J_3 = O(1)\,(1-x)^\gamma \left\{ \left(s+\tfrac{1}{n}\right)^{2(\alpha-\gamma)+2} + \frac{n\left(s+\frac{1}{n}\right)^{\alpha-2\gamma+\frac{3}{2}}}{\left(\sqrt{1-x}+\frac{1}{n}\right)^{\alpha+\frac{1}{2}}} \right\} =$$

$$= O(1)\,(1-x)^\gamma \left\{ \frac{1}{n^{2(\alpha+1-\gamma)}} + n\left(\sqrt{1-x}+\tfrac{1}{n}\right) \frac{1}{\left(\sqrt{1-x}+\frac{1}{n}\right)^{2\gamma}} \right\} =$$

$$= O(1)\,(1-x)^\gamma \left\{ 1 + \frac{1}{\left(\sqrt{1-x}+\frac{1}{n}\right)^{2\gamma}} \right\} =$$

$$= O(1)\,(1-x)^\gamma \frac{1}{\left(\sqrt{1-x}+\frac{1}{n}\right)^{2\gamma}} = O(1)\,\left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}} \right)^{2\gamma}.$$

Finally we get the estimate

$$J_3 = O(1)\left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma}, \tag{4.5}$$

which holds uniformly in $x \in [0,1]$ and $n \in \mathbb{N}$.

4.1.4. Estimation of J_4. First we remark that $J_4 = 0$ if $0 \le s \le \frac{1}{n}$, so we suppose that $s \in \left[\frac{1}{n}, \frac{\pi}{2}\right]$, i.e. $x = \cos s \in \left[0, 1 - \frac{c}{n^2}\right] =: I_n$. The expression J_4 may be simplified (see the estimation of J_2) by using the relation

$$|x-y| \sim |t^2 - s^2| \sim s|t-s| \sim \sqrt{1-x}|t-s|$$
$$\left(\tfrac{1}{n} \le s \le \tfrac{\pi}{2}, \quad t \in \left[0, s-\tfrac{1}{n}\right]\right).$$

Namely, we have (uniformly in $x \in I_n$ and $n \in \mathbb{N}$)

$$J_4 = w^{(\gamma,\delta)}(x) \int_0^{s-\frac{1}{n}} |K_n^{(\alpha,\beta)}(x,\cos t)| w^{(\alpha-\gamma,\beta-\delta)}(\cos t) \sin t \, \mathrm{d}t \sim$$

$$\sim n(1-x)^{\gamma-\frac{1}{2}} \int_0^{s-\frac{1}{n}} |P_{n+1}(x)P_n(\cos t) - P_n(x)P_{n+1}(\cos t)| \, \frac{t^{2(\alpha-\gamma)+1}}{s-t} \, \mathrm{d}t.$$

Using the identity (4.3) and the estimate (3.3) we obtain

$$J_4 = O(n)(1-x)^{\gamma-\frac{1}{2}} \left\{ (1-x)|\overline{P}_n(x)| \int_0^{s-\frac{1}{n}} |P_n(\cos t)| \frac{t^{2(\alpha-\gamma)+1}}{s-t} \, \mathrm{d}t + \right.$$

$$\left. + |P_n(x)| \int_0^{s-\frac{1}{n}} t^2 \, |\overline{P}_n(\cos t)| \frac{t^{2(\alpha-\gamma)+1}}{s-t} \, \mathrm{d}t \right\} =$$

$$= O(\sqrt{n})(1-x)^{\gamma+\frac{1}{2}} |\overline{P}_n(x)| \int_0^{s-\frac{1}{n}} \frac{t^{\alpha-2\gamma+\frac{1}{2}}}{s-t} \, \mathrm{d}t +$$

$$+ O(\sqrt{n})(1-x)^{\gamma-\frac{1}{2}} |P_n(x)| \int_0^{s-\frac{1}{n}} \frac{t^{\alpha-2\gamma+\frac{3}{2}}}{s-t} \, \mathrm{d}t =: J_{41} + J_{42}$$
$$\left(\tfrac{1}{n} \le s = \arccos x \le \tfrac{\pi}{2}, \quad n \in \mathbb{N}\right).$$

Since $\gamma < \frac{\alpha}{2} + \frac{3}{4}$, thus $\alpha - 2\gamma + \frac{1}{2} > -1$ we have by using Lemma 2 and $s \sim \sqrt{1-x}$ that

$$J_{41} = O(\sqrt{n})(1-x)^{\gamma+\frac{1}{2}} \left|\overline{P}_n(x)\right| \left(s + \tfrac{1}{n}\right)^{\alpha-2\gamma+\frac{1}{2}} \log(ns+1) =$$
$$= O(\sqrt{n}) \left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma} \left|\overline{P}_n(x)\right| \left(\sqrt{1-x} + \tfrac{1}{n}\right)^{\alpha+\frac{3}{2}} \log(ns+1) =$$
$$= O(1) \left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma} \log\left(n\sqrt{1-x}+1\right)$$
$$\big(x \in I_n, \quad n \in \mathbb{N}\big).$$

Similarly,

$$J_{42} = O(\sqrt{n})(1-x)^{\gamma-\frac{1}{2}} |P_n(x)| \left(s + \tfrac{1}{n}\right)^{\alpha-2\gamma+\frac{3}{2}} \log(ns+1) =$$
$$= O(\sqrt{n}) \left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma} \left|P_n(x)\right| \frac{\left(\sqrt{1-x} + \frac{1}{n}\right)^{\alpha+\frac{3}{2}}}{\sqrt{1-x}} \log(ns+1) =$$
$$= O(1) \left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma} \log\left(n\sqrt{1-x}+1\right)$$
$$\big(x \in I_n, \quad n \in \mathbb{N}\big).$$

Summarizing the above formulas we obtain

$$J_4 = O(1) \left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma} \log\left(n\sqrt{1-x}+1\right) \qquad (4.6)$$
$$\big(x \in I_n, \quad n \in \mathbb{N}\big).$$

4.1.5. The final upper estimate. Using (4.2), (4.4), (4.5) and (4.6) we have

$$L_n^{(\alpha,\beta),(\gamma,\delta)}(x) = O(1) \left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma} \Big(\log\left(n\sqrt{1-x}+1\right) +$$
$$\sqrt{n}\left(\sqrt{1-x}+\tfrac{1}{n}\right)^{\alpha+\frac{1}{2}} \left(|P_n(x)| + |P_{n+1}(x)|\right) + 1\Big)$$
$$(x \in [0,1],\ n \in \mathbb{N},\ n > N).$$

Let $\bar{x} \in (0,1)$ be the closest number to 1 for which

$$P_n(\bar{x}) = \frac{1}{2} P_n(1) \sim n^\alpha$$

holds. If $x \in [0,\bar{x}]$ then

$$1-x \geq 1-\bar{x} = \frac{P_n(1)-P_n(\bar{x})}{P_n'(\xi)} \sim \frac{1}{n^2} \qquad \big(\xi \in (\bar{x},1)\big) \tag{4.7}$$

(see (3.2)). Thus

$$\log\left(n\sqrt{1-x}+1\right) \geq c.$$

If $x \in (\bar{x},1]$ then $P_n(x) \sim n^\alpha$, so

$$\sqrt{n}\left(\sqrt{1-x}+\tfrac{1}{n}\right)^{\alpha+\frac{1}{2}}\left(|P_n(x)|+|P_{n+1}(x)|\right) \geq c.$$

This means that also

$$L_n^{(\alpha,\beta),(\gamma,\delta)}(x) = O(1)\left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma}\Big(\log\left(n\sqrt{1-x}+1\right)+$$

$$\sqrt{n}\left(\sqrt{1-x}+\tfrac{1}{n}\right)^{\alpha+\frac{1}{2}}\left(|P_n(x)|+|P_{n+1}(x)|\right)\Big)$$

$$(x \in [0,1],\ n \in \mathbb{N},\ n > N)$$

is true.

From this we have uniformly in $x \in [-1,1]$ and $n \in \mathbb{N}$, $n > N$ that

$$L_n^{(\alpha,\beta),(\gamma,\delta)}(x) = O(1)\left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma}\left(\frac{\sqrt{1+x}}{\sqrt{1+x}+\frac{1}{n}}\right)^{2\delta}\phi_n^{(\alpha,\beta)}(x),$$

where

$$\phi_n^{(\alpha,\beta)}(x) = \log\Big(n\sqrt{1-x^2}+1\Big)+$$

$$+\sqrt{n}\left(\sqrt{1-x}+\tfrac{1}{n}\right)^{\alpha+\frac{1}{2}}\left(\sqrt{1+x}+\tfrac{1}{n}\right)^{\beta+\frac{1}{2}}\Big(|P_n^{(\alpha,\beta)}(x)|+|P_{n+1}^{(\alpha,\beta)}(x)|\Big).$$

Thus the upper estimation in (2.8) is proved.

4.2. Lower estimation of $L_n^{(\alpha,\beta),(\gamma,\delta)}(x)$. Because of symmetry, it is enough to consider $x \in [0,1]$. We shall give three different lower estimations for the weighted Lebesgue function.

4.2.1. The first lower estimation. If $\alpha,\beta > -1$ and $\gamma,\delta \geq 0$, then there exists a constant $c > 0$ independent of x and n such that

$$L_n^{(\alpha,\beta),(\gamma,\delta)}(x) \geq c\, w^{(\gamma,\delta)}(x) \qquad \big(x \in [0,1],\ n \in \mathbb{N}\big). \tag{4.8}$$

Indeed, using the orthogonality of Jacobi polynomials we have

$$\int_{-1}^{1} K_n^{(\alpha,\beta)}(x,y) w^{(\alpha,\beta)}(y)\,\mathrm{d}y = 1 \quad \big(x \in [0,1],\ n \in \mathbb{N}\big).$$

Therefore

$$\begin{aligned} L_n^{(\alpha,\beta),(\gamma,\delta)}(x) &= w^{(\gamma,\delta)}(x) \int_{-1}^{1} \big|K_n^{(\alpha,\beta)}(x,y)\big| \frac{w^{(\alpha,\beta)}(y)}{(1-y)^{\gamma}(1+y)^{\delta}}\,\mathrm{d}y \geq \\ &\geq c\, w^{(\gamma,\delta)}(x) \int_{-1}^{1} \big|K_n^{(\alpha,\beta)}(x,y)\big| w^{(\alpha,\beta)}(y)\,\mathrm{d}y \geq \\ &\geq c\, w^{(\gamma,\delta)}(x) \int_{-1}^{1} K_n^{(\alpha,\beta)}(x,y) w^{(\alpha,\beta)}(y)\,\mathrm{d}y = c\, w^{(\gamma,\delta)}(x). \end{aligned}$$

4.2.2. The second lower estimation. If $\alpha, \beta > -1$ and $\gamma, \delta \geq 0$, then there exists a constant $c > 0$ independent of x and n such that

$$L_n^{(\alpha,\beta),(\gamma,\delta)}(x) \geq c\, w^{(\gamma,\delta)}(x) \sqrt{n}\,\big(|P_n(x)| + |P_{n+1}(x)|\big) \qquad (x \in [0,1],\ n \in \mathbb{N}). \tag{4.9}$$

In [1, p. 18] it was proven that

$$\int_{\frac{2\pi}{3}}^{\frac{5\pi}{6}} |K_n^{(\alpha,\beta)}(x,\cos t)|\,\mathrm{d}t \geq c\sqrt{n}\,\big(|P_n(x)| + |P_{n+1}(x)|\big), \qquad (x \in [0,1],\ n \in \mathbb{N}),$$

from which (4.9) follows immediately.

4.2.3. The third lower estimation. It is clear that

$$L_n^{(\alpha,\beta),(\gamma,\delta)}(x) \geq \geq w^{(\gamma,\delta)}(x) \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} |K_n^{(\alpha,\beta)}(x,\cos t)| w^{(\alpha-\gamma,\beta-\delta)}(\cos t)\sin t\,\mathrm{d}t \tag{4.10}$$

for all $x = \cos s \in [0,1]$ and $R > 0$. Using the ideas of [1], we shall give a lower estimation for the right hand side of (4.10) with a suitable number $R > 1$.

Since

$$w^{(\alpha-\gamma,\beta-\delta)}(\cos t)\sin t \sim t^{2\alpha-2\gamma+1}$$

$$\big(s \in [0, \tfrac{\pi}{2}], \quad t \in [s, \tfrac{2\pi}{3}]\big),$$

we obtain from (4.10) that

$$L_n^{(\alpha,\beta),(\gamma,\delta)}(x) \geq c\,(1-x)^{\gamma} \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} |K_n^{(\alpha,\beta)}(x,\cos t)| \cdot t^{2\alpha-2\gamma+1}\,\mathrm{d}t. \tag{4.11}$$

The estimation the above integral is performed in several steps.

Step 1. From (3.7) it follows that

$$F_n(x,y) := P_{n+1}(y)P_n(x) - P_n(y)P_{n+1}(x) = \frac{2n+\alpha+\beta+2}{4(n+1)} \times$$
$$\times \Big\{(1-x^2)\widetilde{P}_{n-1}(x)P_n(y) - (1-y^2)\widetilde{P}_{n-1}(y)P_n(x) + (y-x)P_n(x)P_n(y)\Big\},$$

so by (3.6) we have uniformly for all $x \in [0,1]$ and $n \in \mathbb{N}$ that

$$\big|K_n^{(\alpha,\beta)}(x,y)\big| = \lambda_n^{(\alpha,\beta)} \left|\frac{F_n(x,y)}{x-y}\right| \geq$$

$$\geq c\,n \left|\frac{(1-x^2)\widetilde{P}_{n-1}(x)P_n(y) - (1-y^2)\widetilde{P}_{n-1}(y)P_n(x)}{x-y} - P_n(x)P_n(y)\right| \geq$$

$$\geq c_1\,n \left|\frac{(1-x^2)\widetilde{P}_{n-1}(x)P_n(y) - (1-y^2)\widetilde{P}_{n-1}(y)P_n(x)}{x-y}\right| - c_2\,n\,|P_n(x)|\,|P_n(y)|.$$

Since $|x-y| = |\cos s - \cos t| \sim t(t-s)$ we have

$$\int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} |K_n^{(\alpha,\beta)}(x,\cos t)| \cdot t^{2\alpha-2\gamma+1}\,\mathrm{d}t \geq$$

$$\geq c_1 \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \Big|(1-x^2)\widetilde{P}_{n-1}(x)P_n(y) - (1-y^2)\widetilde{P}_{n-1}(y)P_n(x)\Big| \frac{t^{2\alpha-2\gamma}}{t-s}\,\mathrm{d}t -$$

$$-c_2\,n|P_n(x)| \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} |P_n(\cos t)|t^{2\alpha-2\gamma+1}\,\mathrm{d}t.$$

Therefore by (3.5) we get uniformly for all $x \in [0,1]$ and $n \in \mathbb{N}$ that

$$L_n^{(\alpha,\beta),(\gamma,\delta)}(x) \geq \tag{4.12}$$

$$\geq c_1\, n\,(1-x)^{\gamma} \int\limits_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \left|(1-x^2)\widetilde{P}_{n-1}(x)P_n(y)-(1-y^2)\widetilde{P}_{n-1}(y)P_n(x)\right| \frac{t^{2\alpha-2\gamma}}{t-s}\,\mathrm{d}t-$$

$$-c_2\,\sqrt{n}(1-x)^{\gamma}|P_n(x)|.$$

Step 2. For the estimation of the integral

$$I := \int\limits_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \left|(1-x^2)\widetilde{P}_{n-1}(x)P_n(y)-(1-y^2)\widetilde{P}_{n-1}(y)P_n(x)\right| \frac{t^{2\alpha-2\gamma}}{t-s}\,\mathrm{d}t$$

we use the asymptotic formula (3.4) for the Jacobi polynomials

$$P_n(y) = P_n^{(\alpha,\beta)}(y) \quad \text{and} \quad \widetilde{P}_{n-1}(y) = P_{n-1}^{(\alpha+1,\beta+1)}(y),$$

which gives

$$P_n^{(\alpha,\beta)}(\cos t) = \frac{k^{(\alpha,\beta)}(t)}{\sqrt{n}}\left(\cos(Nt+\nu)+\frac{O(1)}{n\sin t}\right),$$

$$P_{n-1}^{(\alpha+1,\beta+1)}(\cos t) = \frac{k^{(\alpha+1,\beta+1)}(t)}{\sqrt{n-1}}\left(\cos(\overline{N}t+\overline{\nu})+\frac{O(1)}{n\sin t}\right) =$$

$$= \frac{2\,k^{(\alpha,\beta)}(t)}{\sqrt{n-1}\,\sin t}\left(\cos(\overline{N}t+\overline{\nu})+\frac{O(1)}{(n-1)\sin t}\right),$$

where

$$\overline{N} = n-1+\frac{(\alpha+1)+(\beta+1)+1}{2} = N$$

and

$$\overline{\nu} = -\frac{2(\alpha+1)+1}{4}\,\pi = \nu-\frac{\pi}{2}.$$

We have

$$(1-x^2)\widetilde{P}_{n-1}(x)P_n(y)-(1-y^2)\widetilde{P}_{n-1}(y)P_n(x) =$$

$$= \frac{k^{(\alpha,\beta)}(t)}{\sqrt{n}}\Big\{(1-x^2)\widetilde{P}_{n-1}(x)\cos(Nt+\nu)-2\sqrt{\tfrac{n}{n-1}}P_n(x)\sin t\cdot\sin(Nt+\nu)\Big\}+$$

$$+O\left(\frac{1}{n^{3/2}}\right)(1-x^2)\widetilde{P}_{n-1}(x)\cdot\frac{k^{(\alpha,\beta)}(t)}{\sin t}+O\left(\frac{1}{(n-1)^{3/2}}\right)P_n(x)\cdot k^{(\alpha,\beta)}(t).$$

If $0 < s + \frac{R}{n} \le t \le \frac{2\pi}{3}$, then

$$k^{(\alpha,\beta)}(t) = \frac{1}{\sqrt{\pi}}\Big(\sin\tfrac{t}{2}\Big)^{-\alpha-\frac{1}{2}}\Big(\cos\tfrac{t}{2}\Big)^{-\beta-\frac{1}{2}} \sim t^{-\alpha-\frac{1}{2}}.$$

Therefore

$$I \ge \frac{c_1}{\sqrt{n}} \int\limits_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \Big|(1-x^2)\widetilde{P}_{n-1}(x)\cos(Nt+\nu) -$$

$$-2\sqrt{\tfrac{n}{n-1}}P_n(x)\sin t\cdot\sin(Nt+\nu)\Big|\,\frac{t^{\alpha-2\gamma-\frac{1}{2}}}{t-s}\,\mathrm{d}t -$$

$$-\frac{c_2}{n^{3/2}}\left\{(1-x^2)|\widetilde{P}_{n-1}(x)| \int\limits_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \frac{t^{\alpha-2\gamma-\frac{3}{2}}}{t-s}\,\mathrm{d}t + |P_n(x)| \int\limits_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \frac{t^{\alpha-2\gamma-\frac{1}{2}}}{t-s}\,\mathrm{d}t\right\}.$$

Step 3. Using the above inequality and (4.12) we have

$$L_n^{(\alpha,\beta),(\gamma,\delta)}(x) \ge c_1\sqrt{n}\,(1-x)^{\gamma}\times \tag{4.13}$$

$$\times\int\limits_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \Big|(1-x^2)\widetilde{P}_{n-1}(x)\cos(Nt+\nu) - 2\sqrt{\tfrac{n}{n-1}}P_n(x)\sin t\cdot\sin(Nt+\nu)\Big|\times$$

$$\times\frac{t^{\alpha-2\gamma-\frac{1}{2}}}{t-s}\,\mathrm{d}t - c_2\sqrt{n}\,(1-x)^{\gamma}|P_n(x)| - c_3\,\varrho_1(n,x),$$

where

$$\varrho_1(n,x) = \frac{(1-x)^{\gamma}}{\sqrt{n}}\times$$

$$\times\left\{(1-x^2)|\widetilde{P}_{n-1}(x)| \int\limits_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \frac{t^{\alpha-2\gamma-\frac{3}{2}}}{t-s}\,\mathrm{d}t + |P_n(x)| \int\limits_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \frac{t^{\alpha-2\gamma-\frac{1}{2}}}{t-s}\,\mathrm{d}t\right\}.$$

Since $t \ge \frac{R}{n}$ we have

$$\varrho_1(n,x) \le c\,\frac{\sqrt{n}}{R}(1-x)^{\gamma}\times$$

$$\times\left\{(1-x^2)|\widetilde{P}_{n-1}(x)| \int\limits_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \frac{t^{\alpha-2\gamma-\frac{1}{2}}}{t-s}\,\mathrm{d}t + |P_n(x)| \int\limits_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \frac{t^{\alpha-2\gamma+\frac{1}{2}}}{t-s}\,\mathrm{d}t\right\}.$$

Using Lemma 1, $s \sim \sqrt{1-x}$ and (3.3) we get uniformly for all $x \in [0,1]$ and $n \in \mathbb{N}$ that

$$\varrho_1(n,x) \le c \left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}} \right)^{2\gamma} \times$$
$$\times \left\{ \frac{1}{R} \left[\log\left(n\sqrt{1-x}+1\right)+1 \right] + \sqrt{n}\left(\sqrt{1-x}+\tfrac{1}{n}\right)^{\alpha+\frac{1}{2}} |P_n(x)| \right\}.$$

STEP 4. Now, we consider the integral in (4.13) and write $\sin s = \sqrt{1-x^2}$ instead of $\sin t$. Then by the Lagrange mean value theorem we have

$$\sin t = \sin s + \tau = \sqrt{1-x^2} + \tau$$

with $|\tau| \le t - s$. Thus we obtain an error term in the integral, which we shall denote by $\varrho_2(n,x)$. Therefore we have uniformly in $x \in [0,1]$ and $n \in \mathbb{N}$ that

$$L_n^{(\alpha,\beta),(\gamma,\delta)}(x) \ge c_1 \sqrt{n}\,(1-x)^\gamma \sqrt{1-x^2} \times$$
$$\times \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \left| \sqrt{1-x^2}\,\widetilde{P}_{n-1}(x)\cos(Nt+\nu) - 2\sqrt{\frac{n}{n-1}} P_n(x) \sin(Nt+\nu) \right| \times$$
$$\times \frac{t^{\alpha-2\gamma-\frac{1}{2}}}{t-s}\, \mathrm{d}t - c_2\, \varrho_2(n,x) - c_3\, \varrho_1(n,x) - c_4 \sqrt{n}(1-x)^\gamma\, |P_n(x)|,$$

where

$$\varrho_2(n,x) = 2\sqrt{n}\,(1-x)^\gamma \frac{n}{n-1} |P_n(x)| \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} |\sin(Nt+\nu)| t^{\alpha-2\gamma-\frac{1}{2}}\, \mathrm{d}t \le$$
$$\le c\sqrt{n}\,(1-x)^\gamma |P_n(x)| \left(\sqrt{1-x}+\tfrac{1}{n}\right)^{\alpha-2\gamma+\frac{1}{2}} \le c \left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}} \right)^{2\gamma}$$

(using $s \sim \sqrt{1-x}$ and (3.3)).

Let

$$\psi := \arg\left(\sqrt{1-x^2}\,\widetilde{P}_{n-1}(x) + i2\sqrt{\frac{n}{n-1}} P_n(x) \right).$$

Then we have uniformly in $x \in [0,1]$ and $n \in \mathbb{N}$ that

$$L_n^{(\alpha,\beta),(\gamma,\delta)}(x) \ge c_1\,(1-x)^\gamma \times$$
$$\times \left(n(1-x^2)\left((1-x^2)\widetilde{P}_{n-1}^2(x) + \frac{4n}{n-1} P_n^2(x) \right) \right)^{\frac{1}{2}} \times$$

$$\times \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} |\cos(Nt+\nu+\psi)| \frac{t^{\alpha-2\gamma-\frac{1}{2}}}{t-s}\, \mathrm{d}t -$$

$$-c_2\, \varrho_2(n,x) - c_3\, \varrho_1(n,x) - c_4 \sqrt{n}(1-x)^\gamma \, |P_n(x)|.$$

Step 5. Now we will estimate the integral

$$B := \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \big|\cos(Nt+\nu+\psi)\big| \frac{t^{\alpha-2\gamma-\frac{1}{2}}}{t-s}\, \mathrm{d}t.$$

Since $|\cos t| \ge \cos^2 t = \frac{1+\cos(2t)}{2}$ it follows that

$$B \ge \frac{1}{2} \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \Big(1+\cos 2(Nt+\nu+\psi)\Big) \frac{t^{\alpha-2\gamma-\frac{1}{2}}}{t-s}\, \mathrm{d}t.$$

Using Lemma 1 we have

$$\int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \frac{t^{\alpha-2\gamma-\frac{1}{2}}}{t-s}\, \mathrm{d}t \ge c\left(s+\frac{R}{n}\right)^{\alpha-2\gamma-\frac{1}{2}} \left[\log\left(\frac{ns}{R}+1\right)+1\right] \ge$$

$$\ge c\left(s+\frac{R}{n}\right)^{\alpha-2\gamma-\frac{1}{2}} \Big[\log(ns+1)+1-\log R\Big],$$

and by the second mean value theorem

$$\int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \cos 2(Nt+\nu+\psi) \frac{t^{\alpha-2\gamma-\frac{1}{2}}}{t-s}\, \mathrm{d}t = \frac{(s+\frac{R}{n})^{\alpha-2\gamma-\frac{1}{2}}}{R/n} \times$$

$$\times \int_{s+\frac{R}{n}}^{\xi} \cos 2(Nt+\nu+\psi)\, \mathrm{d}t \le c\, \left(s+\tfrac{R}{n}\right)^{\alpha-2\gamma-\frac{1}{2}} \quad \Big(\xi \in (s+\tfrac{R}{n}, \tfrac{2\pi}{3})\Big).$$

Then we get

$$B \ge c_1 \left(s+\frac{R}{n}\right)^{\alpha-2\gamma-\frac{1}{2}} \Big[\log(ns+1)+1-c_2\Big].$$

STEP 6. From this we obtain

$$L_n^{(\alpha,\beta),(\gamma,\delta)}(x) \geq c_1\,(1-x)^{\gamma}\left(n(1-x^2)\left((1-x^2)\widetilde{P}_{n-1}^2(x)+\frac{4n}{n-1}P_n^2(x)\right)\right)^{\frac{1}{2}}\times$$

$$\times\left(s+\frac{R}{n}\right)^{\alpha-2\gamma-\frac{1}{2}}\Big[\log(ns+1)+1-c_2\Big]-$$

$$-c_3\,\varrho_2(n,x)-c_4\,\varrho_1(n,x)-c_5\sqrt{n}(1-x)^{\gamma}\,|P_n(x)|.$$

$$\big(x\in[0,1],\ n\in\mathbb{N},\ n>N\big).$$

By (3.3) and $s\sim\sqrt{1-x}$ we have

$$C(x):=(1-x)^{\gamma}\left(s+\frac{R}{n}\right)^{\alpha-2\gamma-\frac{1}{2}}\times$$

$$\times\Big\{n(1-x^2)\Big((1-x^2)\widetilde{P}_{n-1}^2(x)+\frac{4n}{n-1}P_n^2(x)\Big)\Big\}^{\frac{1}{2}}\leq$$

$$\leq c_1\left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma}\leq c_2,$$

which means that

$$L_n^{(\alpha,\beta),(\gamma,\delta)}(x)\geq c_1\,C(x)\Big[\log(n\sqrt{1-x}+1)+1\Big]-$$

$$-c_2\left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma}\left[\frac{1}{R}\big(\log(n\sqrt{1-x}+1)+1\big)+\right.$$

$$\left.+\sqrt{n}\left(\sqrt{1-x}+\tfrac{1}{n}\right)^{\alpha+\frac{1}{2}}|P_n(x)|+1\right]-c_3\sqrt{n}\,(1-x)^{\gamma}\,|P_n(x)|$$

$$(x\in[0,1],\ n\in\mathbb{N},\ n>N).$$

Let $\bar{x}\in(0,1)$ be the closest number to 1 for which

$$P_n(\bar{x})=\frac{1}{2}P_n(1)\sim n^{\alpha}$$

holds. If $x\in[0,\bar{x}]$ then by (4.7) we have

$$s\sim\sqrt{1-x}\geq\sqrt{1-\bar{x}}\geq\frac{c}{n},$$

thus

$$\left(s+\frac{R}{n}\right)^{\alpha-2\gamma-\frac{1}{2}}\geq c\,s^{\alpha-2\gamma-\frac{1}{2}},$$

which means that

$$C(x) \geq c\, s^{\alpha-\frac{1}{2}}\Big\{n(1-x^2)\Big((1-x^2)\widetilde{P}_{n-1}^2(x)+\frac{4n}{n-1}P_n^2(x)\Big)\Big\}^{\frac{1}{2}}.$$

It is proved in [1, p. 21] that

$$s^{\alpha-\frac{1}{2}}\Big\{n(1-x^2)\Big((1-x^2)\widetilde{P}_{n-1}^2(x)+\frac{4n}{n-1}P_n^2(x)\Big)\Big\}^{\frac{1}{2}} > c \quad (x\in[0,\bar{x}]),$$

so for every $x \in [0,\bar{x}]$ and $n \in \mathbb{N}$, $n > N$ we have

$$L_n^{(\alpha,\beta),(\gamma,\delta)}(x) \geq c_1\Big[\log\big(n\sqrt{1-x}+1\big)+1\Big] - c_2\Bigg\{\sqrt{n}(1-x)^{\gamma}(x)|P_n(x)|+ \\ +\left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma}\Big(\frac{1}{R}\big[\log\big(n\sqrt{1-x}+1\big)+1\big]+ \\ +\sqrt{n}\left(\sqrt{1-x}+\tfrac{1}{n}\right)^{\alpha+\frac{1}{2}}|P_n(x)|+1\Big)\Bigg\}.$$

Here

$$c_1 - \frac{c_2}{R}\left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma} \geq c_1 - \frac{c_2}{R} =: c_3 \geq c_3\left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma}.$$

The number R can be chosen such that $c_3 > 0$. Then we have

$$L_n^{(\alpha,\beta),(\gamma,\delta)}(x) \geq c_3\left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma}\big[\log\big(n\sqrt{1-x}+1\big)+1\big]- \\ -c_2\sqrt{n}(1-x)^{\gamma}(x)|P_n(x)| - c_2\left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma}- \\ -c_2\left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma}\sqrt{n}\left(\sqrt{1-x}+\tfrac{1}{n}\right)^{\alpha+\frac{1}{2}}|P_n(x)|$$

for all $x \in [0,\bar{x}]$ and $n \in \mathbb{N}$, $n > N$. If $x \in [\bar{x},1]$ then

$$1-x \leq 1-\bar{x} \sim \frac{1}{n^2}$$

(see (4.7)), and so

$$\left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma}\big[\log\big(n\sqrt{1-x}+1\big)+1\big] \leq c\left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma} \leq$$

$$\leq c\left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma}\sqrt{n}\left(\sqrt{1-x}+\tfrac{1}{n}\right)^{\alpha+\frac{1}{2}}|P_n(x)|$$

(since $P_n(x)\sim n^\alpha$ on this interval), which means that with a suitable $c_4>0$ we have

$$\begin{aligned}L_n^{(\alpha,\beta),(\gamma,\delta)}(x)\geq c_3\left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma}\left[\log\left(n\sqrt{1-x}+1\right)+1\right]-\\ -c_2\sqrt{n}(1-x)^\gamma(x)|P_n(x)|-c_2\left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma}-\\ -c_4\left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma}\sqrt{n}\left(\sqrt{1-x}+\tfrac{1}{n}\right)^{\alpha+\frac{1}{2}}|P_n(x)|\end{aligned}\tag{4.14}$$

for all $x\in[0,1]$ and $n\in\mathbb{N}$, $n>N$.

4.2.4. The final lower estimation. From (4.8) we have

$$L_n^{(\alpha,\beta),(\gamma,\delta)}(x)\geq c_6\,(1-x)^\gamma\quad(x\in[0,1],\ n\in\mathbb{N}).\tag{4.15}$$

(4.9), (4.14) and (4.15) imply

$$\begin{aligned}c_3\left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma}\left[\log\left(n\sqrt{1-x}+1\right)+1\right]\leq L_n^{(\alpha,\beta),(\gamma,\delta)}(x)+\\ +c_2\sqrt{n}(1-x)^\gamma(|P_n(x)|+|P_{n+1}(x)|)+c_2\left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma}+\\ c_4\sqrt{n}\left(\frac{\sqrt{1-x}}{\sqrt{1-x}+\frac{1}{n}}\right)^{2\gamma}\left(\sqrt{1-x}+\tfrac{1}{n}\right)^{\alpha+\frac{1}{2}}(|P_n(x)|+|P_{n+1}(x)|)\leq\\ \leq L_n^{(\alpha,\beta),(\gamma,\delta)}(x)+\frac{c_2}{c}L_n^{(\alpha,\beta),(\gamma,\delta)}(x)+\frac{c_2}{c_6}L_n^{(\alpha,\beta),(\gamma,\delta)}(x)\left(\sqrt{1-x}+\tfrac{1}{n}\right)^{-2\gamma}+\\ +\frac{c_4}{c}L_n^{(\alpha,\beta),(\gamma,\delta)}(x)\left(\sqrt{1-x}+\tfrac{1}{n}\right)^{\alpha-2\gamma+\frac{1}{2}}.\end{aligned}$$

Hence we obtain

$$c_3(1-x)^\gamma\left[\log\left(n\sqrt{1-x}+1\right)+1\right]\leq c_7\,L_n^{(\alpha,\beta),(\gamma,\delta)}(x)$$

$$(x\in[0,1],\ n\in\mathbb{N},\ n>N).$$

Since (by (3.3))

$$\sqrt{n}\left(\sqrt{1-x}+\tfrac{1}{n}\right)^{\alpha+\frac{1}{2}}\left(|P_n(x)|+|P_{n+1}(x)|\right)\le c$$

$$(x\in[0,1],\ n\in\mathbb{N}),$$

we have

$$L_n^{(\alpha,\beta),(\gamma,\delta)}(x)\ge c\,(1-x)^{\gamma}\Big(\log\left(n\sqrt{1-x}+1\right)+$$

$$+\sqrt{n}\left(\sqrt{1-x}+\tfrac{1}{n}\right)^{\alpha+\frac{1}{2}}\left(|P_n(x)|+|P_{n+1}(x)|\right)\Big)\ge$$

$$\ge c\,w^{(\gamma,\delta)}(x)\,\phi_n^{(\alpha,\beta)}(x),$$

where

$$\phi_n^{(\alpha,\beta)}(x)=\log\left(n\sqrt{1-x^2}+1\right)+\sqrt{n}\left(\sqrt{1-x}+\tfrac{1}{n}\right)^{\alpha+\frac{1}{2}}\times$$

$$\times\left(\sqrt{1+x}+\tfrac{1}{n}\right)^{\beta+\frac{1}{2}}\left(|P_n^{(\alpha,\beta)}(x)|+|P_{n+1}^{(\alpha,\beta)}(x)|\right).$$

The above estimate holds uniformly in $x\in[0,1]$ and $n\in\mathbb{N}$.

Theorem is proved. ■

5. Proof of Corollary

Since $L_n^{(\alpha,\beta),(\gamma,\delta)}(\pm1)=0$ we have

$$\max_{x\in[-1,1]}L_n^{(\alpha,\beta),(\gamma,\delta)}(x)=L_n^{(\alpha,\beta),(\gamma,\delta)}(x_0)$$

with $x_0\in(-1,1)$.

From Theorem and (3.3) it follows that

$$L_n^{(\alpha,\beta),(\gamma,\delta)}(x_0)\le c_1\cdot 1\cdot\left(\log\left(n+1\right)+c_2\right)\le c_3\log\left(n+1\right)$$

and

$$L_n^{(\alpha,\beta),(\gamma,\delta)}(x_0)\ge c_4\,w^{(\gamma,\delta)}(x_0)\,\log\left(n\sqrt{1-x_0^2}+1\right)\ge$$

$$\ge c_5\log\left(c_6 n+1\right)\ge c_7\log\left(n+1\right),$$

where the c_i $(i=1\dots7)$ constants are positive and independent of n. This proves the statement. ■

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ITERATING THE TAU-FUNCTION

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Dedicated to Professor Antal Járai on the occasion of his 60th birthday

Abstract. For every natural number greater than 2, the sequence generated by iterating the tau-function is a strictly monotone decreasing sequence, it stabilizes and at the end reaches 2. The second but last value of the sequence is an odd prime. The question of Imre Kátai is what is the asymptotic distribution of these primes, if any.
Our goal was to analyze every tau-iteration sequence of all natural numbers up to a given bound. We also analyzed the tau-iteration sequence for randomly chosen set of large numbers. For calculating the tau-function, efficient factorization methods are necessary.

Tau-function. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_r^{\alpha_r}$, where $r \in \mathbb{N}, \alpha_i > 0$ integer, $p_i > 0$ prime and $p_i \neq p_j$ if $i \neq j$. Let $\tau(n)$ denote the number of positive divisors of n. Then $\tau(n) = (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_r + 1)$.

It is evident that $\tau(1) = 1, \tau(p) = 2$ and $\tau(n) < n$ if $n \geq 3$.

Tau-iteration. Consider the iterated sequence n, $\tau(n)$, $\tau^{(2)}(n) = \tau(\tau(n))$, $, \ldots,$ where $n > 2$. This is a strictly monotone decreasing sequence until reaching 2 and stabilizing (it cannot reach 1). The value before 2 is an odd prime. We will call this number lasttau(n) from now on.

n	$\tau(n)$	$\tau^{(2)}(n)$	$\tau^{(3)}(n)$	lasttau(n)
$64 = 2^6$	7	2	2	7
$2541 = 3 \cdot 7 \cdot 11^2$	12	6	4	3
$3003 = 3 \cdot 7 \cdot 11 \cdot 13$	2^4	5	2	5

Table 1 – Examples for the iteration

As it is clear from the examples, the most difficult part is the first factorization. Since we want to work with 50–60-digit long numbers, we have to find efficient methods of tolerable running times.

Small factors $(2, 3, \ldots, 9973)$ can be found using trial division. Beyond that the Pollard ρ method is used up to 10^6.

For finding even larger factors, we use elliptic curves. Roughly speaking, the running time of the elliptic curve factorization depends only on the length of the second largest prime factor. This method is appropriate for finding factors of about 20–30 digits.

To guarantee that each found factor is prime, the Miller–Rabin primality test is used after these methods.

Elliptic curves. An elliptic curve over $\mathbb{R}$ is the set of all (x, y) pairs on the plane satisfying $y^2 = x^3 + ax + b$, where a and b are real constants and $4a^3 + 27b^2 \neq 0$.

It is obvious that if any point (x, y) is on the curve, then so is $(x, -y)$. The condition for the constants guarantees that a definite tangent exists at every point of the curve. If a (non-vertical) line intersects the curve at two points, (x_1, y_1) and (x_2, y_2), then it intersects the curve at a third point (x_3, y_3) as well. If slope of the line is $\lambda = (y_1 - y_2)/(x_1 - x_2)$ then it is not hard to prove that $x_3 = \lambda^2 - x_1 - x_2$ and $y_3 = \lambda(x_3 - x_1) + y_1$. We can define the addition operation by the formula $(x_1, y_1)+(x_2, y_2) = (x_3, -y_3)$. If the line is tangent to the curve then we consider the line to intersect the curve at two equal points, i. e., $x_1 = x_2$ and $y_1 = y_2$. In this case $\lambda = (3x_1^2+a)/(2y_1)$. If the line is vertical we consider the third intersection point to be in the infinity; this point will be the zero element of the addition. With this addition operation the points of the elliptic curve form an Abelian group.

We can define elliptic curves over any field having characteristic different from 2 and 3. Even more generally, we can define "elliptic curves" but only with a partial addition operation above a commutative ring with identity element, for example, above $\mathbb{Z}/n\mathbb{Z}$ if $\gcd(n, 6) = 1$ and $\gcd(n, 4a^3 + 27b^2) = 1$. For any prime divisor p of n we also get an elliptic curve modulo p. If an addition is defined over $\mathbb{Z}/n\mathbb{Z}$ then it is also defined for any prime divisor p of n. A key observation here is that for any prime divisor p of n, doing the addition modulo n and reducing the result modulo p is the same as reducing the addends modulo p first and then adding the results modulo p. To factorize n we use "elliptic curves" over $\mathbb{Z}/n\mathbb{Z}$. Roughly speaking, for some point P on the curve, we calculate $k! \cdot P$ for a rather large k. During this calculation the gcd operation to compute λ will with high probability find a non-trivial factor of n.

We can use projective representation: Let the points of the curve be represented as equivalence classes of triplets (X, Y, Z) above $\mathbb{Z}/n\mathbb{Z}$. Point (X, Y, Z)

is equivalent to all points (cX, cY, cZ) where c has an inverse modulo n. The zero element of the "curve" is the equivalence class of $(0, 1, 0)$. In this representation the equation of the curve becomes the homogeneous equation

$$ZY^2 = X^3 + aXZ^2 + bZ^3.$$

First we tried the approach described as follows. We select a random curve above $\mathbb{Z}/n\mathbb{Z}$ with a random point P on it by choosing random x, y, a values and calculating b from them. Then we check that $\gcd(n, 4a^3 + 27b^2) = 1$ holds. If it does, we calculate $k! \cdot P$ for increasing values of k. If it is not successful, we have found one of the divisors of n.

We carry out the multiplication by $k!$ iteratively, by multiplying $Q = (k-1)! \cdot P$ by k. We calculate kQ by another iteration starting from Q and $2Q$. The basic idea is to use only the X and Z coordinates. Let i be the number represented by the first l bits of multiplier k. After the lth step we have the X and Z coordinates of the points iQ and $(i+1)Q$. If the next bit, i. e., the $l+1$st bit of k, is zero then we calculate the X and Z coordinates of the points $2iQ$ and $(2i+1)Q$. If the next bit is one then we calculate the X and Z coordinates of $(2i+1)Q$ and $(2i+2)Q$. Therefore we need only two operations: duplication and the calculation of the X and Z coordinates of $(2i+1)Q$ from the X and Z coordinates of iQ, $(i+1)Q$ and Q.

The above approach could be more efficient with changing the curve parameter determination and calculation of coordinates of the new points. Therefore we switched to the representation proposed by Montgomery [1]:

Let the curve equation in homogeneous coordinates be

$$Y^2Z = X^3 + aX^2Z + bXZ^2 + cZ^3, \tag{1}$$

the two points of the curve $P_1 = (u_1/w_1^2, v_1/w_1^3)$ and $P_2 = (u_2/w_2^2, v_2/w_2^3)$, where $u_1/w_1^2 \neq u_2/w_2^2$.

Then $P_3 = P_1 + P_2$, where $P_3 = (u_3/w_3^2, v_3/w_3^3)$ can be determined the following way:

$$\begin{aligned} u_3 &= (v_2w_1^3 - v_1w_2^3)^2 - aw_1^2w_2^2(u_2w_1^2 - u_1w_2^2)^2 \\ &\quad - (u_1w_2^2 + u_2w_1^2)(u_1w_2^2 - u_2w_1^2)^2, \\ v_3 &= -v_1w_2^3(u_2w_1^2 - u_1w_2^2)^3 - (v_2w_1^3 - v_1w_2^3)u_3 \\ &\quad + w_2^2(u_2w_1^2 - u_1w_2^2)^2u_1(v_2w_1^3 - v_1w_2^3), \\ w_3 &= w_1w_2(u_2w_1^2 - u_1w_2^2). \end{aligned}$$

For the duplication $2P_1 = (u_3/w_3^2, v_3/w_3^3)$, the corresponding coordinates have to be determined as well:

$$\begin{aligned} u_3 &= (3u_1^2 + 2au_1w_1^2 + bw_1^4)^2 - 4(aw_1^2 + 2u_1)v_1^2, \\ v_3 &= -8v_1^4 - (3u_1^2 + 2au_1w_1^2 + bw_1^4)(u_3 - 4u_1v_1^2), \\ w_3 &= 2v_1w_1. \end{aligned}$$

In this approach the calculation of kQ where $Q = (k-1)!P$ is simply done by employing the left-to-right binary method using only duplication and addition of Q.

It seems that the determination of the coordinates requires a lot of multiplication. If we determine the starting point and the parameters of the curve in an appropriate way, the above calculations can be simplified. Let the starting point of the curve be $(1, \alpha, -1)$, where the constants of the curve (1) are $a = 0$, $b = 0$, and $c = \alpha^2 - 2$. With this selection, we can save many calculations. There is only one curve parameter, α, which is selected by random for each curve.

The effieciency of the factorization depends on the number of iterations and the number of curves. The suggested values are the following [10]:

Digits	**Number of iterations**	**Number of curves**
15	2000	25
20	11000	90
25	50000	300
30	250000	700
35	1000000	1800
40	3000000	5100
45	11000000	10600
50	43000000	19300
55	110000000	49000
60	260000000	124000
65	850000000	210000
70	2900000000	340000

Table 2 – Suggested values for number of iterations and curves

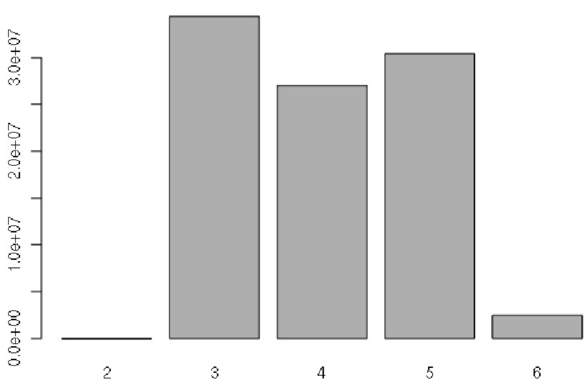
These values served well as good starting points for selecting the actual parameters. During the tests we had to tune them for finding the given length of factors.

With this simple flow control, we could find the lasttau(n) values:

procedure *lasttau*
  $t, last, i \leftarrow \tau(factors), -1, 1$
  while $(t \neq 2)$
    $last \leftarrow t$
    $ECM(t, factors)$
    $t \leftarrow \tau(factors)$
    $i \leftarrow i + 1$
  end
end

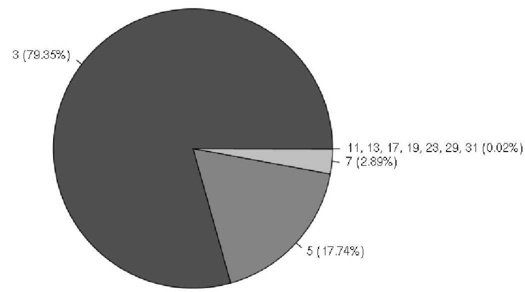
The implementation of the described methods has been done in C and C++ languages, with GNU GMP [12] multi-word arithmetic and with Condor workload management system. The program was run on a cluster of 64-bit AMD processors for several months.

In the next figure we can see how many times it is necessary to iterate the τ function for numbers up to 10^8 to get the lasttau(n) values. We can see that the most frequent value is 3 and it is never required to iterate more than 6 times.



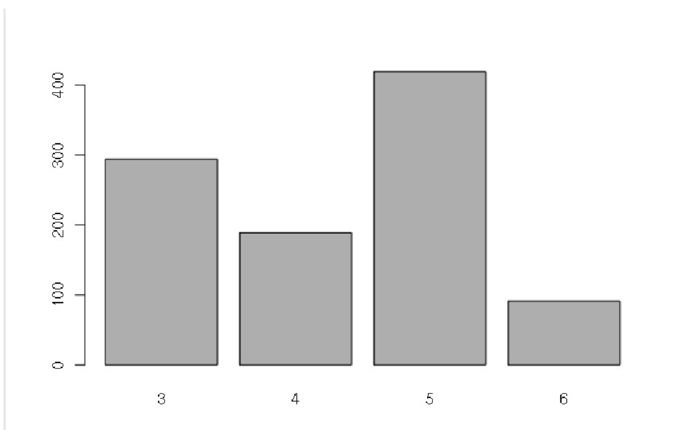
Required number of iterations for lasttau(n) calculations up to $n = 10^8$

The next diagram shows the distribution of lasttau values up to $n = 10^8$. The biggest lasttau value is 31. The occurrences of 3, 5 and 7 are the highest.



The ratio of lasttau(n) values up to $n = 10^8$

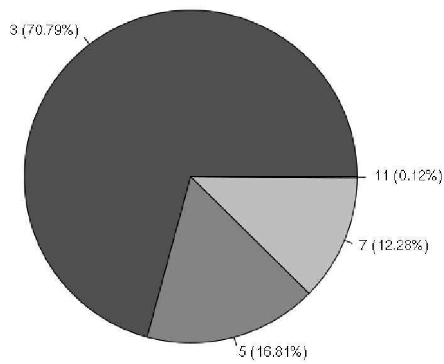
Let us see these ratios for numbers around 10^{50}. We chose randomly 1000 numbers and the distribution is the following:



Required number of iterations for calculating lasttau(n) for n around 10^{50}

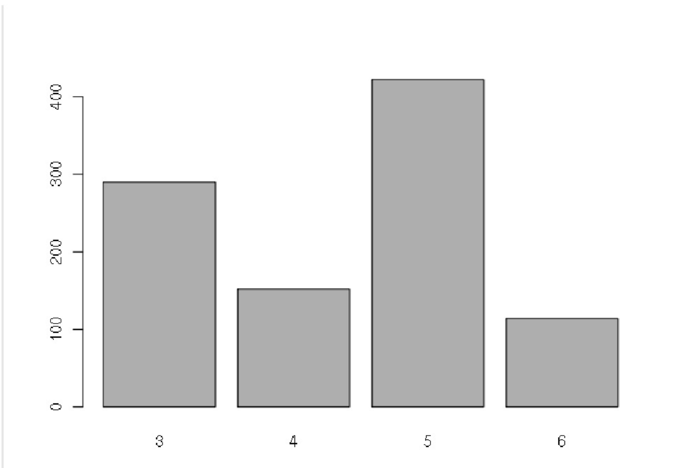
We can see that in this random sample the most frequent τ-iteration length is 5 and the most infrequent is 6.

The next diagram shows that the greatest lasttau value is 11 and the occurrence ratio is very similar to the case of smaller numbers.



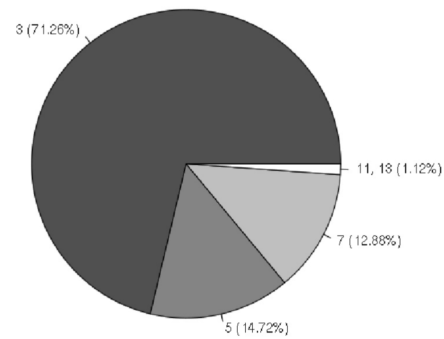
Ratio of lasttau(n) values for n around 10^{50}

Next, we chose the numbers in the interval $[10^{70}, 10^{70} + 1000)$. The distribution is still very similar to before. The most frequent iteration length in this case is also 5, and the most infrequent is also 6.



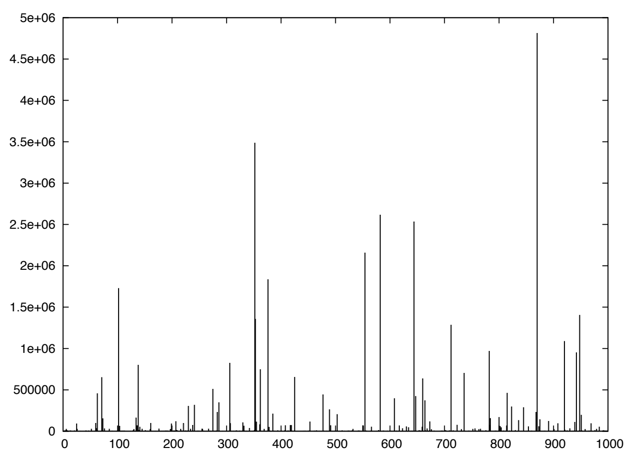
Required number of iterations for calculating lasttau(n)
between 10^{70} and $10^{70} + 1000$

If we analyze the occurrences of lasttau(n) values, we will see that 11 and 13 are the most frequented ones. The distribution of smaller primes is very similar to previous samples.



Ratio of lasttau(n) values between 10^{70} and $10^{70} + 1000$

The last diagram shows the time of factorization of 1000 numbers in seconds. We can see that there are extremely high values, and sometimes it was done very quickly. It depends on the number of curves that we are not able to determine any factor.



Factorization time for numbers between 10^{70} and $10^{70} + 1000$

Let us have a closer look at some numbers of this sample. In Tables 3 and 4 we can see for each n considered what its factors are, the value of lasttau(n) value (L), the number of iterations necessary (I), and the time the calculation took in minutes.

Number	Factors	L	I	Time
$10^{70}+1$	29, 101, 281, 421, 3541, 27961, 3471301, 13489841, 121499449, 60368344121, 848654483879497562821	3	6	0
$10^{70}+2$	2, 3, 2417, 728771, 331844753, 1315441529, 2167576895034805670716583728798525811120513	3	5	0
$10^{70}+3$	7, 103, 13869625520110957004160887656033287101248266296809986130374479889043	3	4	0
$10^{70}+4$	2^2, 465761, 708845197, 735374140501601, 16734221902863133, 615334987861198431900041	3	6	2
$10^{70}+5$	3, 5, 23, 109, 307, 297674527070399026203749, 2909875173333171111479969227134609531967	3	5	142
$10^{70}+6$	2, 1663, 147227767, 2384032997, 8565954590598526685670743287535875319826645623119	3	5	0
$10^{70}+7$	1621, 252073541474195849351629035309353, 24473141552192478478666014277117939	3	4	424
$10^{70}+8$	2^3, 3^2, 17, 16156512259, 56753899267649747956763, 8909949913446915151529019420144601	3	5	177
$10^{70}+9$	233, 176144029, 243655462971284241469045332244275022188090622768643397118037	3	4	0
$10^{70}+10$	2, 5, 7, 11, 13, 47, 139, 2531, 31051, 143574021480139, 549797184491917, 24649445347649059192745899	13	3	0
$10^{70}+11$	3, 53, 433, 525404597, 8990767439, 531094485851013487759, 57896532578451563713869329	3	5	2
$10^{70}+12$	2^2, 8539, 119855917, 194568691, 12554532897290271411337319110714625138715593690191	3	5	0
$10^{70}+13$	8009, 1097408550449, 1667771943738042971066279, 682207835299330859583636902467	5	3	36
$10^{70}+14$	2, 3, 79, 191, 716848837, 3049769839726051129, 50523524701987179815032320340722278177	3	5	46
$10^{70}+15$	5, 19, 211, 58676451232029416603, 2067397236615734055061, 4112502436486078502075149	7	3	104
$10^{70}+16$	2^4, 241, 2593360995850622406639004149377593360995850622406639004149377593361	3	5	0
$10^{70}+17$	3^4, 7, 409, 43121477514305550165370866267361784884197272135332445030896538639	3	5	0
$10^{70}+18$	2, 3881, 1454569, 183339272189671009, 4830994366338537465366958647013901853403609	3	5	4
$10^{70}+19$	674810659, 22393508323, 661753013393221095707734371385920431993559884567867	3	4	0
$10^{70}+20$	2^2, 3, 5, 127, 503, 3323, 101281, 29937550596856922549, 258941975440540758891541779098500261	3	6	1
$10^{70}+21$	11, 44059614698317, 20633201522882013969861187610480521738596357429720481883	3	4	0
$10^{70}+22$	2, 3697, 5281, 2446719944677759, 1216639765372690618513, 86031623194884730077302869	7	3	100
$10^{70}+23$	3, 13, 9781, 7558907689, 35376899347, 1717124714191, 57091504446753490897682552668649	3	5	0
$10^{70}+24$	2^3, 7, 227, 563, 1397261590843800685664207858869872469140080504623818056420305229	7	3	0

Table 3 – Detailed results 1.

Number	Factors	L	I	Time
$10^{70}+976$	2^4, 7, 73, 146477, 260671, 33695203523, 224454548779651, 423545885811836655883732110196 1	5	4	7
$10^{70}+977$	3, 17, 14105606257880525512331, 1390074469596114285346094366863270257693201001 7	5	3	163
$10^{70}+978$	2, 11, 167, 1181, 3192803, 7218364816995637767695300105652423536519898305161100563 79	7	3	0
$10^{70}+979$	10427, 2177056848782317, 44052530107408314444800361130633863072156095958318 1	3	4	0
$10^{70}+980$	2^2, 3^2, 5, 241, 179487843307, 28393367065721666614008346 7, 45233325335895776847343238 09	3	6	426
$10^{70}+981$	59, 997, 808789, 3174287, 1161081043, 57030721932015325310947921649073543009340003	7	3	0
$10^{70}+982$	2, 263, 477130943, 428411743423, 7274023506249233, 127861738834758114255050444476 61	7	3	4
$10^{70}+983$	3, 7, 41, 43, 139, 3862987, 503025899216462846428056315124594918973335288268239975697	3	5	0
$10^{70}+984$	2^3, 19, 83, 8992096609, 4263142668216287, 206769981405813702429808441435963274082 53	3	5	1
$10^{70}+985$	5, 13, 2426789, 2186649450090759728942714 9, 28991818714734167497668706967583599 29	3	5	844
$10^{70}+986$	2, 3, 109, 70849621, 310760837, 925713014519639, 75020865170392960011074730724814605 3	3	5	5
$10^{70}+987$	29, 31, 8467, 114356185229687879, 114881762293484246515752756224565769182533915 41	3	5	1
$10^{70}+988$	2^2, 67, 3731343283582089552238805970149253731343283582089552238805970149254 1	3	5	0
$10^{70}+989$	3^4, 11, 37, 5835672122537, 519792116996283095187631301358767804022228453757834 91	3	5	0
$10^{70}+990$	2, 5, 7, 8392231, 170225465501536906149100451187703075788615855375218886542633 47	3	5	0
$10^{70}+991$	3393413, 741705178385908897 73, 3973122628778431985384125858191532227257375 9	3	4	80
$10^{70}+992$	2^5, 3, 2383770887, 3087434689256921902709, 141535856089772506993151206512000193 19	3	6	101
$10^{70}+993$	71, 1747, 7121, 249881, 5872082217973913357287, 771582550672089772879565046423694 7	7	3	153
$10^{70}+994$	2, 17, 23, 7481, 656497188317, 260375862632602746799038841350378448231798320913857 1	7	3	0
$10^{70}+995$	3, 5, 97, 935096727371, 734988374197375313532128288304873565261817243875896755 9	3	5	0
$10^{70}+996$	2^2, 1303, 103823791160342357633, 184798614023848590441592899381179719634126855 51	3	5	10
$10^{70}+997$	7, 47, 1036751, 8644661, 339142074212058129442258382995426753628913895053366586 3	3	5	0
$10^{70}+998$	2, 3^2, 13, 16831280293, 253902507659003876127387136817867828034364032886127012837 9	3	5	2
$10^{70}+1000$	2^3, 5^3, 11, 909090909090909090909090909090909090909090909090909090909090909 1	7	3	0

Table 4 – Detailed results 2.

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BIORTHOGONAL SYSTEMS TO RATIONAL FUNCTIONS

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Dedicated to Professor Antal Járai on his 60th birthday

Abstract. In this paper we start from a given rational function system and take the linear space spanned by it. Then in this linear space we construct a rational function system that is biorthogonal to the original one. By means of biorthogonality expansions in terms of the original rational functions can be easily given. For the discrete version we need to choose the points of discretization and the weight function in the discrete scalar product in a proper way. Then we obtain that the biorthogonality relation holds true for the discretized systems as well.

1. Introduction

There is a wide range of applications of rational function systems. For instance in system, control theories they are effectively used for representing the transfer function, see e.g. [1], [4], [5]. Another area where they have been found to be very efficient is signal processing [8]. Recently we have been using them for

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representing and decomposing ECG signals [3]. In several cases the so called Malmquist–Takenaka orthogonal systems are generated and used in applications. There are, however, applications when the result should be expressed by the original rational functions rather than by the terms of the orthonormed system generated by them. Then it makes sense to use the corresponding biorthogonal system. This is the basic motivation behind our construction.

Let us take basic rational functions of the form

$$r_{a,n}(z) := \frac{1}{(1-\overline{a}z)^n} \qquad (|a|<1,\ |z|\le 1,\ n\in\mathbb{P}). \tag{1}$$

($\mathbb{P}$ stands for the set of positive integers.) They form a generating system for the linear space of rational functions that are analytic on the closed unit disc $\overline{\mathbb{D}} = \{z\in\mathbb{C} : |z|\le 1\}$, where $\mathbb{D} := \{z\in\mathbb{C} : |z|<1\}$ stands for the open unit disc. Indeed, by partial fraction decomposition any analytic function can be written as a finite linear combination of such functions. $a^* := 1/\overline{a} = a/|a|^2$ is the pole of $r_{a,n}$ the order of which is n. On the basis of the relation $a^*\overline{a} = 1$ the parameter a will be called inverse pole.

In our construction we will use the following modified basic functions

$$\phi_{a,n}(z) := \frac{z^{n-1}}{(1-\overline{a}z)^n} \qquad (z\in\overline{\mathbb{D}},\ a\in\mathbb{D},\ n\in\mathbb{P})\,. \tag{2}$$

If $a\neq 0$ then this modification makes no difference in the generated subspaces, i.e.

$$\operatorname{span}\{r_{a,k} : 1\le k\le n\} = \operatorname{span}\{\phi_{a,k} : 1\le k\le n\} \qquad (n\in\mathbb{P},\, a\neq 0)\,.$$

It is easy to see that the transition between the system of basic and the system of modified basic functions is very simple. We note that, however, if $a=0$ then the two subspaces are different. Indeed, in this special special case we receive the set of polynomials of order $(n-1)$ on the right side.

Let the set of rational functions that are analytic on $\overline{\mathbb{D}}$ be denoted by $\mathfrak{R}$. It is actually the set of linear combinations of modified basic functions given in (2). $\mathfrak{R}$ will be considered as the normed subspace of the Hardy space $H^2(\mathbb{D})$. Recall that $H^2(\mathbb{D})$ is the collection of functions $F:\mathbb{D}\to\mathbb{C}$ which are analytic on $\mathbb{D}$, and for which

$$\|F\|_{H^2} := \sup_{0\le r<1}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|F(re^{it})|^2\,dt\right)^{1/2} < \infty$$

holds. It is known that for any $F\in H^2(\mathbb{D})$ the limit

$$F(e^{it}) := \lim_{r\to 1-0} F(re^{it})$$

exists for a.e. $t \in \mathbb{I} := [-\pi, \pi)$. The radial limit function defined on the torus $\mathbb{T}$ belongs to $L^2(\mathbb{T})$. This way a scalar product can be defined on $H^2(\mathbb{D})$ as follows

$$\langle F, G \rangle := \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{it})\overline{G}(e^{it})\, dt \qquad (F, G \in H^2(\mathbb{T}))\,.$$

Then $H^2(\mathbb{D})$ becomes a Hilbert space since the norm induced by this scalar product is equivalent to the original $\|\cdot\|_{H^2}$ norm.

Let $\mathfrak{b} := (b_n \in \mathbb{D},\ n \in \mathbb{N})$ be a sequence of inverse poles. Taking the segment $b_0, b_1, \cdots, b_n$ we count how many times the value of b_n occurs in that. That number will be called the multiplicity of b_n and denoted by ν_n. In other words ν_n is the number of indices $j \le n$ for which $b_j = b_n$. Then we introduce the following subspaces of $\mathfrak{R}$ and of $H^2(\mathbb{D})$ generated by $\mathfrak{b}$

$$\mathfrak{R}_n^{\mathfrak{b}} := \operatorname{span}\{\phi_{b_k,\nu_k} : 0 \le k < n\} \qquad (n \in \mathbb{P}), \qquad \mathfrak{R}^{\mathfrak{b}} := \bigcup_{n=0}^{\infty} \mathfrak{R}_n^{\mathfrak{b}} \subset \mathfrak{R}.$$

We note that $\mathfrak{R}^{\mathfrak{b}}$ is everywhere dense in the Hilbert space $H^2(\mathbb{D})$, i.e. the system $\{\phi_{b_n,m_n} : \ n \in \mathbb{N}\}$ is closed in $H^2(\mathbb{D})$, if and only if ([7], [11])

$$\sum_{n=0}^{\infty} (1 - |b_n|) = \infty\,.$$

By means of the Cauchy integral formula the scalar product of a function $F \in H^2(\mathbb{D})$ and a modified basic function $\phi_{a,k}$ in (2) can be written in an explicit form. Indeed, by definition

$$\begin{aligned} \langle F, \phi_{a,k} \rangle &= \frac{1}{2\pi} \int_{\mathbb{I}} \frac{F(e^{it})e^{-i(k-1)t}}{(1 - ae^{-it})^k}\, dt = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{F(\zeta)}{(\zeta - a)^k}\, d\zeta = \\ &= \frac{F^{(k-1)}(a)}{(k-1)!} \qquad (a \in \mathbb{D},\ k \in \mathbb{P})\,. \end{aligned} \tag{3}$$

Using this formula one can give an explicit form for the members of the so called Malmquist–Takenaka (MT) system. The Malmquist–Takenaka system $(\Phi_n,\, n \in \mathbb{N})$ is generated from $(\phi_{b_k,m_k},\, k \in \mathbb{N})$ by Gram-Schmidt orthogonalization is of the form [12]:

$$\Phi_n(z) := \frac{\sqrt{1 - |b_n|^2}}{1 - \bar{b}_n z} \prod_{k=0}^{n-1} B_{b_k}(z) \qquad (z \in \overline{\mathbb{D}},\, n \in \mathbb{N}), \tag{4}$$

where

$$B_b(z) := \frac{z - b}{1 - \bar{b} z} \qquad (z \in \overline{\mathbb{D}},\ b \in \mathbb{D}) \tag{5}$$

is the Blaschke function of parameter b. The Blaschke functions enjoy several nice propeties. For instance they are bijections on the disc $\mathbb{D}$ and on the torus $\mathbb{T}$, they define a metric on $\mathbb{D}$ as follows

$$\rho(z_1, z_2) := |B_{z_1}(z_2)| = \frac{|z_1 - z_2|}{|1 - \overline{z}_1 z_2|} \qquad (z_1,\, z_2 \in \mathbb{D})\,.$$

Moreover the maps ϵB_b $(b \in \mathbb{D}, \epsilon \in \mathbb{T})$ can be identified with the congruences in the Poincaré model of the hyperbolic plane.
The orthogonal expansions with respect to Malmquist–Takenaka systems generated by a sequence of inverse poles turned to be very useful in several applications. On the other hand there are problems when the expansion in terms of the generating basic or modified basic functions would be more useful. This is the case for example in system identification when a partial fraction representation of the transfer function is taken, and the poles should be determined [10]. In such cases a biorthogonal system is needed to deduce such an expansion. In the next section we construct a biorthogonal system to a finite system of modified basic functions. The elements of the biorthogonal system are in the subspace generated by the basic functions. In Section 3 we define a set of points of discretization. By means of that and a proper weight function we prove a discrete type biorthogonality as well. We note that a similar problem was addressed in [9] except that equidistant subdivision was taken there and the members of the biorthogonal system were polynomials.

2. Rational biorthogonal systems

Let $\mathfrak{b}$ be a sequence of inverse poles in $\mathbb{D}$ and fix $N \in \mathbb{P}$. Let $a_0, a_1, \cdots, a_n$ denote the distinct elements in $\{b_0, \ldots, b_{N-1}\}$. Then m_j will stand for the number of occurrences of a_j in $\{b_0, \ldots, b_{N-1}\}$. We will use the simplified notations $\phi_{\ell j} := \phi_{a_\ell, j}$, and $\mathfrak{R}_N := \mathfrak{R}_N^{\mathfrak{b}}$. Then the following equations hold

$$\begin{aligned}
&\mathfrak{R}_N = \operatorname{span}\{\phi_{\ell j} \,:\, 1 \le j \le m_\ell, 0 \le \ell \le n\} \\
&\{b_k \,:\, 0 \le k < N\} = \{a_j \,:\, 0 \le j \le n\}\,, \\
&m_0 + m_1 + \cdots + m_n = N.
\end{aligned}$$

In this section we will construct a system $\{\Psi_{\ell j} \,:\, 1 \le j \le m_\ell, k, \ell = 0, 1, \ldots, n\}$ within $\mathfrak{R}_N$ which is biorthogonal to the generating system $\{\phi_{\ell j} \,:\, 1 \le j \le m_\ell, 0 \le \ell \le n\}$. In notation

i) $\operatorname{span}\{\Psi_{\ell j} : 1 \le j \le m_\ell,\ \ell = 0, 1, \cdots, n\} = \mathfrak{R}_N,$

ii) $\langle \Psi_{\ell j}, \phi_{ki} \rangle = \delta_{ij}\delta_{k\ell} \qquad (1 \le i \le m_k,\ 1 \le j \le m_\ell, k,\ \ell = 0, 1, \cdots, n)\,.$

Then the operator P_N of projection onto $\mathfrak{R}_N$ can be expressed as a biorthogonal expansion

$$P_N f = \sum_{k=0}^{n} \sum_{i=1}^{m_k} \langle f, \Psi_{ki} \rangle \phi_{ki} \,.$$

In the construction of the explicit form of the biorthogonal system the formula in (3), that relates biorthogonality with Hermite interpolation, will play a key role. Using the Blaschke functions defined in (5) we introduce the function Ω_n as follows

$$\Omega_{\ell n}(z) := \frac{1}{(1-\overline{a}_\ell z)^{m_\ell}} \prod_{i=0, i\neq \ell}^{n} B_{a_i}^{m_i}(z) \qquad (0 \le \ell \le n)\,. \tag{6}$$

We will show that the members of the biorthogonal system can be written in the form

$$\Psi_{\ell j}(z) = P_{\ell j}(z) \frac{\Omega_{\ell n}(z)}{\Omega_{\ell n}(a_\ell)}\,, \tag{7}$$

where

$$P_{\ell j}(z) = \sum_{s=0}^{m_\ell - 1} \frac{P_{\ell j}^{(s)}(a_\ell)}{s!} (z - a_\ell)^s \tag{8}$$

is a polynomial of order $(m_\ell - 1)$.
Indeed, by (3) we have

$$\langle \Psi_{\ell j}, \phi_{ki} \rangle = \frac{\Psi_{\ell j}^{(i-1)}(a_k)}{(i-1)!} \qquad (1 \le i, j \le m_k)\,. \tag{9}$$

It follows from the definition of $\Omega_{\ell n}$ in (6) that if $k \neq \ell$ then a_k is a root of the nominator of $\Psi_{\ell j}$ of order exactly m_k. Therefore the scalar product product is 0, and orthogonality holds in (9) for $k \neq \ell$. In case $k = \ell$ biorthogonality is equivalent to

$$\langle \Psi_{\ell j}, \phi_{\ell i} \rangle = \frac{\Psi_{\ell j}^{(i-1)}(a_\ell)}{(i-1)!} = \delta_{ij} \qquad (1 \le i, j \le m_\ell)\,. \tag{10}$$

Set

$$\omega_{\ell n}(z) = \frac{\Omega_{\ell n}(a_\ell)}{\Omega_{\ell n}(z)}\,. \tag{11}$$

We note that $\omega_{\ell n}$ is analytic in a proper neighborhood of a_ℓ since $\Omega_{\ell n}(a_\ell) \neq 0$. By definition, see (7), we have

$$P_{\ell j}(z) = \Psi_{\ell j}(z) \omega_{\ell n}(z)\,.$$

Using the product rule of differentiation and the condition (10) we obtain

$$P_{\ell j}^{(s)}(a_\ell) = \sum_{r=0}^{s} \binom{s}{r} \Psi_{\ell j}^{(r)}(a_\ell)\omega_{\ell n}^{(s-r)}(a_\ell) = \binom{s}{j-1}(j-1)!\,\omega_{\ell n}^{(s-j+1)}(a_\ell)$$

for the coefficients of the polynomial $P_{\ell j}$ in (8). Hence

$$\frac{P_{\ell j}^{(s)}(a_\ell)}{s!} = \begin{cases} 0, & (0 \le s < j-1)\,; \\ \dfrac{\omega_{\ell n}^{(s-j+1)}(a_\ell)}{(s-j+1)!}, & (j-1 \le s < m_\ell)\,. \end{cases} \tag{12}$$

For the calculation of the derivatives of $\omega_{\ell n}$ we will use the following logarithmic formula for the Blaschke functions, for definition see (5),

$$\begin{aligned} \frac{d}{dz}\log(B_a(z)) &= \frac{d}{dz}[\log(z-a) - \log(1-\overline{a}z)] \\ &= \frac{1}{z-a} + \frac{\overline{a}}{1-\overline{a}z} = \frac{1}{z-a} - \frac{1}{z-a^*} \qquad (a^* := 1/\overline{a}). \end{aligned} \tag{13}$$

Thus

$$\begin{aligned} \frac{d}{dz}\log(\Omega_{\ell n}(z)) &= \frac{d}{dz}[-m_\ell \log(1-\overline{a}_\ell z) + \sum_{i=1, i\ne\ell}^{n} m_i \log(B_{a_i}(z))] = \\ &= -\frac{m_\ell}{z-a_\ell^*} + \sum_{i=1, i\ne\ell}^{n} \left(\frac{m_i}{z-a_i} - \frac{m_i}{z-a_i^*}\right). \end{aligned} \tag{14}$$

Since

$$\frac{\omega_{\ell n}'(z)}{\omega_{\ell n}(z)} = \frac{d}{dz}\log(\omega_{\ell n}(z)) = -\frac{d}{dz}\log(\Omega_{\ell n}(z))$$

we can conclude by (14) that

$$\omega_{\ell n}'(z) = \omega_{\ell n}(z)\rho_{\ell n}(z) \tag{15}$$

with

$$\rho_{\ell n}(z) := \frac{m_\ell}{z-a_\ell^*} - \sum_{i=1, i\ne\ell}^{n} m_i \left(\frac{1}{z-a_i} - \frac{1}{z-a_i^*}\right).$$

This provides a recursion process for the calculation of the derivatives of $\omega_{\ell n}$. As an example, the second and third derivatives are shown below:

$$\begin{aligned} \omega_{\ell n}^{(2)} &= \omega_{\ell n}'\ \rho_{\ell n} + \omega_{\ell n}\ \rho_{\ell n}' = \omega_{\ell n}(\rho_{\ell n}^2 + \rho_{\ell n}'), \\ \omega_{\ell n}^{(3)} &= \omega_{\ell n}'(\rho_{\ell n}^2 + \rho_{\ell n}') + \omega_{\ell n}(2\rho_{\ell n}\ \rho_{\ell n}' + \rho_{\ell n}^{(2)}) = \omega_{\ell n}(\rho_{\ell n}^3 + 3\rho_{\ell n}\ \rho_{\ell n}' + \rho_{\ell n}^{(2)}). \end{aligned}$$

where the terms $\rho_{\ell n}^{(j)}(z)$ are

$$\rho_{\ell n}^{(j)}(z) = (-1)^j j! \left(\frac{m_\ell}{(z-a_\ell^*)^{j+1}} - \sum_{i=1, i\neq \ell}^{n} m_i \left(\frac{1}{(z-a_i)^{j+1}} - \frac{1}{(z-a_i^*)^{j+1}} \right) \right).$$

In summary, we have proved the following theorem.

Theorem 1. *Let* $\Omega_{\ell n}$, *and* $\omega_{\ell n}$ *be defined as in* (6), *and* (11). *Then the systems*

$$\begin{aligned} \phi_{ki}(z) &:= \frac{z^{i-1}}{(1-\overline{a}_k z)^i}, \\ \Psi_{\ell j}(z) &:= \frac{\Omega_{\ell n}(z)(z-a_\ell)^{j-1}}{\Omega_{\ell n}(a_\ell)} \sum_{s=0}^{m_\ell - j} \frac{\omega_{\ell n}^{(s)}(a_\ell)}{s!}(z-a_\ell)^s \end{aligned}$$

$(z \in \overline{\mathbb{D}},\, 1 \leq i \leq m_k,\, 1 \leq j \leq m_\ell, 0 \leq k, \ell \leq n)$ *are biorthogonal to each other with respect to the scalar product in* $H^2(\mathbb{D})$.
The two systems span the same linear space.
The derivatives of $\omega_{\ell n}$ *can be calculated by recursion based on the relation in* (15).

3. Discrete rational biorthogonal systems

In this section we introduce a discrete scalar product in $\mathfrak{R}_N$ as follows

$$[F,G]_N := \sum_{z\in\mathbb{T}_N} F(z)\overline{G}(z)\rho_N(z) \qquad (F,G \in \mathfrak{R}_N)\,, \tag{16}$$

where the discrete set $\mathbb{T}_N \subset \mathbb{T}$ with number of elements equals to N, and the positive weight function ρ_N on it will be defined later.
The Blaschke function B_a admits a representation on the unit circle of the form

$$B_a(e^{it}) = e^{i\beta_a(t)} \qquad (t \in \mathbb{R})\,, \tag{17}$$

where $\beta_a : \mathbb{R} \to \mathbb{R}$ is strictly increasing for which $\beta_a(t+2\pi) = \beta_a(t) + 2\pi$ holds. Moreover,

$$\beta_a'(t) = \frac{1-r^2}{1-2r\cos(t-\alpha)+r^2} \qquad (t \in \mathbb{R},\, a = re^{i\alpha} \in \mathbb{D})\,. \tag{18}$$

Indeed, let us continue (13) to obtain

$$\frac{d}{dz}\log(B_a(e^{it})) = ie^{it}\left(\frac{1}{e^{it}-re^{i\alpha}} - \frac{1}{e^{it}-\frac{1}{r}e^{i\alpha}}\right)$$
$$= i\frac{1-r^2}{1-2r\cos(t-\alpha)+r^2}\,.$$

Hence (17) and (18) follow. Then by the definition of $\{a_0, \cdots, a_n\}$ at the beginning of Section 2 we have that the Blaschke products can be written as

$$\prod_{k=0}^{N-1} B_{b_k}(e^{it}) = \prod_{j=0}^{n} e^{im_j\beta_{a_j}(t)} = e^{i\theta_N(t)} \qquad (t \in \mathbb{R})\,,$$

where

$$\theta_N(t) := \sum_{j=0}^{n} m_j\beta_{a_j}(t) \quad (t \in \mathbb{R})\,.$$

θ_N is strictly increasing and $\theta_N(t+2\pi) = \theta_N(t) + 2N\pi$. Therefore, for any $t_0 \in \mathbb{I}$ and $k = 1, 2, \cdots, N-1$ there exists exactly one $t_k \in (t_0, t_0 + 2\pi)$ for which

$$\theta_N(t_k) = 2\pi k + \theta_N(t_0) \qquad (k = 0, 1, \cdots, N-1) \tag{19}$$

holds.
Then the set of discretization $\mathbb{T}_N$ and the weight function ρ_N in (16) are defined as follows

$$\mathbb{T}_N := \{e^{it_k} : k = 0, 1, \cdots, N-1\}\,, \qquad \rho_N(e^{it}) = \frac{1}{\theta'_N(t)}\,.$$

Then the following theorem holds for this discrete model and the rational functions.

Theorem 2. *The MT-system* Φ_n $(n = 0, 1, \cdots, N-1)$ *is orthonormed system with respect to the scalar product in* (16)*, i.e.*

$$[\Phi_k, \Phi_\ell]_N = \delta_{k\ell} \qquad (0 \le k, \ell < N)\,.$$

The $\Psi_{\ell j}$*, and* $\phi_{\ell j}$ $(1 \le j \le m_\ell, 0 \le \ell \le n)$ *systems are biorthogonal to each other with respect to the scalar product in* (16)*, i.e.*

$$[\Psi_{\ell r}, \phi_{ks}]_N = \delta_{k\ell}\delta_{rs} \qquad (1 \le r \le m_\ell, 1 \le s \le m_k, 0 \le k, \ell \le n)\,.$$

Proof. For the proof we will use the following closed form the Dirichlet kernels of the MT-systems [2] (or see e.g. [6], pp. 320, [4], pp. 82):

$$D_N(t,\tau) := \sum_{j=0}^{N-1} \Phi_j(e^{it})\overline{\Phi}_j(e^{i\tau}) = \frac{e^{i(\theta_N(t)-\theta_N(\tau))}-1}{e^{i(t-\tau)}-1} \qquad (t,\tau\in\mathbb{R}, t\neq\tau)\,. \tag{20}$$

By the definition of t_k, see (19), we have

$$D_N(t_k,t_\ell)=0 \qquad (k\neq\ell,\, 0\le k,\ell<N)\,.$$

In the special case $t=\tau$ one can deduce from the continuity of the kernel and from (20) that

$$D_N(t,t)=\lim_{\tau\to t} D_n(t,\tau) = \lim_{\tau\to t}\left(\frac{e^{i\tau}}{e^{i\theta_N(\tau)}}\cdot\frac{e^{i\theta_N(t)}-e^{i\theta_N(\tau)}}{t-\tau}\cdot\left(\frac{e^{it}-e^{i\tau}}{t-\tau}\right)^{-1}\right) = \\ = \theta_N'(t)\,.$$

This along with (20) imply

$$\sum_{j=0}^{N-1} u_{jk}\overline{u}_{j\ell} = \frac{D_N(t_k,t_\ell)}{D_N(t_k,t_k)} = \delta_{k\ell} \quad (0\le k,\ell<N),$$

for the matrix

$$u_{jk} := \frac{\Phi_j(t_k)}{\sqrt{D_N(t_k,t_k)}} \quad (0\le k,\ell<N)\,.$$

This means that the matrix is unitarian. Taking the adjoint matrix we have

$$\sum_{j=0}^{N-1} u_{kj}\overline{u}_{\ell j} = \sum_{j=0}^{N-1}\frac{\Phi_k(t_j)\overline{\Phi}_\ell(t_j)}{D_N(t_j,t_j)} = [\Phi_k,\Phi_\ell]_N = \delta_{k\ell} \qquad (0\le k,\ell<N)\,.$$

The first part of our theorem on the discrete orthogonality of the MT-sytems is proved.

The proof of the second part of our theorem follows from the equivalence of the scalar products $\langle\cdot,\cdot\rangle$ and $[\cdot,\cdot]_N$ in the subspace $\mathfrak{R}_N$:

$$\langle F,G\rangle = [F,G]_N \qquad (F,G\in\mathfrak{R}_N)\,.$$

Indeed, if $F,G\in\mathfrak{R}_N$ then they can be expressed as linear combinations of the Φ_k $(k=0,1,\cdots,N-1)$ MT-functions:

$$F=\sum_{k=0}^{N-1}\lambda_k\Phi_k, \quad G=\sum_{k=0}^{N-1}\mu_k\Phi_k.$$

Since, as it has already been shown, the MT-functions are orthonormed with respect to both scalar products we have

$$\langle F, G\rangle = \sum_{k=0}^{N-1}\sum_{\ell=0}^{N-1} \lambda_k \overline{\mu}_\ell \langle \Phi_k, \Phi_\ell\rangle = \sum_{k=0}^{N-1} \lambda_k \overline{\mu}_k =$$
$$= \sum_{k=0}^{N-1}\sum_{\ell=0}^{N-1} \lambda_k \overline{\mu}_\ell [\Phi_k, \Phi_\ell]_N = [F, G]_N.$$

Hence our statement on discrete biorthogonality follows by Theorem 1. ■

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AN INTERPLAY BETWEEN JENSEN'S AND PEXIDER'S FUNCTIONAL EQUATIONS ON SEMIGROUPS

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Dedicated to Professor Antal Járai on his 60-th birthday

Abstract. Let $(S,+)$ and $(G,+)$ be two commutative semigroups. Assuming that the latter one is cancellative we deal with functions $f : S \longrightarrow G$ satisfying the Jensen functional equation written in the form

$$2f(x+y) = f(2x) + f(2y)\,.$$

It turns out that functions $f, g, h : S \longrightarrow G$ satisfying the functional equation of Pexider

$$f(x+y) = g(x) + h(y)$$

must necessarily be Jensen. The validity of the converse implication is also studied with emphasis placed on a very special Pexider equation

$$\varphi(x+y) + \delta = \varphi(x) + \varphi(y)\,,$$

where δ is a fixed element of G. Plainly, the main goal is to express the solutions of both: Jensen and Pexider equations in terms of semigroup homomorphisms.

Bearing in mind the algebraic nature of the functional equations considered, we were able to establish our results staying away from topological tools.

1. Introduction

We will investigate the very classical functional equations of Jensen, i.e.

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2},$$

and of Pexider, i.e.

$$f(x+y) = g(x) + h(y),$$

where f, g, h are functions defined and assuming values in some abstract algebraic structures. These equations have very rich literature; the basic facts concerning that topic may be found (among others) in the well known monographs of J. Aczél [1] and M. Kuczma [2]. It is also commonly known that in the case where both the domain and the target spaces of functions considered are linear spaces, the general solution of the Jensen and of the Pexider equations may be expressed in terms of additive functions. Let us recall that a function a is called additive provided it satisfies the Cauchy functional equation

$$a(x+y) = a(x) + a(y).$$

In classical situations Jensen functions are represented as the sum of an additive map and a constant function. The same can be told about solutions of the Pexider equation. The question we are faced is: to what extent these representations remain valid and/or what kind of potentially new phenomena may occur while dealing with more abstract algebraic structures. In particular, regarding the Jensen equation, the category of not necessarily commutative groups was taken into account in the papers of C.T. Ng [3], [4] and H. Stetkaer [6]. In the present paper we will concentrate on semigroups as potential domains and codomains. In some cases, we try also to get rid of the 2-divisibility assumption dealing with a version of the Jensen equation which does not require the feasibility of such division. On the other hand, we try to keep the strictly algebraic character of our studies avoiding, in particular, any topological structures. This aspect distinguishes our approach from the one applied, for instance, in the paper of W. Smajdor [5]. The basic results from this paper will be generalized considerably just due to the fact that, bearing in mind the algebraic nature of the functional equations considered, we were able to stay away from topological tools.

2. Some lemmas

We start with a simpler case when the target space of functions considered is a group.

Lemma 1. *Let $(S,+)$ be a commutative semigroup and let $(G^*,+)$ be an Abelian group. Then a function $f : S \to G^*$ satisfies the Jensen functional equation*

$$2f(x+y) = f(2x) + f(2y), \qquad x, y \in S, \tag{1}$$

if and only if there exist an additive map $A : S \to G^*$ and a constant $b \in G^*$ such that

$$f(2x) = A(x) + b, \quad x \in S, \qquad \text{and} \qquad 2f(x) = A(x) + 2b, \quad x \in S+S.$$

Proof. Assume (1) and define a function $\varphi : S \to G^*$ by the formula

$$\varphi(x) := f(2x) - 2f(x), \quad x \in S.$$

Then by (1) we obtain

$$\begin{aligned} 2f(x+y+z) &= f(2(x+y)) + f(2z) = \varphi(x+y) + 2f(x+y) + f(2z) = \\ &= \varphi(x+y) + f(2x) + f(2y) + f(2z), \end{aligned}$$

as well as,

$$\begin{aligned} 2f(x+y+z) &= f(2x) + f(2(y+z)) = f(2x) + \varphi(y+z) + 2f(y+z) = \\ &= f(2x) + \varphi(y+z) + f(2y) + f(2z), \end{aligned}$$

for all $x, y, z \in S$, whence

$$\varphi(x+y) = \varphi(y+z), \qquad x, y, z \in S.$$

In particular, setting $z = y$, due to the commutativity of the binary law in S,

$$\varphi(2y) = \varphi(x+y) = \varphi(2x), \quad x, y \in S.$$

Therefore, $\varphi(t) \equiv \text{const} =: c$ on the set $S+S$. In view of (1) and the definition of φ, this implies

$$f(2x) + c + f(2y) + c = 2f(x+y) + c + c = f(2(x+y)) + c, \quad x, y \in S,$$

stating that the map $A(x) := f(2x) + c$, $x \in S$, is additive. By setting $b := -c$ we derive the first part of our assertion. For $x \in S+S$ one has $x = y+z$, $y, z \in \in S$ whence, by (1),

$$\begin{aligned} A(x) + 2b &= A(y+z) + 2b = A(y) + b + A(z) + b = \\ &= f(2y) + f(2z) = 2f(y+z) = 2f(x). \end{aligned}$$

This ends the proof of the necessity, and since the sufficiency is obvious, the proof is completed. ■

Corollary 1. *Let all the assumptions of* Lemma 1 *be satisfied. If, moreover, the division by* 2 *is uniquely performable in* $(G^*, +)$, *then* $f : S \to G^*$ *satisfies equation* (1) *if and only if there exist an additive map* $A^* : S \to G^*$ *and a constant* $b \in G^*$ *such that*

$$f(x) = \begin{cases} A^*(x) + b, & \text{for } x \in S+S \\ \text{arbitrary}, & \text{on } S \setminus (S+S). \end{cases}$$

Proof. By virtue of the second part of the assertion of Lemma 1 it suffices to put $A^*(x) := \frac{1}{2}A(x)$, $x \in S$. ■

Lemma 2. *Let all the assumptions of* Lemma 1 *be satisfied. If functions* $f, g, h : S \to G^*$ *satisfy the Pexider functional equation*

$$f(x+y) = g(x) + h(y), \quad x, y \in S, \tag{2}$$

then there exist an additive map $A : S \to G^*$ *and constants* $b, c \in G^*$ *such that*

$$\begin{cases} f(2x) = A(x) + b, & x \in S; \\ 2g(x) = A(x) + b - c, & x \in S; \\ 2h(x) = A(x) + b + c, & x \in S\ ; \\ 2f(x) = A(x) + 2b, & x \in S+S. \end{cases} \tag{$*$}$$

Conversely, every triple (f, g, h) *satisfying conditions* $(*)$ *yields a solution to the equation*

$$2f(x+y) = 2g(x) + 2h(y), \qquad x, y \in S. \tag{3}$$

Proof. (Necessity.) We shall first show that f satisfies (1). Indeed, for all $x, y \in S$ we have

$$\begin{aligned} 2f(x+y) &= f(x+y) + f(y+x) = g(x) + h(y) + g(y) + h(x) = \\ &= f(2x) + f(2y). \end{aligned}$$

On account of Lemma 1, there exists an additive map $A : S \to G^*$ and a constant $b \in G^*$ such that

$$f(2x) = A(x) + b, \quad x \in S, \qquad \text{and} \quad 2f(x) = A(x) + 2b, \quad x \in S + S.$$

Since

$$g(x) + h(y) = f(x + y) = f(y + x) = g(y) + h(x), \qquad x, y \in S,$$

we get

$$h(x) - g(x) = h(y) - g(y) \equiv \text{const} =: c.$$

Consequently,

$$h(x) = g(x) + c, \quad x \in S\,,$$

and, therefore, for every $x, y \in S$ we have

$$f(x + y) = g(x) + h(y) = g(x) + g(y) + c,$$

whence

$$A(x) + b = f(2x) = 2g(x) + c, \quad x \in S,$$

and

$$2h(x) = 2g(x) + 2c = A(x) + b + c, \quad x \in S,$$

as claimed.

(Sufficiency.)

$$2g(x)+2h(y) = A(x)+b-c+A(x)+b+c = A(x+y)+2b = 2f(x+y), \quad x, y \in S,$$

which completes the proof. ■

Corollary 2. *Let $(S, +)$ be a commutative semigroup and let $(G^*, +)$ be an Abelian group uniquely 2-divisible. Then the triple (f, g, h) of functions from S into G^* yields a solution to equation (2) if and only if*

$$f(x) = \begin{cases} A^*(x) + 2b^* & \textit{for } x \in S + S \\ \textit{arbitrary} & \textit{on } S \setminus (S + S); \end{cases}$$

$$g(x) = A^*(x) + b^* - c^*, \quad x \in S;$$

$$h(x) = A^*(x) + b^* + c^*, \quad x \in S,$$

where $A^* : S \to G^*$ is additive and b^*, c^* are arbitrary constants from G^*.

Proof. In the light of Lemma 2 it suffices to put $A^* := \frac{1}{2}A$, $b^* := \frac{1}{2}b$, $c^* := \frac{1}{2}c$. ■

3. Main results

In what follows, we shall apply these results to deal with the case where G is a cancellative semigroup.

Theorem 1. *Let $(S,+)$ be a commutative semigroup and let $(G,+)$ stand for an Abelian cancellative semigroup. A map $f : S \to G$ satisfies Jensen's functional equation* (1) *if and only if there exist elements $\beta,\ \gamma \in G$ such that*

$$\begin{cases} f(x+y)+\beta = f(x)+f(y)+\gamma & \textit{for } \ x,y \in 2S; \\ f(2x)+\beta = 2f(x)+\gamma & \textit{for } \ x \in S+S; \\ f \textit{ is arbitrary} & \textit{on } \ S \setminus (S+S)\ . \end{cases}$$

Proof. We embed the semigroup $(G,+)$ into a group $(G^*,+)$ of equivalence classes determined by the relation

$$(u,v) \sim (x,y) :\Longleftrightarrow u+y = v+x\,.$$

Clearly, we identify an element x from G with the class $[(2x,x)]$. Moreover, we have also

$$-[(x,y)] = [(y,x)], \quad \text{as well as} \quad 0 = [(x,x)]\,.$$

Finally, we put

$$f^*(x) := [(2f(x), f(x))], \quad x \in S.$$

Equation (1) may equivalently be written in the form

$$4f(x+y)+f(2x)+f(2y) = 2f(x+y)+2f(2x)+2f(2y), \qquad x,y \in S.$$

This allows us to write

$$\begin{aligned} 2f^*(x+y) \quad &= [(4f(x+y), 2f(x+y))] = \\ &= [(2f(2x)+2f(2y), f(2x)+f(2y))] = \\ &= f^*(2x)+f^*(2y). \end{aligned}$$

On account of Lemma 1 we infer that there exist an additive map $A : S \to G^*$ and a constant $b \in G^*$ such that

$$f^*(2x) = A(x)+b,\ x \in S, \qquad 2f^*(x) = A(x)+2b,\ x \in S+S.$$

Let $b = [(\beta,\gamma)]$. Then, for all $x,y \in S$, one has

$$f^*(2x+2y)+b = A(x+y)+2b = A(x)+A(y)+b+b = f^*(2x)+f^*(2y),$$

i.e.

$$[(2f(2x+2y)+\beta, f(2x+2y)+\gamma)] = [(2f(2x)+2f(2y), f(2x)+f(2y))],$$

whence

$$2f(2x+2y)+f(2x)+f(2y)+\beta = f(2x+2y)+2f(2x)+2f(2y)+\gamma, \quad x,y \in S,$$

i.e.

$$f(2x+2y)+\beta = f(2x)+f(2y)+\gamma, \quad x,y \in S$$

or, equivalently,

$$f(x+y)+\beta = f(x)+f(y)+\gamma, \quad \text{for all } x,y \in 2S.$$

Let now $x \in S+S$. Then $x = y+z,\ y,z \in S$ whence by (1):

$$\begin{aligned} 2f(x)+\gamma &= 2f(y+z)+\gamma = f(2y)+f(2z)+\gamma = f(2y+2z)+\beta = \\ &= f(2x)+\beta, \end{aligned}$$

as claimed.

Clearly, equation (1) leaves the values of f on $S \setminus (S+S)$ undetermined.

(Sufficiency). Let $x,y \in S$. Then $x+y \in S+S$ and we have

$$f(2(x+y))+\beta = 2f(x+y)+\gamma \quad \text{and} \quad f(2x+2y)+\beta = f(2x)+f(2y)+\gamma,$$

whence

$$2f(x+y) = f(2x)+f(2y), \qquad x,y \in S.$$

This finishes the proof. ■

Corollary 3. *Let* $(S,+),(G,+)$ *and* f *be the same as in* Theorem 1. *Then the function*

$$a_f(x) := f(2x)+\beta+\gamma, \qquad x \in S,$$

enjoys the property

$$a_f(x+y)+2\beta = a_f(x)+a_f(y), \quad x,y \in S.$$

Proof.

$$\begin{aligned} a_f(x+y)+2\beta &= f(2x+2y)+2\beta+\beta+\gamma = f(2x)+f(2y)+2\beta+2\gamma = \\ &= a_f(x)+a_f(y), \end{aligned}$$

for all $x,y \in S$. ■

Theorem 2. *Let* $(S,+)$ *be a commutative semigroup and let* $(G,+)$ *stand for an Abelian cancellative semigroup. If functions* $f,g,h : S \to G$ *satisfy the Pexider equation* (2)*, then each of them satisfies the Jensen equation* (1)*. Moreover, there exist a map* $\psi : S \to G$ *and constants* $\alpha,\beta,\gamma,\delta,\varepsilon \in G$ *such that*

$$\psi(x+y)+\varepsilon = 2f(x+y)+\alpha, \qquad x,y \in S, \tag{4}$$

$$\psi(x+y) = 2g(x+y)+\beta = 2h(x+y)+\gamma, \qquad x,y \in S, \tag{5}$$

and

$$\psi(x+y)+\delta = \psi(x)+\psi(y)\,, \quad x,y \in S. \tag{6}$$

Conversely, if $\alpha,\beta,\gamma,\delta,\varepsilon \in G$ *are arbitrary constants satisfying condition*

$$\beta+\gamma+\varepsilon = \alpha+\delta \tag{7}$$

and equalities (4), (5) *and* (6) *are fulfilled, then*

$$2f(2x+2y) = 2g(2x)+2h(2y), \quad x,y \in S. \tag{8}$$

Proof. Equation (2) implies that

$$2f(x+y) = f(x+y)+f(y+x) = g(x)+h(y)+g(y)+h(x) = f(2x)+f(2y),$$

for all $x,y \in S$, i.e. f satisfies Jensen equation (1). Therefore

$$f(2x)+f(2y) = 2f(x+y) = 2g(x)+2h(y), \quad x,y \in S.$$

Fix $u,v \in S$ arbitrarily and put $x = u+v$. Then

$$f(2u+2v)+f(2y) = 2g(u+v)+2h(y),$$

and by virtue of (2) we get

$$g(2u)+h(2v)+g(y)+h(y)+g(2v) = 2g(u+v)+2h(y)+g(2v),$$

whence also

$$g(2u)+g(2v)+f(2v+y) = 2g(u+v)+f(2v+y)$$

follows, i.e. $g(2u)+g(2v) = 2g(u+v)$. Analogously, we check that h is a Jensen function.

On account of Theorem 1, there exist constants $\beta_f, \gamma_f, \beta_g, \gamma_g, \beta_h, \gamma_h \in G$ such that

$$\varphi(x+y) + \beta_\varphi = \varphi(x) + \varphi(y) + \gamma_\varphi, \qquad x, y \in 2S, \tag{9}$$

and

$$\varphi(2x) + \beta_\varphi = 2\varphi(x) + \gamma_\varphi, \qquad x \in S + S, \tag{10}$$

where $\varphi \in \{f, g, h\}$. Let us define the functions $a_\varphi : S \to G,\ \varphi \in \{f, g, h\}$ by the formulas

$$a_\varphi(x) := \varphi(2x) + \beta_\varphi + \gamma_\varphi, \qquad x \in S.$$

Since φ is Jensen function we obtain by (10) that

$$a_\varphi(x+y) + 2\beta_\varphi = a_\varphi(x) + a_\varphi(y), \qquad x, y \in S. \tag{11}$$

According to (2) we have

$$\begin{aligned} a_g(x) + a_h(y) &= g(2x) + \beta_g + \gamma_g + h(2y) + \beta_h + \gamma_h = \\ &= f(2x + 2y) + \beta_g + \gamma_g + \beta_h + \gamma_h = \\ &= g(2y) + \beta_g + \gamma_g + h(2x) + \beta_h + \gamma_h = \\ &= a_g(y) + a_h(x), \end{aligned}$$

whence

$$a_g(x) + a_h(y) = a_g(y) + a_h(x), \quad x, y \in S.$$

Thus, there exist constants $\lambda, \mu \in G$ such that

$$a_g(x) + \lambda = a_h(x) + \mu, \quad x \in S. \tag{12}$$

Now, setting

$$\psi(x) := a_g(x) + \lambda = a_h(x) + \mu, \quad x \in S,$$

by virtue of (11), for all $x, y \in S$, we infer that

$$\psi(x)+\psi(y) = a_g(x)+\lambda+a_g(y)+\lambda = a_g(x+y)+2\beta_g+2\lambda = \psi(x+y)+2\beta_g+\lambda,$$

and it suffices to put $\delta := 2\beta_g + \lambda$ to obtain (6). It follows from (11), the definition of a_g and (10) that $\psi(x+y) = a_g(x+y)+\lambda = g(2(x+y))+\beta_g+\gamma_g+\lambda = 2g(x+y)+2\gamma_g+\lambda$, for all $x, y \in S$, which coincides with the first equality in (5) on setting $\beta := 2\gamma_g + \lambda$. The other one may be derived similarly. Finally, by (4), (2) and (10)

$$\begin{aligned} \psi(x+y) + \delta + \beta_f &= \psi(x) + \psi(y) + \beta_f = a_g(x) + \lambda + a_h(y) + \mu + \beta_f = \\ &= g(2x) + \beta_g + \gamma_g + \lambda + h(2y) + \beta_h + \gamma_h + \mu + \beta_f = \\ &= f(2(x+y)) + \beta_g + \gamma_g + \lambda + \beta_h + \gamma_h + \mu + \beta_f = \\ &= 2f(x+y) + \gamma_f + \beta_g + \gamma_g + \lambda + \beta_h + \gamma_h + \mu, \end{aligned}$$

and it sufficies to put $\varepsilon := \delta + \beta_f$ as well as $\alpha := \gamma_f + \beta_g + \gamma_g + \lambda + \beta_h + \gamma_h + \mu$ to arrive at (4).

Conversely, let $\alpha, \beta, \gamma, \delta, \varepsilon \in G$ be arbitrary constants satisfying (7) and assume that equalities (4), (5) and (6) are fulfilled. Then

$$\begin{aligned} 2g(2x) + 2h(2y) + \beta + \gamma + \varepsilon &= \psi(2x) + \psi(2y) + \varepsilon \\ &= \psi(2x + 2y) + \delta + \varepsilon = 2f(2x + 2y) + \alpha + \delta\,, \end{aligned}$$

which jointly with (7) implies (8) and finishes the proof. ■

As we see in our considerations the functional equation (6) (see also (11)) plays a crucial role. Thus the problem of solving this equation seems to be a basic one.

Theorem 3. *Let $(S, +)$ be a commutative semigroup and let $(G, +)$ stand for an Abelian cancellative semigroup. Given a fixed element $\delta \in G$, if a map $\psi : S \to G$ satisfies the equation*

$$\psi(x + y) + \delta = \psi(x) + \psi(y), \qquad x, y \in S, \tag{13}$$

then the set $S_\delta := \psi^{-1}(G + \delta)$ is either empty or $(S_\delta, +)$ yields a subsemigroup of $(S, +)$ and there exists a homomorphism $H : S_\delta \to G$ such that

$$\psi(x) = H(x) + \delta, \qquad x \in S_\delta. \tag{14}$$

If, moreover, there exists a $y_0 \in S$ such that $\psi(y_0) \in G + 2\delta$, then $S + y_0 \subset S_\delta$ and there exists an $\eta \in G$ such that

$$\psi(x) + \eta = H(x + y_0), \qquad x \in S. \tag{15}$$

In particular, such a representation takes place provided that ψ is a surjection from S onto G.

Proof. Assume that $S_\delta \neq \emptyset$ and take arbitrary $x, y \in S_\delta$. Then there exist $w, z \in G$ such that $\psi(x) = w + \delta$ and $\psi(y) = z + \delta$. By (13) we infer that

$$\psi(x + y) + \delta = \psi(x) + \psi(y) = w + \delta + z + \delta$$

whence

$$\psi(x + y) = w + z + \delta \in G + \delta.$$

This means that $x + y \in S_\delta$ and proves that $(S_\delta, +)$ forms a subsemigroup of $(S, +)$. It follows from the definition of S_δ that there exists a function $H : S_\delta \to G$ fulfilling equality (14). For all $x, y \in S_\delta$ we have

$$H(x + y) + 2\delta = \psi(x + y) + \delta = \psi(x) + \psi(y) = H(x) + \delta + H(y) + \delta.$$

which states that H is a homomorphism.

If for some $y_0 \in S$ we have $\psi(y_0) = \eta + 2\delta$ with some $\eta \in G$, then $y_0 \in S_\delta$. Consequently

$$\eta + 2\delta = \psi(y_0) = H(y_0) + \delta,$$

whence

$$H(y_0) = \eta + \delta \in G + \delta. \tag{16}$$

According to (13) we get

$$\psi(x + y_0) + \delta = \psi(x) + \psi(y_0) = \psi(x) + \eta + 2\delta, \quad x \in S,$$

and since G is cancellative,

$$\psi(x + y_0) = \psi(x) + \eta + \delta \in G + \delta, \qquad x \in S. \tag{17}$$

Therefore $x + y_0 \in S_\delta$, $x \in S$, or, equivalently,

$$S + y_0 \subset S_\delta.$$

On account of (14) we obtain

$$\psi(x + y_0) = H(x + y_0) + \delta, \qquad x \in S.$$

By virtue of (17) we get (15). It is easily seen that (15) takes place provided ψ is surjective. ∎

Corollary 4. *Let $(S, +)$ be a commutative semigroup and let $(G, +)$ stand for an Abelian cancellative monoid. Assume that $\psi : S \to G$ is a surjection of S onto G satisfying equation* (13), *$S_\delta := \psi^{-1}(G + \delta) \neq \emptyset$ and y_0 is a fixed element of S such that $\psi(y_0) \in G + 2\delta$. Then $(S_\delta, +)$ is a subsemigroup of $(S, +)$ and there exists a homomorphism H mapping S_δ into G such that*

$$\psi(x) = H(x + y_0), \quad x \in S,$$

and

$$H(S + y_0) = G, \qquad H(y_0) = \delta$$

Proof. Going back to the proof of Theorem 3, take $y_0 \in S$ such that $\psi(y_0) = 2\delta$ there. Then $\eta = 0$ and consequently $\psi(x) = H(x + y_0)$, $x \in S$, and $H(y_0) = \delta$. The equality $H(S + y_0) = G$ is obvious. ∎

Remark 1. Let $(S, +), (G, +)$ be the same as in Theorem 3. If $\psi : S \to G$ satisfies equation (13) and there exist $u, v \in S$ such that $\psi(u) = 2\psi(v)$, then the set $S_\delta = \psi^{-1}(G + \delta)$ is nonvoid.

In fact, $\psi(u) = 2\psi(v) = \psi(v) + \psi(v) = \psi(2v) + \delta \in G + \delta$.

Lemma 3. *Let $(S,+)$ be a commutative semigroup and let $(G,+)$ be an Abelian cancellative semigroup in which the division by 2 is uniquely performable. If $\psi : S \to G$ satisfies equation (13), then for an arbitrary positive integer n and each $x \in S$ the following equality*

$$\psi(x) + \frac{1}{2^n}\delta = \frac{1}{2^n}\psi(2^n x) + \delta \tag{18}$$

holds true.

Proof. (Induction.) Putting $y = x$ in (13) we obtain

$$\psi(2x) + \delta = 2\psi(x), \qquad x \in S\,,$$

whence (18) follows immediately for $n = 1$. Assume (18) for a positive integer n and each $x \in S$. Then

$$\frac{1}{2}\psi(2x) + \frac{1}{2^{n+1}}\delta = \frac{1}{2^{n+1}}\psi(2^{n+1}x) + \frac{1}{2}\delta, \quad x \in S,$$

as well as

$$\frac{1}{2}\psi(2x) + \delta + \frac{1}{2^{n+1}}\delta = \frac{1}{2^{n+1}}\psi(2^{n+1}x) + \frac{1}{2}\delta + \delta, \quad x \in S.$$

Applying (18) for $n = 1$ we obtain

$$\psi(x) + \frac{1}{2}\delta + \frac{1}{2^{n+1}}\delta = \frac{1}{2^{n+1}}\psi(2^{n+1}x) + \frac{1}{2}\delta + \delta$$

and, consequently,

$$\psi(x) + \frac{1}{2^{n+1}}\delta = \frac{1}{2^{n+1}}\psi(2^{n+1}x) + \delta\,,$$

which ends the proof. ∎

Corollary 5. *Under the assumptions of* Lemma 3 *we have*

$$\psi(x) \in \bigcap_{n=1}^{\infty}\left(G + \left(1 - \frac{1}{2^n}\right)\delta\right), \quad x \in S.$$

Proof. Fix an $x \in S$ and a positive integer n. On account of Lemma 3 we have

$$\psi(x) + \frac{1}{2^n}\delta = \frac{1}{2^n}\psi(2^n x) + \frac{1}{2^n}\delta + \left(1 - \frac{1}{2^n}\right)\delta, \quad x \in S,\, n \in \mathbb{N},$$

whence

$$\psi(x) = \frac{1}{2^n}\psi(2^n x) + \left(1 - \frac{1}{2^n}\right)\delta, \quad x \in S,\ n \in \mathbb{N},$$

which finishes the proof. ∎

Theorem 4. *Let $(S, +)$ be a commutative semigroup and let $(G, +)$ be a semigroup that is Abelian uniquely 2-divisible and cancellative. Assume that $\delta \in G$ is such that*

$$\bigcap_{n\in\mathbb{N}} \left(G + \left(1 - \frac{1}{2^n}\right)\delta\right) \subset G + \delta. \tag{19}$$

Then a map $\psi : S \to G$ satisfies (13) if and only if there exists a homomorphism $H : S \to G$ such that

$$\psi(x) = H(x) + \delta, \qquad x \in S.$$

Proof. It follows from (19) and Corollary 5, that

$$\psi(x) \in G + \delta, \qquad x \in S.$$

Therefore

$$\psi(x) = H(x) + \delta, \qquad x \in S,$$

where $H : S \to G$ is a function. Applying (19) we obtain

$$H(x + y) + 2\delta = \psi(x + y) + \delta = \psi(x) + \psi(y) = H(x) + \delta + H(y) + \delta,$$

which implies that $H(x + y) = H(x) + H(y),\ x, y \in S$. Since the suffiency is obvious, the proof has been finished. ∎

Theorem 5. *Let $(S, +), (G, +)$ be two commutative uniquely 2-divisible semigroups. Assume that $(G, +)$ is cancellative and such that condition (19) is fulfilled for every $\delta \in G$. Then $f : S \to G$ satisfies Jensen functional equation (1) if and only if there exists an additive function $H : S \to G$ such that*

$$f(x + y) = H(x) + f(y), \qquad x, y \in S.$$

Proof. By Theorem 1 there exist constants $\beta, \gamma \in G$ such that

$$f(x + y) + \beta = f(x) + f(y) + \gamma, \quad x, y \in 2S = S.$$

Putting $\psi(x) := f(x)+\gamma,\ x\in S,$ we note that

$$\psi(x+y)+\beta = f(x+y)+\gamma+\beta = f(x)+f(y)+2\gamma = \psi(x)+\psi(y), \quad x,y\in S,$$

i.e. equation (13) is satisfied with $\delta=\beta$. On account of Theorem 4, there exists an additive map $H : S\to G$ such that

$$\psi(x) = H(x)+\beta, \quad x\in S.$$

Therefore

$$f(x)+\gamma = H(x)+\beta, \qquad x\in S\,,$$

and hence

$$f(x+y)+\beta = f(x)+f(y)+\gamma = H(x)+\beta+f(y), \quad x,y\in S\,,$$

yielding

$$f(x+y) = H(x)+f(y), \quad x,y\in S\,,$$

as claimed.

Conversely, for all $x,y\in S$, one has

$$f(2x)+f(2y) = H(x)+f(x)+H(y)+f(y) = f(x+y)+f(y+x) = 2f(x+y)\,,$$

which completes the proof. ■

4. Generalizations of W. Smajdor's results

W. Smajdor [5] defines an *abstract convex cone* as a cancellative Abelian monoid $(G,+)$ provided that a map $[0,\infty)\times G\ni(\lambda,s)\to\lambda s\in G$ is given such that

$$1s=s,\ \lambda(\mu s)=(\lambda\mu)s,\ \lambda(s+t)=\lambda s+\lambda t,\ (\lambda+\mu)s=\lambda s+\mu s,$$

$$s,t\in G,\ \lambda,\mu\in[0,\infty).$$

Under the additional assumption that G is endowed with a complete metric ϱ such that

$$\varrho(s+t,s+t')=\varrho(t,t'),\ s,t,t'\in G, \quad \varrho(\lambda s,\lambda t)=\lambda\varrho(s,t),\ \lambda\in[0,\infty), s,t\in G\,,$$

W. Smajdor's main result (see Theorem 1 of [5]) states that any function f mapping an Abelian 2-divisible semigroup $(S,+)$ into $(G,+)$ satisfies the Jensen equation if and only if there exists an additive map $a : S \to G$ such that the equality $f(x+y) = a(x) + f(y)$ holds true for all $x, y \in S$.

The occurrence of a topology (actually: metric topology) in the target cone in Smajdor's theorem seems to be artificial bearing in mind the strictly algebraic nature of the problem considered. Our Theorem 5 generalizes her result by avoiding any topological structure in the target space. In fact, the only thing we need is to show that under W. Smajdor's assumptions condition (19), i.e. the inclusion

$$\bigcap_{n\in\mathbb{N}} \left(G + \left(1 - \frac{1}{2^n}\right)\delta \right) \subset G + \delta$$

is fulfilled for every δ from G. As a matter of fact, we shall achieve that with the aid of considerably weaker requirements.

Proposition. *Given a cancellative semigroup $(G, +)$ uniquely divisible by 2 and admitting a complete metric ϱ such that*

$$\varrho(x+z, y+z) = \varrho(x, y),\ x, y, z \in G, \quad \varrho(2x, 2y) = 2\varrho(x, y),\ x, y \in G\,,$$

there exists a neutral element 0 in G, i.e. $(G, +)$ is necessarily a monoid. Moreover, for every δ from G condition (19) holds true.

Proof. The binary law "+" has to be continuous; in fact, if

$$G \ni x_n \longrightarrow x_0 \in G \quad \text{and} \quad G \ni y_n \longrightarrow y_0 \in G,$$

then

$$\begin{aligned}\varrho(x_n + y_n, x_0 + y_0) \le \varrho(x_n + y_n, x_n + y_0) + \varrho(x_n + y_0, x_0 + y_0) = \\ = \varrho(y_n, y_0) + \varrho(x_n, x_0) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty\,.\end{aligned}$$

In particular the map $G \ni x \longrightarrow 2x \in G$ is continuous. Fix $\delta \in G$ arbitrarily. Then

$$\left(\frac{1}{2^n}\delta\right)_{n\in\mathbb{N}} \quad \text{is a Cauchy sequence}\,.$$

Indeed, for all positive integers n, k one has

$$\varrho\left(\frac{1}{2^{n+k}}\delta, \frac{1}{2^n}\delta\right) \le \sum_{j=0}^{k-1} \frac{1}{2^{n+j}}\varrho\left(\frac{1}{2}\delta, \delta\right) \le \frac{1}{2^{n-1}}\varrho\left(\frac{1}{2}\delta, \delta\right).$$

Since ρ is complete the sequence $(\frac{1}{2^n}\delta)_{n\in\mathbb{N}}$ converges to an $x_0 \in G$. Then also

$$2x_0 = 2\lim_{n\to\infty} \frac{1}{2^{n+1}}\delta = \lim_{n\to\infty} \frac{1}{2^n}\delta = x_0\,,$$

whence, for every $x \in G$, we get

$$x + x_0 = x + 2x_0 = (x + x_0) + x_0 \quad \text{and} \quad x_0 + x = 2x_0 + x = x_0 + (x_0 + x)\,,$$

which, by means of the cancellativity assumption, states that x_0 is zero element in G.

Now, in order to show the inclusion (19), fix an arbitrary x from the intersection $\bigcap_{n\in\mathbb{N}}(G + (1 - \frac{1}{2^n})\delta)$. Then, for every $n \in \mathbb{N}$ one may find a $g_n \in G$ such that

$$x + \frac{1}{2^n}\delta = g_n + \delta\,.$$

Since the addition is continuous and the sequence $\left(\frac{1}{2^n}\delta\right)_{n\in\mathbb{N}}$ converges to the neutral element x_0, the sequence $(g_n + \delta)_{n\in\mathbb{N}}$ tends to x. Therefore, x belongs to $G + \delta$ since, obviously, the set $G + \delta$ is closed as a complete subspace of G. This completes the proof. ■

Remark 2. Condition (19) is automatically satisfied in any Abelian, uniquely 2-divisible group $(G, +)$. Actually, for any $\delta \in G$ the inclusion

$$G + \left(1 - \frac{1}{2^n}\right)\delta = G - \frac{1}{2^n}\delta + \delta \subset G + \delta$$

is satisfied for every $n \in \mathbb{N}$.

Another example of an Abelian, uniquely 2-divisible monoid in which condition (19) holds true reads as follows. Let $a : \mathbb{R} \to \mathbb{R}$ be a discontinuous additive function and let

$$G := \{x \in \mathbb{R} :\, a(x) \geq 0\}.$$

Equipped with the usual addition, the set G yields a commutative semigroup with 0 as the neutral element. For any $\delta \in G$ and for every $n \in \mathbb{N}$ we have

$$G + \left(1 - \frac{1}{2^n}\right)\delta = \left\{y \in \mathbb{R} :\, a(y) \geq \left(1 - \frac{1}{2^n}\right)a(\delta)\right\}\,,$$

whence

$$\bigcap_{n\in\mathbb{N}}\left(G + \left(1 - \frac{1}{2^n}\right)\delta\right) = \bigcap_{n\in\mathbb{N}}\left\{y \in \mathbb{R} :\, a(y) \geq \left(1 - \frac{1}{2^n}\right)a(\delta)\right\} =$$

$$= \{y \in \mathbb{R} :\, a(y) \geq a(\delta)\} = G + \delta\,.$$

Noteworthy is the fact that in the case where $a(\delta) > 0$ the shift $G + \delta$ fails to coincide with G itself.

Finally, each uniquely 2-divisible topological monoid $(G,+;0)$ such that for every $\delta \in G$ the shift $G+\delta$ is closed and $\lim_{n\longrightarrow\infty} 2^{-n}\delta = 0$ enjoys the property (19) (cf. the proof of the Proposition).

The following example shows that, in general, condition (19) need not be fulfilled. Indeed, let $G=(0,\infty)$ and let $\delta>0$ be fixed. Then G equipped with the usual addition is a uniquely 2-divisible commutative semigroup and

$$\bigcap_{n\in\mathbb{N}} \left((0,\infty)+\left(1-\frac{1}{2^n}\right)\delta\right) = \bigcap_{n\in\mathbb{N}} \left(\left(1-\frac{1}{2^n}\right)\delta,\infty\right) = \\ = [\delta,\infty) \not\subset G+\delta = (\delta,\infty)\,.$$

We terminate this paper with the following generalization of Theorem 2 in [5] by W. Smajdor.

Theorem 6. *Let $(S,+),(G,+)$ be two commutative uniquely 2-divisible semigroups. Assume that $(G,+)$ is cancellative and such that condition (19) is fulfilled for every $\delta \in G$. If $f,g,h : S \to G$ fulfil the Pexider equation (2) then there exists a homomorphism $H : S \to G$ such that*

$$f(x+y)=H(x)+f(y),\ g(x+y)=H(x)+g(y),\ h(x+y)=H(x)+h(y),$$

for all $x,y\in S$.

Proof. On account of Theorem 2 we infer that f,g and h are Jensen functions. It follows from Theorem 5 that there exist additive functions H_f, H_g and H_h such that for all $x,y\in S$ the equalities

$$f(x+y)=H_f(x)+f(y),\ g(x+y)=H_g(x)+g(y),\ h(x+y)=H_h(x)+h(y),$$

hold true. Thus, for arbitrary $x,y\in S$ we have

$$\begin{aligned} H_f(x+y)+f(x+y) &= f(2x+2y) = g(x+y)+h(x+y) = \\ &= H_g(x)+g(y)+H_h(y)+h(x) = \\ &= H_g(x)+H_h(y)+f(x+y) \end{aligned}$$

which leads to

$$H_f(x+y)=H_g(x)+H_h(y), \quad x,y\in S.$$

Moreover,

$$H_g(x)+H_h(x)+2H_f(y) = H_f(2x)+2H_f(y) = 2H_f(x+y) = 2H_g(x)+2H_h(y),$$

whence

$$H_h(x)+2H_f(y) = H_g(x)+2H_h(y), \qquad x,y\in S.$$

Fix $y_0 \in S$ arbitrarily and put $\alpha := 2H_f(y_0)$, $\beta := 2H_h(y_0)$ to get the relationship

$$H_h(x) + \alpha = H_g(x) + \beta, \quad x \in S.$$

Similarly, by fixing an x_0 from S and setting $\gamma := \frac{1}{2}H_h(x_0)$, $\delta := \frac{1}{2}H_g(x_0)$ we arrive at

$$H_f(y) + \gamma = H_h(y) + \delta, \quad y \in S.$$

Now, with the aid of the embedding technics applied in the proof of Theorem 1, (we omit the details of that standard procedure) we deduce that the corresponding functions H_f^*, H_g^* and H_h^* mapping S into the group G^* are pairwise equal. This, in turn, forces the functions H_f, H_g and H_h to be pairwise equal, as well. Therefore, we finish the proof by setting $H := H_f = H_g = H_g$. ∎

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MEAN VALUES OF MULTIPLICATIVE FUNCTIONS ON THE SET OF $\mathcal{P}_k + 1$, WHERE $\mathcal{P}_k$ RUNS OVER THE INTEGERS HAVING k DISTINCT PRIME FACTORS

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Dedicated to the 60th anniversary of Professor Antal Járai

Abstract. We investigate the limit behaviour of

$$\sum_{\substack{n \le x \\ n \in \mathcal{P}_k}} g(n+1)$$

as x tends to infinity where g is multiplicative with values in the unit disc and $\mathcal{P}_k$ runs over the integers having k distinct prime factors. We let k vary in the range $2 \le k \le \epsilon(x) \log\log x$ where $\epsilon(x)$ is an arbitrary function tending to zero as x tends to infinity.

Throughout this work n denotes a positive integer and $P(n)$, $p(n)$ denote the largest and the smallest prime factors of n, respectively. p, q with or without suffixes will always denote prime numbers. As usual, the number of primes up to x will be denoted by $\pi(x)$, and $\log_k x := \log(\log_{k-1} x)$ for all positive integers k where $\log_1 x = \log x$ means the natural logarithm of x. If

$$(1) \qquad n = p_1^{r_1} \cdot p_2^{r_2} \cdots p_k^{r_k}, \qquad p_1 < p_2 < \ldots < p_k, \quad r_i, i = 1, \ldots, k$$

are positive integers, p_i, $i = 1, \ldots, k$ are distinct primes then let $\omega(n) := k$. A typical integer n for which $\omega(n) = k$ will be denoted by π_k. We denote the set of integers having k distinct prime factors with $\mathcal{P}_k$, that is

$$\mathcal{P}_k := \{\pi_k \in \mathbb{N}\}.$$

The set of integers in $\mathcal{P}_k$ up to x is denoted by $\mathcal{P}_k(x)$. We introduce the counting function for the set $\mathcal{P}_k$ in arithmetic progressions. If $(d,l)=1$ then let

$$\pi_k(x,d,l) = \sum_{\substack{\pi_k \le x \\ \pi_k \equiv l \pmod d}} 1.$$

In the special case $d=l=1$ we use $\pi_k(x)$ instead of $\pi_k(x,1,1)$.

An arithmetical function $g:\mathbb{N}\to\mathbb{C}$ is said to be *multiplicative* if $g(nm) = g(n)g(m)$ holds for all integers n,m with $(n,m)=1$. It is called *additive* if $g(nm)=g(n)+g(m)$ for $(n,m)=1$ and is called *strongly additive* if additionally $g(p^\alpha)=g(p)$ holds for all p and $\alpha\in\mathbb{N}$.

In the middle of the twentieth century Delange did some pioneering work concerning mean value estimations for multiplicative functions on the set $\mathbb{N}$. One of his results was the following (See [2])

Theorem (Delange). *Let g be a multiplicative function with $|g(n)|\le 1$, satisfying*

$$\sum_p \frac{1-\operatorname{Re} g(p)}{p} < \infty.$$

Then

$$\frac{1}{x}\sum_{n\le x} g(n) = \prod_{p\le x}\left(1-\frac{1}{p}\right)\left(1+\sum_{m\ge 1}\frac{g(p^m)}{p^m}\right) + o(1)$$

as x tends to infinity.

Although this result provides sufficient condition for multiplicative functions to have zero mean value, the full description of such multiplicative functions was given by Wirsing [12] for real and by Halász [4] for complex multiplicative functions of modulus ≤ 1. The result of Halász extends Delange's theorem in the following way:

Theorem (Delange, Wirsing, Halász). *Let g be a multiplicative function with $|g(n)|\le 1$, satisfying*

$$\sum_p \frac{1-\operatorname{Re} g(p)p^{-i\tau}}{p} < \infty$$

for some real τ. Then

$$\frac{1}{x}\sum_{n\le x} g(n) = \frac{x^{i\tau}}{1+i\tau}\prod_{p\le x}\left(1-\frac{1}{p}\right)\left(1+\sum_{m\ge 1}\frac{g(p^m)}{p^{m(1+i\tau)}}\right) + o(1)$$

as x tends to infinity. On the other hand, if there is no such τ then

$$\frac{1}{x}\sum_{n\le x} g(n) = o(1) \quad (x\to\infty).$$

Kátai in [7, 8] began to investigate the mean behaviour of multiplicative functions on the set of shifted primes. Through the contribution of Hildebrand [6] and Timofeev [11] it turned out that the situation is basically different from the case of the whole set of natural numbers. Their result is

Theorem (Kátai, Hildebrand, Timofeev)**.** *Let g be a multiplicative function with $|g(n)|\le 1$ and suppose that there are a real τ and a primitive character χ_d modulo d for some modulus d such that*

$$\sum_p \frac{1-\operatorname{Re}\chi_d(p)f(p)p^{-i\tau}}{p}$$

converges. Then

$$\frac{1}{\pi(x)}\sum_{n\le x} f(p+1) = \frac{\mu(d)}{\varphi(d)}\,\frac{x^{i\tau}}{1+i\tau}\times$$

$$\times\prod_{\substack{p\le x\\ p\nmid d}}\left(1+\sum_{r\ge 1}\frac{\chi_d(p^r)f(p^r)p^{-ri\tau}-\chi_d(p^{r-1})f(p^{r-1})p^{-(r-1)i\tau}}{\varphi(p^r)}\right)+o(1)$$

as $x\to\infty$, which is not necessarily $o(1)$ as x tends to infinity, if χ_d is a real character.

The main result of this paper is

Theorem 1. *Let $g(n)$ be a multiplicative function of modulus one, such that there are a primitive character χ (mod d) for some fixed d and a real τ such that*

$$\sum_p \frac{1-\operatorname{Re}\chi(p)g(p)p^{-i\tau}}{p}$$

converges. Let furthermore $\epsilon(x)$ be an arbitrary function tending to zero as x tends to infinity. Then

$$\pi_k(x)^{-1}\sum_{\substack{n\le x\\ \omega(n)=k}} g(n+1) =$$

$$= \frac{x^{i\tau}}{1+i\tau}\,\frac{\mu(d)}{\varphi(d)}\prod_{\substack{p\le x\\ p\nmid d}}\left(1-\frac{1}{p-1}+\sum_{\alpha\ge 1}\frac{g(p^\alpha)p^{-i\alpha\tau}\chi(p^\alpha)}{p^\alpha}\right)+o(1) \qquad (x\to\infty)$$

uniformly for all k, if $1\le k\le \epsilon(x)\log\log x$.

We will use the method of [3] since as we deduce the results from the analogoue for $D\mathcal{P}+1$ where $\mathcal{P}$ denotes the set of primes.

Let

$$M(x,f,D) := \sum_{Dp+1\le x} f(Dp+1).$$

Theorem 2. *Let $f(n)$ be a multiplicative function of modulus 1. Let furthermore d be a positive integer. Suppose that there is a real τ such that the series*

$$\sum_p \frac{|\chi(p)f(p)p^{i\tau}-1|^2}{p} \tag{2}$$

converges for some primitive character $\chi \pmod d$. Let $0<\epsilon<1/2$. Then

$$\left(\pi\left(\frac{x-1}{D}\right)\right)^{-1} M(x,f,D) =$$

$$= \frac{x^{i\tau}}{1+i\tau}\frac{\mu(d)}{\varphi(d)}\prod_{\substack{p\le x\\ p\nmid dD}}\left(1-\frac{1}{p-1}+\sum_{\alpha\ge1}\frac{f(p^\alpha)p^{-i\alpha\tau}\chi(p^\alpha)}{p^\alpha}\right)+o(1)\quad (x\to\infty)$$

holds uniformly for all $x>2$ and $D\le x^{1/2-\epsilon}$ with $(d,D)=1$.

As an application of Theorem 2 we are able to analyze the mean behavior of multiplicative functions on the set $\mathcal{P}_k+1$ in some cases. We need the following

Lemma 1. *Let $\epsilon(x)\to 0$ as $x\to\infty$. Then there exist sequences $y_x\to\infty$, $\delta_x\to 0$ as $x\to\infty$ such that*

$$P(n) > x^{1-\delta_x},\qquad y_x < p(n),\qquad n \text{ is square-free} \tag{3}$$

hold for all but $o(\pi_k(x))$ elements of $\mathcal{P}_k(x)$, uniformly for all

$$2\le k\le \epsilon(x)\log\log x \qquad \text{as}\quad x\to\infty.$$

Proof. The following sets have zero relative density in $\mathcal{P}_k$.

1. If $A_1=\{n\in\mathcal{P}_k,\ n\le x : \exists\ p^2|n\}$, then we have

$$\#A_1 \le \sum_{\substack{p^\alpha\le x^{1/2}\\ \alpha\ge2}} \pi_{k-1}\left(\frac{x}{p^\alpha}\right) + \sum_{\substack{p^\alpha > x^{1/2}\\ \alpha\ge2}} \frac{x}{p^\alpha} \ll \pi_k(x)\frac{k}{\log\log x}\sum_{\substack{p^\alpha\le x^{1/2}\\ \alpha\ge2}}\frac{1}{p^\alpha} + \mathcal{O}(x^{3/4}).$$

Here we used that

$$\frac{\pi_{k-1}(x)}{\pi_k(x)} \sim \frac{k}{\log\log x} (\to 0)\quad (x\to\infty)$$

holds uniformly for $2 \le k \le \epsilon(x)\log\log x$. This is a direct consequence of the asymptotic estimation

$$(4)\qquad \pi_k(x) = \frac{x}{\log x}\frac{\log\log^{k-1} x}{(k-1)!}\left(1+\mathcal{O}\left(\frac{1}{\log\log x}\right)\right),$$

which is uniform for $1 \le k \le \epsilon(x)\log\log x$ (see for example in [9]).

2. If $A_2 = \{n \in \mathcal{P}_k,\ n \le x : p(n) < y_x\}$, then we have

$$\#A_2 \le \sum_{\substack{p^\alpha \le x^{1/2}\\ p<y_x}} \pi_{k-1}\left(\frac{x}{p^\alpha}\right) + \sum_{\substack{p^\alpha > x^{1/2}\\ \alpha\ge 2}} \frac{x}{p^\alpha} \ll \pi_k(x)\frac{k}{\log\log x}\sum_{p<y_x}\frac{1}{p} + \mathcal{O}(x^{3/4}).$$

By means of these last two steps we can assume that $p(n) > y_x$, and n is square-free. Finally we have

$$\begin{aligned}
\sum_{\substack{\pi_k \le x\\ P(\pi_k)\le x^{1-\delta_x}}} 1 \ll \sum_{\pi_k \le x^{1/2}} 1 + \sum_{\substack{x^{1/2}\le \pi_k \le x\\ P(\pi_k)\le x^{1-\delta_x}}} 1 \ll&\\
\ll x^{1/2} + \frac{1}{\log x}\sum_{\substack{x^{1/2}\le \pi_k \le x\\ P(\pi_k)\le x^{1-\delta_x}}} \log \pi_k \ll&\\
\ll \frac{1}{\log x}\sum_{p\le x^{1-\delta_x}} \pi_{k-1}\left(\frac{x}{p}\right)\log p + x^{1/2} \ll&\\
\ll \frac{x}{\log x}\frac{\log^{k-2}\log x}{(k-2)!}\sum_{p\le x^{1-\delta_x}}\frac{\log p}{p\log(x/p)} + x^{1/2} \ll&\\
\ll \frac{1}{\delta_x}\pi_k(x)\frac{k}{\log\log x}&
\end{aligned}$$

and the proof is finished. ■

Proof of Theorem 1. The case $k=1$ was proved by Kátai, Hildebrand and Timofeev, and is included in Theorem 2. Therefore we can suppose that $k \ge 2$. Let $U_k(x)$ be the set of those elements of $\mathcal{P}_k(x)$, for which (3) holds true. Let S_x be the set of those π_{k-1}, for which there exists at least one prime $p > P(\pi_{k-1})$ such that $\pi_{k-1}p \in U_k(x)$. Let $p^* = p_{\pi_{k-1}}$ be the smallest p with this property. Then $\pi_{k-1}p \in U_k(x)$ for all $p^* \le p \le \frac{x}{\pi_{k-1}}$. Using Lemma 1 we have that $\pi_{k-1} < x^{\lambda_x}$, with an appropriate $\lambda_x \to 0$, as x tends to infinity. Further,

$$P(\pi_{k-1}) < p, \quad \text{and} \quad p(\pi_{k-1}) > y_x,$$

where $y_x \to \infty$ as $x \to \infty$, slowly. We obtain

$$
\begin{aligned}
\sum_{\substack{n \le x \\ \omega(n)=k}} g(n+1) &= \sum_{\pi_{k-1} \in S_x} \sum_{p^*_{\pi_{k-1}} \le p \le \frac{x}{\pi_{k-1}}} g(\pi_{k-1}p+1) + o(\pi_k(x)) = \\
&= \sum_{\pi_{k-1} \in S_x} M(g, x, \pi_{k-1}) - \sum_{\pi_{k-1} \in S_x} \sum_{p \le p^*_{\pi_{k-1}}} g(\pi_{k-1}p+1) + o(\pi_k(x))
\end{aligned} \tag{5}
$$

as $x \to \infty$.

Let

$$\psi(x, D) := \frac{x^{i\tau}}{1+i\tau} \frac{\mu(d)}{\varphi(d)} \prod_{\substack{p \le x \\ p \nmid dD}} \left(1 - \frac{1}{p-1} + \sum_{\alpha \ge 1} \frac{f(p^\alpha) p^{-i\alpha\tau} \chi(p^\alpha)}{p^\alpha} \right).$$

Note that using Lemma 1 we have $y_x \le p(\pi_{k-1})$, therefore in our case π_{k-1} and d are coprimes for large x. Furthermore,

$$\sum_{\pi_{k-1} \in S_x} \pi(p^*_{\pi_{k-1}}) \ll x^{1/2} + \sum_{\pi_{k-1} \in S_x} \sum_{P(\pi_{k-1}) < p < p^*_{\pi_{k-1}}} 1 \tag{6}$$

which, by the definition of S_x, equals $o(\pi_k(x))$ as x tends to infinity. Thus, the second sum on the most right hand side of (5) is $o(\pi_k(x))$. For the estimation of the first sum here we apply Theorem 2 and we deduce

$$\sum_{\substack{n \le x \\ \omega(n)=k}} g(n+1) = \sum_{\pi_{k-1} \in S_x} \psi(x, \pi_{k-1}) \pi(\frac{x}{\pi_{k-1}}) + o(\pi_k(x)) \quad (x \to \infty).$$

Defining $K(x, D)$ by the identity

$$\psi(x, 1) = \psi(x, D) K(x, D),$$

such that

$$K(x, D) = \prod_{\substack{p \le x \\ p \mid D}} \left(1 - \frac{1}{p-1} + \sum_{\alpha \ge 1} \frac{f(p^\alpha) p^{-i\alpha\tau} \chi(p^\alpha)}{p^\alpha} \right)$$

holds, we have that the left hand side of (5) equals

$$
\begin{aligned}
&\psi(x, 1) \sum_{\pi_{k-1} \in S_x} \pi\left(\frac{x}{\pi_{k-1}}\right) + \\
&+ \sum_{\pi_{k-1} \in S_x} \pi\left(\frac{x}{\pi_{k-1}}\right) \psi(x, \pi_{k-1}) [1 - K(x, \pi_{k-1})] + o(\pi_k(x)) \quad (x \to \infty).
\end{aligned}
$$

Since $y_x \le p(\pi_{k-1})$, and since

$$K(x,\pi_{k-1}) = \exp\Bigg[\sum_{\substack{p\le x\\ p|\pi_{k-1}}} \frac{f(p^\alpha)\chi(p^\alpha)p^{i\tau}-1}{p} + \mathcal{O}\Bigg(\sum_{\substack{p\le x\\ p|\pi_{k-1}}} \frac{1}{p^2}\Bigg)\Bigg],$$

the right hand side of (5) equals

$$\psi(x,1)\sum_{\pi_{k-1}\in S_x} \pi\left(\frac{x}{\pi_{k-1}}\right) + o(1)\sum_{\pi_{k-1}\in S_x} \pi\left(\frac{x}{\pi_{k-1}}\right) + o(\pi_k(x)) \quad (x\to\infty).$$

By the same argument as in the estimation of (5) and then using (6) again we obtain

$$\pi_k^{-1}(x)\sum_{\pi_{k-1}\in S_x} \pi\left(\frac{x}{\pi_{k-1}}\right) \to 1 \quad (x\to\infty)$$

and the assertion follows. ■

In order to show Theorem 2 we need an analogoue of the Turán–Kubilius inequality.

Lemma 2. *Let $0\le\epsilon<1$ and let $0<\theta_x$ be an arbitrary sequence tending to zero as x tends to infinity. Let D be a positive integer, and let $x\ge 2D$. Let h be a real strongly additive function and*

$$h_x(n) = \sum_{\substack{p^\alpha||n\\ p\le(\frac{x-1}{D})^{1-\theta_x}}} h(p).$$

Then

$$\frac{1}{\pi(\frac{x-1}{D})}\sum_{p\le(x-1)/D}\Bigg|h_x(Dp+1) - \sum_{\substack{q\le x\\ q\nmid D}} \frac{h(q)}{\varphi(q)}\Bigg|^2 \ll \frac{1}{\theta_x}\sum_{q\le x}\frac{|h(q)|^2}{q} \tag{7}$$

uniformly for all x and all $D\le x^\epsilon$.

Proof. With $x_D := (x-1)/D$ let

$$h_{1,x}(n) := \sum_{\substack{p^\alpha||n\\ p\le x_D^{1/8}}} h(p) \quad \text{and} \quad h_{2,x}(n) := \sum_{\substack{p^\alpha||n\\ x_D^{1/8}<p\le x_D^{1-\theta_x}}} h(p).$$

Further, define

$$A(y) := \sum_{\substack{p\le y\\ q\nmid D}} \frac{h(p)}{\varphi(p)} \quad \text{and} \quad B^2(y) := \sum_{p\le y}\frac{|h(p)|^2}{p}.$$

The left hand side of (7) is $\ll \Sigma_1 + \Sigma_2 + \Sigma_3$, where

$$\Sigma_1 = \frac{1}{\pi(x_D)} \sum_{p \le x_D} |h_{1,x}(Dp+1) - A(x_D^{1/8})|^2,$$

$$\Sigma_2 = \frac{1}{\pi(x_D)} \sum_{p \le x_D} |h_{2,x}(Dp+1)|^2,$$

$$\Sigma_3 = \frac{1}{\pi(x_D)} \sum_{p \le x_D} |A(x) - A(x_D^{1/8})|^2.$$

Using the Cauchy–Schwarz inequality we have

$$\Sigma_3 \ll \Bigg(\sum_{x_D^{1/8} \le p \le x} \frac{1}{p} \Bigg) \Bigg(\sum_{x_D^{1/8} \le p \le x} \frac{|h(p)|^2}{p} \Bigg) \ll \sum_{p \le x} \frac{|h(p)|^2}{p}.$$

In order to estimate Σ_2 note that a positive integer, $n \le x$, can have at most a bounded number of distinct prime divisors $q > x_D^{1/8}$. Thus, using the Brun–Titchmarsh inequality (Theorem I.4.9 in [10]) we deduce

$$\Sigma_2 = \frac{1}{\pi(x_D)} \sum_{p \le x_D} \Bigg| \sum_{q \mid Dp+1} h_{2,x}(q) \Bigg|^2 \ll \frac{1}{\pi(x_D)} \sum_{\substack{q \le x_D^{1-\theta_x} \\ q \nmid D}} |h(q)|^2 \pi(x_D, q, l_{D,q}) \ll$$

$$\ll \frac{x_D}{\pi(x_D)} \sum_{q \le x_D^{1-\theta_x}} \frac{|h(q)|^2}{q \log(\frac{x_D}{q})} \ll$$

$$\ll \frac{1}{\theta_x} \sum_{q \le x_D^{1-\theta_x}} \frac{|h(q)|^2}{q}.$$

Here we used that if $Dp + 1 = aq$ then there exists a unique residue class $l_{D,q}$ (mod q) such that $p \equiv l_{D,q} \pmod q$ holds.

It remains to estimate Σ_1. Performing the multiplications we obtain

$$\sum_{p \le x_D} \Big| h_{1,x}(Dp+1) - A(x_D^{1/8}) \Big|^2 = S_1 - 2S_2 + S_3,$$

where

$$S_1 = \sum_{p \le x_D} |h_{1,x}(Dp+1)|^2,$$

$$S_2 = A(x_D^{1/8}) \sum_{p \le x_D} h_{1,x}(Dp+1),$$

$$S_3 = A(x_D^{1/8})^2 \pi(x_D).$$

Further,

$$
(8)\qquad
\begin{aligned}
S_1 = \sum_{p\le x_D}\Big(\sum_{q|Dp+1} h_{1,x}(q)\Big)^2 &= \sum_{\substack{q\le x_D\\ q\nmid D}} h_{1,x}^2(q)\pi(x_D,q,l_{D,q})+\\
&+\sum_{\substack{q_1,q_2\le x_D\\ q_1\ne q_2,\ q_1\nmid D,\ q_2\nmid D}} h_{1,x}(q_1)h_{1,x}(q_2)\pi(x_D,q_1q_2,l_{D,q_1q_2}).
\end{aligned}
$$

Since $h_{1,x}(q)=0$ for $q>x_D^{1/8}$, the Brun–Titchmarsh theorem is applicable and we deduce that the first term on the right hand side of (8) does not exceed $c\pi(x_D)B^2(x)$.

The second term on the right hand side of (8) equals

$$
(9)\qquad
\begin{aligned}
&\sum_{\substack{q_1,q_2\le x_D^{1/8}\\ q_1\ne q_2,\ q_1\nmid D,\ q_2\nmid D}} h_{1,x}(q_1)h_{1,x}(q_2)\frac{\pi(x_D)}{\varphi(q_1q_2)}+\\
&+\sum_{\substack{q_1,q_2\le x_D^{1/8}\\ q_1\ne q_2,\ q_1\nmid D,\ q_2\nmid D}} h_{1,x}(q_1)h_{1,x}(q_2)\{\pi(x_D,q_1q_2,l_{D,q_1q_2})-\frac{\pi(x_D)}{\varphi(q_1q_2)}\}.
\end{aligned}
$$

Let T_1,T_2 be the sums in (9). We have

$$
\frac{T_1}{\pi(x_D)} = A^2(x_D^{1/8}) - \sum_{\substack{q_1\le x_D^{1/8}\\ q_1\nmid D}} \frac{h_{1,x}^2(q_1)}{\varphi^2(q_1)} = A^2(x_D^{1/8}) + \mathcal{O}(B^2(x)).
$$

For T_2 we use the Cauchy–Schwarz inequality to obtain

$$
\begin{aligned}
T_2^2 \ll &\sum_{\substack{q_1,q_2\le x_D^{1/8}\\ q_1\ne q_2,\ q_1\nmid D,\ q_2\nmid D}} \frac{h_{1,x}^2(q_1)}{\varphi(q_1)}\frac{h_{1,x}^2(q_2)}{\varphi(q_2)}\times\\
&\times\sum_{\substack{q_1,q_2\le x_D^{1/8}\\ q_1\ne q_2,\ q_1\nmid D,\ q_2\nmid D}} \varphi(q_1q_2)\left\{\pi(x_D,q_1q_2,l_{D,q_1q_2})-\frac{\pi(x_D)}{\varphi(q_1q_2)}\right\}^2 \ll\\
\ll &B^4(x)\sum_{\substack{q_1,q_2\le x_D^{1/8}\\ q_1\ne q_2,\ q_1\nmid D,\ q_2\nmid D}} \varphi(q_1q_2)\left\{\pi(x_D,q_1q_2,l_{D,q_1q_2})-\frac{\pi(x_D)}{\varphi(q_1q_2)}\right\}^2.
\end{aligned}
$$

Using the Brun–Titchmarsh inequality

$$
T_2^2 \ll B^4(x)\pi(x_D)\sum_{\substack{q_1,q_2\le x_D^{1/8}\\ q_1\ne q_2,\ q_1\nmid D,\ q_2\nmid D}} \left|\pi(x_D,q_1q_2,l_{D,q_1q_2})-\frac{\pi(x_D)}{\varphi(q_1q_2)}\right|,
$$

and an application of the Bombieri–Vinogradov theorem (Chapter 28. in [1]) shows

$$T_2 \ll B^2(x)\frac{\pi(x_D)}{\log^A x_D},$$

where $A > 0$ is an arbitrary large costant. Since by the Cauchy–Schwarz inequality we have

$$A(y) = \sum_{\substack{q\le y\\ q\nmid D}} \frac{h(q)}{\varphi(q)} \ll \left(\sum_{q\le y}\frac{h^2(q)}{q}\right)^{1/2} \log\log^{1/2} y \ll B(y)\log\log^{1/2} y,$$

for $y \ge e^2$, in a similar way as in the estimation of T_2 we deduce

$$\begin{aligned} S_2 - A^2(x_D^{1/8})\pi(x_D) \ll & A(x_D^{1/8})B(x)\frac{\pi(x_D)}{\log^A x_D} \ll \\ \ll & B^2(x)\log\log x_D \frac{\pi(x_D)}{\log^A x_D} \ll \\ \ll & B^2(x)\pi(x_D), \end{aligned}$$

and the proof is finished. ■

Lemma 3. *Let D, q be two coprime positive integers and let $(l_{D,q} =)l_D$ be the unique residue class satisfying $Dl_D \equiv 1 \pmod q$. Let further $0 < \epsilon < 1/2$ and $x_D := (x-1)/D$ whenever $x > 2$ and let $a > \frac{1-2\epsilon}{1+2\epsilon}$. Then*

$$\sum_{\substack{q > x_D^a\\ q\ \text{prime},\ q\nmid D}} q\pi^2(x_D, q, l_D) \ll \pi^2(x_D) \tag{10}$$

holds uniformly for all $x > 2$ and $D \le x^{1/2-\epsilon}$. The constant implied by $\ll$ depends on a.

Proof. The sum on the left hand side of (10) equals

$$\sum_{q > x_D^a} q \sum_{\substack{a_1 q = Dp_1 + 1\\ a_1 \le x/q}} \sum_{\substack{a_2 q = Dp_2 + 1\\ a_2 \le x/q}} 1 \le 2x \sum_{\substack{a_1 \le x x_D^{-a}\\ (a_1, D) = 1}} \frac{1}{a_1} \sum_{\substack{a_2 < a_1\\ (a_2, D) = 1}} \sum_{\substack{a_1 q = Dp_1 + 1 \le x\\ a_2 q = Dp_2 + 1 \le x}} 1. \tag{11}$$

Denote the inner sum by $(\Sigma(a_1, a_2) =)\Sigma$. It is nonempty only if $a_1 \equiv a_2 \pmod D$. Suppose, a_1, a_2 is fixed and

$$q = Dn + l_{a_1 D}.$$

Then

$$Dp_1 + 1 = a_1 Dn + a_1 l_{a_1 D}, \quad Dp_2 = a_2 Dn + a_2 l_{a_1 D}.$$

Thus, the primes we want to count in Σ satisfy

$$q = Dn + l_{a_1 D},$$
$$p_1 = a_1 n + t_{Da_1}, \quad p_2 = a_2 n + t_{Da_2},$$

where

$$a_1 l_{a_1 D} - Dt_{Da_1} = 1 \quad \text{and} \quad a_2 l_{a_1 D} - Dt_{Da_2} = 1.$$

It follows,

$$\Sigma \ll \#\Big\{ n \le \frac{x_D}{a_1} : q = Dn + l_{a_1 D},\ p_1 = a_1 n + t_{Da_1},\ p_2 = a_2 n + t_{Da_2} \text{ primes} \Big\}.$$

Let

$$E = Da_1 a_2 (a_1 - a_2),$$

and let $\varrho(p)$ be the number of solutions of

$$(Dn + l_{a_1 D})(a_1 n + t_{Da_1})(a_2 n + t_{Da_2}) \equiv 0 \pmod p.$$

Since $E \le x_D^A$ for some appropriate $A > 0$, by Theorem 5.7 of [5]

$$\Sigma \ll \frac{x_D}{a_1 \log^3 \frac{x_D}{a_1}} \prod_p (1 - \frac{\varrho(p) - 1}{p - 1})(1 - \frac{1}{p})^{-2}.$$

Noting that $(D, a_1 a_2) = 1$ we have

$$\varrho(p) = \begin{cases} 1 & \text{if } p|D,\ p|\frac{a_1 - a_2}{D} \text{ or } p|a_1,\ p|a_2 \\ 2 & \text{if } p|D,\ p \nmid \frac{a_1 - a_2}{D} \text{ or } p|a_1 a_2,\ p \nmid (a_1, a_2) \\ 3 & \text{otherwise.} \end{cases}$$

Now, making use of the inequality $\log(1 - z) = 1 + z + \mathcal{O}(z^2)$ which holds uniformly for all real numbers $|z| \le 1/2$ we obtain

$$\prod_p \left(1 - \frac{\varrho(p) - 1}{p - 1}\right)\left(1 - \frac{1}{p}\right)^{-2} \ll$$

$$\ll \prod_{p|D} \left(1 + \frac{1}{p}\right) \prod_{p|\frac{a_1 - a_2}{D}} \left(1 + \frac{1}{p}\right) \prod_{p|a_1} \left(1 + \frac{2}{p}\right) \prod_{p|a_2} \left(1 + \frac{2}{p}\right).$$

Thus, the right hand side of (11) is at most

$$c\frac{x^2}{D}\prod_{p|D}\left(1+\frac{1}{p}\right)\sum_{\substack{a_1\le xx_D^{-a}\\(a_1,D)=1}}\frac{1}{a_1^2\log^3\frac{x_D}{a_1}}\prod_{p|a_1}\left(1+\frac{2}{p}\right)\times$$
$$\times\sum_{\substack{a_2\le a_1\\a_1\equiv a_2 \pmod D}}\prod_{p|\frac{a_1-a_2}{D}}\left(1+\frac{1}{p}\right)\prod_{p|a_2}\left(1+\frac{2}{p}\right).$$

Since $|ab|\le a^2+b^2$ holds for all real a,b we deduce

$$\sum_{\substack{a_2\le a_1\\a_1\equiv a_2 \pmod D}}\prod_{p|\frac{a_1-a_2}{D}}\left(1+\frac{1}{p}\right)\prod_{p|a_2}\left(1+\frac{2}{p}\right)\ll$$
$$\ll\sum_{\substack{a_2\le a_1\\a_1\equiv a_2 \pmod D}}\left\{\sum_{d|\frac{a_1-a_2}{D}}\frac{2^{\omega(d)}\mu^2(d)}{d}+\sum_{d|a_2}\frac{4^{\omega(d)}\mu^2(d)}{d}\right\}\ll$$
$$\ll\sum_{d\le\frac{a_1}{D}}\frac{2^{\omega(d)}\mu^2(d)}{d}\sum_{\substack{a_2\le a_1\\a_2\equiv a_1 \pmod D\\\frac{a_1-a_2}{D}\equiv 0 \pmod d}}1+\sum_{\substack{d\le a_1\\(d,D)=1}}\frac{4^{\omega(d)}\mu^2(d)}{d}\sum_{\substack{a_2\le a_1\\a_2\equiv 0 \pmod d\\a_1\equiv a_2 \pmod D}}1\ll$$
$$\ll\frac{a_1}{D}.$$

Since $a>\frac{1-2\epsilon}{1+2\epsilon}$ and $a_1\le xx_D^{-a}$ we have $\log\frac{x_D}{a_1}\gg_a\log x\gg\log x_D$. Further,

$$\sum_{\substack{a_1\le xx_D^{-a}\\(a_1,D)=1}}\frac{1}{a_1}\prod_{p|a_1}\left(1+\frac{2}{p}\right)=\prod_{\substack{p\le xx_D^{-a}\\p\nmid D}}\left(1+\frac{1}{p}\left(1+\frac{2}{p}\right)\right)\ll$$
$$\ll\prod_{p\le xx_D^{-a}}\left(1+\frac{1}{p}\right)\prod_{p|D}\left(1+\frac{1}{p}\right)^{-1}\ll$$
$$\ll\log x_D\prod_{p|D}\left(1+\frac{1}{p}\right)^{-1}.$$

Thus, the right hand side of (11) does not exceed

$$c\frac{x_D^2}{\log^3 x_D}\prod_{p|D}\left(1+\frac{1}{p}\right)\sum_{\substack{a_1\le xx_D^{-a}\\(a_1,D)=1}}\frac{1}{a_1}\prod_{p|a_1}\left(1+\frac{2}{p}\right)\ll\pi^2(x_D),$$

which proves the assertion. ■

Proof of Theorem 2. First suppose that $\tau = 0$. We set $r = \log\log x$, and $x_D = \frac{x-1}{D}$. Let

$$K_D(x) := \{Dp + 1 \le x : p \text{ prime}\}.$$

We have

$$\#\{n \in K_D(x) \mid \exists q^2|n,\ q > y\} \le$$

$$\le \sum_{y<q<(\frac{x-1}{D})^a} \pi\left(\frac{x-1}{D}, q^2, l_q\right) + \frac{x-1}{D} \sum_{q \ge (\frac{x-1}{D})^a} \frac{1}{q^2} = \delta(y)\pi\left(\frac{x-1}{D}\right), \tag{12}$$

where $\delta(y) \to 0$ $(y \to \infty)$. Let f^* be a multiplicative function defined by

$$f^*(p^\alpha) = \begin{cases} f(p^\alpha), & \text{if } p \le r \\ f(p), & \text{if } r < p \le x_D^{1-\vartheta_x} \\ \overline{\chi}(p), & \text{otherwise.} \end{cases}$$

Since $\chi(q) \ne 0$ for $q > d$, there exists a function $g(q) \in [-\pi, \pi)$ such that $f(q) = \chi(q)e^{ig(q)}$. By (12)

$$\left|\sum_{Dp+1 \le x} \{f(Dp+1) - f^*(Dp+1)\}\right| \le$$

$$\sum_{\substack{Dp+1 \le x \\ \exists q^2|Dp+1,\ q>r}} 1 + \sum_{\substack{Dp+1 \le x \\ \exists q|Dp+1,\ q>x_D^{1-\vartheta_x}}} |e^{i\tilde{g}(Dp+1)} - 1| \le$$

$$\le \sum_{\substack{Dp+1 \le x \\ \exists q|Dp+1,\ q>x_D^{1-\vartheta_x}}} |e^{i\tilde{g}(Dp+1)} - 1| + o(\pi(x_D)) \quad (x \to \infty),$$

where

$$\tilde{g}(p^\alpha) = \begin{cases} g(p), & \text{if } x_D^{1-\vartheta_x} < q, \quad \alpha = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\sum_{\substack{Dp+1 \le x \\ \exists q|Dp+1,\ q>x_D^{1-\vartheta_x}}} |e^{i\tilde{g}(Dp+1)} - 1| \le \sum_{\substack{Dp+1 \le x \\ \exists q|Dp+1,\ q>x_D^{1-\vartheta_x}}} |\tilde{g}(Dp+1)|$$

$$\le \sum_{x_D^{1-\vartheta_x} < q \le x} |g(q)|\pi(x_D, q, t_D),$$

where $(t_{D,q} =)t_D$ is the unique residue class satisfying

$$Dt_D \equiv -1 \pmod q.$$

Applying the Cauchy–Scwarz inequality then using Lemma 3 we obtain

$$\sum_{x_D^{1-\vartheta_x}<q\le x} |g(q)|\pi(x_D,q,t_D) \ll$$

$$\ll \Bigg(\sum_{x_D^{1-\vartheta_x}<q\le x} \frac{g(q)^2}{q}\Bigg)^{1/2} \Bigg(\sum_{x_D^{1-\vartheta_x}<q\le x} q\pi^2(x_D,q,t_D)\Bigg)^{1/2} \ll$$

$$\ll \pi(x_D)\Bigg(\sum_{x_D^{1-\vartheta_x}<q\le x} \frac{g(q)^2}{q}\Bigg)^{1/2}.$$

Noting that

$$|g(q)|^2 \ll |f(q)\overline{\chi}(q)-1|^2,$$

by (2) we obtain

$$\sum_{Dp+1\le x}\{f(Dp+1)-f^*(Dp+1)\} = o(\pi(x_D)) \quad (x\to\infty). \tag{13}$$

Let f_r be a further multiplicative function defined by

$$f_r(p^\alpha) = \begin{cases} f(p^\alpha), & \text{if } p\le r\\ \overline{\chi}(p), & \text{if } r<p.\end{cases}$$

Next we give an alternative representation of $M(x,f_r,D)$. It can be written as follows

(14)
$$\sum_{Dp+1\le x} f_r(Dp+1) = \sum_{\substack{m\le x+1\\ P(m)\le r\\ (D,m)=1}} f(m) \sum_{\substack{p\le x_D\\ p\equiv l_D \pmod m\\ (\frac{Dp+1}{m},\mathcal{P}(r))=1}} \overline{\chi}\left(\frac{Dp+1}{m}\right) + Err(x,r),$$

where

$$\mathcal{P}(r) := \prod_{p\le r} p,$$

and $(l_{D,m} =)l_D$ is the unique residue class satisfying

$$Dl_D \equiv -1 \pmod{m},$$

and by (12)

$$Err(x,r) \ll \sum_{\substack{Dp+1\le x\\ \exists q^2|Dp+1,\ r<q}} 1 = o(\pi(x_D)) \quad (x\to\infty).$$

Furthermore, $(\frac{Dp+1}{m}, \mathcal{P}(r)) = 1$. Hence, $\frac{Dp+1}{m}$ is always odd and there is at most one prime p satisfying $Dp+1 = m\frac{Dp+1}{m}$ if D and m have the same parity. The contribution of these integers to the sum on the right hand side of (14) is at most

$$\sum_{\substack{m\le x\\ P(m)\le r}} 1 \ll x \exp\left(-\frac{1}{2}\frac{\log x}{\log r}\right),$$

which inequality is well known in number theory (Theorem III.5.1 in [10]). The sum over the integers $m > e^r$ on the right hand side of (14) is at most

$$\sum_{\substack{e^r\le m\le\sqrt{x}\\ P(m)\le r}} \pi(x_D, m, l_D) + \sum_{\substack{\sqrt{x}\le m\le x\\ P(m)\le r}} \frac{x_D}{m} = \Sigma_1 + \Sigma_2.$$

Using the Brun–Titchmarsh theorem we obtain

$$\begin{aligned}\Sigma_1 \ll \pi(x_D) \sum_{\substack{e^r\le m\le\sqrt{x}\\ P(m)\le r}} \frac{1}{\varphi(m)} &\ll \frac{\pi(x_D)}{r} \sum_{\substack{m\le x\\ P(m)\le r}} \frac{\log m}{\varphi(m)} \ll \\ &\ll \frac{\pi(x_D)}{r} \sum_{p\le r}\sum_{\alpha} \log p^\alpha \sum_{\substack{mp^\alpha\le x\\ P(m)\le r,\ (m,p)=1}} \frac{1}{\varphi(p^\alpha m)} \ll \\ &\ll \frac{\pi(x_D)\log r}{r} \sum_{p\le r} \frac{\log p}{p} \ll \\ &\ll \pi(x_D)\frac{\log^2 r}{r}.\end{aligned}$$

Further, using the inequality $|\log(1-y) - y| \le 2y^2$, which is valid for all real

y with $|1-y| \le 1/2$ we have

$$\Sigma_2 \ll x_D x^{-1/8} \sum_{\substack{m\le x\\ P(m)\le r}} \frac{1}{m^{3/4}} \ll x_D x^{-1/4} \prod_{p\le r} \left(1-\frac{1}{p^{3/4}}\right)^{-1} \ll$$
$$\ll x_D x^{-1/4} \exp\left(\sum_{p\le r} \frac{1}{p^{3/4}}\right) \ll$$
$$\ll x_D x^{-1/4} e^r.$$

The inner sum on the right hand side of (14) equals

$$\sum_{\substack{Dp\le x\\ Dp\equiv -1 \pmod m}} \overline{\chi}_d\left(\frac{Dp+1}{m}\right) \sum_{\delta|(\frac{Dp+1}{m},\mathcal{P}(r))} \mu(\delta) =$$
$$= \sum_{\substack{\delta|\mathcal{P}(r)\\ (\delta,Dd)=1}} \mu(\delta) \sum_{\substack{Dp\le x\\ Dp+1\equiv 0 \pmod{\delta m}}} \overline{\chi}_d\left(\frac{Dp+1}{m}\right) =$$
$$= \sum_{\substack{\delta|\mathcal{P}(r)\\ (\delta,Dd)=1}} \mu(\delta) \sum_{\substack{b=1\\ (b,d)=1}}^{d} \overline{\chi}_d(b) J(x,m,\delta,b),$$

where

$$J_m(x,m,\delta,b) := \#\left\{p\le x_D \,:\, Dp+1\equiv 0 \pmod{\delta m},\ \tfrac{Dp+1}{m}\equiv b \pmod d\right\}.$$

Note that $J_m(x,m,\delta,b) \ll 1$ for all b with $(bm-1,d)\ne 1$. There is a unique $l_\delta \pmod d$ such that $\delta l_\delta \equiv b \pmod d$, therefore

$$Dp+1 = c\delta m \quad \text{and} \quad Dp+1 = mb+tdm,$$

implies

$$Dp+1 \equiv m l_\delta \delta \pmod{m\delta d}.$$

Thus,

$$J_m(x,m,\delta,b) = \#\{p\le x_D \,:\, Dp+1\equiv m\delta l_\delta \pmod{\delta d m}\}.$$

We arrive at

$$M(x,f_r,D) = {\sum_{\substack{m\le e^r\\ P(m)\le r}}}' f(m) \sum_{\substack{b=1\\ (b,d)=1\\ (bm-1,d)=1}}^{d} \overline{\chi}_d(b) \sum_{\substack{\delta|\mathcal{P}(r)\\ (\delta,Dd)=1}} \mu(\delta)\pi(x_D,\delta dm, m\delta l_\delta) +$$
$$+ o(\pi(x_D)) \quad (x\to\infty), \tag{15}$$

where Σ' indicates that m and D are of opposite parity. The right hand side of (15) equals

$$\sum_{\substack{m\le e^r\\ P(m)\le r}}{}' f(m) \sum_{\substack{b=1\\(b,d)=1\\(bm-1,d)=1}}^{d} \overline{\chi}_d(b) \sum_{\substack{\delta|\mathcal{P}(r)\\(\delta,Dd)=1}} \mu(\delta)\frac{\pi(x_D)}{\varphi(\delta dm)}+$$
$$+\mathcal{O}\left(\sum_{\substack{\delta|\mathcal{P}(r)\\(\delta,Dd)=1}}\sum_{\substack{m\le e^r\\P(m)\le r}}\left|\pi(x_D,\delta dm, m\delta l_\delta)-\frac{\pi(x_D)}{\varphi(\delta dm)}\right|\right)=M+Err_2(x,r).$$

Applying the Cauchy–Schwarz inequality and then the Brun–Titchmarsh theorem we obtain that $Err_2^2(x,r)$ is at most

$$c\left(\sum_{\delta\le\frac{\sqrt{x}}{\sqrt{D}\log^A x}} 4^{\omega(\delta)}\max_{(l,\delta)=1}\left|\pi(x_D,\delta,l)-\frac{\pi(x_D)}{\varphi(\delta)}\right|\right)^2 \ll$$
$$\ll \sum_{\delta\le\frac{\sqrt{x}}{\sqrt{D}\log^A x}}\frac{16^{\omega(\delta)}}{\varphi(\delta)}\sum_{\delta\le\frac{\sqrt{x}}{\sqrt{D}\log^A x}}\varphi(\delta)\max_{(l,\delta)=1}\left|\pi(x_D,\delta,l)-\frac{\pi(x_D)}{\varphi(\delta)}\right|^2\ll$$
$$\ll\prod_{p\le x}\left(1+\frac{16}{p}\right)\pi(x_D)\sum_{\delta\le\frac{\sqrt{x}}{\sqrt{D}\log^A x}}\max_{(l,\delta)=1}\left|\pi(x_D,\delta,l)-\frac{\pi(x_D)}{\varphi(\delta)}\right|,$$

which by the Bombieri–Vinogradov theorem does not exceed $\frac{\pi^2(x_D)}{\log^A x}$, where $A>0$ is an arbitrary large fixed constant.

Since

$$\varphi(\delta dm)=\delta dm\prod_{p|dm}\left(1-\frac{1}{p}\right)\prod_{\substack{p|\delta\\p\nmid dm}}\left(1-\frac{1}{p}\right)=\varphi(dm)\delta\prod_{\substack{p|\delta\\p\nmid dm}}\left(1-\frac{1}{p}\right),$$

we have

$$\sum_{\substack{\delta|\mathcal{P}(r)\\(\delta,Dd)=1}}\frac{\mu(\delta)}{\varphi(\delta md)}=\frac{1}{\varphi(dm)}\prod_{\substack{p\le r\\p\nmid Dd\\p|dm}}\left(1-\frac{1}{p}\right)\prod_{\substack{p\le r\\p\nmid Dd\\p\nmid dm}}\left(1-\frac{1}{p-1}\right)=$$
$$=\frac{1}{\varphi(dm)}\prod_{\substack{p\le r\\p\nmid Dd\\p|m}}\left(1-\frac{1}{p}\right)\prod_{\substack{p\le r\\p\nmid Dd\\p\nmid m}}\left(1-\frac{1}{p-1}\right).$$

Further, by the inclusion-exclusion principle and by the orthogonality relation of the Dirichlet characters we have

$$\sum_{\substack{b=1\\(b(bm-1),d)=1}}^{d}\overline{\chi}(b)=\sum_{(b,d)=1}\overline{\chi}(b)\sum_{k|d}\frac{\mu(k)}{\varphi(k)}\sum_{\chi\pmod k}\chi_k(bm).$$

Thus,

$$\begin{aligned}\frac{1}{\pi(x_D)}M=&\sum_{(b,d)=1}\overline{\chi}(b)\sum_{k|d}\frac{\mu(k)}{\varphi(k)}\sum_{\chi\pmod k}\chi_k(b)\times\\ &\times\sideset{}{'}\sum_{\substack{m\\P(m)\le r}}\frac{f(m)\chi_k(m)}{\varphi(dm)}\prod_{\substack{p\le r\\p\nmid Dd\\p|m}}\left(1-\frac{1}{p}\right)\prod_{\substack{p\le r\\p\nmid Dd\\p\nmid m}}\left(1-\frac{1}{p-1}\right)+Err_3(r),\end{aligned}\tag{16}$$

where

$$\begin{aligned}Err_3(r)&\ll\sum_{\substack{m>e^r\\P(m)\le r}}\frac{1}{\varphi(dm)}\prod_{\substack{p\le r\\p\nmid Dd\\p|m}}\left(1-\frac{1}{p}\right)\prod_{\substack{p\le r\\p\nmid Dd\\p\nmid m}}\left(1-\frac{1}{p-1}\right)\ll\\ &\ll\prod_{\substack{p\le r\\p\nmid Dd}}\left(1-\frac{1}{p}\right)\sum_{\substack{m>e^r\\P(m)\le r}}\frac{1}{\varphi(m)}\ll\\ &\ll\frac{\log^2 r}{r}.\end{aligned}$$

Keeping in mind that m and D has opposite parity

$$\sideset{}{'}\sum_{\substack{m\\P(m)\le r}}\frac{f(m)\chi_k(m)}{\varphi(dm)}\prod_{\substack{p\le r\\p\nmid Dd\\p|m}}\left(1-\frac{1}{p}\right)\prod_{\substack{p\le r\\p\nmid Dd\\p\nmid m}}\left(1-\frac{1}{p-1}\right)\tag{17}$$

can be written as

$$\prod_{\substack{p\le r\\p\nmid Dd}}\left(1-\frac{1}{p-1}+\sum_{\alpha\ge1}\frac{f(p^\alpha)\chi_k(p^\alpha)}{p^\alpha}\right)\prod_{\substack{p\le r\\p\nmid 2D\\p|d}}\left(1+\sum_{\alpha\ge1}\frac{f(p^\alpha)\chi_k(p^\alpha)}{p^\alpha}\right).$$

Thus, the first term on the right hand side of (16) equals

$$
\begin{aligned}
&\sum_{\substack{b=1\\(b,d)=1}}^{d} \frac{\overline{\chi}(b)}{\varphi(d)} \sum_{k|d} \frac{\mu(k)}{\varphi(k)} \times \\
(18)\qquad &\times \sum_{\chi \pmod k} \chi_k(b) \prod_{\substack{p\le r\\ p\nmid dD}} \left(1 - \frac{1}{p-1} + \sum_{\alpha\ge 1} \frac{f(p^\alpha)\chi_k(p^\alpha)}{p^\alpha}\right) \times \\
&\times \prod_{\substack{p\le r\\ p|d,\ p\nmid 2D}} \left(1 + \sum_{\alpha\ge 1} \frac{f(p^\alpha)\chi_k(p^\alpha)}{p^\alpha}\right).
\end{aligned}
$$

Since the character induced by $\chi_k \cdot \overline{\chi}$ is not the principal character if $\chi_k \neq \chi$ we obtain using Dirichlet's theorem in arithmetic progressions that

$$
\sum_{z\le p\le r} \frac{|1 - \chi_k \cdot \overline{\chi}(p)|^2}{p} \gg \log\left(\frac{\log r}{\log z}\right) \gg \log\left(\frac{\log_3 x}{\log_4 x}\right),
$$

if $z = \log_3 x$. Here we used that $\chi_k \cdot \overline{\chi}(p)$ is at most a $\varphi(d)$-th root of unity. Further,

$$
|\chi_k(p) f(p) - 1|^2 \gg |1 - \overline{\chi}(p)\chi_k(p)|^2 - |1 - \overline{\chi}(p) f(p)|^2,
$$

therefore

$$
\begin{aligned}
&\sum_{z\le p\le r} \frac{|1 - \chi_k(p) f(p)|^2}{p} \gg \\
&\gg \sum_{z\le p\le r} \frac{|1 - \chi_k(p)\overline{\chi}(p)|^2}{p} + \mathcal{O}\left(\sum_{z\le p\le r} \frac{|1 - \overline{\chi}(p) f(p)|^2}{p}\right) \gg \\
&\gg \log\left(\frac{\log_3 x}{\log_4 x}\right) + o(1) \quad (x\to\infty).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\Big|\prod_{\substack{p\le r\\ p\nmid dD}} (1 - \frac{1}{p-1} + \sum_{\alpha\ge 1} \frac{f(p^\alpha)\chi_k(p^\alpha)}{p^\alpha})\Big| &\ll \Big|\exp\Big(\sum_{p\le r} \frac{f(p)\chi_k(p) - 1}{p}\Big)\Big| \ll \\
&\ll \exp\Big(-\sum_{z\le p\le r} \frac{1 - \operatorname{Re} f(p)\chi_k(p)}{p}\Big) = \\
&= o(1) \quad (x\to\infty).
\end{aligned}
$$

Putting it back into (18) we deduce

$$\begin{aligned}\frac{1}{\pi(x_D)}M(x,f_r,D) = \frac{\mu(d)}{\varphi(d)} \prod_{\substack{p\le r\\ p\nmid dD}} \Big(1-\frac{1}{p-1}+\sum_{\alpha\ge 1}\frac{f(p^\alpha)\chi(p^\alpha)}{p^\alpha}\Big)\times \\ \times \prod_{\substack{p\le r\\ p\mid d,\ p\nmid 2D}} \Big(1+\sum_{\alpha\ge 1}\frac{f(p^\alpha)\chi(p^\alpha)}{p^\alpha}\Big) + o(1) \quad (x\to\infty).\end{aligned} \tag{19}$$

Since $\chi(p^\alpha)=0$ for all $p \mid d$, introducing the notation

$$P(y) := \prod_{\substack{p\le y\\ p\nmid dD}} \Big(1-\frac{1}{p-1}+\sum_{\alpha\ge 1}\frac{f(p^\alpha)\chi(p^\alpha)}{p^\alpha}\Big),$$

we proved that

$$\pi(x_D)^{-1}M(x,f_r,D) = \frac{\mu(d)}{\varphi(d)}P(r)+o(1) \quad (x\to\infty). \tag{20}$$

Here we note that if (2) converges for $\tau=0$ then $1 \ll |P(r)| \le 1$. Now we can prove that

$$\frac{\mu(d)}{\varphi(d)}P(x_D)$$

is a good approximation of the sum $M(x,f,D)$. Now

$$\begin{aligned}\Big|\pi^{-1}(x_D)M(x,f,D)-\frac{\mu(d)}{\varphi(d)}P(x_D)\Big| \le \\ \le \Big|\pi^{-1}(x_D)M(x,f^*,D)-\pi^{-1}(x_D)M(x,f_r,D)\frac{P(x_D)}{P(r)}\Big|+ \\ +\pi(x_D)^{-1}|M(x,f^*,D)-M(x,f,D)\Big|+ \\ +\Big|\frac{\mu(d)}{\varphi(d)}P(x_D)-\pi^{-1}(x_D)M(x,f_r,D)\frac{P(x_D)}{P(r)}\Big|,\end{aligned}$$

therefore by (13) and by (20) we have to show that

$$\pi^{-1}(x_D)\Big|M(x,f^*,D)-M(x,f_r,D)\frac{P(x_D)}{P(r)}\Big| = o(1) \quad (x\to\infty). \tag{21}$$

We note that, if $d<r$, then

$$|f^*(p^\alpha)| = |f_r(p^\alpha)| = 1.$$

Hence there is a strongly additive function $g_r^*(p) \in (-\pi, \pi]$ with

$$f_r^*(n) = f^* \cdot \overline{f}_r(n) = e^{ig_r^*(n)}.$$

We note that if

$$p \le r, \quad \text{or} \quad p > x_D^{1-\vartheta_x}, \quad \text{then} \quad g_r^*(p) = 0.$$

By Lemma 2 we have

$$\sum_{Dp+1 \le x} \Big| g_r^*(Dp+1) - \sum_{\substack{q \le x_D \\ q \nmid D}} \frac{g_r^*(q)}{q} \Big|^2 \ll \frac{1}{\vartheta_x} \pi(x_D) \sum_{p \le x_D} \frac{|g_r^*(p)|^2}{p}. \tag{22}$$

Let

$$A(x) := \sum_{\substack{p \le x_D \\ p \nmid D}} \frac{g_r^*(p)}{p}.$$

We obtain that the left hand side of (21) is at most

$$\frac{c}{\pi(x_D)} \Big| \sum_{Dp+1 \le x} f^*(Dp+1) - f_r(Dp+1) \frac{P(x_D)}{P(r)} \Big| \ll$$

$$\ll \frac{1}{\pi(x_D)} \sum_{Dp+1 \le x} \Big| f_r^*(Dp+1) - \frac{P(x_D)}{P(r)} \Big| \ll$$

$$\ll \frac{1}{\pi(x_D)} \sum_{Dp+1 \le x} \big| f_r^*(Dp+1) - \exp[iA(x)] \big| + \Big| \exp[iA(x)] - \frac{P(x_D)}{P(r)} \Big| =$$

$$= \Sigma_1' + \Sigma_2'.$$

Using the Cauchy–Schwarz inequality again we obtain

$$\Sigma_1' = \pi(x_D)^{-1} \sum_{Dp+1 \le x} \Big| \exp\big[i\big(g_r^*(Dp+1) - A(x) \big) \big] - 1 \Big| \le$$

$$\le \pi(x_D)^{-1/2} \Big(\sum_{Dp+1 \le x} |g_r^*(Dp+1) - A(x)|^2 \Big)^{1/2}.$$

Thus, by (22) we deduce that Σ_1 is at most

$$\left(\frac{c}{\vartheta_x} \sum_{\substack{p \le x_D \\ p \nmid D}} \frac{|g_r^*(p)|^2}{p} \right)^{1/2}.$$

Further,

$$|g_r^*(p)|^2 \ll |f_r^*(p)-1|^2 = |f(p)-f_r(p)|^2,$$

therefore

$$\Sigma_1' \ll \left(\frac{1}{\vartheta_x}\sum_{r\le p\le x}\frac{|\chi(p)f(p)-1|^2}{p}\right)^{1/2},$$

which according to condition (2) tends to zero as $r\to\infty$ with a suitable choice of ϑ_x.

We have to estimate Σ_2'. It can be written as

$$\left|1-\prod_{\substack{r<p\le x_D\\ p\nmid D}}\Big(1-\frac{1}{p-1}+\sum_{m\ge1}\frac{f(p^m)\chi(p^m)}{p^m}\Big)\exp\Big(-i\sum_{\substack{r<p\le x_D\\ p\nmid D}}\frac{g_r^*(p)}{p}\Big)\right|,$$

which equals

$$\left|1-\exp\Big[\mathcal{O}\Big(\sum_{r<p\le x_D}\frac{|f(p)\chi(p)-1|^2}{p}+\sum_{r<p}\frac{1}{p^2}\Big)\Big]\right|,$$

which again tends to zero as $x\to\infty$, such that (21) follows. Finally we note that $x^{1-\varepsilon}<x_D$, therefore we have

$$|P(x_D)-P(x)| \ll \left|\prod_{x_D<p\le x}\left(1+\frac{f(p)\chi(p)-1}{p}+\mathcal{O}\Big(\frac{1}{p^2}\Big)\right)-1\right| =$$
$$= \left|\exp\left(\sum_{x_D<p\le x}\frac{f(p)\chi(p)-1}{p}+\mathcal{O}\Big(\frac{1}{p^2}\Big)\right)-1\right|,$$

which tends to zero as $x\to\infty$ inasmuch as

$$\left|\sum_{x_D<p\le x}\frac{f(p)\chi(p)-1}{p}\right| \ll$$
$$\ll \left(\sum_{x_D<p\le x}\frac{1}{p}\right)^{1/2}\left(\sum_{x_D<p\le x}\frac{|f(p)\chi(p)-1|^2}{p}\right)^{1/2} = o(1) \quad (x\to\infty). \tag{23}$$

We proved Theorem 2 in the case $\tau=0$.

Now consider the case of an arbitrary τ. We proved that

$$\pi(x_D)^{-1}M(x,f(n)n^{-i\tau},D) = \frac{\mu(d)}{\varphi(d)}\prod_{\substack{p\le x\\ p\nmid dD}}\Big(1-\frac{1}{p-1}+\sum_{\alpha\ge 1}\frac{f(p^\alpha)p^{-i\alpha\tau}\chi(p^\alpha)}{p^\alpha}\Big)+ \\ +o(1) =: \psi(x)+o(1)$$

as $x\to\infty$. Using a summation by parts we obtain that

$$\sum_{Dp+1\le x} f(Dp+1) = x^{i\tau}\sum_{Dp+1\le x} f(Dp+1)(Dp+1)^{-i\tau} \\ - i\tau\int_2^x \sum_{Dp+1\le u} f(Dp+1)(Dp+1)^{-i\tau}u^{i\tau-1}\,\mathrm{d}u. \tag{24}$$

If $D<x^\varepsilon$, then $D<x^{\gamma\varepsilon'}$ with some other $\varepsilon<\varepsilon'<1$ and an appropriate $0\le\gamma<1$. Therefore the estimation

$$\pi\left(\frac{u-1}{D}\right)^{-1} M(u,f(n)n^{-i\tau},D) = \\ = \frac{\mu(d)}{\varphi(d)}\prod_{\substack{p\le u\\ p\nmid dD}}\Big(1-\frac{1}{p-1}+\sum_{\alpha\ge 1}\frac{f(p^\alpha)p^{-i\alpha\tau}\chi(p^\alpha)}{p^\alpha}\Big)+o(1) \quad (x\to\infty)$$

remains valid in the range $x^\gamma<u<x$. Thus, we can estimate the integral on the right hand side of (24) in this range as

$$\int_{x^\gamma}^x \sum_{Dp+1\le u} f(Dp+1)(Dp+1)^{-i\tau}u^{i\tau-1}\,\mathrm{d}u = \\ = \frac{\mu(d)}{\varphi(d)}\int_{x^\gamma}^x \pi(u_D)\psi(u)u^{i\tau-1}\,\mathrm{d}u + o(1)\int_{x^\gamma}^x \frac{1}{D\log u}\,\mathrm{d}u \quad (x\to\infty). \tag{25}$$

Now if $x^\gamma\le u\le x$, then as in (23) we have

$$|\psi(x)-\psi(u)| = o(1)$$

as $x\to\infty$. Therefore the right hand side of (25) equals

$$\pi(x_D)\frac{x^{i\tau}}{1+i\tau}\frac{\mu(d)}{\varphi(d)}\psi(x)+o(\pi(x_D)) \quad (x\to\infty).$$

Using the trivial bound

$$|M(u,f(n)n^{i\tau},D)|\le\pi(u_D),$$

we have that the integral on the right hand side of (24) in the range $2 \le u \le x^\gamma$ is not more than

$$\mathcal{O}\left(\frac{1}{D}\int\limits_{2D+1}^{x^\gamma} \frac{1}{\log(u/D)}\,\mathrm{d}u\right) \ll \int\limits_{2}^{x^\gamma/D} \frac{1}{\log(u)}\,\mathrm{d}u = o(\pi(x_D)) \quad (x \to \infty).$$

In summary we have

$$\sum_{Dp+1\le x} f(Dp+1) = \pi(x_D)\frac{x^{i\tau}}{1+i\tau}\,\frac{\mu(d)}{\varphi(d)}\,\psi(x) + o(\pi(x_D)) \quad (x \to \infty),$$

as asserted. ■

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A CHARACTERIZATION OF THE RELATIVE ENTROPIES

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Dedicated to Professor Antal Járai on his sixtieth birthday

Abstract. In this note we give a characterization of a family of relative entropies on open domain depending on a real parameter α, which is based on recursivity and semisymmetry. In cases $\alpha = 1$ and $\alpha = 0$ we use a weak regularity assumption additionally while in the other cases no regularity assumptions are made at all.

1. Introduction and preliminaries

Throughout this paper $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{R}_+$ will denote the sets of all positive integers, real numbers, and positive real numbers, respectively. For all $2 \leq n \in \mathbb{N}$ let

$$\Gamma_n^\circ = \left\{ (p_1, \ldots, p_n) \in \mathbb{R}^n \;\middle|\; p_i \in \mathbb{R}_+, \quad i = 1, \ldots, n, \quad \sum_{i=1}^{n} p_i = 1 \right\}$$

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and

$$\Gamma_n = \left\{ (p_1, \ldots, p_n) \in \mathbb{R}^n \;\middle|\; p_i \geq 0, \quad i = 1, \ldots, n, \quad \sum_{i=1}^{n} p_i = 1 \right\}.$$

Furthermore, for a fixed $\alpha \in \mathbb{R}$, define the function $D_n^\alpha(\cdot|\cdot) : \Gamma_n^\circ \times \Gamma_n^\circ \to \mathbb{R}$ by

$$D_n^\alpha(p_1, \ldots, p_n | q_1, \ldots, q_n) = -\sum_{i=1}^{n} p_i \ln_\alpha \left(\frac{q_i}{p_i} \right), \tag{1.1}$$

where

$$\ln_\alpha(x) = \begin{cases} \dfrac{x^{1-\alpha} - 1}{1 - \alpha}, & \text{if } \alpha \neq 1 \\ \ln(x), & \text{if } \alpha = 1 \end{cases} \qquad (x > 0).$$

The sequence (D_n^α) is called the Shannon relative entropy (or Kullback–Leibler entropy or Kullback's directed divergence) if $\alpha = 1$, and the Tsallis relative entropy if $\alpha \neq 1$, respectively. (D_n^1) is introduced and extensively discussed in Kullback [12] and Aczél–Daróczy [2], respectively. For $0 \leq \alpha \neq 1$, (D_n^α) was introduced and discussed in Shiino [15], Tsallis [17], and Rajagopal–Abe [14] from physical point of view, and in Furuichi–Yanagi–Kuriyama [8] and Furuichi [7] from mathematical point of view, respectively. In [7] and also in Hobson [9], several fundamental properties of (D_n^α) are listed and it is proved that some of them together determine (D_n^α), up to a constant factor.

In this note, we follow the method of the basic references [2] and Ebanks–Sahoo–Sander [6] of investigating characterization problems of information measures. We prove a characterization theorem similar to those of [9] and [7], and we point out that the regularity conditions (say, continuity) can be avoided if $\alpha \notin \{0, 1\}$, and can essentially be weakened if $\alpha \in \{0, 1\}$.

In what follows, a sequence (I_n) of real-valued functions $I_n, (n \geq 2)$ on $\Gamma_n^\circ \times \Gamma_n^\circ$ or on $\Gamma_n \times \Gamma_n$ is called a *relative information measure* on the open or closed domain, respectively. In the closed domain case, however, the expressions $\frac{0}{0+0}$, $\frac{0}{0+\ldots+0}$, 0^α, $0^{1-\alpha}$, $\ln_\alpha \frac{0}{0}$ can appear. Therefore, throughout the paper, the conventions

$$\frac{0}{0+0} = \frac{0}{0+\ldots+0} = 0^\alpha = 0^{1-\alpha} = \ln_\alpha \frac{0}{0} = 0$$

are always adapted (see also [3]).

Our characterization theorem for the Shannon and the Tsallis relative entropies will be based on the following two properties.

Definition 1.1. Let $\alpha \in \mathbb{R}$. The relative information measure (I_n) is α–*recursive* on the open or closed domain, if for any $n \geq 3$ and

$$(p_1, \dots, p_n),\ (q_1, \dots, q_n) \in \Gamma_n^\circ \ \text{ or } \ \Gamma_n,$$

respectively, the identity

$$\begin{aligned} &I_n\left(p_1, \dots, p_n | q_1, \dots, q_n\right) = \\ &= I_{n-1}\left(p_1 + p_2, p_3, \dots, p_n | q_1 + q_2, q_3, \dots, q_n\right) + \\ &+ (p_1 + p_2)^\alpha (q_1 + q_2)^{1-\alpha} I_2\left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2} \middle| \frac{q_1}{q_1 + q_2}, \frac{q_2}{q_1 + q_2}\right) \end{aligned}$$

holds. We say that (I_n) is 3-*semisymmetric* on the open or closed domain, if

$$I_3\left(p_1, p_2, p_3 | q_1, q_2, q_3\right) = I_3\left(p_1, p_3, p_2 | q_1, q_3, q_2\right)$$

is fulfilled for all $(p_1, p_2, p_3), (q_1, q_2, q_3) \in \Gamma_3^\circ$ or Γ_3, respectively.

The following lemma shows how the initial element of an α–recursive relative information measure (I_n) determines (I_n) itself.

Lemma 1.2. *Let $\alpha \in \mathbb{R}$ and assume that the relative information measure (I_n) is α–recursive on the open domain and define the function $f :]0,1[^2 \to \mathbb{R}$ by*

$$f(x, y) = I_2(1 - x, x | 1 - y, y) \qquad \big(x, y \in]0, 1[\big).$$

Then, for all $n \geq 3$ and for arbitrary, $(p_1, \dots, p_n), (q_1, \dots, q_n) \in \Gamma_n^\circ$

$$\begin{aligned} &I_n(p_1, \dots, p_n | q_1, \dots, q_n) = \\ &= \sum_{i=2}^{n} (p_1 + p_2 + \ldots + p_i)^\alpha (q_1 + q_2 + \ldots + q_i)^{1-\alpha} \times \\ &\times f\left(\frac{p_i}{p_1 + p_2 + \ldots + p_i}, \frac{q_i}{q_1 + q_2 + \ldots + q_i}\right) \end{aligned}$$

holds.

Proof. The proof runs by induction on n. If we use the α–recursivity of (I_n) and the definition of the function f, we obtain that

$$\begin{aligned} &I_3(p_1, p_2, p_3 | q_1, q_2, q_3) = \\ &= I_2(p_1 + p_2, p_3 | q_1 + q_2, q_3) + (p_1 + p_2)^\alpha (q_1 + q_2)^{1-\alpha} \times \\ &\times I_2\left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2} \middle| \frac{q_1}{q_1 + q_2}, \frac{q_2}{q_1 + q_2}\right) = \\ &= \sum_{i=2}^{3} (p_1 + \ldots + p_i)^\alpha (q_1 + \ldots + q_i)^{1-\alpha} f\left(\frac{p_i}{p_1 + \ldots p_i}, \frac{q_i}{q_1 + \ldots + q_i}\right) \end{aligned}$$

is fulfilled for all $(p_1, p_2, p_3), (q_1, q_2, q_3) \in \Gamma_3^\circ$, that is, the statement is true for $n = 3$. Assume now that the statement holds for some $3 < n \in \mathbb{N}$. We will prove that in this case the proposition holds also for $n+1$. Let $(p_1, \dots, p_{n+1}), (q_1, \dots, q_{n+1}) \in \Gamma_{n+1}^\circ$ be arbitrary. Then the α–recursivity and the induction hypothesis together imply that

$$I_{n+1}(p_1, \dots, p_{n+1}|q_1, \dots, q_{n+1}) = I_n(p_1 + p_2, \dots, p_{n+1}|q_1 + q_2, \dots, q_{n+1}) +$$

$$+(p_1 + p_2)^\alpha (q_1 + q_2)^{1-\alpha} I_2 \left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2} \middle| \frac{q_1}{q_1 + q_2}, \frac{q_2}{q_1 + q_2} \right) =$$

$$= \sum_{n=3}^{n+1} ((p_1 + p_2) + p_3 \dots + p_i)^\alpha ((q_1 + q_2) + p_3 + \dots + q_i)^{1-\alpha} \times$$

$$\times f \left(\frac{p_i}{(p_1 + p_2) + \dots + p_i}, \frac{q_i}{(q_1 + q_2) + \dots + q_i} \right) +$$

$$+(p_1 + p_2)^\alpha (q_1 + q_2)^{1-\alpha} I_2 \left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2} \middle| \frac{q_1}{q_1 + q_2}, \frac{q_2}{q_1 + q_2} \right) =$$

$$= \sum_{i=2}^{n+1} (p_1 + p_2 + \dots + p_i)^\alpha (q_1 + q_2 + \dots + q_i)^{1-\alpha} \cdot$$

$$\cdot f \left(\frac{p_i}{p_1 + p_2 + \dots + p_i}, \frac{q_i}{q_1 + q_2 + \dots + q_i} \right),$$

that is, the statement holds also for $n+1$, which ends the proof. ■

2. The characterization

We begin with the following

Theorem 2.1. *For any $\alpha \in \mathbb{R}$ the relative entropy (D_n^α) is an α–recursive relative information measure on the open domain.*

Proof. In the proof, we will use the identities

$$\ln_\alpha(xy) = \ln_\alpha(x) + \ln_\alpha(y) + (1 - \alpha) \ln_\alpha(x) \ln_\alpha(y),$$

$$\ln_\alpha \left(\frac{1}{x} \right) = -x^{\alpha - 1} \ln_\alpha(x)$$

several times, which hold for all $\alpha \in \mathbb{R}$ and $x, y \in \mathbb{R}_+$. Let $n \geq 3$ and

$$(p_1, \ldots, p_n), (q_1, \ldots, q_n) \in \Gamma_n^\circ$$

be arbitrary. Then

$$(p_1+p_2)^\alpha (q_1+q_2)^{1-\alpha} D_2\left(\frac{p_1}{p_1+p_2}, \frac{p_2}{p_1+p_2}\middle| \frac{q_1}{q_1+q_2}, \frac{q_2}{q_1+q_2}\right) =$$

$$= (p_1+p_2)^\alpha (q_1+q_2)^{1-\alpha} \times$$

$$\times \left(-\frac{p_1}{p_1+p_2} \ln_\alpha\left(\frac{p_1+p_2}{q_1+p_2}\frac{q_1}{p_1}\right) - \frac{p_2}{p_1+p_2}\ln_\alpha\left(\frac{p_1+p_2}{q_1+q_2}\frac{q_2}{p_2}\right)\right) =$$

$$= (p_1+p_2)^\alpha (q_1+q_2)^{1-\alpha}\left(-\ln_\alpha\left(\frac{p_1+p_2}{q_1+q_2}\right) + \left(1 + (1-\alpha)\ln_\alpha\left(\frac{p_1+p_2}{q_1+q_2}\right)\right)\times\right.$$

$$\left.\times\left(-\frac{p_1}{p_1+p_2}\ln_\alpha\left(\frac{q_1}{p_1}\right) - \frac{p_2}{p_1+p_2}\ln_\alpha\left(\frac{q_2}{p_2}\right)\right)\right) =$$

$$= (p_1+p_2)\ln_\alpha\left(\frac{q_1+q_2}{p_1+p_2}\right) + \left[\left(\frac{q_1+q_2}{p_1+p_2}\right)^{1-\alpha} - (1-\alpha)\ln_\alpha\left(\frac{q_1+q_2}{p_1+p_2}\right)\right]\times$$

$$\times\left[-p_1\ln_\alpha\frac{q_1}{p_1} - p_2\ln_\alpha\frac{q_2}{p_2}\right] =$$

$$= (p_1+p_2)\ln_\alpha\left(\frac{q_1+q_2}{p_1+p_2}\right) - p_1\ln_\alpha\left(\frac{q_1}{p_1}\right) - p_2\ln_\alpha\left(\frac{q_2}{p_2}\right) =$$

$$= D_n(p_1, \ldots, p_n | q_1, \ldots, q_n) - D_{n-1}(p_1+p_2, \ldots, p_n | q_1+q_2+\ldots, q_n).$$

Therefore the relative entropy (D_n^α) is α–recursive, indeed. ■

Obviously (D_n^α) is 3-semisymmetric, and for arbitrary $\gamma \in \mathbb{R}$, (γD_n^α) is α–recursive and 3-semisymmetric, as well. Before dealing with the converse we need two lemmas about *logarithmic* functions. A function $\ell : \mathbb{R}_+ \to \mathbb{R}$ is logarithmic if $\ell(xy) = \ell(x) + \ell(y)$ for all $x, y \in \mathbb{R}_+$. If a logarithmic function ℓ is bounded above or below on a set of positive Lebesgue measure then $\ell(x) = c\ln(x)$ for all $x \in \mathbb{R}_+$ with some $c \in \mathbb{R}$ (see [11], Theorem 5 and Theorem 8 on pages 311, 312). The concept of *real derivation* will also be needed. The function $d : \mathbb{R} \to \mathbb{R}$ is a real derivation if it is *additive*, i.e. $d(x+y) = d(x)+d(y)$ for all $x, y \in \mathbb{R}$, and satisfies the functional equation $d(xy) = xd(y) + yd(x)$ for all $x, y \in \mathbb{R}$. It is somewhat surprising that there are non-identically zero real derivations (see [11], Theorem 2 on page 352). If d is a real derivation then the function $x \mapsto \frac{d(x)}{x}, x \in \mathbb{R}_+$ is logarithmic. Therefore it is easy to see that the real derivation is identically zero if it is bounded above or below on a set of positive Lebesgue measure.

Lemma 2.2. *Suppose that the logarithmic function* $\ell : \mathbb{R}_+ \to \mathbb{R}$ *satisfies the equality*

$$x\ell(x) + (1-x)\ell(1-x) = 0 \qquad \big(x \in]0,1[\big). \tag{2.1}$$

Then there exists a real derivation $d : \mathbb{R} \to \mathbb{R}$ *such that*

$$x\ell(x) = d(x) \qquad (x \in \mathbb{R}_+). \tag{2.2}$$

Proof. Let $x, y \in \mathbb{R}_+$. Then, by (2.1) and by using the properties of the logarithmic function, we have that

$$\begin{aligned} 0 &= \frac{x}{x+y}\ell\left(\frac{x}{x+y}\right) + \frac{y}{x+y}\ell\left(\frac{y}{x+y}\right) = \\ &= \frac{x}{x+y}\big(\ell(x) - \ell(x+y)\big) + \frac{y}{x+y}\big(\ell(y) - \ell(x+y)\big) = \\ &= \frac{1}{x+y}\big(x\ell(x) + y\ell(y) - (x+y)\ell(x+y)\big). \end{aligned}$$

This shows that the function $x \mapsto x\ell(x)$, $x \in \mathbb{R}_+$ is additive on $\mathbb{R}_+$. Hence, by the well-known extension theorem (see e.g. [11], Theorem 1 on page 471), there exists an additive function $d : \mathbb{R} \to \mathbb{R}$ such that (2.2) holds. Since ℓ is logarithmic, this implies that $d(xy) = xd(y) + yd(x)$ holds for all $x, y \in \mathbb{R}_+$. On the other hand, d is odd. Therefore this equation holds also for all $x, y \in \mathbb{R}$, that is, d is a real derivation. ■

Lemma 2.3. *Suppose that* $\ell : \mathbb{R}_+ \to \mathbb{R}$ *is a logarithmic function and the function* g_0 *defined on the interval* $]0,1[$ *by*

$$g_0(x) = x\ell(x) + (1-x)\ell(1-x)$$

is bounded on a set of positive Lebesque measure. Then there exist a real number β *and a real derivation* $d : \mathbb{R} \to \mathbb{R}$ *such that*

$$x\ell(x) + \beta x \ln(x) = d(x) \qquad (x \in \mathbb{R}_+). \tag{2.3}$$

Proof. Define the function g on the interval $[0,1]$ by $g(0) = g(1) = 0$, and for $x \in]0,1[$ by

$$g(x) = \begin{cases} -\dfrac{g_0(x)}{\ell(2)}, & \text{if } \ell(2) \neq 0 \\ g_0(x) - x\log_2(x) - (1-x)\log_2(1-x), & \text{if } \ell(2) = 0. \end{cases}$$

Then g is a symmetric information function (see [2], (3.5.33) Theorem on page 100) which, by our assumption, is bounded on a set of positive Lebesgue measure. Therefore, applying a theorem of Diderrich [5], we obtain that

$$g(x) = -x\log_2(x) - (1-x)\log_2(1-x) \qquad \big(x \in]0,1[\big).$$

For a short proof of Diderrich's theorem see also [13] in which an idea of Járai [10] proved to be very efficient. Taking into consideration the definition of g and applying Lemma 2.2, we get (2.3) with suitable $\beta \in \mathbb{R}$. ■

Now we are ready to prove our main result.

Theorem 2.4. *Let $\alpha \in \mathbb{R}$, (I_n) be an α-recursive and 3-semisymmetric relative information measure on the open domain, and*

$$f(x,y) = I_2(1-x, x|1-y, y) \qquad \big(x, y \in]0,1[\big).$$

Furthermore, suppose that

$$I_2(p_1,p_2|p_1,p_2) = 0 \qquad \big((p_1,p_2) \in \Gamma_2\big). \tag{2.4}$$

If $\alpha \notin \{0,1\}$ then $(I_n) = (\gamma D_n^\alpha)$ for some $\gamma \in \mathbb{R}$.

If $\alpha = 1$ and there exists a point $(u,v) \in]0,1[^2$ such that the function $f(\cdot, v)$ is bounded on a set of positive Lebesgue measure and the function $f(u,\cdot)$ is bounded above or below on a set of positive Lebesgue measure then $(I_n) = (\gamma D_n^1)$ for some $\gamma \in \mathbb{R}$.

And finally, if $\alpha = 0$ and there exists a point $(u,v) \in]0,1[^2$ such that the function $f(\cdot, v)$ is bounded above or below on a set of positive Lebesgue measure and the function $f(u,\cdot)$ is bounded on a set of positive Lebesgue measure then $(I_n) = (\gamma D_n^0)$ for some $\gamma \in \mathbb{R}$.

Proof. Applying Theorem 4.2.3. on page 87 of [6] with $M(x,y) = x^\alpha y^{1-\alpha}$, $x, y \in \mathbb{R}_+$, and taking into consideration Lemma 1.2.12. on page 16 of [6], (see also [1]), we have that

$$I_n\left(p_1,\dots,p_n|q_1,\dots,q_n\right) = bp_1^\alpha q_1^{1-\alpha} + c\sum_{i=2}^n p_i^\alpha q_i^{1-\alpha} - b \tag{2.5}$$

in case $\alpha \notin \{0,1\}$,

$$I_n\left(p_1,\dots,p_n|q_1,\dots,q_n\right) = \sum_{i=1}^n p_i(\ell_1(p_i) + \ell_2(q_i)) + c(1-p_1) \tag{2.6}$$

in case $\alpha = 1$, and

$$I_n\left(p_1, \ldots, p_n | q_1, \ldots, q_n\right) = \sum_{i=1}^{n} q_i(\ell_1(p_i) + \ell_2(q_i)) + c(1 - q_1) \tag{2.7}$$

in case $\alpha = 0$ for all $n \geq 2, (p_1, \ldots, p_n), (q_1, \ldots, q_n) \in \Gamma_n^\circ$ with some $b, c \in \mathbb{R}$ and logarithmic functions $\ell_1, \ell_2 : \mathbb{R}_+ \to \mathbb{R}$.

Now we utilize our further conditions on (I_n). In case $\alpha \notin \{0, 1\}$, (2.5) with $n = 2$ and (2.4) imply that $0 = bp_1 + cp_2 - b$ for all $(p_1, p_2) \in \Gamma_2$ whence $b = c$ follows. Thus, by (2.5), we obtain that $(I_n) = (\gamma D_n^\alpha)$ with $\gamma = (\alpha - 1)^{-1}$. In case $\alpha = 1$, (2.6) with $n = 2$ and (2.4) imply that

$$0 = p_1\ell(p_1) + p_2\ell(p_2) + c(1 - p_1) \qquad \big((p_1, p_2) \in \Gamma_2\big),$$

where $\ell = \ell_1 + \ell_2$. Therefore $c = 0$, and, by Lemma 2.2 we get that $x\ell_2(x) = -x\ell_1(x) + d_1(x)$ for all $x \in \mathbb{R}_+$ and for some real derivation $d_1 : \mathbb{R} \to \mathbb{R}$. Thus

$$f(x, y) = x\ell_1\left(\frac{x}{y}\right) + (1 - x)\ell_1\left(\frac{1 - x}{1 - y}\right) + \left(\frac{x}{y} - \frac{1 - x}{1 - y}\right) d_1(y) \quad \big(x, y \in]0, 1[\big).$$

Since the function $f(\cdot, v)$ is bounded on a set of positive Lebesque measure, we get that the function $x \mapsto x\ell_1(x) + (1 - x)\ell_1(1 - x)$, $x \in]0, 1[$ has the same property. Thus, by Lemma 2.3,

$$x\ell_1(x) + \beta x \ln(x) = d_2(x) \qquad (x \in \mathbb{R}_+)$$

for some $\beta \in \mathbb{R}$ and derivation $d_2 : \mathbb{R} \to \mathbb{R}$. Hence

$$f(x, y) = -\beta x \ln\left(\frac{x}{y}\right) - \beta(1 - x) \ln\left(\frac{1 - x}{1 - y}\right) - \left(\frac{x}{y} - \frac{1 - x}{1 - y}\right) (d_2(y) - d_1(y))$$
$$\big(x, y \in]0, 1[\big).$$

$f(u, \cdot)$ is bounded above or below on a set of positive Lebesgue measure for some $u \in]0, 1[$ thus the derivation $d_2 - d_1$ has the same property, so $d_2 - d_1 = 0$. Therefore

$$f(x, y) = -\beta x \ln\left(\frac{x}{y}\right) - \beta(1 - x) \ln\left(\frac{1 - x}{1 - y}\right) \qquad \big(x, y \in]0, 1[\big)$$

and the statement follows from Lemma 1.2 with a suitable $\gamma \in \mathbb{R}$. The case $\alpha = 0$ can be handled similarly by interchanging the role of the distributions $(p_1, \ldots, p_n)$ and $(q_1, \ldots, q_n)$ and of the logarithmic functions ℓ_1 and ℓ_2, respectively. ■

3. Connections to known characterizations

In this section we discuss the connection between our characterization theorem and other statements known from the literature in this subject. Here we deal especially with the results of Hobson [9] and Furuichi [7] which were the main motivations of our paper. They considered the relative information measures on closed domain thus the comparison is not obvious.

We begin with some definitions.

Definition 3.1. The relative information measure (I_n) on the closed domain is said to be *expansible*, if

$$I_{n+1}\left(p_1,\ldots,p_n,0|q_1,\ldots,q_n,0\right)=I_n\left(p_1,\ldots,p_n|q_1,\ldots,q_n\right)$$

is satisfied for all $n\geq 2$ and $(p_1,\ldots,p_n),(q_1,\ldots,q_n)\in\Gamma_n$, and it is called *decisive*, if $I_2(1,0|1,0)=0$. Let $\alpha\in\mathbb{R}$ be arbitrarily fixed. We say that the relative information measure (I_n) satisfies the *generalized additivity* on the closed (resp. open) domain if for all $n,m\geq 2$ and for arbitrary

$$(p_{1,1},\ldots,p_{1,m},\ldots,p_{n,1},\ldots,p_{n,m}),\ (q_{1,1},\ldots,q_{1,m},\ldots,q_{n,1},\ldots,q_{n,m})\in \\ \in\Gamma_{nm}\quad(\text{or }\Gamma^{\circ}_{nm})$$

$$I_{nm}\left(p_{1,1},\ldots,p_{1,m},\ldots,p_{n,1},\ldots,p_{n,m}|q_{1,1},\ldots,q_{1,m},\ldots,q_{n,1},\ldots,q_{n,m}\right)= \\ =I_n(P_1,\ldots,P_n|Q_1,\ldots Q_n)+\sum_{i=1}^{n}P_i^{\alpha}Q_i^{1-\alpha}I_m\left(\frac{p_{i,1}}{P_i},\ldots,\frac{p_{i,m}}{P_i}|\frac{q_{i,1}}{Q_i},\ldots,\frac{q_{i,m}}{Q_i}\right)$$

is fulfilled, where $P_i=\sum_{j=1}^{m}p_{i,j}$ and $Q_i=\sum_{j=1}^{m}q_{i,j}$, $i=1,\ldots,n$.

A lengthy but simple calculation shows that the relative information measure (D_n^{α}) fulfills all of the above listed criteria. As well as Hobson [9] and Furuichi [7], we would like to investigate the converse direction. More precisely, the question is whether the generalized additivity property determines (D_n^{α}) up to a multiplicative constant. In general this is not true. Let us observe that in case we consider the generalized additivity on the open domain Γ_n° then this property is insignificant for I_n if n is a prime. Nevertheless, on the closed domain this property is well–treatable. In this case we can prove the following.

Lemma 3.2. *If the relative information measure (I_n) on the closed domain is expansible and satisfies the general additivity property with a certain $\alpha\in\mathbb{R}$, then it is also decisive and α–recursive.*

Proof. Firstly, we will show that the generalized additivity and the expansibility imply that the relative information measure (I_n) is decisive. Indeed, if we use the generalized additivity with the choice $n = m = 2$ and $(p_1, p_2, p_3, p_4) = (q_1, q_2, q_3, q_4) = (1, 0, 0, 0)$, then we get that

$$I_4(1, 0, 0, 0|1, 0, 0, 0) = I_2(1, 0|1, 0) + I_2(1, 0|1, 0)$$

holds. On the other hand, (I_n) is expansible, therefore

$$I_4(1, 0, 0, 0|1, 0, 0, 0) = I_2(1, 0|1, 0).$$

Thus $I_2(1, 0|1, 0) = 0$ follows, so (I_n) is decisive.

Now we will prove the α–recursivity of (I_n).
Let $(r_1, \ldots, r_n), (s_1, \ldots, s_n) \in \Gamma_n$ and use the generalized additivity with the following substitution

$$p_{1,1} = r_1, \quad p_{1,2} = r_2, \quad p_{i,1} = r_{i+1}, i = 2, \ldots, n-1, \quad p_{i,j} = 0 \quad \text{otherwise}$$

and

$$q_{1,1} = s_1, \quad q_{1,2} = s_2, \quad q_{i,1} = s_{i+1}, i = 2, \ldots, n-1, \quad q_{i,j} = 0 \quad \text{otherwise}$$

to derive

$$I_{nm}(\ r_1, r_2, 0, \ldots, 0, r_3, 0, \ldots, 0, r_n, 0, \ldots, 0|s_1, s_2, 0, \ldots, 0, s_3, 0, \ldots, 0, s_n, 0, \ldots, 0\ ) =$$

$$= I_n(r_1 + r_2, r_3, \ldots, r_n, 0|s_1 + s_2, s_3, \ldots, s_n, 0)+$$

$$+(r_1 + r_2)^\alpha (s_1 + s_2)^{1-\alpha} I_2 \left(\frac{r_1}{r_1 + r_2}, \frac{r_2}{r_1 + r_2} \middle| \frac{s_1}{s_1 + s_2}, \frac{s_2}{s_1 + s_2} \right) +$$

$$+ \sum_{j=3}^{n} r_j^\alpha q_j^{1-\alpha} I_m(1, 0, \ldots, 0|1, 0, \ldots, 0).$$

After using that (I_n) is expansible and decisive, we obtain the α–recursivity. ■

In view of Theorem 2.4. and Lemma 3.2. the following characterization theorem follows easily.

Theorem 3.3. *Let $\alpha \in \mathbb{R}$, (I_n) be an expansible and 3-semisymmetric relative information measure on the closed domain which also satisfies the generalized additivity property on Γ_n with the parameter α and let $f(x, y) = I_2(1 - x, x|1 - y, y)$, $x, y \in]0, 1[$. Additionally, suppose that*

$$I_2(p_1, p_2|p_1, p_2) = 0. \qquad \big((p_1, p_2) \in \Gamma_2\big) \tag{3.1}$$

If $\alpha \notin \{0, 1\}$ *then* $(I_n) = (\gamma D_n^\alpha)$ *for some* $\gamma \in \mathbb{R}$.

If $\alpha = 1$ *and there exists a point* $(u, v) \in]0, 1[^2$ *such that the function* $f(\cdot, v)$ *is bounded on a set of positive Lebesgue measure and the function* $f(u, \cdot)$ *is bounded above or below on a set of positive Lebesgue measure then* $(I_n) = (\gamma D_n^1)$ *for some* $\gamma \in \mathbb{R}$.

And finally, if $\alpha = 0$ *and there exists a point* $(u, v) \in]0, 1[^2$ *such that the function* $f(\cdot, v)$ *is bounded above or below on a set of positive Lebesgue measure and the function* $f(u, \cdot)$ *is bounded on a set of positive Lebesgue measure then* $(I_n) = (\gamma D_n^0)$ *for some* $\gamma \in \mathbb{R}$.

Finally, we remark that the essence of Theorems 2.4. and 3.3. is that, in case $\alpha \notin \{0, 1\}$, the algebraic properties listed in these theorems determine the information measure (D_n^α) up to a multiplicative constant *without any regularity assumption*. Furthermore, if $\alpha \in \{0, 1\}$, then the mentioned algebraic properties with a really mild regularity condition determine (D_n^α) up to a multiplicative constant.

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SOME REMARKS ON THE CARMICHAEL AND ON THE EULER'S φ FUNCTION

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Dedicated to my friend, Professor Antal Járai on his 60th anniversary

Abstract. Several theorems on the iterates of the Carmichael and on the Euler's φ function is presented, some of them without proof.

1. Introduction

We shall formulate several in my opinion new theorems on the divisors of the Carmichael and Euler's totient function.

Some of them can be proved by direct application of sieve theorems. We omit the proof of them. We shall prove only Theorem 6, 10, 11, 12.

1.1. Notations. $\mathcal{P}$ = set of primes; p, π with and without suffixes always denote prime numbers; $\pi(x) = \#\{p \leq x\}$, $\pi(x,k,l) = \#\{p \leq x,\ p \equiv l \pmod{k}\}$.

$\lambda(n)$ = Carmichael function defined for p^α by

$$\lambda(p^\alpha) = \begin{cases} p^{\alpha-1}(p-1), & \text{if } p \geq 3, \text{ or } \alpha \leq 2, \\ 2^{\alpha-2}, & \text{if } p = 2 \text{ and } \alpha \geq 3, \end{cases}$$

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and for $n = p_1^{\alpha_1} \dots p_r^{\alpha_r}$ $(p_i \neq p_j,\ p_i \in \mathcal{P})$

$$\lambda(n) = LCM\left[\lambda(p_1^{\alpha_1}), \dots, \lambda(p_r^{\alpha_r})\right].$$

Here LCM = least common multiple.

Let $\omega(n)$ = number of distinct prime factors of n, $\Omega(n)$ = number of prime power divisors of n.

$$\varphi(n) = \prod_{j=1}^{r} p_j^{\alpha_j - 1}(p_j - 1) \text{ the Euler's totient function.}$$

$P(n)$ = largest prime divisor of n; $p(n)$ = smallest prime divisor of n.

Let $x_1 = \log x,\ x_2 = \log x_1 \dots$.

Let $\lambda^{(k)}(n),\ \varphi^{(k)}(n)$ be the kth iterate of $\lambda(n)$ and of $\varphi(n)$, respectively, i.e. $\lambda^{(0)}(n) = n,\ \varphi^{(0)}(n) = n$, and $\lambda^{(k+1)}(n) = \lambda(\lambda^{(k)}(n)),\ \varphi^{(k+1)}(n) = \varphi(\varphi^{(k)}(n))$.

1.2. In this paper we shall formulate some theorems on λ, φ and on their iterates. Some of these theorems can be proved by known methods which were applied earlier, and we omit their complete proof.

1.3. Let $q \geq 2$ be a fixed prime, $\gamma(n)$ be that exponent, for which $q^{\gamma(n)} \,\|\, \varphi(n)$. M. Wijsmuller [3] investigated the completely additive function β defined on $p \in \mathcal{P}$ by $q^{\beta(p)} \,\|\, p+1$, and proved a global central limit theorem for $\beta(n)$. Her method can be used to prove central limit theorem for $\gamma(n)$ as well. In [1], [2] we developed a method by which we can prove local central limit theorem for $\gamma(n)$ and $\beta(n)$. We are unable to give the asymptotic of $\#\{p \leq x,\ p \in \mathcal{P},\ \gamma(p+1) = k\}$, and that of $\{n \leq x,\ \gamma(n^2+1) = k\}$. Global central limit theorem can be proved for $\gamma(p+1)$, and $\gamma(n^2+1)$.

1.4. Let $\nu(n)$ be defined by $q^{\nu(n)} \,\|\, \lambda(n)$. Let $\mathcal{P}_k := \{p \mid p \in \mathcal{P},\ p \equiv 1 \pmod{q^k}\}$; $\mathcal{P}_k^* = \mathcal{P}_k \backslash \mathcal{P}_{k+1}$. Let furthermore

$$\omega_k(n) = \sum_{\substack{p \mid n \\ p \in \mathcal{P}_k}} 1, \tag{1.1}$$

$$t_k(x) := \prod_{\substack{p \equiv 1 \pmod{q^k} \\ p \leq x}} \left(1 - \frac{1}{p}\right). \tag{1.2}$$

From the Siegel–Walfisz theorem (Lemma 7) one can obtain that

$$\log t_k(x) = -\sum_{\substack{p\le x\\ p\equiv 1(q^k)}} \frac{1}{p} + O\left(\frac{1}{q^k}\right) = -\frac{x_2}{\varphi(q^k)} + O\left(\frac{1}{q^k}\right) \tag{1.3}$$

valid if $1 \le q^k \le c\,x_2$.

The following assertion can be proved by routine application of the asymptotic sieve.

Theorem 1. *Let $q \ge 2$ be a fixed prime,*

$$\alpha_k(x) := \frac{x_2}{\varphi(q^k)}. \tag{1.4}$$

Assume that $k = k(x) \to \infty$ and that $x_2 \cdot q^{-k} \to \infty$. Then

$$\begin{aligned}\frac{1}{\left(1-\frac{1}{q}\right)x}\#\{n\le x,\ (n,q)=1,\ \nu(n)=k,\ \omega_k(n)=r\} =\\ = (1+o_x(1))t_k(x)\sum \frac{1}{\varphi(p_1\cdots p_r)}\end{aligned} \tag{1.5}$$

valid for $0 \le r \le \dfrac{x}{x_3^2}$. The last sum is extended over those $p_1 < \ldots < p_r$ for which $p_i \in \mathcal{P}_k^*$, $p_1 < \ldots < p_r \le x$. In this range of r we have

$$\sum \frac{1}{\varphi(p_1\cdots p_r)} = (1+o_x(1))\left(\frac{x_2}{q^k}\right)^r \cdot \frac{1}{r!}. \tag{1.6}$$

Assume that $q^k/x_2 \to \infty$, $q^k < x^{1/3}$. Then

$$\sum_{n\le x} \omega_k(n) = x\sum_{\substack{p\le x\\ p\in\mathcal{P}_k}} \frac{1}{p} + O(\pi(x,q^k,1)), \tag{1.7}$$

and

$$\sum_{n\le x}\omega_k(n)(\omega_k(n)-1) = \sum_{\substack{p_1\ne p_2\\ p_1p_2\le x\\ p_1,p_2\in\mathcal{P}_k}} \frac{x}{p_1p_2} + O\left(\sum_{\substack{p_1<\sqrt{x}\\ p_1\in\mathcal{P}_k}} \pi\left(\frac{x}{p_1}, q^k, 1\right)\right). \tag{1.8}$$

By using the Brun–Titchmarsh theorem (Lemma 8), we obtain that the error term on the right hand side of (1.8) is less than $(\mathrm{li}\,x)q^{-2k}x_2$. From (1.7), (1.8) we can deduce a Turán–Kubilius type inequality and from that

Theorem 2. *Let $q \in \mathcal{P}$ be fixed, $k = k(x)$ be such that $q^k/x_2 \to \infty$ and that $q^k < c\,x_1^A$ hold with arbitrary constants c, A. Then*

$$\frac{1}{x}\#\{n \le x \mid \nu(n) \ge k\} = (1 + o_x(1)) \sum_{\substack{p \le x \\ p \in \mathcal{P}_k}} \frac{1}{p}, \tag{1.9}$$

furthermore

$$\sum_{\substack{p \le x \\ p \in \mathcal{P}_k}} \frac{1}{p} = \alpha_k(x) + O\left(\frac{1}{q^k}\right). \tag{1.10}$$

Remark. By using the Barban–Linnik–Tshudakov theorem (Lemma 9) (1.9) remains valid up to $q^k < x^\delta$, where δ is a suitable positive constant.

We can prove also the following Theorem 3, 4, 5.

Theorem 3. *Assume that $k = k(x)$ is such a sequence for which $q^k/x_2 \to \infty$ and that $q^k < c\,x_1^A$ with arbitrary constants c, A. Then*

$$\frac{1}{\operatorname{li} x}\#\{p \le x \mid \nu(p+1) \ge k\} = (1 + o_x(1))\alpha_k(x). \tag{1.11}$$

Furthermore

$$\frac{1}{\operatorname{li} x}\#\{p \le x \mid \nu(p+1) \ge k,\ \nu(p-1) \ge l\} = (1 + o_x(1))\alpha_k(x) \cdot \alpha_l(x) \tag{1.12}$$

holds, if additionally $q^l/x_2 \to \infty,\ q^l < c\,x_1^A$.

Remark. One could prove in general that

$$\frac{1}{\operatorname{li} x}\#\{p \le x \mid \nu(p + t_j) \ge k_j,\ j = 1, \dots, h\} = (1 + o_x(1))\alpha_{k_1}(x) \dots \alpha_{k_h}(x)$$

if $t_1, \dots, t_h$ are distinct nonzero integers and $q^{k_j}/x_2 \to \infty,\ q^{k_j} \le c x_1^A$ $(j = 1, \dots, h)$.

Theorem 4. *Let q be an odd prime. Assume that $k = k(x) \to \infty,\ x_2 q^{-k} \to \infty$. Then*

$$\begin{aligned} &\frac{1}{\operatorname{li} x}\#\{p \le x,\ (p+1, q) = 1,\ \nu(p+1) = k,\ \omega_k(p+1) = r\} = \\ &= (1 + o_x(1))(\operatorname{li} x) t_k^*(x) \frac{1}{r!}\left(\frac{x_2^k}{q^k}\right)^r \end{aligned} \tag{1.13}$$

if $0 \le r \le \frac{x_2}{x_3}$. *Here*

$$t_k^*(x) = \prod_{\substack{p<x \\ p\in\mathcal{P}_k}} \left(1 - \frac{1}{p-1}\right). \tag{1.14}$$

Remark. Since

$$\log \frac{t_k^*(x)}{t_k(x)} = O\left(\sum_{p\in\mathcal{P}_k} \frac{1}{p^2}\right) = O\left(\frac{1}{q^k}\right),$$

(1.13) remains valid with $t_k(x)$ instead of $t_k^*(x)$.

Theorem 5. *Let q be an odd prime, $k = k(x)$ be such a sequence for which $x_2 q^{-k} \to \infty$. Let $\rho(m)$ be the number of residue classes $n \pmod m$, for which $n^2 + 1 \equiv 0 \pmod m$.*

Let

$$s_k(x) = \prod_{\substack{p<x \\ p\in\mathcal{P}_k}} \left(1 - \frac{\rho(p)}{p-1}\right). \tag{1.15}$$

Then

$$\begin{aligned} &\frac{1}{x}\#\left\{n \le x,\ (n^2+1, q) = 1,\ \nu(n^2+1) = k, \omega_k(n^2+1) = r\right\} = \\ &= (1 + o_x(1))\left(1 - \frac{\rho(q)}{q}\right) s_k(x) \frac{1}{r!}\left(\sum_{\substack{\pi<x \\ \pi\in\mathcal{P}_k}} \frac{\rho(\pi)}{\pi - 1}\right)^r \end{aligned} \tag{1.16}$$

if $0 \le r \le \frac{x_2}{x_3}$.

1.5. In their paper [6] W.D. Banks, F. Luca, I.E. Shparlinski investigated some arithmetic properties of $\varphi(n), \lambda(n)$, and that of $\xi(n) = \frac{\varphi(n)}{\lambda(n)}$. Among others they investigated the distribution of $P(\xi(n))$. Namely they proved that

$$1 + o(1) \le \frac{1}{x \cdot x_3} \sum_{n\le x} \log P(\xi(n)) \le 2 + o(1), \tag{1.17}$$

and that

$$(0 <) c_1 \le \frac{1}{x x_2^3} \sum_{n \le x} P(\xi(n)) \le c_2 \quad (x \ge 1) \tag{1.18}$$

holds with suitable positive constants.

We can prove that $P(\xi(n))$ is distributed in limit according to the Poisson law.

Let

$$\kappa_q(n) := \sum_{\substack{p\,|n\\ p\equiv 1 \pmod{q^2}}} 1; \quad f_Y(n) := \sum_{q>Y} \kappa_q(n).$$

Since

$$\sum_{n\le x} \kappa_q(n) = \sum_{p\equiv 1 \pmod{q^2}} \left[\frac{x}{p}\right] \le x \sum_{\substack{p\le x\\ p\equiv 1 \pmod{q^2}}} \frac{1}{p} \le \frac{cxx_2}{q^2}$$

holds with a suitable constant c, and

$$\sum_{q\ge Y} \frac{1}{q^2} = \frac{1}{Y\log Y} + O\left(\frac{1}{Y(\log Y)^2}\right),$$

we obtain that

$$\sum_{n\le x} f_Y(n) \le \frac{c\,x\,x_2}{Y\log Y}.$$

If q is an odd prime, $q^2 \mid \lambda(n)$, then either $q^3 \mid n$, or there exists some $p \mid n$ for which $q^2 \mid p-1$. We obtain

$$\#\left\{n\le x \ \mid\ q^2 \mid \lambda(n) \text{ for some } q > x_2^2\right\} \le \frac{c\,x}{x_2x_3}. \tag{1.19}$$

Let

$$f_Y^*(n) = \sum_{Y\le q\le x_2^2} \kappa_q(n),$$

$$\sum\nolimits_1 := \sum_{n\le x} f_Y^*(n), \qquad \sum\nolimits_2 := \sum_{n\le x} f_Y^{*2}(n).$$

From the Siegel–Walfisz theorem one can prove that

$$\sum_{\substack{p\le x\\ p\equiv 1 \pmod{k}}} \frac{1}{p} = \frac{1}{\varphi(k)}x_2 + O\left(\frac{x_3}{\varphi(k)}\right) \text{ if } 1\le k\le x_2^A,$$

where A is an arbitrary constant, whence we deduce that

$$\sum\nolimits_1 = xx_2\ A_{Y,x} + O\left(\frac{xx_3}{Y\log Y}\right),$$

$$A_{Y,x} := \sum_{Y\le q\le x_2^2} \frac{1}{\varphi(q^2)} = \frac{1}{Y\log Y} + O\left(\frac{1}{Y(\log Y)^2}\right).$$

Furthermore $\sum_2 = \sum_{2,1} + \sum_{2,2}$, where

$$\sum\nolimits_{2,1} = \sum_{Y \le q \le x_2^2} \sum_{n \le x} \kappa_q^2(n), \qquad \sum\nolimits_{2,2} = \sum_{\substack{q_1 \ne q_2 \\ Y \le q_1, q_2 \le x_2^2}} \sum_{n \le x} \kappa_{q_1}(n)\kappa_{q_2}(n).$$

In this section q, q_1, q_2 run over the set of primes.

We have

$$\sum\nolimits_{2,1} = \sum\nolimits_1 + \sum_{Y \le q \le x_2^2} \sum_{\substack{p_1 \ne p_2 \\ q^2 / p_j - 1}} \left[\frac{x}{p_1 p_2}\right] = \sum\nolimits_1 + x \sum_{Y \le q \le x_2^2} \frac{x_2^2}{\varphi(q^2)^2} +$$

$$+ O\left(x x_2 x_3 \sum_{q > Y} 1/q^4\right) = \sum\nolimits_1 + O\left(\frac{x x_2^2}{Y^3 \log Y}\right)$$

and

$$\sum\nolimits_{2,2} = x \sum_{\substack{q_1 \ne q_2 \\ q_j \in [Y, x_2^2]}} \sum_{\substack{p_j \equiv 1 \pmod{q_j^2} \\ p_1 p_2 \le x}} \frac{1}{p_1 p_2} + x \sum_{\substack{q_1 \ne q_2 \\ q_j \in [Y, x_2^2]}} \sum_{\substack{p \le x \\ p \equiv 1 \pmod{q_1^2 q_2^2}}} \frac{1}{p} + O(x)$$

whence we obtain that

$$\sum\nolimits_{2,2} = \left(1 + O\left(\frac{x_3}{x_2}\right)\right) x x_2^2 A_{y,x}^2 + O\left(x x_2 \left(\sum_{q > Y} \frac{1}{q^2}\right)^2\right) =$$

$$= x x_2^2 A_{y,x}^2 + O\left(x x_3 x_2 \cdot \frac{1}{Y^2 \log^2 Y}\right).$$

After some easy computation we obtain that

$$\frac{1}{x} \sum_{n \le x} (f_Y^*(n) - x_2 A_{Y,x})^2 \ll \frac{x_2}{Y \log Y} + \frac{x_2 x_3}{(Y \log Y)^2} + \frac{x_2^2}{Y^3 \log Y}. \tag{1.20}$$

From (1.20) we can deduce

Theorem 6. *Let* $\varepsilon_x \to 0$. *Then*

$$x^{-1} \# \left\{ n \le x \mid P(\lambda(n)) \in \left[\varepsilon_x \cdot \frac{x_2}{x_3}, \frac{1}{\varepsilon_x} \cdot \frac{x_2}{x_3}\right]\right\} \to 1 \quad (x \to \infty).$$

Proof. Indeed, choose first $Y = \varepsilon_x \cdot \frac{x_2}{x_3}$, then $Y = \frac{1}{\varepsilon_x} \cdot \frac{x_2}{x_3}$ and apply (1.20). ■

We can prove also

Theorem 7. *Let* $\varepsilon_x \to 0$. *Then*

$$\frac{1}{\operatorname{li} x} \# \left\{ p \le x \;\middle|\; P(\lambda(p-1)) \in \left[\varepsilon_x \cdot \frac{x_2}{x_3}, \frac{1}{\varepsilon_x} \cdot \frac{x_2}{x_3} \right] \right\} \to 1 \quad (x \to \infty).$$

1.6. Assume that $Y = O(x_2^2)$, $Y \ge x_2^{3/2}$, $u(n) := e^{i\tau f_Y^*(n)}$, where $\tau \in \mathbb{R}$. Then u is a strongly multiplicative function, for $p \in \mathcal{P}$

$$u(p) := \begin{cases} e^{i\tau} & \text{if } p \equiv 1 \pmod{q^2} \text{ for some } \; q \in [Y, x_2^2], \\ 1 & \text{otherwise.} \end{cases}$$

Let h be the Moebius transform of u, i.e.

$$h(p) = \begin{cases} e^{i\tau} - 1 & \text{if } q^2 \mid p-1 \text{ for some } q \in [Y, x_2^2], \\ 0 & \text{otherwise,} \end{cases}$$

$h(p^\alpha) = 0$ if $p \in \mathcal{P}$, $\alpha \ge 2$.

Let

$$S_1(x,\tau) := \sum_{n \le x} e^{i\tau f_Y^*(n)}; \; S_2(x,\tau) = \sum_{n \le x} u(n).$$

If $f_Y^*(n) \ne u(n)$ for some n, then there exists a prime divisor p of n, and $q_1, q_2 \in \mathcal{P}$, $q_1, q_2 > Y$, $q_1 \ne q_2$ such that $p \equiv 1 (\operatorname{mod} q_1^2 q_2^2)$.

Then

$$|S_1(x,\tau) - S_2(x,\tau)| \le x \sum_{\substack{q_1, q_2 \in [Y, x_2^2] \\ q_1 \ne q_2}} \; \sum_{\substack{p \equiv 1 \pmod{q_1^2, q_2^2} \\ p \le x}} \frac{1}{p} \ll$$

$$\ll x x_2 \left(\sum_{q > Y} \frac{1}{q^2} \right)^2 \ll x x_2 \left(\frac{1}{Y \log Y} \right)^2 = O\left(\frac{x}{x_2^2} \right).$$

There are several ways to prove that

$$\frac{S_2(x,\tau)}{x} = (1 + o_x(1)) \prod_{\substack{p < x \\ p \equiv 1 \ (q^2) \\ q > Y \\ q \in \mathcal{P}}} \left(1 + \frac{e^{i\tau} - 1}{p} \right) = \tag{1.21}$$

$$= (1 + o_x(1)) \exp\left(\left(e^{i\tau} - 1 \right) \frac{x_2}{Y \log Y} \right).$$

One way to prove (1.21) is to copy the argument of the theorem of H. Delange
for the arithmetical mean of multiplicative functions of moduli 1. (See [7], or [4] pp. 331–336.) Another method is to compute the asymptotic of $\sum_{n\le x} f_Y^{*h}(n)$ for $h = 1, 2, \dots$ and use the Frechlet–Shohat theorem (see J. Galambos [11]). A relevant paper is written by J. Šiaulys [8]. We can prove

Theorem 8. *Let* $\alpha_Y = x_2 \sum_{q>Y} \frac{1}{\varphi(q^2)}$. *Assume that* $\alpha_Y \in [c_1, c_2]$, *where* $c_1 < c_2$ *are arbitrary positive constants. Then*

$$\lim_{x\to\infty} \sup_{\alpha_Y\in[c_1,c_2]} \sup_{k\ge 0} \left| \frac{1}{x} \# \{n \le x \mid f_Y^*(n) = k\} - \frac{\alpha_Y^k}{k!} \exp(-\alpha_Y) \right| = 0. \tag{1.22}$$

Similarly, we have

(1.23)

$$\lim_{x\to\infty} \sup_{\alpha_Y\in[c_1,c_2]} \sup_{k\ge 0} \left| \frac{1}{\operatorname{li} x} \# \{p \le x \mid f_Y^*(p-1) = k\} - \frac{\alpha_Y^k}{k!} \exp(-\alpha_Y) \right| = 0.$$

Assume that Q is such a prime for which $(Q \log Q)/x_2 \in [c_1, c_2]$, where c_1, c_2 are positive constants. We would like to estimate the number of those integers $n \le x$ for which $P(\xi(n)) = Q$. By using the asymptotic sieve one can obtain quite immediately that

$$\frac{1}{x} \#\{n \le x \mid P(\xi(n)) < Q\} = (1 + o_x(1)) \prod_{\substack{p\le x\\ q^2/p-1\\ q\ge Q}} \left(1 - \frac{1}{p}\right).$$

Let

$$\tau(Q, x) = x_2 \cdot \sum_{q\ge Q} \frac{1}{\varphi(q^2)}.$$

Then

$$\frac{1}{x} \#\{n \le x \mid P(\xi(n)) < Q\} = (1 + o_x(1)) \exp(-\tau(Q, x)).$$

Let $\mathcal{B}_{Q,r}$ be the set of those n for which $P(\xi(n)) = Q$, and there exists exactly r distinct prime divisors $p_1, p_2, \dots, p_r$ of n for which $Q^2 \mid p_j - 1$. Then

$$\frac{1}{x} \#\{n \le x \mid n \in \mathcal{B}_{Q,r}\} = (1 + o_x(1)) \exp(-\tau(Q, x)) \cdot \frac{1}{r!} \left\{ \sum_{\substack{p\equiv 1 (Q^2)\\ p\le x}} \frac{1}{p} \right\}^r$$

valid for every fixed $r = 0, 1, 2, \dots$.

We can prove furthermore

Theorem 9. *We have*

$$\frac{1}{\operatorname{li} x}\#\{p \le x \mid p-1 \in \mathcal{B}_{Q,r}\} = (1+o_x(1))\exp(-\tau(Q,x)) \cdot \frac{1}{r!}\left\{\sum_{\substack{p\equiv 1 \pmod{Q^2}\\ p\le x}} 1/p\right\}^r$$

for every fixed $r = 0, 1, 2, \dots$.

1.7. For $p_1, p_2,\ q \in \mathcal{P}$ let

$$f_q(p_1,p_2) = \begin{cases} 1 & \text{if } p_1 \equiv p_2 \equiv 1 \pmod q, \quad p_1 < p_2, \\ 0 & \text{otherwise.} \end{cases} \tag{1.24}$$

Let

$$\Delta_Y(n) := \sum_{q>Y} \sum_{p_1 p_2 \mid n} f_1(p_1,p_2). \tag{1.25}$$

We observe that $\Delta_Y(n) \ne 0$ implies that $q^2 \mid \varphi(n)$ for some $q > Y$. On the other hand, if $q^2 \mid \varphi(n)$, then either $q^3 \mid n$; or $q^2 \mid n$ and $p \mid n$ with some $p \equiv 1 \pmod q$, or $p \mid n$ with some $p \equiv 1 \pmod{q^2}$; or there exist $p_1 \ne p_2$, $p_1 \equiv p_2 \equiv 1 \pmod q$, $q > Y$, and $p_1p_2 \mid n$.

Thus

$$\frac{1}{x}\#\{n \le x \mid \Delta_Y(n) \ne 0\} - \frac{1}{x}\#\{n \le x \mid q^2|\varphi(n) \text{ for some } q > Y\} \ll \frac{x}{Y \log Y}. \tag{1.26}$$

By using our method developed by De Koninck and myself [1], [2] we can compute the asymptotic of $\sum_{n\le x}\Delta_Y^h(n)$ and from the Frechet–Shohat theorem deduce

Theorem 10. *Let* $0 < c_1 < c_2 < \infty$ *be fixed constants,* $\alpha = \alpha_x \in [c_1, c_2]$, $Y = Y_x = \frac{1}{2\alpha} \cdot x_2^2/2x_3$. *Then*

$$x^{-1}\#\{n \le x \mid \Delta_{Y_x}(n) = k\} = (1+o_x(1))\frac{\alpha^k}{k!}e^{-\alpha} \quad (x \to \infty) \tag{1.27}$$

for every fixed $k = 0, 1, 2, \dots$ *uniformly as* $\alpha_x \in [c_1, c_2]$.

Furthermore we obtain that

$$\frac{1}{\operatorname{li} x}\#\{p \le x \mid \Delta_{Y_x}(p-1) = k\} = (1+o_x(1))\frac{\alpha^k}{k!}e^{-\alpha} \quad (x \to \infty) \tag{1.28}$$

for every fixed $k = 0, 1, \dots$ *uniformly as* $\alpha_x \in [c_1, c_2]$.

We shall prove this theorem in Section 3.

The following theorem can be deduced easily from Theorem 10.

Let $\kappa_Y(n)$ be the number of those $q > Y$ for which $q^2 \mid \varphi(n)$.

Theorem 11. *Let Y_x be the same as in Theorem 10. Then*

$$x^{-1}\#\{n \le x \mid \kappa_{Y_x}(n) = k\} = (1 + o_x(1))\frac{\alpha^k}{k!}e^{-\alpha} \quad (x \to \infty), \tag{1.29}$$

and

$$\frac{1}{\operatorname{li} x}\#\{p \le x \mid \kappa_{Y_x}(p-1) = k\} = (1 + o_x(n))\frac{\alpha^k}{k!}e^{-\alpha} \quad (x \to \infty). \tag{1.30}$$

Remark. By using our method we can determine the distribution of

$$\delta_Y^{(k,r)}(n) = \delta_Y(n) = \#\{q > Y,\ q \in \mathcal{P},\ q^r \mid \varphi_k(n)\}$$

and that of $\delta_Y^{(k,r)}(p-1)$, where $Y_x = \alpha\ (x_2^{kr}/x_3)^{1/(r-1)}$. We shall prove it in another paper.

1.8. In a paper of F. Luca and C. Pomerance [17] the conjecture of Erdős, namely that $\varphi(n - \varphi(n)) < \varphi(n)$ holds on a set of asymptotic density 1 is proved.

They deduce that

$$\left|\frac{\varphi(n - \varphi(n))}{n - \varphi(n)} - \frac{\varphi(n)}{n}\right| < \varepsilon_n \tag{1.31}$$

holds for almost all n, with a sequence $\varepsilon_n \to 0$, which implies the conjecture of Erdős. Namely they prove (1.31) with $\varepsilon_n = 2\frac{\log\log\log n}{\log\log n}$ but this is not necessary for obtaining Erdős conjecture.

By their method one can prove that

$$\left|\frac{f_i(n \pm f_j(n))}{n \pm f_j(n)} - \frac{f_i(n)}{n}\right| < \varepsilon_n \tag{1.32}$$

holds on a set of asymptotic density 1, where $\varepsilon_n \to 0$, and $f_1(n), f_2(n)$ can take the values $\varphi(n),\ \sigma(n) : (f_1, f_2) = (\varphi, \varphi);\ (\varphi, \sigma), (\sigma, \varphi), (\sigma, \sigma)$.

We can prove (1.32) also, if n runs over the set of shifted primes. We shall give a complete proof only in the case $f_1 = f_2 = \varphi$, $\pm = -$, and over the set of prime $+1'$s.

Theorem 12. *There exists a suitable sequence $\varepsilon_p \to 0$ $(p \in \mathcal{P}, p \to \infty)$ such that*

$$\left| \frac{\varphi(p-1-\varphi(p-1))}{p-1-\varphi(p-1)} - \frac{\varphi(p-1)}{p-1} \right| < \varepsilon_p$$

holds for $p \in \mathcal{P}$ with the possible exception of $o_x(1)\pi(x)$ of $p \in \mathcal{P}$ up to x.

1.9. J.-M. De Koninck and F. Luca [17] investigated

$$H(n) := \frac{\sigma(\varphi(n))}{\varphi(\sigma(n))}.$$

In particular, they obtain the maximal and minimal orders of $H(n)$, its average order, and also proved some density theorems.

Since

$$H(n) = \frac{\sigma(\varphi(n))}{\varphi(n)} \cdot \frac{\sigma(n)}{\varphi(\sigma(n))} \cdot \frac{\varphi(n)}{\sigma(n)},$$

therefore

$$\log H(n) = \kappa_1(n) + \kappa_2(n) + \kappa_3(n),$$

where

$$\kappa_1(n) = \sum_{p^\alpha \,\|\, \varphi(n)} \log\left(1 + \frac{1}{p} + \cdots + \frac{1}{p^\alpha}\right),$$

$$\kappa_2(n) = \sum_{p \mid \sigma(n)} \log \frac{1}{1-\frac{1}{p}},$$

$$\kappa_3(n) = \sum_{p^\alpha \,\|\, n} \log \frac{1-\frac{1}{p}}{1+\frac{1}{p}+\cdots+\frac{1}{p^\alpha}}.$$

By using a known theorem of P. Erdős one can prove that

$$\left| \kappa_j(n) - \sum_{p < x_2/x_3^2} \log \frac{1}{1-\frac{1}{p}} \right| < \varepsilon_x \quad (j = 1, 2)$$

holds for all but at most $o(x)$ integers $n \le x$, where $\varepsilon_x \to 0$ $(x \to \infty)$. Since $\kappa_3(n)$ is an additive function satisfying the conditions of the Erdős–Wintner theorem, we obtain immediately that

$$\frac{1}{x}\#\left\{ n \le x \;\middle|\; \log H(n) - \sum_{p < x_2/x_3^2} \log \frac{1}{1-\frac{1}{p}} < y \right\} = F_x(y)$$

tends to $F(y)$, where F is the distribution function defined as

$$F(y)=\lim_{x\to\infty}\frac{1}{x}\#\{n\le x\mid \kappa_3(n)<y\}.$$

Erdős proved that F is a continuous singular function.

Distribution of H on the set of shifted primes, on polynomial values, and on prime places of polynomial values can be proved similarly. Let

$$s(x)=\prod_{p<x}\left(1-\frac{1}{p}\right)^{-1}.$$

Then $s(x)=e^{\gamma}x_1\ (1+o_x(1))$.

Theorem 13. *Let $k,l\ge 0$, $f_{k,l}^{(1)}(n):=\sigma_k(\varphi_l(n))$, $f_{k,l}^{(2)}(n)=\varphi_k(\sigma_l(n))$. Then for every $n\le x$ dropping at most $o(x)$ integers*

$$\frac{\sigma_k(n)}{\sigma_{k-1}(n)}=s(x_2^{k-1})(1+o_x(1))\quad (k\ge 2), \tag{1.33}$$

$$\frac{\varphi_k(n)}{\varphi_{k-1}(n)}=\frac{1}{s(x_2^{k-1})}(1+o_x(1))\quad (k\ge 2), \tag{1.34}$$

and for $k,l\ge 1$

$$\frac{f_{k,l}^{(1)}(n)}{f_{k-1,l}^{(1)}(n)}=\frac{1}{s(x_2^{k+l-1})}(1+o_x(1))\quad (k\ge 1), \tag{1.35}$$

$$\frac{f_{k,l}^{(2)}(n)}{f_{k-1,l}^{(2)}(n)}=s(x_2^{k+l-1})(1+o_x(1))\quad (k\ge 1). \tag{1.36}$$

Furthermore the relations (1.33), (1.34), (1.35), (1.36) *are valid on the set of shifted primes $p+a$ $(a\ne 0)$, with the exception of no more than $o(\operatorname{li}x)$ integers $p+a$ up to x.*

This theorem is an immediate consequence of the following

Theorem 14. *Let $k,l\ge 1$. Then, with the exception of at most $\delta_x x$ integers $n\le x$, for the others*

α) $p^{\alpha}\mid\varphi_k(n)$, $p^{\alpha}\mid\sigma_k(n)$ if $p^{\alpha}\le x_2^{k-\varepsilon_x}$, *and*

$$\sum_{\substack{p\mid\varphi_k(n)\\ p>x_2^{k+\varepsilon_x}}}\frac{1}{p}<\varepsilon_x;\quad \sum_{\substack{p\mid\varphi_k(n)\\ p>x_2^{k+\varepsilon_x}}}\frac{1}{p}<\varepsilon_x,$$

β) $p^\alpha \mid f_{k+l}^{(1)}(n)$, $p^\alpha \mid f_{k+l}^{(2)}(n)$ if $p^\alpha \le x_2^{k+l-\varepsilon_x}$,
and

$$\sum_{\substack{p\mid f_{k+l}^{(1)}(n)\\ p>x_x^{k+l+\varepsilon_x}}} \frac{1}{p} < \varepsilon_x; \quad \sum_{\substack{p\mid f_{k+l}^{(2)}(n)\\ p>x_2^{k+l+\varepsilon_x}}} \frac{1}{p} < \varepsilon_x,$$

where $\varepsilon_x \to 0$. *Here* $\delta_x \to 0$.

The same assertions hold if n *runs over the set of shifted primes, i.e. dropping no more than* $\delta_x \mathrm{li}\, x$ *integers* $p+a \le x$ (a fix, $a \ne 0$), *for the other* $p+a$ *the relations* α), β) *hold true.*

Remark. Theorem 14. α) for $k=1$ is due to Erdős [11], for arbitrary k is given in [12]. The proof of β), can be proved similarly. One can use the method using in the papers [13], [15], [16].

From Theorem 13, 14 and from Erdős–Wintner theorem (see in [5]) we can deduce several generalizations of the theorem of De Koninck and Luca [16].

Examples.

1. The function

$$\nu_k(n) = \frac{\varphi_k(n)}{n} \cdot (k-1)!(\log\log\log n)^{k-1} \cdot e^{(k-1)\gamma}$$

has a limit distribution, which is the same as the limit distribution of $\dfrac{\varphi(n)}{n}$.

2. The function

$$\mu_k(n) = \frac{\sigma_k(n)}{n} \frac{(\log\log\log n)^{-(k-1)}}{(k-1)!} e^{-(k-1)\gamma}$$

is distributed in limit as $\dfrac{\sigma(n)}{n}$.

3. The function

$\nu_k(p+a)$ is distributed in limit as $\dfrac{\varphi(p+a)}{p+a}$;

$\mu_k(p+a)$ is distributed in limit as $\dfrac{\sigma(p+a)}{p+a}$.

Here $a \ne 0$, p runs over the set of primes.

4. The function

$$\rho_{k,l}^{(1)}(n) := \frac{f_{k,l}^{(1)}(n)}{n}(\log\log\log n)^{l-1-k}\frac{(l-1)!}{l(l+1)\dots(l+k-1)}e^{l-1-k}\gamma$$

is distributed in limit as $\frac{\varphi(n)}{n}$;

the function

$$\rho_{k,l}^{(2)}(n) = \frac{f_{k,l}^{(2)}(n)}{n}\cdot\frac{l(l+1)\dots(l+k-1)}{(l-1)!}e^{(k-l+1)\gamma}\cdot(\log\log\log n)^{k-l+1}$$

is distributed in limit as $\frac{\sigma(n)}{n}$.

5. Let $a \neq 0$, fixed integer. The functions

$$\rho_{k,l}^{(1)}(p+a); \qquad \rho_{k,l}^{(2)}(p+a)$$

are distributed in limit as $\frac{\varphi(p+a)}{p+a}$, $\frac{\sigma(p+a)}{p+a}$ respectively. Here p runs over the set of primes

2. Lemmata

We shall use Selberg's sieve theorem as it is formulated in Elliott ([4], Chapter 2, Lemma 2.1).

Lemma 1. *Let a_n $(n = 1,\dots,N)$ be integers, $f(n) \geq 0$. Let $r > 0$, and $p_1 < p_2 < \dots < p_s \leq r$ be rational primes. Set $Q = p_1 \dots p_s$. If $d \mid Q$ then let*

$$\sum_{\substack{n=1\\ a_n\equiv 0 \pmod d}}^{N} f(n) = \eta(d)X + R(N,d),$$

where X, R are real numbers, $X \geq 0$, and $\eta(d_1 d_2) = \eta(d_1)\cdot\eta(d_2)$ whenever d_1 and d_2 are coprime divisors of Q.
Assume that for each prime p, $0 \leq \eta(p) < 1$. Let

$$I(N,Q) := \sum_{\substack{n=1\\ (a_n,Q)=1}}^{N} f(n).$$

Then the estimate

$$I(N,Q)=\{1+2\Theta_1 H\}\times\prod_{p|Q}(1-\eta(p))+2\Theta_2\sum_{\substack{d|Q\\ d\le z^3}}3^{\omega(d)}|R(N,d)|$$

holds uniformly for $r\ge 2$, $\max(\log r,S)\le\frac{1}{8}\log z$, *where* $|\Theta_1|\le 1$, $|\Theta_2|\le 1$ *and*

$$H=\exp\left(-\frac{\log z}{\log r}\left\{\log\left(\frac{\log z}{S}\right)-\log\log\left(\frac{\log z}{S}\right)-\frac{2S}{\log z}\right\}\right),$$

$$S=\sum_{p|Q}\frac{\eta(p)}{1-\eta(p)}\log p.$$

The next lemma can be found in Halberstam and Richert [5], Corollary 2.4.1.

Lemma 2. *Let* k *be a positive integer,* l,a,b *be nonzero integers,* $k\le x$. *Then*

$$\#\{p\le x\mid p\equiv l\pmod k,\ \ ap+b\in\mathcal{P},\ p\in\mathcal{P}\}\le$$
$$\le c\prod_{p|kab}\left(1-\frac{1}{p}\right)^{-1}\cdot\frac{x}{\varphi(k)\log^2\frac{x}{k}},$$

where c *is an absolute constant.*

Lemma 3 (E. Bombieri and A.I. Vinogradov). *For fixed* $A>0$, *there exists* $B=B(A)>0$ *such that*

$$\sum_{k\le\frac{\sqrt{x}}{x_1^B}}\max_{(l,k)=1}\max_{2\le y\le x}\left|\pi(y,k,l)-\frac{li\,y}{\varphi(k)}\right|\ll\frac{x}{x_1^A}.$$

For a proof see [4].

Lemma 4. *Let* f *be a multiplicative non-negative function which for suitable* A *and* B *satisfies*

(i) $$\sum_{p\le y}f(p)\log p\le Ay\quad(y\ge 0),$$

(ii) $$\sup_p\sum_{\nu\ge 2}\frac{f(p^\nu)}{p^\nu}\log p^\nu\le B.$$

Then, for $x > 1$,

$$\sum_{n\le x} f(n) \le (A+B+1)\frac{x}{x_1}\sum_{n\le x}\frac{f(n)}{n}.$$

This assertion is Theorem 5 in Tenenbaum [4], Part III. Chapter 5.

Lemma 5. *We have for $l = 1, 2,\ 1 \le k \le x$*

$$\sum_{\substack{p\le x\\ p\equiv l (\mathrm{mod}\, k)}} \frac{1}{p} \le c\frac{x_2}{\varphi(k)}.$$

(See [5].)

Lemma 6 (Frechet and Shohat [9]). *Let $F_n(u)$ $(n = 1, 2, \dots)$ be a sequence of distribution functions. For each non-negative integer l let*

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} u^l \,\mathrm{d}F_n(u)$$

exist. Then there exists a subsequence $F_{n_k}(u)$, $n_1 < n_2 < \dots$ which converges weakly to the limiting distribution $F(u)$ satisfying

$$\alpha_l = \int_{-\infty}^{\infty} u^l \,\mathrm{d}F(u), \quad (l = 0, 1, \dots).$$

Moreover, if the sequence of moments α_l determines $F(u)$ uniquely, then the sequence $F_n(u)$ converges to $F(u)$ weakly.

Lemma 7 (Siegel and Walfisz). *We have*

$$\pi(x, k, l) = \frac{li\, x}{\varphi(k)}\left(1 + O\left(e^{-c\sqrt{x_1}}\right)\right)$$

uniformly as $(k, l) = 1,\ 1 \le k \le x_1^A$. A is an arbitrary constant.

(See in [4].)

Lemma 8 (Brun–Titchmarsh). *We have*

$$\pi(x, k, l) \le \frac{cx}{\varphi(k)\log x/k},$$

if $1 \le k < x,\ (k, l) = 1$. c is an absolute constant.

(See in [18].)

Lemma 9 (Barban, Linnik and Tshudakov [10]). *Let q be an odd prime. Then*

$$\pi(x, q^r, l) = \frac{li\, x}{\varphi(q^r)}\left(1 + O\left(e^{-c\sqrt{x_1}}\right)\right)$$

uniformly as $(l, q) = 1,\ q^r \le x^{1/3}$.

3. Proof of Theorem 10 and Theorem 11

First we prove the relation (1.27). Let

$$\delta_q(n) = \sum_{p_1p_2|n} f_q(p_1, p_2), \tag{3.1}$$

$$\Delta_Y^*(n) = \sum_{Y<q\le x_2^2}\ \sum_{p_1p_2|n} f_q(p_1, p_2). \tag{3.2}$$

We observe that

$$\begin{aligned}\#\{n \le x \mid \Delta_{x_2}^2(n) \ne 0\} &\le \sum_{n\le x} \Delta_{x_2^2}(n) \le \\ &\le \sum_{q\ge x_2^2}\ \sum_{\substack{p_1p_2\le x\\ p_j\equiv 1(q)}} \left[\frac{x}{p_1p_2}\right] \le cx x_2^2 \sum_{q\ge x_2^2} \frac{1}{q^2} = O\left(\frac{x}{x_3}\right).\end{aligned} \tag{3.3}$$

Let $r \ge 1$, and

$$\tau_r(n) = \Delta_{Y_x}^*(n)\left(\Delta_{Y_x}^*(n) - 1\right)\dots\left(\Delta_{Y_x}^*(n) - (r-1)\right). \tag{3.4}$$

If $z_1, z_2, \dots, z_M \in \{0, 1\}$, then

$$\sum_{i_1<i_2<\dots<i_r} z_{i_1}z_{i_2}\dots z_{i_r} = \frac{T(T-1)\dots(T-(r-1))}{r!}, \tag{3.5}$$

$$T = z_1 + z_2 + \dots + z_m. \tag{3.6}$$

The relation (3.5) can be proved by using induction on r.

We can write

$$\tau_r(n) = \sum_{\substack{\pi_j,\pi_j',q_j \\ \pi_j\pi_j'|n}} \prod_{j=1}^{r} f_{q_j}(\pi_j,\pi_j'), \tag{3.7}$$

where $\pi_j,\pi_j',q_j \in \mathcal{P},\ q_j \in [Y_x, x_2^2]$.

Let $\tau_r(n) = \tau_r^{(1)}(n)+\tau_r^{(2)}(n)$, where in $\tau_r^{(1)}(n)$ we sum over those π_j,π_j' ($j = 1,\dots,r$) for which $\{\pi_u,\pi_u'\} \cap \{\pi_v,\pi_v'\} = \emptyset$ if $u \neq v$, and in $\tau_r^{(2)}(n)$ we sum over the others.

We have

$$\sum\nolimits_2 := \sum_{n\le x} \tau_r^{(2)}(n) \le \sum_{q_j,\pi_j,\pi_j'}^{*} \left[\frac{x}{LCM(\pi_1,\pi_1',\dots,\pi_r,\pi_r')}\right]$$

where $*$ indicates that no more than $(2r-1)$ distinct primes occur among $\pi_1,\pi_1',\dots,\pi_r,\pi_r'$.

By using Lemma 3 we obtain that

$$\frac{1}{x}\sum\nolimits_2 \ll x_2^{2r-1}\left\{\sum_{q>Y_x}\frac{1}{q^2}\right\}^r \ll x_2^{2r-1}\cdot\frac{1}{(Y_x\log Y_x)^r} = o_x(1).$$

Let

$$\sum\nolimits_1 := \sum_{n\le x} \tau_r^{(1)}(n).$$

Then

$$\sum\nolimits_1 = \sum_{\substack{\pi_j,\pi_j',q_j \\ \pi_j<\pi_j'}} \left[\frac{x}{\pi_1\pi_1'\cdots\pi_r\pi_r'}\right], \tag{3.8}$$

where in the right hand side $\pi_1,\pi_1'\dots,\pi_r,\pi_r'$ are distinct primes $q_j|\pi_j-1$, $q_j|\pi_j'-1$ and $q_j\in[Y_x,x_2^2]$.

By using our method in [1] one can obtain that

$$\frac{1}{x}\sum\nolimits_1 = (1+o_x(1))\frac{1}{2^r}\sum_{p_1p_2\cdots p_{2r}\le x}\frac{1}{p_1p_2\cdots p_{2r}}\left\{\sum_{Y_x\le q\le x_2^2}\frac{1}{(q-1)^2}\right\}^r. \tag{3.9}$$

Since

$$\sum_{Y_x\le q\le x_2^2}\frac{1}{(q-1)^2} = (1+o_x(1))\frac{1}{Y_x\log Y_x} = (1+o_x(1))\frac{2\alpha}{x_2^2},$$

and

$$\sum_{p_1\cdots p_{2r}\le x}\frac{1}{p_1\cdots p_{2r}}=(1+o_x(1))x_2^{2r},$$

we obtain that

$$\frac{1}{x}\sum\nolimits_1=(1+o_x(1))\alpha^r, \tag{3.10}$$

and so

$$\frac{1}{x}\sum_{n\le x}\tau_r(n)=(1+o_x(1))\alpha^r \quad (x\to\infty)$$

uniformly as $\alpha=\alpha_x\in[c_1,c_2],\ 0<c_1<c_2<\infty$.

By the Frechet–Shohat theorem and that $\frac{\alpha^r}{r!}$ are the factorial moments of the Poisson-distribution, furthermore taking into consideration (1.26), we obtain (1.27).

The proof of (1.28) is similar, somewhat more complicated.

Let $r\ge 1$ be fixed. Count those primes $p\le x$ for which there exists such a couple of primes $\pi<\pi'$ for which $\pi\pi'\mid p-1$ and $\pi\equiv 1 \pmod q$, $\pi'\equiv 1 \pmod q$, $q>Y_x$, furthermore $\pi'>x^{1/4r}$. We shall apply Lemma 2. We write $p-1$ as $a\pi\pi'$. Let a,π,q be fixed, $\pi\equiv 1 \pmod q$. Since $\pi'>x^{1/4r}$, therefore $a\pi<x^{1-1/4r}$. We have

$$\#\{p\le x\mid p-1=a\pi\pi';\ p,\pi'\in\mathcal{P},\ p'\equiv 1 \pmod q\}\le c\frac{x}{a\pi q\log^2\frac{x}{a\pi q}}.$$

Let us sum over $q<x^{1/8r}$, a, $\pi\equiv 1 \pmod q$. Since $a\pi q\le x^{1-1/8r}$, therefore this sum is

$$\le\sum_{q\ge Y_x}\frac{c(\operatorname{li}x)x_2}{q^2}=o_x(1)\operatorname{li}x.$$

The contribution of those π,π' for which $q\ge x^{1/8r}$ is

$$\le\sum_{q\ge x^{1/8r}}\sum_{\pi\pi'\le x}\left[\frac{x}{\pi\pi'}\right]\le xx_2^2\sum_{q\ge x^{1/8r}}\frac{1}{q^2}=o(\operatorname{li}x).$$

Let

$$\tilde\Delta_Y(n)=\sum_{Y<q}\sum_{\substack{p_1p_2\mid n\\ p_1<p_2<x^{1/4r}}}f_q(p_1,p_2).$$

By using the Brun–Titchmarsh inequality (Lemma 8), we obtain that

$$\frac{1}{\operatorname{li}x}\#\{p\le x\mid\tilde\Delta_{x_2^2}(p-1)\ne 0\}=o(\operatorname{li}x).$$

Let $\tilde{\Delta}^*_Y(n) = \tilde{\Delta}_Y(n) - \tilde{\Delta}_{x_2^2}(n)$, and

$$\tilde{\tau}_r(n) = \tilde{\Delta}^*_Y(n)(\tilde{\Delta}^*_Y(n) - 1) \cdots (\tilde{\Delta}^*_Y(n) - (r-1)). \tag{3.11}$$

Let $\tilde{\tau}_r(n) = \tilde{\tau}_r^{(1)}(n) + \tilde{\tau}_r^{(2)}(n)$. Arguing as earlier, we deduce that

$$\sum_{p \le x} \tau_r^{(2)}(p-1) = o(\operatorname{li} x),$$

and that

$$\sum_{p \le x} \tau_r^{(1)}(p-1) = \sum_{\substack{\pi_j, \pi_j', q_j \\ \pi_j < \pi_j' < x^{1/8r}}} \pi(x, \pi_1\pi_1' \ldots \pi_r\pi_r', 1).$$

By using the Bombieri–Vinogradov theorem (Lemma 3) we can continue the proof as we did in the proof of (1.27).

Now we prove Theorem 11.

It is clear that $\Delta_{Y_x}(n) \ge \kappa_{Y_x}(n)$. It is enough to prove that

$$x^{-1}\#\{n \le x \mid \kappa_{Y_x}(n) \ne \Delta_{Y_x}(n)\} \to 0 \quad (x \to \infty), \tag{3.12}$$

and that

$$\frac{1}{\operatorname{li} x}\#\{p \le x \mid \kappa_{Y_x}(p-1) \ne \Delta_{Y_x}(p-1)\} \to 0 \quad (x \to \infty). \tag{3.13}$$

If $\kappa_{Y_x}(n) \ne \Delta_{Y_x}(n)$, then there exists $q > Y_x$ and $\pi_1 < \pi_2 < \pi_3$, $\pi_j \in \mathcal{P}$, $q \mid \pi_j - 1$ $(j = 1,2,3)$ such that $\pi_1\pi_2\pi_3 \mid n$. Thus (3.12) is less than

$$\sum_{q > Y} \sum_{\substack{\pi_1\pi_2\pi_3 \\ q \mid \pi_j - 1}} \frac{x}{\pi_1\pi_2\pi_3} \ll x \cdot x_2^3 \sum_{q > Y_x} \frac{1}{q^3} \ll \frac{x x_2^3}{Y_x^2 \log Y_x} = o(x).$$

(3.14) can be proved similarly. We have to overestimate the size of those $p \le x$ for which there exists $q > Y_x$ and primes $\pi_1 < \pi_2 < \pi_3$ such that $\pi_1\pi_2\pi_3 \mid p-1$, and $q \mid \pi_j - 1$ $(j = 1,2,3)$.

We can drop the contribution of those primes $p \le x$ for which $q > x_2^2$, say. Now we may assume that $q \le x_2^2$. By using the Brun–Titchmarsh inequality, we can drop also the contribution of those primes p for which $\pi_1\pi_2\pi_3 < x^{1-\delta}$, where δ is a fixed positive constant. It remains the case when $p - 1 = a\pi_1\pi_2\pi_3$, $\pi_1\pi_2\pi_3 \ge x^{1-\delta}$, $\pi_j \equiv 1 \pmod q$, $\pi_j \in [Y_x, x_2^2]$. From Lemma 5 we obtain that the number of these primes is $o(\operatorname{li} x)$.

4. Proof of Theorem 12

Let $e(n)=\dfrac{\varphi(n)}{n}$, $\log\dfrac{1}{e(n)}=t(n)=\sum\limits_{q\mid n}\log\dfrac{1}{1-\frac{1}{q}}$.

Let $\delta_x\to 0$ slowly, $t(n)=t_1(n)+t_2(n)+t_3(n)+t_4(n)$ where

$$t_1(n)=\sum_{\substack{q\mid n\\ q<x_2^{1-\delta_x}}} t(q);\quad t_2(n)=\sum_{\substack{x_2^{1-\delta_x}<q<x_2^{1+\delta_x}\\ q\mid n}} t(q),$$

$$t_3(n)=\sum_{\substack{q\mid n\\ x_2^{1+\delta}<q<x_1}} t(q);\quad t_4(n)=\sum_{\substack{q>x_1\\ q\mid n}} t(q).$$

It is clear that $\max\limits_{n\le x}\ t_2(n)=o_x(1)$, $\max\limits_{n\le x}\ t_4(n)=o_x(1)$.

By using sieve theorems one can prove that for all but $o(\mathrm{li}\,x)$ of primes $p\le x$, $q\mid\varphi(p-1)$ holds for all $q<x_2^{1-\delta_x}$, if $\delta_x\to 0$ sufficiently slowly. This implies that $t_1(p-1)=t_1(p-1-\varphi(p-1))$ for all but $o(\pi(x))$ of primes $p\le x$.

Since

$$(4.1)\quad \sum_{p\le x} t_3(p-1)\ll\sum_{x_1\ge q>x_2^{1+\delta_x}}(1/q)\ \pi(x,q,1)\ll \mathrm{li}\,x\cdot\sum_{q>x_2^{1+\delta_x}}1/q^2=o(\mathrm{li}\,x)$$

we obtain that $t_3(p-1)=o_x(1)$ holds for all but $o(\pi(x))$ primes $p\le x$.

Now we shall prove that $t_3(p-1-\varphi(p-1))=o_x(1)$ holds for all but $o(\pi(x))$ of primes $p\le x$.

Let us write each $p-1$ as Qm, where Q is the largest prime factor of $p-1$. The size of those $p\le x$ for which $P(p-1)<x^{\delta_x}$, or $P(p-1)>x^{1-\delta_x}$ is $o(\mathrm{li}\,x)$. This is wellknown, easy consequence of sieve theorems. We shall drop all these primes. Starting from (4.1) it is enough to prove that

$$\sum_{\substack{p\le x\\ P(p-1)\in\left[x^{\delta_x},x^{1-\delta_x}\right]}} t_3^*(p-1-\varphi(p-1))=o(\mathrm{li}\,x),$$

where

$$t_3^*(p-1-\varphi(p-1))=\sum_{\substack{p-1-\varphi(p-1)\equiv 0(q)\\ q\nmid p-1}}1/q.$$

Let $Q \in \mathcal{P},\ S_Q = \{p \le x,\ p-1 = Qm,\ P(p-1) = Q\}$. Observe that if $p-1 = Qm$, $q \mid p-1-\varphi(p-1)$, then $Q(m-\varphi(m)) + \varphi(m) \equiv 0 \pmod q$. If $q \mid m - \varphi(m)$, then the above equation has a solution Q only if $q \mid \varphi(m)$, and so if $q \mid m$. Such kind of q's are excluded in t_3^*.

Hence

$$\sum := \sum_{P(p-1)\in[x^{\delta_x}, x^{1-\delta_x}]} t_3^*(p-1-\varphi(p-1)) \ll$$

$$\ll \sum_{x_1 \ge q > x_1^{1+\delta_x}} \frac{1}{q} \sum_{\substack{m \le x^{1-\delta_x} \\ q \nmid m}} \#\{Q \in \mathcal{P}, Qm \le$$

$$\le x, Q(m-\varphi(m)) + \varphi(m) \equiv O(q)\} \ll$$

$$\ll \sum_{x_2^{1+\delta_x} \le q < x_1} \frac{1}{q} \sum_{\substack{m \le x_2^{1-\delta_x} \\ q \nmid m}} \#\{p, Q \in \mathcal{P},\ p = Qm+1,\ Q(m-\varphi(m)) +$$

$$+ \varphi(m) \equiv 0 \pmod q\}.$$

Let us apply Lemma 1 with substituting in it $x \to \frac{x}{n}$, $p \to Q$, $k \to q$. We have

$$\sum \ll \sum_{x_2^{1+\delta_x} < q < x_1} \frac{1}{q^2} \sum_{m < x^{1-\delta_x}} \frac{x}{m \log^2 \frac{x}{mq}}.$$

The right hand side is clearly $o(\operatorname{li} x)$.

We are almost ready. Let $e_j(n) := e^{t_j(n)}$. Then $e(n) = e_1(n)e_2(n)e_3(n)e_4(n)$. We have to consider

$$u_{p-1} := e(p-1-\varphi(p-1)) - e(p-1).$$

We proved that $e_j(p-1-\varphi(p-1)) = 1 + o_x(1)$, $e_j(p-1) = 1 + o_x(1)$ hold for all but $o(\operatorname{li} x)$ primes $p \le x$, for $j = 2, 3, 4$, and claimed that $e_1(p-1) = e_1(p-1-\varphi(p-1))$ is satisfied for all but $o(\operatorname{li} x)$ primes $p \le x$.

The proof of the theorem is completed. ■

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FUNCTIONAL EQUATIONS RELATED TO HOMOGRAPHIC FUNCTIONS

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Dedicated to the sixtieth birthday of Professor Antal Járai

Abstract. A functional equation in two variables related to homographic functions is introduced. The solutions are established with the aid of some results on functional equations in a single variable. A conjecture on a general solution is presented.

1. Introduction

We consider the functional equation

$$\frac{\alpha\left(\frac{3x+y}{4}\right)-\alpha\left(x\right)}{\alpha\left(\frac{x+y}{2}\right)-\alpha\left(x\right)}\left(3-2\frac{\alpha\left(\frac{x+y}{2}\right)-\alpha\left(x\right)}{\alpha\left(y\right)-\alpha\left(x\right)}\right)=1,$$

in two variables where the unknown function α is continuous and strictly monotonic in a real interval. It is easy to verify that any homographic function is a solution. In section 2 we present some motivation. In section 3 we show that this equation is a consequence of a more complicated functional equation in three variables $(*)$ appearing in connection with the problem of existence of discontinuous Jensen affine functions in the sense of Beckenbach with respect

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to the two parameter family of functions $\{b\alpha + c : b, c \in \mathbb{R}\}$, and related to the invariance of double ratios of four points.

In section 4, applying an M. Laczkovich theorem [4], we prove that if a continuous function satisfies this equation in any interval (a_0, ∞) then it is a homographic function.

In section 5, assuming some local regularity conditions, we consider some related functional equations in a single variable. A possible application of the celebrated regularity theorems of A. Járai [1] is mentioned.

2. Some motivations

In order to present a problem leading to the considered equation, take a continuous and strictly monotonic function α defined on an interval I and consider a two parameter family of functions defined by

$$\mathcal{F}_\alpha := \{b\alpha + c : a, b \in \mathbb{R}\}.$$

The family $\mathcal{F}_\alpha$ has the property: for every two points $(x_1, y_1), (x_2, y_2) \in I \times \mathbb{R}$, $x_1 \neq x_2$, there is a unique function $b\alpha + c$ in $\mathcal{F}_\alpha$ such that

$$b\alpha(x_1) + c = y_1, \qquad b\alpha(x_2) + c = y_2;$$

more precisely, the real numbers

$$b = \frac{y_1 - y_2}{\alpha(x_1) - \alpha(x_2)}, \qquad c = \frac{\alpha(x_1)y_2 - \alpha(x_2)y_1}{\alpha(x_1) - \alpha(x_2)}$$

are uniquely determined. Following a more general idea due to Beckenbach, we say that a function $f : I \to \mathbb{R}$ is *convex with respect the family* $\mathcal{F}_\alpha$, briefly, $\mathcal{F}_\alpha$-*convex*, if for all $x_1, x_2 \in I$, $x_1 < x_2$, we have

$$f(x) \leq b\alpha(x) + c, \qquad x_1 < x < x_2,$$

where

$$b = \frac{f(x_1) - f(x_2)}{\alpha(x_1) - \alpha(x_2)}, \qquad c = \frac{\alpha(x_1)f(x_2) - \alpha(x_2)f(x_1)}{\alpha(x_1) - \alpha(x_2)},$$

$\mathcal{F}_\alpha$-*concave*, if the reversed inequality is satisfied, and $\mathcal{F}_\alpha$-*affine* if it is both $\mathcal{F}_\alpha$-convex and $\mathcal{F}_\alpha$-concave.

Note that a function f is $\mathcal{F}_\alpha$-affine iff $f \in \mathcal{F}_\alpha$.

Adopting the idea of Jensen, we say that a function $f : I \to \mathbb{R}$ is *Jensen $\mathcal{F}_\alpha$-convex* if, for all $x_1, x_2 \in I$,

$$f\left(\frac{x_1+x_2}{2}\right) \le b\alpha\left(\frac{x_1+x_2}{2}\right) + c,$$

where b and c are given by the above formula; *Jensen $\mathcal{F}_\alpha$-concave* if the reverse inequality is satisfied, and *Jensen $\mathcal{F}_\alpha$-affine* if it is both *Jensen $\mathcal{F}_\alpha$-convex* and *Jensen $\mathcal{F}_\alpha$-concave,* that is if, for all $x_1, x_2 \in I$,

$$f\left(\frac{x_1+x_2}{2}\right) = \frac{f(x_1)-f(x_2)}{\alpha(x_1)-\alpha(x_2)}\alpha\left(\frac{x_1+x_2}{2}\right) + \frac{\alpha(x_1)f(x_2)-\alpha(x_2)f(x_1)}{\alpha(x_1)-\alpha(x_2)}$$

or, equivalently

$$f\left(\frac{x_1+x_2}{2}\right) = \frac{\alpha\left(\frac{x_1+x_2}{2}\right)-\alpha(x_2)}{\alpha(x_1)-\alpha(x_2)} f(x_1) + \frac{\alpha(x_1)-\alpha\left(\frac{x_1+x_2}{2}\right)}{\alpha(x_1)-\alpha(x_2)} f(x_2).$$

For $\alpha := \mathrm{id}|_I$ one gets the classical notions of convex, concave, affine and Jensen convex, Jensen concave and Jensen affine functions. It is known since Hamel that there are discontinuous Jensen affine functions and that every Jensen affine function $f : I \to \mathbb{R}$ is of the form $f(x) = A(x)+a$, $x \in I$, where A is an additive function and $a \in \mathbb{R}$ which, in general, does not belong to $\mathcal{F}_\alpha$. In this context a natural question arises: determine all functions $\alpha : I \to \mathbb{R}$ which admit the discontinuous Jensen $\mathcal{F}_\alpha$-affine functions.

In [7] it was shown that this problem leads to the following, quite complicated, functional equation of three variables

$$(*)\qquad \begin{aligned} &\frac{\alpha\left(\frac{x+2y+z}{4}\right)-\alpha(y)}{\alpha\left(\frac{x+z}{2}\right)-\alpha(y)} \cdot \frac{\alpha\left(\frac{x+z}{2}\right)-\alpha(z)}{\alpha(x)-\alpha(z)} = \\ &= \frac{\alpha\left(\frac{x+2y+z}{4}\right)-\alpha\left(\frac{y+z}{2}\right)}{\alpha\left(\frac{x+y}{2}\right)-\alpha\left(\frac{y+z}{2}\right)} \cdot \frac{\alpha\left(\frac{x+y}{2}\right)-\alpha(y)}{\alpha(x)-\alpha(y)} \end{aligned}$$

for all $x, y, z \in I$, $(x+z-2y)(x-z)(x-y) \neq 0$.

Note that this equation can be written as the equality of the following two double ratios:

$$\begin{aligned} &\frac{\alpha\left(\frac{x+2y+z}{4}\right)-\alpha(y)}{\alpha\left(\frac{x+z}{2}\right)-\alpha(y)} : \frac{\alpha\left(\frac{x+2y+z}{4}\right)-\alpha\left(\frac{y+z}{2}\right)}{\alpha\left(\frac{x+y}{2}\right)-\alpha\left(\frac{y+z}{2}\right)} = \\ &= \frac{\alpha\left(\frac{x+y}{2}\right)-\alpha(y)}{\alpha(x)-\alpha(y)} : \frac{\alpha\left(\frac{x+z}{2}\right)-\alpha(z)}{\alpha(x)-\alpha(z)}. \end{aligned}$$

Taking into account that for all admissible $x, y, z \in I$,

$$\frac{\frac{x+2y+z}{4} - y}{\frac{x+z}{2} - y} : \frac{\frac{x+2y+z}{4} - \frac{y+z}{2}}{\frac{x+y}{2} - \frac{y+z}{2}} = \frac{1}{4} = \frac{\frac{x+y}{2} - y}{x-y} : \frac{\frac{x+z}{2} - z}{x-z}$$

we conclude that any homographic function α satisfies equation $(*)$.

In [7] it was proved that a continuous and monotonic function satisfies $(*)$ if, and only if α is any homographic function. This fact implies that a family $\mathcal{F}_\alpha$ admits discontinuous Jensen affine functions in the Beckenbach sense iff α is a homographic function. In [7], as an application, an answer to a more general question posed by Zs. Páles [8] is given.

3. A functional equation related to equation $(*)$

We prove the following

Theorem 1. *Let $I \subset \mathbb{R}$ be an interval. If a continuous function $\alpha : I \to \mathbb{R}$ satisfies equation $(*)$, then it is strictly monotonic and*

$$\frac{\alpha\left(\frac{3x+y}{4}\right) - \alpha(x)}{\alpha\left(\frac{x+y}{2}\right) - \alpha(x)} \left(3 - 2\frac{\alpha\left(\frac{x+y}{2}\right) - \alpha(x)}{\alpha(y) - \alpha(x)}\right) = 1, \qquad x, y \in I,\ x \neq y. \tag{1}$$

Proof. Equation $(*)$ implies that α is one-to-one. The continuity of α implies that it is strictly monotonic. By the continuity of α, letting $x \to y$ in $(*)$, we infer that, for every $y \in I$, the limit

$$\varphi(y) := \lim_{x \to y} \frac{\alpha\left(\frac{x+y}{2}\right) - \alpha(y)}{\alpha(x) - \alpha(y)} \tag{2}$$

exists and, for all $y \neq z$,

$$\frac{\alpha\left(\frac{3y+z}{4}\right) - \alpha(y)}{\alpha\left(\frac{y+z}{2}\right) - \alpha(y)} \frac{\alpha\left(\frac{y+z}{2}\right) - \alpha(z)}{\alpha(y) - \alpha(z)} = \frac{\alpha\left(\frac{3y+z}{4}\right) - \alpha\left(\frac{y+z}{2}\right)}{\alpha(y) - \alpha\left(\frac{y+z}{2}\right)} \varphi(y). \tag{3}$$

Similarly, letting $y \to x$ in $(*)$, we infer that, for every $x \in I$, the limit

$$\psi(x) := \lim_{y \to x} \frac{\alpha\left(\frac{x+y}{2}\right) - \alpha(y)}{\alpha(x) - \alpha(y)} \tag{4}$$

exists and, for all $x \neq z$,

$$\frac{\alpha\left(\frac{3x+z}{4}\right)-\alpha\left(x\right)}{\alpha\left(\frac{x+z}{2}\right)-\alpha\left(x\right)}\frac{\alpha\left(\frac{x+z}{2}\right)-\alpha\left(z\right)}{\alpha\left(x\right)-\alpha\left(z\right)}=\frac{\alpha\left(\frac{3x+z}{4}\right)-\alpha\left(\frac{x+z}{2}\right)}{\alpha\left(x\right)-\alpha\left(\frac{x+z}{2}\right)}\psi(x).$$

Thus

$$\psi=\varphi. \tag{5}$$

Hence, letting $x \to z$ in $(*)$, making use of the definitions of φ and ψ and the identity

$$\alpha\left(\frac{x+2y+z}{4}\right)=\alpha\left(\frac{\frac{x+y}{2}+\frac{y+z}{2}}{2}\right)$$

we get

$$\frac{\alpha\left(\frac{y+z}{2}\right)-\alpha\left(y\right)}{\alpha\left(z\right)-\alpha\left(y\right)}\varphi(z)=\varphi\left(\frac{y+z}{2}\right)\frac{\alpha\left(\frac{z+y}{2}\right)-\alpha\left(y\right)}{\alpha\left(z\right)-\alpha\left(y\right)},$$

for $y \neq z$, whence

$$\varphi(z)=\varphi\left(\frac{y+z}{2}\right), \qquad y\neq z,$$

and, consequently, φ is a constant function in I.

Letting $x \to y$ in the identity

$$\frac{\alpha\left(\frac{x+y}{2}\right)-\alpha\left(y\right)}{\alpha\left(x\right)-\alpha\left(y\right)}+\frac{\alpha(x)-\alpha\left(\frac{x+y}{2}\right)}{\alpha\left(x\right)-\alpha\left(y\right)}=1$$

and making use of (2), (4) we get $\varphi+\psi=1$, whence by (5),

$$\varphi=\frac{1}{2}.$$

Now, from (3), we get

$$\frac{\alpha\left(\frac{3y+z}{4}\right)-\alpha\left(y\right)}{\alpha\left(\frac{y+z}{2}\right)-\alpha\left(y\right)}\frac{\alpha\left(\frac{y+z}{2}\right)-\alpha\left(z\right)}{\alpha\left(y\right)-\alpha\left(z\right)}=\frac{1}{2}\frac{\alpha\left(\frac{3y+z}{4}\right)-\alpha\left(\frac{y+z}{2}\right)}{\alpha\left(y\right)-\alpha\left(\frac{y+z}{2}\right)}$$

for $y \neq z$. Since

$$\frac{\alpha\left(\frac{3y+z}{4}\right)-\alpha\left(\frac{y+z}{2}\right)}{\alpha\left(y\right)-\alpha\left(\frac{y+z}{2}\right)}=1-\frac{\alpha\left(\frac{3y+z}{4}\right)-\alpha\left(y\right)}{\alpha\left(\frac{y+z}{2}\right)-\alpha(y)}$$

we get

$$\frac{\alpha\left(\frac{3y+z}{4}\right)-\alpha\left(y\right)}{\alpha\left(\frac{y+z}{2}\right)-\alpha\left(y\right)}\frac{\alpha\left(\frac{y+z}{2}\right)-\alpha\left(z\right)}{\alpha\left(y\right)-\alpha\left(z\right)}=\frac{1}{2}\left(1-\frac{\alpha\left(\frac{3y+z}{4}\right)-\alpha\left(y\right)}{\alpha\left(\frac{y+z}{2}\right)-\alpha(y)}\right)$$

that is, for $y \neq z$,

$$\frac{\alpha\left(\frac{3y+z}{4}\right)-\alpha(y)}{\alpha\left(\frac{y+z}{2}\right)-\alpha(y)}\left(\frac{\alpha\left(\frac{y+z}{2}\right)-\alpha(z)}{\alpha(y)-\alpha(z)}+\frac{1}{2}\right)=\frac{1}{2}.$$

Since

$$\frac{\alpha\left(\frac{y+z}{2}\right)-\alpha(z)}{\alpha(y)-\alpha(z)}=1-\frac{\alpha(y)-\alpha\left(\frac{y+z}{2}\right)}{\alpha(y)-\alpha(z)}$$

we get, for all $y, z \in I$, $y \neq z$,

$$\frac{\alpha\left(\frac{3y+z}{4}\right)-\alpha(y)}{\alpha\left(\frac{y+z}{2}\right)-\alpha(y)}\left(\frac{3}{2}-\frac{\alpha\left(\frac{y+z}{2}\right)-\alpha(y)}{\alpha(z)-\alpha(y)}\right)=\frac{1}{2}\,,$$

which was to be shown. ■

Remark 1. Let $A:\mathbb{R}\to\mathbb{R}$ be an arbitrary additive function and $a,b,c,d\in$ $\in\mathbb{R}$ be such that $ad-bc\neq 0$. Then it is easy to check that the function α given by

$$\alpha(x):=\frac{aA(x)+b}{cA(x)+d}$$

is a solution of equation (1) (as well as of equation $(*)$).

Remark 2. A function $\alpha : I\to\mathbb{R}$ satisfies equation (1) iff so does the function $h\circ\alpha$, where h is an arbitrary nonconstant homographic function.

Remark 3. Let $k,m,p,q\in\mathbb{R}$, $kp\neq 0$ be arbitrarily fixed. A function $\alpha : I\to\mathbb{R}$ satisfies equation (1) iff the function $\beta(x)=k\alpha(px+q)+m$ satisfies equation (1) with α replaced by β and the interval I replaced by $J:=\{x\in\mathbb{R}: px+q\in I\}$.

Remark 4. Interchanging x and y in (1) and then eliminating $\alpha\left(\frac{y+z}{2}\right)$ from both equations we obtain the functional equation

$$[\alpha(x)-\alpha(x)]\left[\alpha\left(\frac{3x+y}{4}\right)-\alpha\left(\frac{x+3y}{4}\right)\right]=$$

$$=8\left[\alpha(y)-\alpha\left(\frac{3x+y}{4}\right)\right]\left[\alpha\left(\frac{3x+y}{4}\right)-\alpha(x)\right],$$

$$x,y\in I,$$

which can be written in the form

$$8\alpha(x)\alpha(y)+\alpha(x)\alpha\left(\frac{3x+y}{4}\right)+\alpha(y)\alpha\left(\frac{x+3y}{4}\right)+$$

$$+8\alpha\left(\frac{3x+y}{4}\right)\alpha\left(\frac{x+3y}{4}\right)=9\alpha(x)\alpha\left(\frac{x+3y}{4}\right)+9\alpha(y)\alpha\left(\frac{3x+y}{4}\right),$$

whence

$$\frac{8\alpha(x)\,\alpha(y)}{\alpha\left(\frac{x+3y}{4}\right)\alpha\left(\frac{3x+y}{4}\right)}+\frac{\alpha(x)-9\alpha(y)}{\alpha\left(\frac{x+3y}{4}\right)}+\frac{\alpha(y)-9\alpha(x)}{\alpha\left(\frac{3x+y}{4}\right)}+8=0.$$

Remark 5. Interchanging x and y in (1) we obtain the simultaneous system of functional equations

$$\alpha\left(\frac{3x+y}{4}\right)=\frac{\alpha\left(\frac{x+y}{2}\right)[3\alpha(x)-\alpha(y)]-2\alpha(x)\,\alpha(y)}{2\alpha\left(\frac{x+y}{2}\right)+\alpha(x)-3\alpha(y)}$$

$$\alpha\left(\frac{x+3y}{4}\right)=\frac{\alpha\left(\frac{x+y}{2}\right)[3\alpha(y)-\alpha(x)]-2\alpha(x)\,\alpha(y)}{2\alpha\left(\frac{x+y}{2}\right)+\alpha(y)-3\alpha(x)},$$

which can be iterated.

4. Main result

In this section we need the following result which is a special case of M. Laczkovich theorem [4].

Lemma 1. (M. Laczkovich [4]) *Let p, q, A, B be positive and such that $\frac{\log p}{\log q}$ is irrational. If λ_1, λ_2 are the roots of the equation*

$$Ap^{\lambda}+Bq^{\lambda}=1$$

then every nonnegative measurable solution $f:(0,\infty)\to(0,\infty)$ of the functional equation

$$f(x)=Af(px)+Bf(qx),\qquad x>0,$$

is of the form

$$f(x)=rx^{\lambda_1}+sx^{\lambda_2},\qquad x>0.$$

Remark 6. If $A+B=1$ then the condition of positivity of the solution can be replaced by a weaker condition of the boundedness below.

Lemma 2. *Let p, A be positive numbers and $p<1$. If for some $\delta>0$, a function $f:(0,\infty)\to\mathbb{R}$ is strictly increasing and positive in an interval $(0,\delta)$ and satisfies the functional equation*

$$f(x)=(1+A)f(px)-Af(p^2x),\qquad x>0,$$

then f is positive in $(0,\infty)$.

Proof. Suppose that f satisfies the assumptions of the lemma. Putting $\varphi(x) := f(x) - f(px)$ for $x > 0$ we get

$$\varphi(x) = f(x) - f(px) = A\left[f(px) - f(p^2x)\right] = A\varphi(px),$$

whence, by induction,

$$\varphi(x) = A^n\varphi(p^nx), \qquad n \in \mathbb{N}, \ \ x > 0.$$

Take an arbitrary $x > 0$. Since $p < 1$, there is an $n_0 \in \mathbb{N}$ such that $p^nx \in (0, \delta)$ for all $n \in \mathbb{N}$, $n \geq n_0$. Since f is increasing in $(0, \delta)$, we get

$$\varphi(x) = A^n\varphi(p^nx) = A^n\left[f(p^nx) - f(p^{n+1}x)\right] > 0, \qquad n \geq n_0,$$

whence

$$\varphi(x) > 0, \qquad x > 0,$$

and, consequently,

$$f(x) > f(px), \qquad x > 0.$$

Hence, by induction,

$$f(x) > f(p^nx), \qquad x > 0, \ \ n \in \mathbb{N}.$$

Since f is strictly increasing and positive in $(0, \delta)$, letting $n \to \infty$ we get $f(x) > 0$ for all $x > 0$ which was to be shown. ∎

The main result reads as follows.

Theorem 2. *Let $a_0 \in \mathbb{R}$ be fixed. A continuous function $\alpha : (a_0, \infty) \to \mathbb{R}$ satisfies equation* (1) *if and only if, α is a homographic function, i.e.*

$$\alpha(x) = \frac{ax + b}{cx + d}, \qquad x > a_0,$$

for some $a, b, c, d \in \mathbb{R}$, $ad \neq bc$.

Proof. Suppose that a continuous function $\alpha : (a_0, \infty) \to \mathbb{R}$ satisfies equation (1). By (1) it must be strictly monotonic in (a_0, ∞). Without loss of generality we can assume that α is strictly increasing. Take arbitrary $x_0 > 0$ and define $\beta : (0, \infty) \to \mathbb{R}$, $\beta(x) := \alpha(x + x_0) - \alpha(x_0)$. Of course β is continuous, strictly increasing, $\beta(0) = 0$ and, by Remarks 2 and 3, β satisfies equation (1), that is

$$\frac{\beta\left(\frac{3x+y}{4}\right) - \beta(x)}{\beta\left(\frac{x+y}{2}\right) - \beta(x)}\left(3 - 2\frac{\beta\left(\frac{x+y}{2}\right) - \beta(x)}{\beta(y) - \beta(x)}\right) = 1, \qquad x, y > 0, \ x \neq y.$$

Setting $y = 0$ we get

$$\frac{\beta\left(\frac{3x}{4}\right) - \beta(x)}{\beta\left(\frac{x}{2}\right) - \beta(x)} \left(3 - 2\frac{\beta\left(\frac{x}{2}\right) - \beta(x)}{-\beta(x)}\right) = 1, \qquad x > 0,$$

which, after simple calculation, can be written in the equivalent form

$$\frac{3}{\beta\left(\frac{3x}{4}\right)} = \frac{1}{\beta\left(\frac{x}{2}\right)} + \frac{2}{\beta(x)}, \qquad x > 0.$$

It follows that the function $f : (0, \infty) \to (0, \infty)$,

$$f(x) := \frac{1}{\beta(x)}, \qquad x > 0,$$

is decreasing and satisfies the functional equation

$$f(x) = \tfrac{1}{3} f\left(\tfrac{2}{3}x\right) + \tfrac{2}{3} f\left(\tfrac{4}{3}x\right), \qquad x > 0.$$

Put $p = \frac{2}{3}$, $q = \frac{4}{3}$, $A = \frac{1}{3}$, $B = \frac{2}{3}$. Note that $\frac{\log p}{\log q}$ is irrational and the only solutions of the equation $Ap^\lambda + Bq^\lambda = 1$, that is

$$\frac{1}{3}\left(\frac{2}{3}\right)^\lambda + \frac{2}{3}\left(\frac{4}{3}\right)^\lambda = 1$$

are the numbers $\lambda_1 = 0$ and $\lambda_2 = -1$. By Lemma 1 there are $r, s \in \mathbb{R}$, such that

$$f(x) = rx^0 + sx^{-1} = r + \frac{s}{x}, \qquad x > 0.$$

Thus, by the definition of f,

$$\beta(x) = \frac{1}{f(x)} = \frac{x}{rx + s}, \qquad x > 0,$$

where, obviously, $s \neq 0$. Now the definition of β implies that

$$\alpha(x + x_0) = \alpha(x_0) + \frac{x}{rx + s}, \qquad x > 0.$$

It follows that α is a homographic function in the interval (x_0, ∞), i.e.

$$\alpha(x) = \frac{ax + b}{cx + d}, \qquad x > x_0,$$

for some $a, b, c, d \in \mathbb{R}$, $ad \neq bc$. Since $x_0 > a_0$ is arbitrarily chosen, the proof is completed. ■

5. Some related functional equations

Assume that a one-to-one function α satisfies equation (1) in the interval I. Take an $x_0 \in I$ and define a function β by

$$\beta(x) = \alpha(x + x_0) - \alpha(x_0), \qquad x \in J := I - x_0. \tag{6}$$

In view of Remark 3 the function β satisfies equation (1) in the interval J, i.e.

$$\frac{\beta\left(\frac{3x+y}{4}\right) - \beta(x)}{\beta\left(\frac{x+y}{2}\right) - \beta(x)}\left(3 - 2\frac{\beta\left(\frac{x+y}{2}\right) - \beta(x)}{\beta(y) - \beta(x)}\right) = 1, \qquad x, y \in J,\ x \neq y. \tag{7}$$

Since $\beta(0) = 0$, setting here $x = 0$ and then replacing y by x we get

$$\frac{\beta\left(\frac{x}{4}\right)}{\beta\left(\frac{x}{2}\right)}\left(3 - 2\frac{\beta\left(\frac{x}{2}\right)}{\beta(x)}\right) = 1, \qquad x \in J,\ x \neq 0.$$

It follows that $\varphi : J \to \mathbb{R}$ defined by

$$\varphi(x) := \frac{\beta\left(\frac{x}{2}\right)}{\beta(x)}\ , \qquad x \neq 0, \tag{8}$$

satisfies the functional equation

$$\varphi\left(\frac{x}{2}\right)[3 - 2\varphi(x)] = 1, \qquad x \in J,\ x \neq 0.$$

If the limit $\eta := \lim_{x\to 0} \varphi(x)$ exists then, obviously, $\eta \neq 0$. Setting $\varphi(0) := \eta$, we see that φ satisfies the functional equation

$$\varphi(x) = \frac{3}{2} - \frac{1}{2\varphi\left(\frac{x}{2}\right)}, \qquad x \in J. \tag{9}$$

Theorem 3. *Let $J \subset \mathbb{R}$ be an interval such that $0 \in J$. Suppose that $\varphi : J \to \mathbb{R}$ satisfies equation* (9)*. Then either $\varphi(0) = 1$ or $\varphi(0) = \frac{1}{2}$. Moreover,*

1. *if $\varphi(0) = 1$ and*
$$\varphi(x) = 1 + 0(x), \qquad x \to 0,$$
then φ satisfies (9) *iff $\varphi \equiv 1$ in J;*

2. *if $\varphi(0) = \frac{1}{2}$ and, for some $p \in \mathbb{R}$,*
$$\varphi(x) = \frac{1}{2} + px + 0(x^2), \qquad x \to 0,$$

then φ satisfies (9) *iff*

$$\varphi(x) = \frac{4px+1}{4px+2}, \qquad x \in J. \tag{10}$$

Proof. Setting $x = 0$ in (9) we get $\eta = \frac{3}{2} - \frac{1}{2\eta}$ for $\eta := \varphi(0)$, whence either $\eta = 1$ or $\eta = \frac{1}{2}$.

Putting $f(x) = \frac{x}{2}$ for $x \in J$ and $H(y) := \frac{3}{2} - \frac{1}{2y}$ for all $y \in \mathbb{R}$ we can write equation (9) in the form

$$\varphi(x) = H(\varphi[f(x)]), \qquad x \in J.$$

In the case when $\eta = 1$ we have $H'(\eta) = \frac{1}{2}$, whence, by the continuity of H' at the point $\eta = 1$ we infer that there exists a $\theta \in [\frac{1}{2}, 1)$ and $\delta > 0$ such that

$$|H(y_1) - H(y_2)| \le \theta\, |y_1 - y_2| \tag{11}$$

for all $y \in (\eta - \delta, \eta + \delta)$. Since $0 \le f(x) \le sx$ for all $x \in J$ with $s = \frac{1}{2}$ and $s\theta < 1$, by applying a general uniqueness theorem [5, Theorem 1] (cf. also [4], p. 200-201), we conclude that there exists at most one continuous solution φ such that $\varphi(0) = 1$. Since the constant function $\varphi \equiv 1$ satisfies equation (9), the first part of the theorem is proved.

In the case when $\eta = \frac{1}{2}$ we have $H'(\eta) = 2$. By the continuity of H' there exists $\theta \in [2, 4)$ and $\delta > 0$ such that (11) is fulfilled for all $y \in (\eta - \delta, \eta + \delta)$ and $s^2\theta = \frac{1}{2} < 1$. Since the function (10) is a solution of (9) and

$$\varphi(x) = \frac{1}{2} + px - \frac{4px^2}{4px+1} = \frac{1}{2} + px + 0(x^2), \qquad x \to 0,$$

the uniqueness of φ follows from the already cited theorem in [5]. This completes the proof. ■

Now applying this result we prove

Theorem 4. *Let $I \subset \mathbb{R}$ be an interval. Suppose that the function $\alpha : I \to \mathbb{R}$ satisfies equation* (1). *If for some $x_0 \in I$ there exists the limit*

$$\eta := \lim_{x \to 0} \frac{\alpha\left(\frac{x}{2} + x_0\right) - \alpha(x_0)}{\alpha\left(x + x_0\right) - \alpha(x_0)},$$

then $\eta = \frac{1}{2}$. If moreover, for some $p \in \mathbb{R}$,

$$\frac{\alpha\left(\frac{x}{2} + x_0\right) - \alpha(x_0)}{\alpha\left(x + x_0\right) - \alpha(x_0)} = \frac{1}{2} + px + 0(x^2), \qquad x \to 0,$$

and α is continuous at least at one point $x_1 \in I$, $x_1 \neq x_0$, then

$$\alpha(x) = \frac{ax+b}{cx+d}, \qquad x \in I,$$

for some $a, b, c, d \in \mathbb{R}$, $ad \neq bc$.

Proof. Suppose that $\alpha : I \to \mathbb{R}$ satisfies equation (1). Take an $x_0 \in I$, put $J := I - x_0$ and define the function $\beta : J \to \mathbb{R}$ by (6). By Remark 3, β satisfies equation (7). According to what we have observed at the beginning of this section, the function φ defined by (8) satisfies equation (9) and

$$\varphi(x) := \frac{\alpha\left(\frac{x}{2}+x_0\right)-\alpha(x_0)}{\alpha\left(x+x_0\right)-\alpha(x_0)} \qquad x \in J.$$

By the first statement of Theorem 3 either $\eta = 1$ or $\eta = \frac{1}{2}$. Assume first that $\eta = 1$. Then

$$\frac{\beta\left(\frac{x}{2}\right)}{\beta\left(x\right)} = 1, \qquad x \in J,$$

would imply that β and, consequently α, would be constant function. This is a contradiction, as every function satisfying (1) must be one-to-one.

Consider the case when $\eta = \frac{1}{2}$. Now from Theorem 3 we get

$$\frac{\beta\left(\frac{x}{2}\right)}{\beta\left(x\right)} = \frac{4px+1}{4px+2}, \qquad x \in J,$$

or equivalently, setting $q := 4p$,

$$\beta\left(\frac{x}{2}\right) = \frac{qx+1}{qx+2}\beta\left(x\right), \qquad x \in J, \tag{12}$$

for some $q \in \mathbb{R}$, $q \neq 0$, which can be written in the form

$$\left(\frac{x}{2}+1\right)\beta\left(\frac{x}{2}\right) = \frac{1}{2}\left(x+1\right)\beta\left(x\right), \qquad x \in J. \tag{13}$$

Setting $y = 0$ in (1) we get

$$\frac{\beta\left(\frac{3x}{4}\right)-\beta\left(x\right)}{\beta\left(\frac{x}{2}\right)-\beta\left(x\right)}\left(3+2\frac{\beta\left(\frac{x}{2}\right)-\beta\left(x\right)}{\beta\left(x\right)}\right) = 1, \qquad x \in J,\ x \neq 0.$$

Applying here (12) we obtain

$$\frac{\beta\left(\frac{3x}{4}\right)-\beta\left(x\right)}{\frac{qx+1}{qx+2}\beta\left(x\right)-\beta\left(x\right)}\left(3+2\frac{\frac{qx+1}{qx+2}\beta\left(x\right)-\beta\left(x\right)}{\beta\left(x\right)}\right) = 1, \qquad x \in J,\ x \neq 0,$$

which reduces to the equation

$$\left(q\frac{3}{4}x+1\right)\beta\left(\frac{3}{4}x\right)=\frac{3}{4}\left(qx+1\right)\beta\left(x\right),\qquad x\in J. \tag{14}$$

By (13) and (14) the function $\gamma : J \to \mathbb{R}$ defined by

$$\gamma(x)=\left(qx+1\right)\beta\left(x\right),\qquad x\in J,$$

the simultaneous system of functional equations

$$\gamma\left(\frac{x}{2}\right)=\frac{1}{2}\gamma\left(x\right),\qquad \gamma\left(\frac{3}{4}x\right)=\frac{3}{4}\gamma\left(x\right),\qquad x\in J.$$

It is easy to show (taking into account that $\gamma(0) = 0$), that the function γ can be uniquely extended to the function satisfying this system of equations, respectively in $[0, \infty)$ or $(-\infty, 0]$ or in $\mathbb{R}$ depending on whether x_0 is the left end point of I, the right endpoint of I or the interior point of I. Assume for instance that x_0 is the left end point of I and, for convenience, denote this extension by γ. Since $(\log \frac{1}{2})/(\log \frac{3}{4})$ is irrational and γ is continuous at a point in the interval $(0, \infty)$, we infer that (cf. [6]),

$$\gamma(x)=\gamma(1)x,\qquad x\geq 0.$$

By the definition of γ we get

$$\beta(x)=\frac{\gamma(1)x}{qx+1},\qquad x\in J,$$

whence, by the definition of β we get the result. In the case when x_0 is the right end point of I the argument is analogous. In the case when x_0 is an interior point of I, then, according to the previous cases, α must be a homographic function at least at one of the intervals $I \cap [x_0, \infty)$ and $I\cap (-\infty, x_0]$. In this case equation (1) easily implies that α is a homographic function in the interval I. This completes the proof. ∎

For the discussion the question if the regularity conditions assumed in Theorems 3 and 4 can be omitted consider

Remark 7. Equation (1) is equivalent to the functional equation

$$\alpha(y)=\frac{\alpha\left(x\right)\left[3\alpha\left(\frac{x+y}{2}\right)-\alpha\left(\frac{3x+y}{4}\right)\right]-2\alpha\left(\frac{x+y}{2}\right)\alpha\left(\frac{3x+y}{4}\right)}{2\alpha\left(x\right)+\alpha\left(\frac{x+y}{2}\right)-3\alpha\left(\frac{3x+y}{4}\right)} \tag{15}$$
$$x,y\in I,\quad x\neq y.$$

Proof. Assume that α is one-to-one and satisfies equation (1). From (1), for all $x, y \in I, x \neq y$, we have

$$\alpha(y)\left[2\alpha\left(x\right)+\alpha\left(\frac{x+y}{2}\right)-3\alpha\left(\frac{3x+y}{4}\right)\right]=$$

$$=\alpha\left(x\right)\left[3\alpha\left(\frac{x+y}{2}\right)-\alpha\left(\frac{3x+y}{4}\right)\right]-2\alpha\left(\frac{x+y}{2}\right)\alpha\left(\frac{3x+y}{4}\right).$$

Suppose that $2\alpha\left(x\right)+\alpha\left(\frac{x+y}{2}\right)-3\alpha\left(\frac{3x+y}{4}\right)=0$, that is

$$\alpha\left(\frac{3x+y}{4}\right)=\frac{2}{3}\alpha\left(x\right)+\frac{1}{3}\alpha\left(\frac{x+y}{2}\right)$$

for some $x, y \in I, x \neq y$. Setting this to the right-hand side of the above equality we get $\left[\alpha\left(x\right)-\alpha\left(\frac{x+y}{2}\right)\right]^2=0$, whence $y = x$, as α is one-to-one. Thus equation (1) implies (15). The converse implication is obvious. ∎

Remark 8. Thus equation (15) is of the form

$$\alpha(y)=h\left(\alpha(x),\alpha\left(\frac{x+y}{2}\right)\alpha\left(\frac{3x+y}{4}\right)\right),$$

where

$$h\left(z_1,z_2,z_3\right)=\frac{z_1z_3-3z_1z_2+2z_2z_3}{3z_3-z_2+2z_1}$$

and one could try to employ the celebrated regularity theory due to Antal Járai [1] by the assumption that the unknown function α is monotonic, so it is a.e. differentiable. To get its differentiability one could apply Theorem 17.6 in [1], and then, to get higher regularity, Theorem 15.2. At this background a question arises if the lack of regularity of h at the points (z_1, z_2, z_3) such that $3z_3 - z_2 + 2z_1 = 0$ is a serious difficulty.

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EXPONENTIAL UNITARY DIVISORS

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Dedicated to Professor Antal Járai on his 60th birthday

Abstract. We say that d is an exponential unitary divisor of $n = p_1^{a_1} \cdots p_r^{a_r} > 1$ if $d = p_1^{b_1} \cdots p_r^{b_r}$, where b_i is a unitary divisor of a_i, i.e., $b_i \mid a_i$ and $(b_i, a_i/b_i) = 1$ for every $i \in \{1, 2, \dots, r\}$. We survey properties of related arithmetical functions and introduce the notion of exponential unitary perfect numbers.

1. Introduction

Let n be a positive integer. We recall that a positive integer d is called a unitary divisor of n if $d \mid n$ and $(d, n/d) = 1$. Notation: $d \mid_* n$. If $n > 1$ and has the prime factorization $n = p_1^{a_1} \cdots p_r^{a_r}$, then $d \mid_* n$ iff $d = p_1^{u_1} \cdots p_r^{u_r}$, where $u_i = 0$ or $u_i = a_i$ for every $i \in \{1, 2, \dots, r\}$. Also, $1 \mid_* 1$.

Furthermore, d is said to be an exponential divisor (e-divisor) of $n = p_1^{a_1} \cdots p_r^{a_r} > 1$ if $d = p_1^{e_1} \cdots p_r^{e_r}$, where $e_i \mid a_i$, for any $i \in \{1, 2, \dots, r\}$. Notation: $d \mid_e n$. By convention $1 \mid_e 1$.

Let $\tau^*(n) := \sum_{d \mid_* n} 1$, $\sigma^*(n) := \sum_{d \mid_* n} d$ and $\tau^{(e)}(n) := \sum_{d \mid_e n} 1$, $\sigma^{(e)}(n) := \sum_{d \mid_e n} d$ denote, as usual, the number and the sum of the unitary divisors

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of n and of the e-divisors of n, respectively. These functions are multiplicative and one has

$$\tau^*(n) = 2^{\omega(n)}, \quad \sigma^*(n) = (1+p_1^{a_1})\cdots(1+p_r^{a_r}), \tag{1}$$

$$\tau^{(e)}(n) = \tau(a_1)\cdots\tau(a_r), \quad \sigma^{(e)}(n) = \left(\sum_{d_1|a_1} p_1^{d_1}\right)\cdots\left(\sum_{d_r|a_r} p_r^{d_r}\right), \tag{2}$$

where $\omega(n) := \sum_{p|n} 1$ is the number of distinct prime divisors of n, and $\tau(n) :=$ $:= \sum_{d|n} 1$ stands for the number of divisors of n.

Note that if n is squarefree, then $d \,|_*\, n$ iff $d \mid n$, and $\tau^*(n) = \tau(n)$, $\sigma^*(n) = \sigma(n) := \sum_{d|n} d$.

Closely related to the concepts of unitary and exponential divisors are the unitary convolution and the exponential convolution (e-convolution) of arithmetic functions defined by

$$(f \times g)(n) = \sum_{d|_* n} f(d)g(n/d), \quad n \geq 1, \tag{3}$$

and by $(f \odot g)(1) = f(1)g(1)$,

$$(f \odot g)(n) = \sum_{b_1 c_1 = a_1} \cdots \sum_{b_r c_r = a_r} f(p_1^{b_1}\cdots p_r^{b_r})g(p_1^{c_1}\cdots p_r^{c_r}), \quad n > 1, \tag{4}$$

respectively.

The function $I(n) = 1$ $(n \geq 1)$ has inverses with respect to the unitary convolution and e-convolution given by $\mu^*(n) = (-1)^{\omega(n)}$ and $\mu^{(e)}(n) =$ $= \mu(a_1)\cdots\mu(a_r)$, $\mu^{(e)}(1) = 1$, respectively, where μ is the Möbius function. These are the unitary and exponential analogues of the Möbius function.

Unitary divisors (called block factors) and the unitary convolution (called compounding of functions) were first considered by R. Vaidyanathaswamy [23]. The current terminology was introduced by E. Cohen [1, 2]. The notions of exponential divisor and exponential convolution were first defined by M. V. Subbarao [15]. Various properties of arithmetical functions defined by unitary and exponential divisors, including the functions τ^*, σ^*, μ^*, $\tau^{(e)}$, $\sigma^{(e)}$, $\mu^{(e)}$ and properties of the convolutions (3) and (4) were investigated by several authors.

A positive integer n is said to be unitary perfect if $\sigma^*(n) = 2n$. This notion was introduced by M. V. Subbarao and L. J. Warren [16]. Until now five unitary perfect numbers are known. These are $6 = 2\cdot 3$, $60 = 2^2\cdot 3\cdot 5$, $90 = 2\cdot 3^2\cdot 5$, $87\,360 = 2^6\cdot 3\cdot 5\cdot 7\cdot 13$ and the following number of 24 digits: $146\,361\,946\,186\,458\,562\,560\,000 = 2^{18}\cdot 3\cdot 5^4\cdot 7\cdot 11\cdot 13\cdot 19\cdot 37\cdot 79\cdot 109\cdot 157\cdot 313$.

It is conjectured that there are finitely many such numbers. It is easy to see that there are no odd unitary perfect numbers.

An integer n is called exponentially perfect (e-perfect) if $\sigma^{(e)}(n) = 2n$. This originates from M. V. Subbarao [15]. The smallest e-perfect number is $36 = 2^2 \cdot 3^2$. If n is any squarefree number, then $\sigma^{(e)}(n) = n$, and $36n$ is e-perfect for any such n with $(n, 6) = 1$. Hence there are infinitely many e-perfect numbers. Also, there are no odd e-perfect numbers, cf. [14]. The squarefull e-perfect numbers under 10^{10} are: $2^2 \cdot 3^2$, $2^3 \cdot 3^2 \cdot 5^2$, $2^2 \cdot 3^3 \cdot 5^2$, $2^4 \cdot 3^2 \cdot 11^2$, $2^4 \cdot 3^3 \cdot 5^2 \cdot 11^2$, $2^6 \cdot 3^2 \cdot 7^2 \cdot 13^2$, $2^7 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 13^2$, $2^6 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 13^2$. It is not known if there are infinitely many squarefull e-perfect numbers, see [4, p. 110].

For a survey on results concerning unitary and exponential divisors we refer to the books [10] and [12]. See also the papers [3, 5, 8, 9, 11, 13, 18, 19, 20] and their references.

M.V. Subbarao [15, Section 8] says: ,,We finally remark that to every given convolution of arithmetic functions, one can define the corresponding exponential convolution and study the properties of arithmetical functions which arise therefrom. For example, one can study the exponential unitary convolution, and in fact, the exponential analogue of any Narkiewicz-type convolution, among others."

While such convolutions were investigated by several authors, cf. [7, 6], it appears that arithmetical functions corresponding to the exponential unitary convolution mentioned above were not considered in the literature.

It is the aim of this paper to overcome this shortage. Combining the notions of e-divisors and unitary divisors we consider in this paper exponential unitary divisors (e-unitary divisors). We review properties of the corresponding τ, σ, μ and Euler-type functions. It turns out that the asymptotic behavior of these functions is similar to those of the functions $\tau^{(e)}$, $\sigma^{(e)}$, $\mu^{(e)}$ and $\phi^{(e)}$ (the latter one will be given in Section 3). We define the e-unitary perfect numbers, which were not considered before, and state some open problems.

2. Exponential unitary divisors

We say that d is an exponential unitary divisor (e-unitary divisor) of $n = p_1^{a_1} \cdots p_r^{a_r} > 1$ if $d = p_1^{b_1} \cdots p_r^{b_r}$, where $b_i \mid_* a_i$, for any $i \in \{1, 2, \ldots, r\}$. Notation: $d \mid_{e*} n$. By convention $1 \mid_{e*} 1$.

For example, the e-unitary divisors of $n = p^{12}$, with p prime, are $d = p, p^3, p^4, p^{12}$, while its e-divisors are $d = p, p^2, p^3, p^4, p^6, p^{12}$.

Let $\tau^{(e)*}(n) := \sum_{d|_{e*}n} 1$ and $\sigma^{(e)*}(n) := \sum_{d|_{e*}n} d$ denote the number and the sum of the e-unitary divisors of n, respectively. It is immediate that these functions are multiplicative and we have

$$\tau^{(e)*}(n) = \tau^*(a_1)\cdots\tau^*(a_r) = 2^{\omega(a_1)+\ldots+\omega(a_r)},$$
$$(5)\qquad \sigma^{(e)*}(n) = \left(\sum_{d_1|_* a_1} p_1^{d_1}\right)\cdots\left(\sum_{d_r|_* a_r} p_r^{d_r}\right).$$

If n is e-squarefree, i.e., $n=1$ or $n>1$ and all the exponents in the prime factorization of n are squarefree, then $d\ |_{e*}\ n$ iff $d\ |_e\ n$, and $\tau^{(e)*}(n) = \tau^{(e)}(n)$, $\sigma^{(e)*}(n) = \sigma^{(e)}(n)$.

Note that for any $n>1$ the values $\tau^{(e)*}(n)$ and $\sigma^{(e)*}(n)$ are even.

The corresponding exponential unitary convolution (e-unitary convolution) is given by

$$(f \odot_* g)(1) = f(1)g(1),$$
$$(6)\qquad (f \odot_* g)(n) = \sum_{\substack{b_1c_1=a_1\\(b_1,c_1)=1}} \cdots \sum_{\substack{b_rc_r=a_r\\(b_r,c_r)=1}} f(p_1^{b_1}\cdots p_r^{b_r})g(p_1^{c_1}\cdots p_r^{c_r}),$$

with the notation $n = p_1^{a_1}\cdots p_r^{a_r} > 1$.

The arithmetical functions form a commutative semigroup under (6) with identity μ^2. A function f has an inverse with respect to the e-unitary convolution iff $f(1)\neq 0$ and $f(p_1\cdots p_k)\neq 0$ for any distinct primes $p_1,\ldots,p_k$.

The inverse of the function $I(n)=1$ $(n\geq 1)$ with respect to the e-unitary convolution is the function $\mu^{(e)*}(n) = \mu^*(a_1)\cdots\mu^*(a_r) = (-1)^{\omega(a_1)+\ldots+\omega(a_r)}$, $\mu^{(e)*}(1)=1$.

These properties of convolution (6) are special cases of those of a more general convolution, involving regular convolutions of Narkiewicz-type, mentioned in the Introduction.

Remark. It is possible to define ,,unitary exponential divisors" (in the reverse order) in the following way. An integer d is a unitary exponential divisor (unitary e-divisor) of $n = p_1^{a_1}\cdots p_r^{a_r} > 1$ if $d \mid n$ and the integers d and n/d are exponentially coprime. This means that, denoting $d = p_1^{b_1}\cdots p_r^{b_r}$, we require d and n/d to have the same prime factors as n, i.e., $1\leq b_i < a_i$, and $(b_i, a_i - b_i) = 1$ for any $i\in\{1,2,\ldots,r\}$. This is fulfilled iff n is squarefull, i.e., $a_i\geq 2$ and $(b_i,a_i)=1$ for every $i\in\{1,2,\ldots,r\}$. Hence the number of unitary e-divisors of $n>1$ is $\phi(a_1)\cdots\phi(a_r)$ (ϕ is Euler's function) or 0, depending on whether n is squarefull or not. We do not go into other details here. For exponentially coprime integers cf. [18].

3. Arithmetical functions defined by exponential unitary divisors

As noted before, the functions $\tau^{(e)*}$ and $\sigma^{(e)*}$ are multiplicative. Also, for any prime p, $\tau^{(e)*}(p) = 1$, $\tau^{(e)*}(p^2) = 2$, $\tau^{(e)*}(p^3) = 2$, $\tau^{(e)*}(p^4) = 2$, $\tau^{(e)*}(p^5) = 2$, ..., $\sigma^{(e)*}(p) = p$, $\sigma^{(e)*}(p^2) = p+p^2$, $\sigma^{(e)*}(p^3) = p+p^3$, $\sigma^{(e)*}(p^4) = = p + p^4$, $\sigma^{(e)*}(p^5) = p + p^5$, ... Observe that the first difference compared with the functions $\tau^{(e)}$ and $\sigma^{(e)}$ occurs for p^4 (which is not e-squarefree).

The function $\tau^{(e)*}(n)$ is identical with the function $t^{(e)}(n)$, defined as the number of e-squarefree e-divisors of n and investigated by L. Tóth [20]. According to [20, Th. 4],

$$\sum_{n\le x} \tau^{(e)*}(n) = C_1 x + C_2 x^{1/2} + \mathcal{O}(x^{1/4+\varepsilon}), \tag{7}$$

for every $\varepsilon > 0$, where C_1, C_2 are constants given by

$$C_1 := \prod_p \left(1 + \frac{1}{p^2} + \sum_{a=6}^{\infty} \frac{2^{\omega(a)} - 2^{\omega(a-1)}}{p^a}\right), \tag{8}$$

$$C_2 := \zeta(1/2) \prod_p \left(1 + \sum_{a=4}^{\infty} \frac{2^{\omega(a)} - 2^{\omega(a-1)} - 2^{\omega(a-2)} + 2^{\omega(a-3)}}{p^{a/2}}\right). \tag{9}$$

The error term of (7) was improved to $\mathcal{O}(x^{1/4})$ by Y.-F. S. Pétermann [11, Th. 1] showing that

$$\sum_{n=1}^{\infty} \frac{t^{(e)}(n)}{n^s} = \frac{\zeta(s)\zeta(2s)}{\zeta(4s)} H(s), \quad \operatorname{Re} s > 1, \tag{10}$$

where $H(s) = \sum_{n=1}^{\infty} \frac{h(n)}{n^s}$ is absolutely convergent for $\operatorname{Re} s > 1/6$.

For the maximal order of the function $\tau^{(e)*}$ we have

$$\limsup_{n\to\infty} \frac{\log \tau^{(e)*}(n) \log\log n}{\log n} = \frac{1}{2}\log 2, \tag{11}$$

this is proved (for $t^{(e)}(n)$) in [20, Th. 5]. (11) holds also for the function $\tau^{(e)}$ instead of $\tau^{(e)*}$, cf. [15].

For the maximal order of the function $\sigma^{(e)*}$ we have

Theorem 1.

$$\limsup_{n\to\infty} \frac{\sigma^{(e)*}(n)}{n \log\log n} = \frac{6}{\pi^2} e^{\gamma}, \tag{12}$$

where γ is Euler's constant.

Proof. This is a direct consequence of the following general result of L. Tóth and E. Wirsing [22, Cor. 1]: Let f be a nonnegative real-valued multiplicative function. Suppose that for all primes p we have $\varrho(p) := \sup_{\nu \geq 0} f(p^{\nu}) \leq$ $\leq (1-1/p)^{-1}$ and that for all primes p there is an exponent $e_p = p^{o(1)}$ such that $f(p^{e_p}) \geq 1 + 1/p$. Then

$$\limsup_{n\to\infty} \frac{f(n)}{\log\log n} = e^{\gamma} \prod_p \left(1 - \frac{1}{p}\right) \varrho(p). \tag{13}$$

Apply this for $f(n) = \sigma^{(e)*}(n)/n$. Here $f(p) = 1$, $f(p^2) = 1 + 1/p$ and for $a \geq 2$, $f(p^a) \leq \sigma^{(e)}(p^a)/p^a \leq 1 + 1/p$. Hence $\varrho(p) = 1 + 1/p$ and we can choose $e_p = 2$ for all p. ■

(12) holds also for the function $\sigma^{(e)}$ instead of $\sigma^{(e)*}$. For the function $\mu^{(e)*}$ one has:

Theorem 2. (i) *The Dirichlet series of* $\mu^{(e)*}$ *is of the form*

$$\sum_{n=1}^{\infty} \frac{\mu^{(e)*}(n)}{n^s} = \frac{\zeta(s)}{\zeta^2(2s)} W(s), \quad \operatorname{Re} s > 1, \tag{14}$$

where $W(s) := \sum_{n=1}^{\infty} \frac{w(n)}{n^s}$ *is absolutely convergent for* $\operatorname{Re} s > 1/4$.

(ii)

$$\sum_{n \leq x} \mu^{(e)*}(n) = C_3 x + \mathcal{O}(x^{1/2} \exp(-c(\log x)^{\Delta}), \tag{15}$$

where

$$C_3 := \prod_p \left(1 + \sum_{a=2}^{\infty} \frac{(-1)^{\omega(a)} - (-1)^{\omega(a-1)}}{p^a}\right), \tag{16}$$

and $\Delta = 9/25 - \varepsilon$ *for every* $\varepsilon > 0$, *where* $9/25 = 0.36$, *and* $c > 0$ *are constants*

Proof. A similar result was proved for the function $\mu^{(e)}$ in [20, Th. 2] (with the auxiliary Dirichlet series absolutely convergent for $\operatorname{Re} s > 1/5$). The same proof works out in case of $\mu^{(e)*}$. The error term can be improved assuming the Riemann hypothesis, cf. [20]. ■

The unitary analogue of Euler's arithmetical function, denoted by ϕ^* is defined as follows. Let $(k,n)_* := \max\{d \in \mathbb{N} : d \mid k, d \mid_* n\}$ and let

$$\phi^*(n) := \#\{k \in \mathbb{N} : 1 \leq k \leq n, (k,n)_* = 1\}, \tag{17}$$

which is multiplicative and $\phi^*(p^a) = p^a - 1$ for every prime power p^a $(a \geq 1)$. Why do we not consider here the greatest common unitary divisor of k and n? Because if we do so the resulting function is not multiplicative and its properties are not so close to those of Euler's function ϕ, cf. [21].

Furthermore, for $n = p_1^{a_1} \cdots p_r^{a_r} > 1$ let $\phi^{(e)}(n)$ denote the number of divisors d of n such that d and n are exponentially coprime, i.e., $d = p_1^{b_1} \cdots p_r^{b_r}$, where $1 \leq b_i \leq a_i$ and $(b_i, a_i) = 1$ for any $i \in \{1, \ldots, r\}$. By convention, let $\phi^{(e)}(1) = 1$. This is the exponential analogue of the Euler function, cf. [19]. Here $\phi^{(e)}$ is multiplicative and

$$\phi^{(e)}(n) = \phi(a_1) \cdots \phi(a_r), \quad n > 1. \tag{18}$$

We define the e-unitary Euler function in the following way: for $n = p_1^{a_1} \cdots p_r^{a_r} > 1$ let $\phi^{(e)*}(n)$ denote the number of divisors d of n such that $d = p_1^{b_1} \cdots p_r^{b_r}$, where $1 \leq b_i \leq a_i$ and $(b_i, a_i)_* = 1$ for any $i \in \{1, \ldots, r\}$. By convention, let $\phi^{(e)*}(1) = 1$. Then $\phi^{(e)*}$ is multiplicative and

$$\phi^{(e)*}(n) = \phi^*(a_1) \cdots \phi^*(a_r), \quad n > 1. \tag{19}$$

Theorem 3.

$$\sum_{n \leq x} \phi^{(e)*}(n) = C_4 x + C_5 x^{1/3} + \mathcal{O}(x^{1/4+\varepsilon}), \tag{20}$$

for every $\varepsilon > 0$, where C_4, C_5 are constants given by

$$C_4 := \prod_p \left(1 + \sum_{a=3}^{\infty} \frac{\phi^*(a) - \phi^*(a-1)}{p^a}\right), \tag{21}$$

(22)

$$C_5 := \zeta(1/3) \prod_p \left(1 + \frac{1}{p^{4/3}} + \sum_{a=5}^{\infty} \frac{\phi^*(a) - \phi^*(a-1) - \phi^*(a-3) + \phi^*(a-4)}{p^{a/3}}\right).$$

Proof. A similar result was proved for the function $\phi^{(e)}$ in [19, Th. 1], with error term $\mathcal{O}(x^{1/5+\varepsilon})$, improved to $\mathcal{O}(x^{1/5} \log x)$ by Y.-F. S. Pétermann [11, Th. 1]. The same proof works out in case of $\phi^{(e)*}$. ■

Theorem 4.

$$\limsup_{n \to \infty} \frac{\log \phi^{(e)*}(n) \log \log n}{\log n} = \frac{\log 4}{5}. \tag{23}$$

Proof. We apply the following general result given in [17]: Let F be a multiplicative function with $F(p^a) = f(a)$ for every prime power p^a, where f is positive and satisfies $f(n) = \mathcal{O}(n^\beta)$ for some fixed $\beta > 0$. Then

$$\limsup_{n\to\infty} \frac{\log F(n) \log\log n}{\log n} = \sup_{m\ge 1} \frac{\log f(m)}{m}. \tag{24}$$

Let $F(n) = \phi^{(e)*}(n)$, $f(a) = \phi^*(a)$, $L(m) = (\log f(m))/m$. Here $L(1) = L(2) = 0$, $L(3) = (\log 2)/3 \approx 0.231$, $L(4) = (\log 3)/4 \approx 0.274$, $L(5) = (\log 4)/5 \approx 0.277$, $L(6) = (\log 5)/6 \approx 0.268$, $L(7) = (\log 6)/7 \approx 0.255$, and $L(m) \le (\log m)/m \le (\log 8)/8 \approx 0.259$ for $m \ge 8$, using that $(\log m)/m$ is decreasing. This proves the result. ∎

(23) holds also for the function $\phi^{(e)}$ instead of $\phi^{(e)*}$, cf. [19].

These results show that the asymptotic behavior of the functions $\tau^{(e)*}$, $\sigma^{(e)*}$, $\mu^{(e)*}$ and $\phi^{(e)*}$ is very close to those of the functions $\tau^{(e)}$, $\sigma^{(e)}$, $\mu^{(e)}$ and $\phi^{(e)}$.

This is confirmed also by the next result.

Theorem 5.

$$\begin{aligned} &\sum_{n\le x} \frac{\tau^{(e)*}(n)}{\tau^{(e)}(n)} = \\ &= x \prod_p \left(1 + \sum_{a=4}^{\infty} \frac{2^{\omega(a)}/\tau(a) - 2^{\omega(a-1)}/\tau(a-1)}{p^a}\right) + \mathcal{O}\left(x^{1/4}\log x\right). \end{aligned} \tag{25}$$

A similar asymptotic formula, with the same error term, is valid also for the quotients $\sigma^{(e)*}(n)/\sigma^{(e)}(n)$ and $\phi^{(e)}(n)/\phi^{(e)*}(n)$ (in the reverse order for the last one).

Proof. This follows from the following general result, which may be known. Let g be a complex valued multiplicative function such that $|g(n)| \le 1$ for every $n \ge 1$ and $g(p) = g(p^2) = g(p^3) = 1$ for every prime p. Then

$$\sum_{n\le x} g(n) = x \prod_p \left(1 + \sum_{a=4}^{\infty} \frac{g(p^a) - g(p^{a-1})}{p^a}\right) + \mathcal{O}\left(x^{1/4}\log x\right). \tag{26}$$

In order to obtain (26), which is similar to [20, Th. 1], let $h = g*\mu$ in terms of the Dirichlet convolution. Then h is multiplicative, $h(p) = h(p^2) = h(p^3) = 0$, $h(p^a) = g(p^a) - g(p^{a-1})$ and $|h(p^a)| \le 2$ for every prime p and every $a \ge 4$.

Hence $|h(n)| \le \ell_4(n)2^{\omega(n)}$ for every $n \ge 1$, where $\ell_4(n)$ stands for the characteristic function of the 4-full integers. Note that

$$\ell_4(n)2^{\omega(n)} = \sum_{d^4 e = n} \tau(d)v(e), \tag{27}$$

where the function v is given by

$$\sum_{n=1}^{\infty} \frac{v(n)}{n^s} = \prod_p \left(1 + \frac{2}{p^{5s}} + \frac{2}{p^{6s}} + \frac{2}{p^{7s}} - \frac{1}{p^{8s}} - \frac{2}{p^{9s}} - \frac{2}{p^{10s}} - \frac{2}{p^{11s}}\right), \tag{28}$$

absolutely convergent for $\operatorname{Re} s > 1/5$. We obtain (26) by usual estimates, cf. the proof of [20, Th. 1]. ■

Note also, that $\mu^{(e)}(n)/\mu^{(e)*}(n) = |\mu^{(e)}(n)|$ is the characteristic function of the e-squarefree integers n. Asymptotic formulae for $|\mu^{(e)}(n)|$ were given in [24, Th. 2], [20, Th. 3].

4. Exponential unitary perfect numbers

We call an integer n exponential unitary perfect (e-unitary perfect) if $\sigma^{(e)*}(n) = 2n$.

If n is e-squarefree, then n is e-unitary perfect iff n is e-perfect. Consider the squarefull e-unitary perfect numbers. The first three such numbers given in Introduction, that is $36 = 2^2 \cdot 3^2$, $1\,800 = 2^3 \cdot 3^2 \cdot 5^2$ and $2\,700 = 2^2 \cdot 3^3 \cdot 5^2$ are e-squarefree, therefore also e-unitary perfect. It follows that there are infinitely many e-unitary perfect numbers.

The smallest number which is e-perfect but not e-unitary perfect is $17\,424 = 2^4 \cdot 3^2 \cdot 11^2$.

Theorem 6. *There are no odd e-unitary perfect numbers.*

Proof. Let $n = p_1^{a_1} \cdots p_r^{a_r}$ be an odd e-unitary perfect number. That is

$$\sigma^{(e)*}(p_1^{a_1}) \cdots \sigma^{(e)*}(p_r^{a_r}) = 2p_1^{a_1} \cdots p_r^{a_r}. \tag{29}$$

We can assume that $a_1, \ldots, a_r \ge 2$, i.e. n is squarefull (if $a_i = 1$ for an i, then $\sigma^{(e)*}(p_i) = p_i$ and we can simplify in (29) by p_i).

Now each $\sigma^{(e)*}(p_i^{a_i}) = \sum_{d \mid_* a_i} p_i^d$ is even, since the number of terms is $2^{\omega(a_i)}$, which is even.

From (29) we obtain that $r = 1$ and

$$\sigma^{(e)*}(p_1^{a_1}) = 2p_1^{a_1}. \tag{30}$$

Using that $a_1 \geq 2$ we have

$$2 = \frac{\sigma^{(e)*}(p_1^{a_1})}{p_1^{a_1}} \leq \frac{\sigma^{(e)}(p_1^{a_1})}{p_1^{a_1}} \leq 1 + \frac{1}{p_1} \leq 1 + \frac{1}{3} < 2, \tag{31}$$

which is a contradiction, and the proof is complete. ■

We state the following open problems.

Problem 1. Is there any e-unitary perfect number which is not e-squarefree, therefore not e-perfect?

Problem 2. Is there any e-unitary perfect number which is not divisible by 3?

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CONTINUOUS MAPS ON MATRICES TRANSFORMING GEOMETRIC MEAN TO ARITHMETIC MEAN

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Dedicated to Professor Antal Járai
on the occasion of his sixtieth birthday

Abstract. In this paper we determine the general form of continuous maps between the spaces of all positive definite and all self-adjoint matrices which transform geometric mean to arithmetic mean or the other way round.

In the papers [6, 7] we determined the structure of all bijective maps on the space of all positive semidefinite operators on a complex Hilbert space which preserve the geometric mean, or the harmonic mean, or the arithmetic mean of operators in the sense of Ando [1, 3]. In this short note we consider a related question. The logarithmic function is a continuous function from the set $\mathbb{R}_+$ of all positive real numbers to $\mathbb{R}$ that transforms geometric mean to arithmetic mean. Similarly, the exponential function is a continuous function from $\mathbb{R}$ to $\mathbb{R}_+$ that transforms arithmetic mean to geometric mean. Here we investigate the structure of maps between the spaces of all positive definite and all self-adjoint matrices with the analogous transformation properties.

Let us begin with the necessary definitions. For a given complex Hilbert space H, denote by $\mathcal{S}(H)$ and $\mathcal{P}(H)$ the spaces of all bounded self-adjoint and

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all bounded positive definite (i.e., invertible bounded positive semidefinite) operators on H, respectively. The geometric mean of $A, B \in \mathcal{P}(H)$ in Ando's sense is defined by

$$A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.$$

We remark that Ando defined the geometric mean for all positive semidefinite operators, but in this note we consider only positive definite operators. The most important properties of the operation $\#$ are listed below. Let A, B, C, D be positive semidefinite operators on H.

(i) If $A \leq C$ and $B \leq D$, then $A\#B \leq C\#D$.

(ii) (Transfer property) We have $S(A\#B)S^* = (SAS^*)\#(SBS^*)$ for every invertible bounded linear operator S on H.

(iii) Suppose $A_1 \geq A_2 \geq \ldots \geq 0$, $B_1 \geq B_2 \geq \ldots \geq 0$ and $A_n \to A$, $B_n \to B$ strongly. Then we have that $A_n\#B_n \to A\#B$ strongly.

(iv) $A\#B = B\#A$.

The arithmetic mean of $A, B \in \mathcal{S}(H)$ is defined in the natural way, i.e., by $(A+B)/2$. For a finite dimensional Hilbert space H, our first result describes those continuous maps from $\mathcal{P}(H)$ to $\mathcal{S}(H)$ which transform geometric mean to arithmetic mean.

Theorem 1. *Assume* $2 \leq \dim H < \infty$. *Let* $\phi : \mathcal{P}(H) \to \mathcal{S}(H)$ *be a continuous map satisfying*

$$\phi(A\#B) = \frac{\phi(A) + \phi(B)}{2} \tag{1}$$

for all $A, B \in \mathcal{P}(H)$. *Then there are* $J, K \in \mathcal{S}(H)$ *such that* ϕ *is of the form*

$$\phi(A) = (\log(\det A))J + K, \quad A \in \mathcal{P}(H).$$

Proof. Considering the map $\phi(.) - \phi(I)$ we may and do assume that $\phi(I) = 0$. Inserting $B = I$ into the equality (1) we obtain that $\phi(\sqrt{A}) = \phi(A)/2$. Moreover, we compute

$$0 = \phi(I) = \phi(A\#A^{-1}) = (1/2)(\phi(A) + \phi(A^{-1}))$$

which implies $\phi(A^{-1}) = -\phi(A)$ for every $A \in \mathcal{P}(H)$. For any $A, B, T \in \mathcal{P}(H)$, using the uniqueness of the square root in $\mathcal{P}(H)$, it is easy to check that $T = A^{-1}\#B$ holds if and only if $TAT = B$. From

$$\phi(T) = (1/2)(\phi(A^{-1}) + \phi(B)) = (1/2)(\phi(B) - \phi(A))$$

we obtain $\phi(B) = 2\phi(T) + \phi(A)$. Therefore, we have

$$\phi(TAT) = 2\phi(T) + \phi(A)$$

for any $A, T \in \mathcal{P}(H)$. Pick an arbitrary $X \in \mathcal{S}(H)$ and consider the functional $\varphi_X : A \mapsto \exp(\mathrm{tr}[\phi(A)X])$ on $\mathcal{P}(H)$. It is easy to see that $\varphi_X : \mathcal{P}(H) \to \mathbb{R}$ is a continuous function satisfying

$$\varphi_X(TAT) = \varphi_X(T)\varphi_X(A)\varphi_X(A)$$

for all $A, T \in \mathcal{P}(H)$. In [4, Theorem 2] the structure of such functions has been completely described. It follows from that result that there is a real number c_X such that $\varphi_X(A) = (\det A)^{c_X}$ $(A \in \mathcal{P}(H))$. Therefore, we have

$$\mathrm{tr}[\phi(A)X] = c_X \log(\det A)$$

for all $A \in \mathcal{P}(H)$. It follows from that formula that $c_X \in \mathbb{R}$ depends linearly on X, i.e., $X \mapsto c_X$ is a linear functional on $\mathcal{S}(H)$. By Riesz representation theorem it follows that there is a $J \in \mathcal{S}(H)$ such that $c_X = \mathrm{tr}[XJ]$ for every $X \in \mathcal{S}(H)$. Hence we obtain that

$$\mathrm{tr}[\phi(A)X] = c_X \log(\det A) = \mathrm{tr}[\log(\det A))JX]$$

holds for all $A \in \mathcal{P}(H)$ and $X \in \mathcal{S}(H)$. This gives us that

$$\phi(A) = (\log(\det A))J$$

for every $A \in \mathcal{P}(H)$ and the statement of the theorem follows. ■

Remark 1. One may ask what happens in the infinite dimensional case, i.e., when $\dim H = \infty$. The answer to that question is that ϕ is necessarily constant. In order to see this, just as above, applying the simple and apparent reduction $\phi(I) = 0$, one can follow the first part of the proof to check that for every vector $x \in H$, the continuous functional $\varphi_x : A \mapsto \exp(\langle \phi(A)x, x\rangle)$ maps $\mathcal{P}(H)$ into the set of all positive real numbers and satisfies

$$\varphi_x(TAT) = \varphi_x(T)\varphi_x(A)\varphi_x(A)$$

for all $A, T \in \mathcal{P}(H)$. Lemma in [5] states that then φ_x is necessarily identically 1. This gives us that $\langle \phi(A)x, x\rangle = 0$ for all $x \in H$ and $A \in \mathcal{P}(H)$ which implies $\phi \equiv 0$.

In our second result we consider the reverse problem. We describe the form of all continuous maps from $\mathcal{S}(H)$ to $\mathcal{P}(H)$ which transform arithmetic mean to geometric mean.

Theorem 2. *Assume* $2 \le \dim H < \infty$. *Let* $\phi : \mathcal{S}(H) \to \mathcal{P}(H)$ *be a continuous map satisfying*

$$\phi\left(\frac{A+B}{2}\right) = \phi(A)\#\phi(B) \tag{2}$$

for all $A, B \in \mathcal{S}(H)$. *Then there are a* $T \in \mathcal{P}(H)$, *a collection of mutually orthogonal rank-one projections* P_i *on* H *and a collection of self-adjoint operators* $J_i \in \mathcal{S}(H)$, $i = 1, \dots, \dim H$ *such that* ϕ *is of the form*

$$\phi(A) = T\left(\sum_{i=1}^{\dim H} (\exp(\mathrm{tr}[AJ_i]))P_i\right)T, \quad A \in \mathcal{S}(H).$$

Proof. Using the transfer property we see that considering the transformation $\phi(0)^{-1/2}\phi(.)\phi(0)^{-1/2}$ we may and hence do assume that $\phi(0) = I$. Inserting $B = 0$ into (2) we obtain $\phi(A/2) = \sqrt{\phi(A)}$. We next have

$$I = \phi(0) = \phi(A)\#\phi(-A).$$

It requires easy computation to deduce from this equality that $\phi(-A) = \phi(A)^{-1}$. Setting $T = (A + (-B))/2$ we infer

$$\begin{aligned}\phi(T) = \phi(-B)\#\phi(A) = \phi(B)^{-1}\#\phi(A)\\ = \phi(B)^{-1/2}(\phi(B)^{1/2}\phi(A)\phi(B)^{1/2})^{1/2}\phi(B)^{-1/2}.\end{aligned}$$

Multiplying both sides by $\phi(B)^{1/2}$ and taking square, we deduce

$$\phi(B)^{1/2}\phi(T)\phi(B)\phi(T)\phi(B)^{1/2} = \phi(B)^{1/2}\phi(A)\phi(B)^{1/2}.$$

Again, multiplying both sides by $\phi(B)^{-1/2}$ we obtain $\phi(T)\phi(B)\phi(T) = \phi(A) = \phi(2T + B)$. It follows that

$$\phi(T)\phi(B)\phi(T) = \phi(2T + B)$$

for every $B, T \in \mathcal{S}(H)$. Since $\phi(T)^{1/2} = \phi(T/2)$, we infer

$$\phi(T)^{1/2}\phi(B)\phi(T)^{1/2} = \phi(T + B) = \phi(B + T) = \phi(B)^{1/2}\phi(T)\phi(B)^{1/2}.$$

We learn from [2, Corollary 3] that for any $C, D \in \mathcal{P}(H)$ we have $C^{1/2}DC^{1/2} = D^{1/2}CD^{1/2}$ if and only if $CD = DC$. Therefore, it follows that the range of ϕ is commutative. Let us now identify the operators in $\mathcal{P}(H)$ with $n \times n$ matrices, where $n = \dim H$. By its commutativity, the range of ϕ is simultaneously diagonisable by some unitary matrix U. Considering the transformation $U^*\phi(.)U$ we may and do assume that $\phi(A) = \mathrm{diag}[\phi_1(A), \dots, \phi_n(A)]$

$(A \in \mathcal{S}(H))$, where ϕ_i maps $\mathcal{S}(H)$ into the set of all positive real numbers and satisfies $\phi_i((A+B)/2) = \sqrt{\phi_i(A)\phi_i(B)}$ for every $A, B \in \mathcal{S}(H)$ and $i = 1, \ldots, n$. Using continuity and $\phi(0) = I$, it is easy to see that $\log \phi_i$ is a linear functional on $\mathcal{S}(H)$. Therefore, for every $i = 1, \ldots, n$ we have $J_i \in \mathcal{S}(H)$ such that $\log(\phi_i(A)) = \mathrm{tr}[AJ_i]$ implying $\phi_i(A) = \exp(\mathrm{tr}[AJ_i])$ for all $A \in \mathcal{S}(H)$. Consequently, we obtain

$$\phi(A) = \mathrm{diag}[\exp(\mathrm{tr}[AJ_1]), \ldots, \exp(\mathrm{tr}[AH_n])]$$

for all $A \in \mathcal{S}(H)$, and the proof can be completed in an easy way. ■

Remark 2. As for the case $\dim H = \infty$, we note that for any $T \in \mathcal{P}(H)$, any collection $P_1, \ldots, P_n$ of mutually orthogonal projections with sum I and any collection $J_1, \ldots, J_n$ of self-adjoint trace-class operators on H, the formula

$$\phi(A) = T\left(\sum_{i=1}^{n} (\exp(\mathrm{tr}[AJ_i]))P_i\right)T, \quad A \in \mathcal{S}(H) \tag{3}$$

defines a continuous map from $\mathcal{S}(H)$ to $\mathcal{P}(H)$ which transforms arithmetic mean to geometric mean. With some more effort and refining the continuity assumption on the transformations, one could obtain a result which would show that a "continuous analogue" of the formula (3) (i.e., with integral in the place of the sum) describes the general form of continuous maps from $\mathcal{S}(H)$ to $\mathcal{P}(H)$ that transform arithmetic mean to geometric mean. However, we do not present the precise details here.

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ON THE SIMULTANEOUS NUMBER SYSTEMS OF GAUSSIAN INTEGERS

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Dedicated to Professor Antal Járai on his 60th anniversary

Abstract. In this paper we show that there is no simultaneous number system of Gaussian integers with the canonical digit set. Furthermore we give the construction of a new digit set by which simultaneous number systems of Gaussian integers exist.

1. Introduction

K.-H. Indlekofer, I. Kátai and P. Racskó examined in [1], for what N_1, N_2 will $(-N_1, -N_2, \mathcal{A}_c)$ be a simultaneous number system, where $2 \leq N_1 < N_2$ are rational integers and $\mathcal{A}_c = \{0, 1, \ldots, |N_1||N_2| - 1\}$. The triple $(-N_1, -N_2, \mathcal{A}_c)$ is called a simultaneous number system if there exist $a_j \in \mathcal{A}_c$ $(j = 0, 1, \ldots, n)$ for all z_1, z_2 rational integers so that

$$z_1 = \sum_{j=0}^{n} a_j (-N_1)^j, \quad z_2 = \sum_{j=0}^{n} a_j (-N_2)^j.$$

In the first part of this article we examine the case of Gaussian integers with the canonical digit set (there exist no $Z_1, Z_2 \in \mathbb{Z}[i]$ for which $(Z_1, Z_2, \mathcal{A}_c)$ is a simultaneous number system), and in the second part we give the construction

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of a new digit set by which simultaneous number systems of Gaussian integers exist.

Let Z_1 and Z_2 be Gaussian integers and let $\mathcal{A}$ be a digit set. The triple $(Z_1, Z_2, \mathcal{A})$ is called a simultaneous number system if there exist $a_j \in \mathcal{A}$ ($j = 0, 1, \dots, n$) for all $z_1, z_2 \in \mathbb{Z}[i]$ so that:

$$z_1 = \sum_{j=0}^{n} a_j Z_1^j, \quad z_2 = \sum_{j=0}^{n} a_j Z_2^j. \tag{1.1}$$

Statement 1.1. *If $(Z_1, Z_2, \mathcal{A})$ is a simultaneous number system, then $Z_1 - Z_2$ is unit.*

Proof. Let (z_1, z_2) be an ordered pair which can be written in the form (1.1). We get:

$$z_1 - z_2 = \sum_{j=1}^{n} a_j \left(Z_1^j - Z_2^j \right).$$

It is easy to see, that $Z_1 - Z_2$ is the divisor of all terms of the right hand side of the equation, so it is the divisor of the left hand side of the equation as well. If $(Z_1, Z_2, \mathcal{A})$ is a simultaneous number system, then every ordered pair (z_1, z_2) can be written in the form (1.1). This holds for $(z_1, z_1 - 1)$ as well. Hence we get that $Z_1 - Z_2$ is the divisor of 1, so it is unit. ■

Corollary 1.1. *If $(Z_1, Z_2, \mathcal{A})$ is a simultaneous number system of Gaussian integers, then $Z_1 - Z_2 \in \{\pm 1, \pm i\}$.*

2. The case of canonical digit set

Let $\mathcal{A}_c = \{0, 1, \dots, |Z_1|^2 |Z_2|^2 - 1\}$. If we would like $(Z_1, Z_2, \mathcal{A}_c)$ to be a simultaneous number system, then Z_1 and Z_2 must be of the form $A \pm i$. Otherwise not every ordered pair (x, y) could be written in the form (1.1). Considering the previous Corollary we get that $(Z_1, Z_2, \mathcal{A}_c)$ can be a simultaneous number system, only if $Z_1 = A \pm i$ and $Z_2 = Z_1 \pm 1$. Similarly to the case of number systems of the Gaussian integers we get that $(Z_1, Z_2, \mathcal{A}_c)$ is a simultaneous number system if and only if $(\overline{Z_1}, \overline{Z_2}, \mathcal{A}_c)$ is a simultaneous number system as well. Furthermore $(Z_1, Z_2, \mathcal{A}_c)$ is a simultaneous number system if and only if $(Z_2, Z_1, \mathcal{A}_c)$ is a simultaneous number system as well. Therefore it is enough to examine the case $Z_1 = A + i$ and $Z_2 = Z_1 - 1$.

Theorem 2.1. *$(Z_1, Z_2, \mathcal{A}_c)$ is not a simultaneous number system.*

Statement 2.1. *Let* $Z_1 = -A + i, \quad A \in \mathbb{Z}, \quad A > 0,\ Z_2 = Z_1 - 1,$ *and* $\mathcal{A}_c = \{0, 1, \ldots, |Z_1|^2|Z_2|^2 - 1\}$. *Then* $(Z_1, Z_2, \mathcal{A}_c)$ *is not a simultaneous number system.*

Proof of Statement 2.1. We will show that there are nontrivial periodic elements. If $a = (b, c) \in \mathbb{Z}[i] \times \mathbb{Z}[i]$ then let $d(a) \in \mathcal{A}_c$ be such that $d(a) \equiv \equiv b\,(mod\ Z_1)$ and $d(a) \equiv c\,(mod\ Z_2)$. Furthermore let $J(a) = \left(\frac{b-d(a)}{Z_1}, \frac{c-d(a)}{Z_2}\right)$.

Let $B = \{1, 3, 4, 5, 6, 10, 11, 16\}$. If $A \in B$ then the structure of periodic elements of $(-A + i, -A - 1 + i, \mathcal{A}_c)$ or at least the values of transitions are different from the other cases.

If $A = 1$ then let $p_0 = (0, 0)$ and

$$
\begin{aligned}
&p_1 = (2, 1), && p_2 = (2 + 2i, 2 + i), && p_3 = (3, 1),\\
&p_4 = (-1 - i, 0), && p_5 = (i, 0), && p_6 = (3 + 2i, 2 + i),\\
&p_7 = (4 + 2i, 3 + i), && p_8 = (-1 - 3i, -1 - i), && p_9 = (3i, 1 + i),\\
&p_{10} = (-1, 0), && p_{11} = (3 + 3i, 2 + i), && p_{12} = (2 - i, 1),\\
&p_{13} = (2 + 3i, 2 + i), && p_{14} = (5 + 2i, 3 + i), && p_{15} = (1 - i, 1),\\
&p_{16} = (2 + 4i, 2 + i), && p_{17} = (3 - i, 1), && p_{18} = (1 + 2i, 2 + i),\\
&p_{19} = (5 + 3i, 3 + i), && p_{20} = (-1 - 4i, -1 - i), && p_{21} = (2 + 6i, 3 + 2i),\\
&p_{22} = (3 - 3i, -i), && p_{23} = (1 + 4i, 3 + 2i), && p_{24} = (5 + i, 2),\\
&p_{25} = (-1 - 2i, 0), && p_{26} = (3 + 4i, 2 + i), && p_{27} = (5 + i, 3 + i),\\
&p_{28} = (-2 - 3i, -1 - i), && p_{29} = (3 + 6i, 3 + 2i), && p_{30} = (5 - i, 2),\\
&p_{31} = (-2 - i, 0), && p_{32} = (6 + 2i, 3 + i), && p_{33} = (-2 - 4i, -1 - i),\\
&p_{34} = (4i, 1 + i), && p_{35} = (2 + 4i, 3 + 2i), && p_{36} = (2 - 2i, -i).
\end{aligned}
$$

Then

$$
\begin{aligned}
&J(p_0) = p_0, && J(p_1) = p_2, && J(p_2) = p_1, && J(p_3) = p_4, && J(p_4) = p_5,\\
&J(p_5) = p_6, && J(p_6) = p_7, && J(p_7) = p_8, && J(p_8) = p_9, && J(p_9) = p_3,\\
&J(p_{10}) = p_{11}, && J(p_{11}) = p_{12}, && J(p_{12}) = p_{10}, && J(p_{13}) = p_{14}, && J(p_{14}) = p_{15},\\
&J(p_{15}) = p_{13}, && J(p_{16}) = p_{17}, && J(p_{17}) = p_{18}, && J(p_{18}) = p_{19}, && J(p_{19}) = p_{20},\\
&J(p_{20}) = p_{21}, && J(p_{21}) = p_{22}, && J(p_{22}) = p_{23}, && J(p_{23}) = p_{24}, && J(p_{24}) = p_{25},\\
&J(p_{25}) = p_{16}, && J(p_{26}) = p_{27}, && J(p_{27}) = p_{28}, && J(p_{28}) = p_{29}, && J(p_{29}) = p_{30},\\
&J(p_{30}) = p_{31}, && J(p_{31}) = p_{26}, && J(p_{32}) = p_{33}, && J(p_{33}) = p_{34}, && J(p_{34}) = p_{32},\\
&J(p_{35}) = p_{36}, && J(p_{36}) = p_{35},
\end{aligned}
$$

furthermore $d(p_0) = 0$ and

$$\begin{array}{llllll} d(p_1)=6, & d(p_2)=4, & d(p_3)=1, & d(p_4)=0, & d(p_5)=5, & d(p_6)=9, \\ d(p_7)=0, & d(p_8)=2, & d(p_9)=3, & d(p_{10})=5, & d(p_{11})=4, & d(p_{12})=1, \\ d(p_{13})=9, & d(p_{14})=5, & d(p_{15})=6, & d(p_{16})=4, & d(p_{17})=6, & d(p_{18})=9, \\ d(p_{19})=0, & d(p_{20})=7, & d(p_{21})=2, & d(p_{22})=8, & d(p_{23})=7, & d(p_{24})=2, \\ d(p_{25})=5, & d(p_{26})=9, & d(p_{27})=0, & d(p_{28})=7, & d(p_{29})=7, & d(p_{30})=2, \\ d(p_{31})=5, & d(p_{32})=0, & d(p_{33})=2, & d(p_{34})=8, & d(p_{35})=2, & d(p_{36})=8. \end{array}$$

The structure of periodic elements is shown in Figure 1.

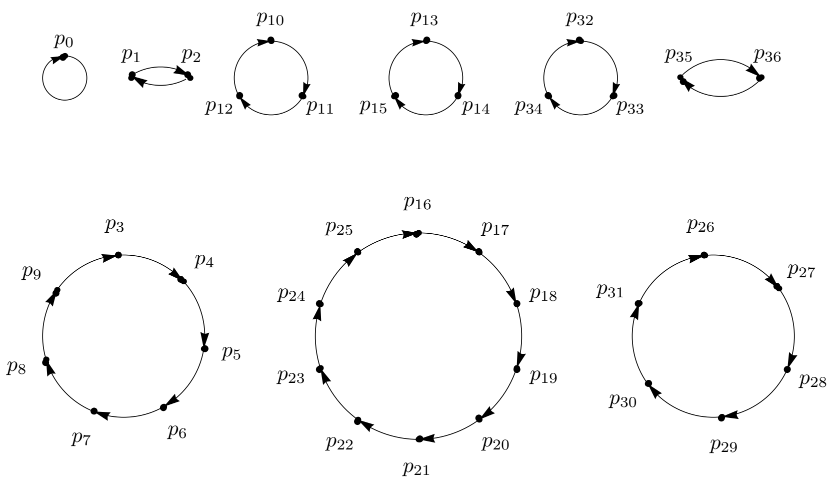


Figure 1. The structure of periodic elements of $(-1+i, -2+i, \mathcal{A}_c)$

If $A = 3$ then let $p_0 = (0,0)$ and

$$\begin{array}{lll} p_1=(9+4i,7+2i), & p_2=(43+13i,34+8i), & p_3=(-2-5i,-2i), \\ p_4=(13+6i,10+3i), & p_5=(39+11i,31+7i) & p_6=(2-3i,3-i), \\ p_7=(47+15i,37+9i), & p_8=(-6-7i,-3-3i), & p_9=(17+8i,13+4i), \\ p_{10}=(35+9i,28+6i), & p_{11}=(6-i,6), & p_{12}=(5+2i,4+i). \end{array}$$

Then

$$\begin{array}{lllll} J(p_0)=p_0, & J(p_1)=p_2, & J(p_2)=p_3, & J(p_3)=p_4, & J(p_4)=p_5, \\ J(p_5)=p_6 & J(p_6)=p_1, & J(p_7)=p_8, & J(p_8)=p_9, & J(p_9)=p_{10}, \\ J(p_{10})=p_{11}, & J(p_{11})=p_{12}, & J(p_{12})=p_7, & & \end{array}$$

furthermore $d(p_0) = 0$ and

$$d(p_1) = 151, \quad d(p_2) = 32, \quad d(p_3) = 43, \quad d(p_4) = 141, \quad d(p_5) = 42,$$
$$d(p_6) = 33, \quad d(p_7) = 22, \quad d(p_8) = 53, \quad d(p_9) = 131, \quad d(p_{10}) = 52,$$
$$d(p_{11}) = 23, \quad d(p_{12}) = 161.$$

The structure of periodic elements is shown in Figure 2.

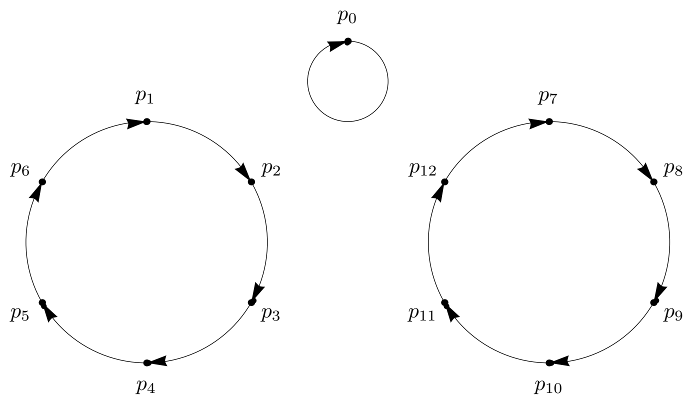


Figure 2. The structure of periodic elements of $(-3+i, -4+i, \mathcal{A}_c)$

If $A = 4$ then let

$$p_0 = (0, 0), \quad p_1 = (55 + 14i, 45 + 9i), \quad p_2 = (58 + 11i, 49 + 8i),$$
$$p_3 = (63 + 13i, 53 + 9i), \quad p_4 = (9 - i, 9).$$

Then

$$J(p_0) = p_0, \quad J(p_1) = p_2, \quad J(p_2) = p_3, \quad J(p_3) = p_4, \quad J(p_4) = p_1,$$

furthermore

$$d(p_0) = 0, \quad d(p_1) = 298, \quad d(p_2) = 323, \quad d(p_3) = 98, \quad d(p_4) = 243.$$

The structure of periodic elements is shown in Figure 3a.

If $A = 5$ then let

$$p_0 = (0, 0), \quad p_1 = (65 + 13i, 55 + 9i), \quad p_2 = (73 + 12i, 63 + 9i),$$
$$p_3 = (137 + 25i, 117 + 18i), \quad p_4 = (25, 24 + i).$$

Then

$$J(p_0) = p_0, \qquad J(p_1) = p_2, \qquad J(p_2) = p_3, \qquad J(p_3) = p_4, \qquad J(p_4) = p_1,$$

furthermore

$$d(p_0) = 0, \qquad d(p_1) = 442, \qquad d(p_2) = 783, \qquad d(p_3) = 262, \qquad d(p_4) = 363.$$

The structure of periodic elements is shown in Figure 3a.

If $A = 6$ then let

$$p_0 = (0, 0), \qquad p_1 = (91 + 13i, 80 + 10i),$$
$$p_2 = (182 + 26i, 160 + 20i), \qquad p_3 = (229 + 38i, 198 + 28i),$$
$$p_4 = (44 + i, 42 + 2i), \qquad p_5 = (139 + 25i, 119 + 18i),$$
$$p_6 = (253 + 38i, 221 + 29i), \qquad p_7 = (-28 - 11i, -20 - 7i).$$

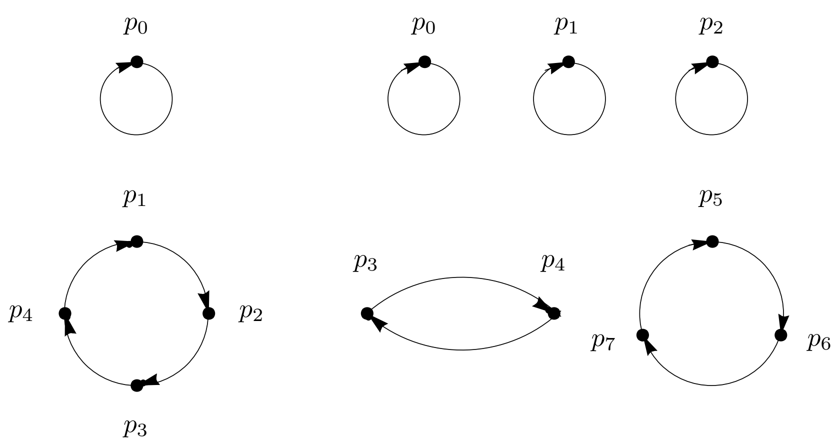
Then

$$J(p_0) = p_0, \qquad J(p_1) = p_1, \qquad J(p_2) = p_2, \qquad J(p_3) = p_4,$$
$$J(p_4) = p_3, \qquad J(p_5) = p_6, \qquad J(p_6) = p_7, \qquad J(p_7) = p_5,$$

furthermore

$$d(p_0) = 0, \qquad d(p_1) = 650, \qquad d(p_3) = 1300, \qquad d(p_3) = 494,$$
$$d(p_4) = 1456, \qquad d(p_5) = 1695, \qquad d(p_6) = 74, \qquad d(p_7) = 831.$$

The structure of periodic elements is shown in Figure 3b.



(a) $A \in \{4, 5\}$ (b) $A = 6$

Figure 3. The structure of periodic elements of $(-A + i, -A - 1 + i, \mathcal{A}_c)$

If $A = 10$ then let $p_0 = (0, 0)$ and

$$p_1 = (439 + 45i, 399 + 37i), \qquad p_2 = (375 + 33i, 345 + 28i),$$
$$p_3 = (813 + 78i, 743 + 65i), \qquad p_4 = (-32 - 11i, -23 - 8i).$$

Then

$$J(p_0) = p_0, \quad J(p_1) = p_2, \quad J(p_2) = p_3, \quad J(p_3) = p_4, \quad J(p_4) = p_1,$$

furthermore

$$d(p_0) = 0, \quad d(p_1) = 4222, \quad d(p_2) = 8583, \quad d(p_3) = 482, \quad d(p_4) = 4403.$$

The structure of periodic elements is shown in Figure 4a.

If $A = 11$ then let

$$p_0 = (0, 0), \qquad p_1 = (408 + 34i, 377 + 29i),$$
$$p_2 = (816 + 68i, 754 + 58i), \qquad p_3 = (1224 + 102i, 1131 + 87i),$$
$$p_4 = (1222 + 112i, 1121 + 94i), \qquad p_5 = (2 - 10i, 10 - 7i).$$

Then

$$J(p_0) = p_0, \quad J(p_1) = p_1, \quad J(p_2) = p_2, \quad J(p_3) = p_3, \quad J(p_4) = p_5, \quad J(p_5) = p_4,$$

furthermore

$$d(p_0) = 0, \qquad d(p_1) = 4930, \qquad d(p_2) = 9860,$$
$$d(p_3) = 14790, \qquad d(p_4) = 1234, \qquad d(p_5) = 13556.$$

The structure of periodic elements is shown in Figure 4b.

If $A = 16$ then let $p_0 = (0, 0)$ and

$$p_1 = (1105 + 65i, 1044 + 58i), \qquad p_2 = (2210 + 130i, 2088 + 116i),$$
$$p_3 = (3315 + 195i, 3132 + 174i), \qquad p_4 = (3586 + 226i, 3375 + 200i),$$
$$p_5 = (4370 + 259i, 4127 + 231i), \qquad p_6 = (-221 - 30i, -194 - 25i).$$

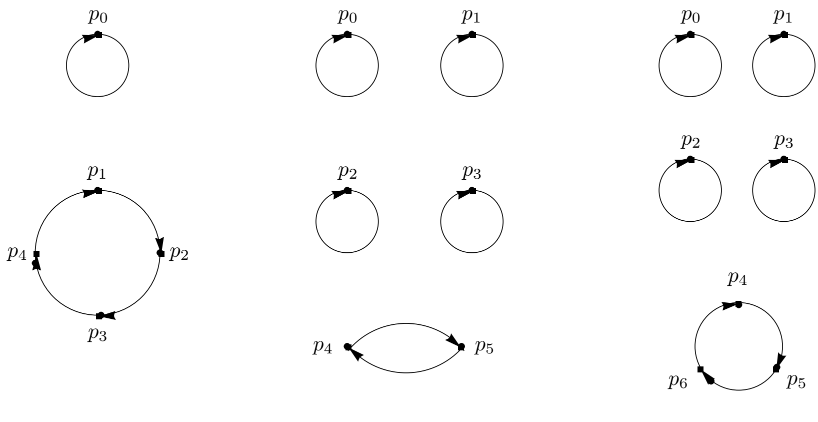
Then

$$J(p_0) = p_0, \qquad J(p_1) = p_1, \qquad J(p_2) = p_2, \qquad J(p_3) = p_3,$$
$$J(p_4) = p_5, \qquad J(p_5) = p_6, \qquad J(p_6) = p_4,$$

furthermore

$$d(p_0) = 0, \qquad d(p_1) = 18850, \qquad d(p_2) = 37700, \qquad d(p_3) = 56550,$$
$$d(p_4) = 73765, \qquad d(p_5) = 804, \qquad d(p_6) = 57381.$$

The structure of periodic elements is shown in Figure 4c.



(a) $A = 10$ (b) $A = 11$ (c) $A = 16$

Figure 4. The structure of periodic elements of $(-A+i, -A-1+i, \mathcal{A}_c)$

We can get the following connections with interpolation from examining a few examples:

Case 1. $A = 5k + 1$. Let

$$a_{11} = 25k^3 + 40k^2 + 22k + 4, \qquad b_{11} = 5k^2 + 6k + 2,$$
$$a_{21} = 25k^3 + 35k^2 + 17k + 3, \qquad b_{21} = 5k^2 + 4k + 1,$$
$$a_{12} = 50k^3 + 80k^2 + 44k + 8, \qquad b_{12} = 10k^2 + 12k + 4,$$
$$a_{22} = 50k^3 + 70k^2 + 34k + 6, \qquad b_{22} = 10k^2 + 8k + 2,$$
$$a_{13} = 75k^3 + 120k^2 + 66k + 12, \qquad b_{13} = 15k^2 + 18k + 6,$$
$$a_{23} = 75k^3 + 105k^2 + 51k + 9, \qquad b_{23} = 15k^2 + 12k + 3,$$
$$a_{14} = 100k^3 + 160k^2 + 88k + 16, \qquad b_{14} = 20k^2 + 24k + 8,$$
$$a_{24} = 100k^3 + 140k^2 + 68k + 12, \qquad b_{24} = 20k^2 + 16k + 4,$$

and

$$p_1 = (a_{11} + b_{11}i, a_{21} + b_{21}i), \qquad p_2 = (a_{12} + b_{12}i, a_{22} + b_{22}i),$$
$$p_3 = (a_{13} + b_{13}i, a_{23} + b_{23}i), \qquad p_4 = (a_{14} + b_{14}i, a_{24} + b_{24}i).$$

Then

$$J(p_1) = p_1, \qquad J(p_2) = p_2, \qquad J(p_3) = p_3, \qquad J(p_4) = p_4,$$

furthermore

$$\begin{aligned}
d(p_1) &=, 125k^4 + 250k^3 + 195k^2 + 70k + 10,\\
d(p_2) &= 250k^4 + 500k^3 + 390k^2 + 140k + 20,\\
d(p_3) &= 375k^4 + 750k^3 + 585k^2 + 210k + 30,\\
d(p_4) &= 500k^4 + 1000k^3 + 780k^2 + 280k + 40.
\end{aligned}$$

Case 2. $A = 5k + 2$. In this case $\mathcal{A}$ is not a suitable digit set since $((5k + 2)^2 + 1, (5k + 3)^2 + 1) = 5$.

Case 3. $A = 5k + 3$. Let

$$\begin{aligned}
a_{11} &= 25k^3 + 115k^2 + 132k + 46, & b_{11} &= 5k^2 + 16k + 10,\\
a_{21} &= 25k^3 + 110k^2 + 117k + 38, & b_{21} &= 5k^2 + 14k + 7,\\
a_{12} &= 100k^3 + 235k^2 + 198k + 58, & b_{12} &= 20k^2 + 34k + 16,\\
a_{22} &= 100k^3 + 215k^2 + 168k + 47, & b_{22} &= 20k^2 + 26k + 10,\\
a_{13} &= 50k^3 + 155k^2 + 154k + 50, & b_{13} &= 10k^2 + 22k + 12,\\
a_{23} &= 50k^3 + 145k^2 + 134k + 41, & b_{23} &= 10k^2 + 18k + 8,\\
a_{14} &= 75k^3 + 195k^2 + 176k + 54, & b_{14} &= 15k^2 + 28k + 14,\\
a_{24} &= 75k^3 + 180k^2 + 151k + 44, & b_{24} &= 15k^2 + 22k + 9,
\end{aligned}$$

and

$$\begin{aligned}
p_1 &= (a_{11} + b_{11}i, a_{21} + b_{21}i), & p_2 &= (a_{12} + b_{12}i, a_{22} + b_{22}i),\\
p_3 &= (a_{13} + b_{13}i, a_{23} + b_{23}i), & p_4 &= (a_{14} + b_{14}i, a_{24} + b_{24}i).
\end{aligned}$$

Then

$$J(p_1) = p_2, \qquad J(p_2) = p_1, \qquad J(p_3) = p_4, \qquad J(p_4) = p_3,$$

furthermore

$$\begin{aligned}
d(p_1) &= 500k^4 + 1500k^3 + 1830k^2 + 1050k + 236,\\
d(p_2) &= 125k^4 + 750k^3 + 1245k^2 + 840k + 206,\\
d(p_3) &= 375k^4 + 1250k^3 + 1635k^2 + 980k + 226,\\
d(p_4) &= 250k^4 + 1000k^3 + 1440k^2 + 910k + 216.
\end{aligned}$$

Case 4. $A = 5k + 4$. Let

$$a_{11} = 25k^3 + 115k^2 + 162k + 72, \qquad b_{11} = 5k^2 + 16k + 12,$$
$$a_{21} = 25k^3 + 110k^2 + 147k + 62, \qquad b_{21} = 5k^2 + 14k + 9,$$
$$a_{12} = 50k^3 + 155k^2 + 169k + 64, \qquad b_{12} = 10k^2 + 22k + 13,$$
$$a_{22} = 50k^3 + 145k^2 + 149k + 54, \qquad b_{22} = 10k^2 + 18k + 9,$$
$$a_{13} = 100k^3 + 310k^2 + 323k + 113, \qquad b_{13} = 20k^2 + 44k + 25,$$
$$a_{23} = 100k^3 + 290k^2 + 283k + 94, \qquad b_{23} = 20k^2 + 36k + 17,$$
$$a_{14} = 75k^3 + 270k^2 + 316k + 121, \qquad b_{14} = 15k^2 + 38k + 24,$$
$$a_{24} = 75k^3 + 255k^2 + 281k + 102, \qquad b_{24} = 15k^2 + 32k + 17,$$

furthermore

$$p_1 = (a_{11} + b_{11}i, a_{21} + b_{21}i), \qquad p_2 = (a_{12} + b_{12}i, a_{22} + b_{22}i),$$
$$p_3 = (a_{13} + b_{13}i, a_{23} + b_{23}i), \qquad p_4 = (a_{14} + b_{14}i, a_{24} + b_{24}i).$$

Then

$$J(p_1) = p_2, \qquad J(p_2) = p_3, \qquad J(p_3) = p_4, \qquad J(p_4) = p_1,$$

furthermore

$$d(p_1) = 250k^4 + 1000k^3 + 1590k^2 + 1180k + 341,$$
$$d(p_2) = 500k^4 + 2000k^3 + 3030k^2 + 2070k + 541,$$
$$d(p_3) = 375k^4 + 1750k^3 + 2985k^2 + 2230k + 621,$$
$$d(p_4) = 125k^4 + 750k^3 + 1545k^2 + 1340k + 421.$$

Case 5. $A = 5k$. Let

$$a_{11} = 25k^3 + 40k^2 + 22k + 3, \qquad b_{11} = 5k^2 + 6k + 2,$$
$$a_{21} = 25k^3 + 35k^2 + 17k + 2, \qquad b_{21} = 5k^2 + 4k + 1,$$
$$a_{12} = 75k^3 + 45k^2 + 16k + 2, \qquad b_{12} = 15k^2 + 8k + 2,$$
$$a_{22} = 75k^3 + 30k^2 + 11k + 2, \qquad b_{22} = 15k^2 + 2k + 1,$$
$$a_{13} = 100k^3 + 85k^2 + 23k + 2, \qquad b_{13} = 20k^2 + 14k + 3,$$
$$a_{23} = 100k^3 + 65k^2 + 13k + 2, \qquad b_{23} = 20k^2 + 6k + 1,$$
$$a_{14} = 50k^3 + 80k^2 + 29k + 3, \qquad b_{14} = 10k^2 + 12k + 3,$$
$$a_{24} = 50k^3 + 70k^2 + 19k + 2, \qquad b_{24} = 10k^2 + 8k + 1,$$

furthermore

$$p_1 = (a_{11} + b_{11}i, a_{21} + b_{21}i), \qquad p_2 = (a_{12} + b_{12}i, a_{22} + b_{22}i),$$
$$p_3 = (a_{13} + b_{13}i, a_{23} + b_{23}i), \qquad p_4 = (a_{14} + b_{14}i, a_{24} + b_{24}i).$$

Then

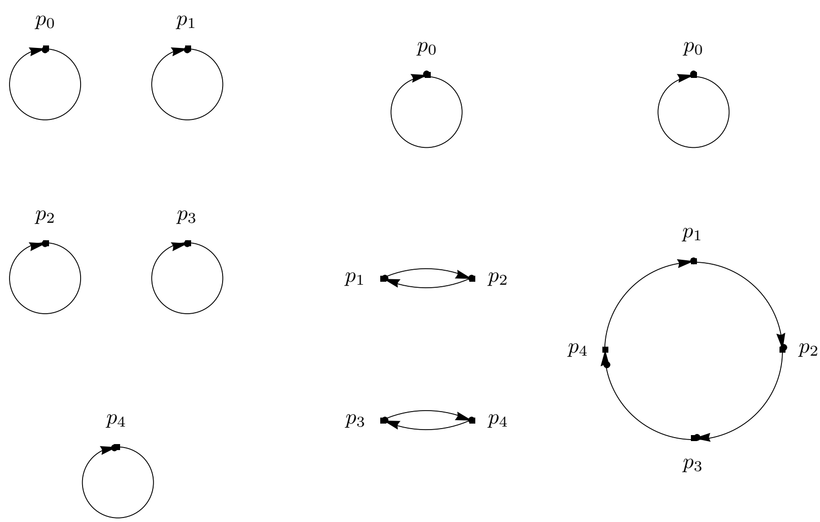
$$J(p_1) = p_2, \qquad J(p_2) = p_3, \qquad J(p_3) = p_4, \qquad J(p_4) = p_1,$$

furthermore

$$\begin{aligned} d(p_1) &= 375k^4 + 250k^3 + 135k^2 + 40k + 5, \\ d(p_2) &= 500k^4 + 500k^3 + 180k^2 + 40k + 5, \\ d(p_3) &= 250k^4 + 500k^3 + 240k^2 + 50k + 5, \\ d(p_4) &= 125k^4 + 250k^3 + 195k^2 + 50k + 5. \end{aligned}$$

The statements can be verified by simple calculations.

The structure of periodic elements is shown in Figure 5. ■



(a) $A \equiv 1 \pmod 5$ (b) $A \equiv 3 \pmod 5$ (c) $A \equiv 0$ or $4 \pmod 5$

Figure 5. The structure of periodic elements of $(-A + i, -A - 1 + i, \mathcal{A}_c)$, if $A \notin B$

Conjecture 2.1. *There are no periodic elements other than the enumerated ones.*

If the conjecture is true, then if $A \notin B$, then the number of nontrivial periodic elements will be 4 and their structure depends on the remainder of A divided by 5. Namely:

- 4 pieces of loops
- 2 pieces of circles with the length of 2
- 1 piece of circle with the length of 4

Statement 2.2. *Let now* $Z_1 = A + i, \quad A \in \mathbb{Z}, \quad A > 0$, $Z_2 = Z_1 + 1$, *and* $\mathcal{A}_c = \{0, 1, \ldots, |Z_1|^2|Z_2|^2 - 1\}$. *Then* $(Z_1, Z_2, \mathcal{A}_c)$ *is not a simultaneous number system.*

Proof of Statement 2.2. If $A \equiv 2\ (mod\ 5)$ then $\mathcal{A}_c$ is not a suitable digit set. Otherwise there would exist nontrivial periodic elements. Let

$$p = (-A^3 + A^2 - A + 1 + A^2 i + i, -A^3 + 2A^2 - 2A + k^2 i - 2Ai + 2i).$$

We get with simple calculations that in this case $J(p) = p$. ■

Proof of Theorem 2.1. Theorem 2.1 follows from Statement 2.1 and Statement 2.2 immediately. ■

We proved that $(Z_1, Z_2, \mathcal{A}_c)$ is not a simultaneous number system for all $Z_1, Z_2 \in \mathbb{Z}[i]$.

3. The case of the new digit set

With the help of K-type digit sets one can define such digit set by which simultaneous number systems of Gaussian integers exist.

Definition 3.1. Let $Z = a + bi$ and $t = |Z|^2$. Then let $E_\alpha^{(\varepsilon,\delta)}$ be the sets of those $d = k + li,\ k, l \in \mathbb{Z}$ for which

$$d\overline{Z} = (k + li)(a - bi) = (ka + bl) + (la - kb)i = r + si$$

satisfy the following conditions:

if $(\varepsilon, \delta) = (1, 1)$, then $r, s \in (-t/2, t/2]$,
if $(\varepsilon, \delta) = (-1, -1)$, then $r, s \in [-t/2, t/2)$,
if $(\varepsilon, \delta) = (-1, 1)$, then $r \in [-t/2, t/2), s \in (-t/2, t/2]$
if $(\varepsilon, \delta) = (1, -1)$, then $r \in (-t/2, t/2], s \in [-t/2, t/2)$.
We call the above constructed coefficient sets *K-type digit sets.*

The K-type digit set was used by G. Steidl in [2], by I. Kátai in [3] and by G. Farkas in [4], [5], [6], [7] and [8]. Now we use them to construct a new digit set by which simultaneous number systems of Gaussian integers exist.

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be K-type digit sets belonging to given $Z_1, Z_2 \in \mathbb{Z}[i]$ Gaussian integers. Define $\mathcal{A}$ in the following way:

$$\mathcal{A} := \bigcup_{a_j \in \mathcal{A}_2} (\mathcal{A}_1 + a_j Z_1).$$

Theorem 3.1. *If $Z_1, Z_2 \in \mathbb{Z}[i]$ are such, that $Z_2 = Z_1 + \varepsilon$, where $\varepsilon \in \in \{\pm 1, \pm i\}$, $\mathcal{A}$ is as defined above and $|Z_1|$ is large enough, then $(Z_1, Z_2, \mathcal{A})$ is a simultaneous number system.*

Remarks.

$$\max_{a \in \mathcal{A}_1} |a| \leq \frac{|Z_1|}{\sqrt{2}}, \qquad \max_{a \in \mathcal{A}_2} |a| \leq \frac{|Z_1| + 1}{\sqrt{2}}.$$

$$M := \max_{a \in \mathcal{A}} |a| \leq \frac{|Z_1|}{\sqrt{2}} + \frac{|Z_1| + 1}{\sqrt{2}} |Z_1| = \frac{|Z_1|}{\sqrt{2}} (|Z_1| + 2).$$

Let $L_1 := \frac{M}{|Z_1| - 1}$, $L_2 := \frac{M}{|Z_2| - 1}$ and $L := \max(L_1, L_2)$. Then

$$L \leq \frac{\frac{|Z_1|}{\sqrt{2}} (|Z_1| + 2)}{|Z_1| - 2}.$$

Lemma 3.1. *If (z_1, z_2) is a periodic element, then $|z_1| \leq L_1$ and $|z_2| \leq L_2$.*

Lemma 3.2. *If $a \in \mathbb{Z}[i]$, $|a| \leq L$, then $a \in \mathcal{A}$.*

Lemma 3.3. *If $z_1 \neq z_2$, $|z_1|, |z_2| \leq L$ and $J(z_1, z_2) = (w_1, w_2)$, then $|w_1 - -w_2| < |z_1 - z_2|$.*

Lemma 3.4. *For every $z_1, z_2 \in \mathbb{Z}[i]$ there exists $a \in \mathcal{A}$ such that $z_1 \equiv \equiv a \ (Z_1)$ and $z_2 \equiv a \ (Z_2)$.*

Proof of Lemma 3.1. The proof is similar to the proof for previous structures. ■

Proof of Lemma 3.2. $\mathcal{A}_2$ is K-type digit set. Therefore $\forall a \in \mathbb{Z}[i]$, if $|a| < \frac{|Z_2|}{2}$ then $a \in \mathcal{A}_2$. From the definition of $\mathcal{A}$ we get that if $|a| < \left(\frac{|Z_2|}{2} - 1\right)|Z_1|$ then $a \in \mathcal{A}$. Consequently we have to solve the following inequality:

$$L < \left(\frac{|Z_2|}{2} - 1\right)|Z_1|, \qquad \frac{\frac{|Z_1|}{\sqrt{2}}(|Z_1|+2)}{|Z_1|-2} < \left(\frac{|Z_2|}{2} - 1\right)|Z_1|,$$

$$\frac{|Z_1|(|Z_1|+2)}{\sqrt{2}(|Z_1|-2)} < \frac{|Z_1|-3}{2}|Z_1|, \qquad 2|Z_1| + 4 < \sqrt{2}(|Z_1|^2 - 5|Z_1| + 6),$$

$$0 < |Z_1|^2 - 7|Z_1| + 2,$$

which is true, if $|Z_1| > \frac{7}{2} + \frac{1}{2}\sqrt{41} \approx 6,7$. ∎

Proof of Lemma 3.3.

$$\left|\frac{z_1 - a}{Z_1} - \frac{z_2 - a}{Z_2}\right| = \left|\frac{z_1 - a}{Z_1} - \frac{z_2 - a}{Z_1} + \frac{z_2 - a}{Z_1} - \frac{z_2 - a}{Z_1 + \varepsilon}\right| \leq$$

$$\leq \frac{|(z_1 - a) - (z_2 - a)|}{|Z_1|} + \frac{|\varepsilon(z_2 - a)|}{|Z_1(Z_1 + \varepsilon)|} = \frac{|(z_1 - a) - (z_2 - a)|}{|Z_1|} + \frac{|z_2 - a|}{|Z_1||Z_1 + \varepsilon|} \leq$$

$$\leq \frac{|(z_1 - a) - (z_2 - a)|}{|Z_1|} + \frac{L + M}{|Z_1||Z_1 + \varepsilon|}.$$

Therefore we have to prove that if $|Z_1|$ is large enough, then

$$\frac{|(z_1 - a) - (z_2 - a)|}{|Z_1|} + \frac{L + M}{|Z_1||Z_1 + \varepsilon|} \leq |z_1 - z_2| = |(z_1 - a) - (z_2 - a)|,$$

or equivalently

$$\frac{L + M}{|Z_1||Z_1 + \varepsilon|} \leq |(z_1 - a) - (z_2 - a)|\left(1 - \frac{1}{|Z_1|}\right).$$

For this it is enough to prove that

$$\frac{L + M}{|Z_1||Z_1 + \varepsilon|} \leq 1 - \frac{1}{|Z_1|}.$$

Multiplying by $|Z_1|$ we get

$$\frac{L + M}{|Z_1| - 1} \leq |Z_1| - 1,$$

$$L + M \leq (|Z_1| - 1)^2.$$

Substituting L and M by their previous estimates we obtain

$$\frac{\frac{|Z_1|}{\sqrt{2}}(|Z_1|+2)}{|Z_1|-2}+\frac{|Z_1|}{\sqrt{2}}(|Z_1|+2)\le(|Z_1|-1)^2,$$

$$\frac{|Z_1|}{\sqrt{2}}(|Z_1|+2)\left(1+\frac{1}{|Z_1|-2}\right)\le(|Z_1|-1)^2.$$

Dividing by $|Z_1|^2$ leads to

$$\frac{1}{\sqrt{2}}\left(1+\frac{2}{|Z_1|}\right)\left(1+\frac{1}{|Z_1|-2}\right)\le\left(1-\frac{1}{|Z_1|}\right).$$

If $|Z_1|$ tends to infinity then the left hand side of the inequality tends to $\frac{1}{\sqrt{2}}$ and the right hand side tends to 1. Then the inequality holds if $|Z_1|$ is large enough,. The inequality is true, if $|Z_1|>4+\frac{5}{2}\sqrt{2}+\frac{1}{2}\sqrt{98+72\sqrt{2}}\approx 14,6$. ■

Proof of Lemma 3.4. Let $a_1\in\mathcal{A}_1$ and $a_2\in\mathcal{A}_2$ be such that $z_1\equiv a_1\ (Z_1)$ and $a_2\equiv\frac{a_1-z_2}{\varepsilon}\ (Z_2)$ hold. Then $a_1+a_2Z_1\in\mathcal{A}$ will be a suitable digit. ■

Proof of Theorem 3.1. The theorem follows from the lemmas immediately. ■

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SYMMETRIC DEVIATIONS AND DISTANCE MEASURES

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Dedicated to Professor Antal Járai on his 60th birthday

Abstract. In this paper we characterize measurable information measures depending upon two probability distributions in a unified manner in order to get most of the existing information measures. Moreover it turns out that our characterization contains new, unexpected information measures.

1. Introduction

In this paper we investigate information measures on the open domain depending upon two probability distributions which are also called deviations (or similarity, affinity or divergence measures). Thus a deviation is a sequence (M_n) of functions, where

$$M_n : \Gamma_n^2 \to \mathbb{R},\ n \in \mathbb{N},\ n \geq 2.$$

Here

$$\Gamma_n = \left\{ P = (p_1, \dots, p_n) \middle| \quad p_i \in I, \sum_{i=1}^{n} p_i = 1 \right\} \tag{1.1}$$

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denotes the set of all discrete n-ary complete positive probability distributions and I denotes the open interval (0,1).

In Shore and Johnson [11] it is shown that each deviation (M_n) which satisfies the four desirable conditions of uniqueness, invariance, system independency and subset independency has a sum form representation

$$M_n(P, Q) = \sum_{i=1}^{n} f(p_i, q_i) \tag{1.2}$$

for some generating function $f : I^2 \to \mathbb{R}$. This result underlines the fact that each known deviation has a sum form, and it is thus natural to assume that a deviation has the sum form property (1.2) for some generating function f.

Many known deviations have a symmetric generating function f that is, $f(p, q) = f(q, p)$ for all $p, q \in I$. If a deviation (M_n) is not symmetric then going over to $M'_n(P, Q) = M_n(P, Q) + M_n(Q, P)$ means that M'_n has a symmetric generating function $f'(p, q) = f(p, q) + f(q, p)$.

The problem of how to characterize all sum form deviations, that is to find some natural conditions which imply the explicite form of the generating function, arises.

In Ebanks et al [3] (see chapter 5) two results were proven for information measures (M_n) depending upon two probability distributions $P, Q \in \Gamma_n$ satisfying a sufficient "fullness" of the range of (M_n) (the range $\{M_n(\Gamma_n^2)|n = 2, 3, ...\}$ has infinite cardinality):

1. For $P, Q \in \Gamma_n, U, V \in \Gamma_m$ we introduce $P * U, Q * U, P * V, Q * V \in \Gamma_{nm}$, where

$$(P * U, Q * V) =$$
$$= ((p_1u_1, ..., p_1u_m, ..., p_nu_1, ..., p_nu_m), (q_1v_1, ..., q_1v_m, ..., q_nv_1, ..., q_nv_m))).$$

Now, if (M_n) has the sum form property with some generating function f and if $M_{nm}(P * U, Q * V) = h(M_n(P, Q), M_m(U, V))$ for some polynomial $h : \mathbb{R}^2 \to \mathbb{R}$ and for all $m, n \geq 2$, then it is shown that h is a symmetric polynomial of degree at most one so that

$$\begin{aligned} &M_{nm}(P * U, Q * V) = \\ &= M_n(P, Q) + M_m(U, V) + \lambda M_n(P, Q) M_m(U, V) \end{aligned} \tag{1.3}$$

for some $\lambda \in \mathbb{R}$.

2. If (M_n) has the sum form property with some generating function f, and there are distributions $P', Q' \in \Gamma_n, U', V' \in \Gamma_m$ such that $I_n(P', Q') \neq 0$, respectively $I_m(U', V') \neq 0$ and

$$M_{nm}(P * U, Q * V) = \\ = A(U,V)M_n(P,Q) + B(P,Q)M_m(U,V) \tag{1.4}$$

for some "weights" A and B, then A and B have the sum form

$$A(U,V) = \sum_{j=1}^{m} M(u_j, v_j) \quad , \quad B(P,Q) = \sum_{i=1}^{n} M'(p_i, q_i) \tag{1.5}$$

for some generating multiplicative functions $M, M' : \mathbb{R}_+^2 \to \mathbb{R}$.

We remark that in Ebanks et al [3] the results in **1.** and **2.** were proven for information measures depending upon k probability distributions, but the special case k = 2 with the notation $(P,Q) * (U,V) = (P * U, Q * V)$ leads exactly to the above (nontrivial) results given in (1.3)–(1.5).

We now assume that the generating function f is symmetric in (1.3) and (1.4) and that $M = M'$ is symmetric so that $M(p,q) = M'(p,q) = M_1(p)M_1(q)$ for some multiplicative function $M_1 : I \to \mathbb{R}$ (since a multiplicative function of two variables is the product of two multiplicative functions in one variable).

Then we form the expression $M_{nm}(P * U, Q * V) + M_{nm}(P * V, Q * U)$ to get

$$M_{nm}(P * U, Q * V) + M_{nm}(P * V, Q * U) = \\ = 2M_n(P,Q) + 2M_m(U,V) + \lambda' M_n(P,Q)M_m(U,V) \tag{1.6}$$

and

$$M_{nm}(P * U, Q * V) + M_{nm}(P * V, Q * U) = \\ = 2A(U,V) \cdot M_n(P,Q) + 2A(P,Q) \cdot M_m(U,V), \tag{1.7}$$

from (1.3) and (1.4) respectively, where $\lambda' = 2\lambda$ and where

$$2A(P,Q) = \sum_{i=1}^{n} 2M_1(p_i)M_1(q_i) \ , \ 2A(U,V) = \sum_{j=1}^{m} 2M_1(u_j)M_1(v_j). \tag{1.8}$$

Thus a common generalization of the deviations given in (1.6) and (1.7) leads to the following class of deviations:

Definition 1.1. A deviation (M_n) is a symmetrically weighted compositive sum form deviation of additive-multiplicative type if (M_n) satisfies

$$M_{nm}(P * U, Q * V) + M_{nm}(P * V, Q * U) = \\ = G_m(U,V)M_n(P,Q) + G_n(P,Q)M_m(U,V) + \lambda M_n(P,Q)M_m(U,V), \tag{1.9}$$

for some $\lambda \in \mathbb{R}$, for all $m, n \geq 2$ and for all $P, Q \in \Gamma_n$, $U, V \in \Gamma_m$ with $P * U, Q * U, P * V, Q * V \in \Gamma_{nm}$, where M_n and G_n have the sum form

$$(1.10) \quad M_n(P,Q) = \sum_{i=1}^{n} f(p_i, q_i), \qquad G_n(P,Q) = \sum_{i=1}^{n} g(p_i, q_i), \quad P, Q \in \Gamma_n$$

for some symmetric functions $f, g : I^2 \to \mathbb{R}$, and where g satisfies

$$(1.11) \quad g(pu, qv) + g(pv, qu) = g(p,q)g(u,v) \quad , \quad p, q, u, v \in I.$$

We say that (M_n) is measurable if f and g are measurable in each variable. Moreover, every symmetric deviation (M_n) satisfying $M_n(P,P) = 0$ is called a distance measure.

Note that (1.9) and (1.10) with $g(p,q) = p + q$ and $g(p,q) = 2M_1(p)M_1(q)$ lead to (1.6) and (1.7), respectively, and that both functions g satisfy (1.11).

Thus the deviations (M_n) given by (1.9) and (1.10) satisfy the following fundamental functional equation

$$(1.12) \quad \begin{aligned} &\sum_{i=1}^{n}\sum_{j=1}^{m}[\, f(p_i u_j, q_i v_j) + f(p_i v_j, q_i u_j) - \\ &-g(u_j, v_j) f(p_i, q_i) - g(p_i, q_i) f(u_j, v_j) - \lambda f(p_i, q_i) f(u_j, v_j)\,] = 0, \end{aligned}$$

where g satisfies (1.11).

In this paper we will present the measurable solutions of (1.11) and (1.12), generalizing the result in Chung et al [2] where the measurable solutions of functional equation (1.6) were given.

Let us finally consider some examples in this introduction.

Kerridge's inaccuracy K_n or the directed divergence F_n is given by

$$(1.13) \quad K_n(P,Q) = -\sum_{i=1}^{n} p_i \log q_i, \qquad F_n(P,Q) = \sum_{i=1}^{n} p_i \log \frac{p_i}{q_i}\,.$$

Note that $K_n(P,P) = H_n(P)$ and $F_n(P,Q) = K_n(P,Q) - K_n(P,P)$, where H_n is the well-known Shannon-entropy. K_n and F_n are indeed errors or deviations due to using $Q = (q_1, \dots, q_n)$ as an estimation of the true probability distribution $P = (p_1, \dots, p_n)$.

A 1-parametric generalization of (F_n) is given by (F_n^α), the directed divergence of degree α,

$$(1.14) \quad F_n^\alpha(P,Q) = \begin{cases} F_n(P,Q) & \alpha = 1 \\ \dfrac{1}{2^{\alpha-1} - 1}\left(\displaystyle\sum_{i=1}^{n} p_i^\alpha q_i^{1-\alpha} - 1\right) & \alpha \in \mathbb{R} \setminus \{1\}. \end{cases}$$

We see immediately that $\lim_{\alpha\to 1} F_n^\alpha = F_n^1 = F_n$. F_n^α is not symmetric in P and Q, but F_n^α can be symmetrized by going over to

$$(1.15)\qquad J_n^\alpha(P,Q) = F_n^\alpha(P,Q) + F_n^\alpha(Q,P) \qquad P,Q \in \Gamma_n$$

so that we arrive at the J-divergence (J_n^α) of degree α, $\alpha \in \mathbb{R}$, which satisfies $J_n^\alpha(P,Q) = J_n^\alpha(Q,P)$. Again we have $\lim_{\alpha\to 1} J_n^\alpha = J_n^1$ (because of $\lim_{\alpha\to 1} F_n^\alpha = = F_n^1$).

A further generalization of J_n^α is given by

$$(1.16)\quad L_n^{\alpha,\gamma}(P,Q) = \begin{cases} 2^{1-\alpha} \sum_{i=1}^{n} (p_i^\alpha - q_i^\alpha) \log \frac{p_i}{q_i} & \alpha = \gamma \\ \frac{1}{2^{\alpha-1} - 2^{\gamma-1}} \sum_{i=1}^{n} (p_i^\alpha - q_i^\alpha) \left(q_i^{\gamma-\alpha} - p_i^{\gamma-\alpha}\right) & \alpha \neq \gamma, \end{cases}$$

the J-divergence of degree (α,γ). We get $L_n^{\alpha,1} = J_n^\alpha$ and $\lim_{\gamma\to\alpha} L_n^{\alpha,\gamma} = L_n^{\alpha,\alpha}$, therefore $L_n^{\alpha,\gamma}$ can be considered as a 2-parametric generalization of J_n^1.

The sequences (J_n^α) and $(L_n^{\alpha,\gamma})$ satisfy (1.9) and (1.10) indeed: In the first case we choose $\lambda = 2^{\alpha-1} - 1$ and $g(p,q) = p + q$ and in the second case $\lambda = 2^{\alpha-1} - 2^{\gamma-1}$ and $g(p,q) = p^\gamma + q^\gamma$, respectively (and the obvious choices for f (see (1.13) and (1.14)). Moreover, $L_n^{\alpha,\gamma}$ is a distance measure since $L_n^{\alpha,\gamma}(P,P) = = 0$.

Note that for example (for $\lambda \neq 0$ and $\gamma = 2\alpha$)

$$(1.17)\qquad \frac{2^{2\alpha-1} - 2^{\alpha-1}}{\lambda} L_n^{\alpha,2\alpha}(P,Q) = \frac{1}{\lambda} \sum_{i=1}^{n} (p_i^\alpha - q_i^\alpha)^2 =: \frac{1}{\lambda} D_n^\alpha(P,Q),$$

i.e. for $\alpha = \frac{1}{2}$ we arrive at Jeffreys distance in Jeffreys [5].

In the following Lemma we finally cite for the convenience of the reader Lemma 2 and Lemma 4 of Riedel and Sahoo [10] which are needed in the proof of Lemma 2.1.

Lemma 1.2. (1) *Let $M : I^2 \to \mathbb{C}$ be a given multiplicative function. The function $f : I^2 \to \mathbb{C}$ satisfies the functional equation*

$$(1.18)\qquad f(pu,qv) + f(pv,qu) = 2M(uv)f(p,q) + 2M(pq)f(u,v)$$

if and only if

$$(1.19)\qquad f(,p,q) = M(p)M(q)\Big[L(p) + L(q) + l(\frac{p}{q}.\frac{p}{q})\Big],$$

where $L : I \to \mathbb{C}$ is an arbitrary logarithmic map and $l : I^2 \to \mathbb{C}$ is a bilogarithmic function.

(2) *Let $M_1, M_2 : I \to \mathbb{C}$ be any two nonzero multiplicative maps with $M_1 \neq M_2$. Then the function $f : I^2 \to \mathbb{C}$ satisfies the functional equation*

$$f(pu,qv) + f(pv,qu) = [M_1(u)M_2(v) + M_1(v)M_2(u)]f(p,q) + [M_1(p)M_2(q) + M_1(q)M_2(p)]f(u,v) \tag{1.20}$$

if and only if

$$f(p,q) = M_1(p)M_2(q)[L_1(p) + L_2(q)] + M_1(q)M_2(p)[L_1(q) + L_2(p)], \tag{1.21}$$

where $L_1, L_2 : I \to \mathbb{C}$ are logarithmic functions.

2. Symmetrically weighted compositive sum form deviations

In order to solve the functional equation (1.11) and (1.12) we first determine the general solution of (1.11) and the corresponding "functional equation without the sums"

$$f(pu,qv) + f(pv,qu) = g(u,v)f(p,q) + g(p,q)f(u,v) + \lambda f(p,q)f(u,v) \tag{2.1}$$

for all $p, q, u, v \in I$.

Lemma 2.1. *The functions $f, g : I^2 \to \mathbb{R}, f \neq 0$ satisfy* (1.11) *and* (2.1) *for all $p, q \in I$ if and only if for all $p, q \in I$:*
in the case $\lambda = 0$

$$\begin{aligned} f(p,q) &= M_1(p)M_2(q)[L_1(p) + L_2(q)] + M_1(q)M_2(p)[L_1(q) + L_2(p)], \\ g(p,q) &= M_1(p)M_2(q) + M_1(q)M_2(p), \qquad M_1 \neq M_2 \end{aligned} \tag{2.2}$$

or

$$\begin{aligned} f(p,q) &= M(p)M(q)[L_3(p) + L_3(q) + l(p,p) + l(q,q) - 2l(p,q)], \\ g(p,q) &= 2M(p)M(q); \end{aligned} \tag{2.3}$$

and in the case $\lambda \neq 0$

$$\begin{aligned} f(p,q) &= \frac{1}{\lambda}([M_3(p)M_4(q) + M_3(q)M_4(p)] - [M_5(p)M_6(q) + M_5(q)M_6(p)]), \\ g(p,q) &= M_5(p)M_6(q) + M_5(q)M_6(p), \end{aligned} \tag{2.4}$$

where $c \neq 0$, $M : \mathbb{R}_+ \to \mathbb{R}$ *and* $M_i : \mathbb{R}_+ \to \mathbb{C}, 1 \leq i \leq 6$ *are multiplicative functions,* $L_1, L_2, L_3 : \mathbb{R}_+ \to \mathbb{R}$ *are logarithmic functions and* $l : \mathbb{R}_+ \to \mathbb{R}$ *is a bilogarithmic function, i.e.* l *is logarithmic in both variables. Moreover,* M_{2i-1} *and* M_{2i} *are both real-valued or* M_{2i} *is the complex conjugate of* M_{2i-1}, $i = 1, 2, 3$.

Finally, if f *and* g *are measurable then* M, M_i, L *and* L_i *are measurable, too.*

Proof. We start with the case $\lambda \neq 0$ in (2.1). By substituting

$$h(p,q) = g(p,q) + \lambda f(p,q)$$

we obtain from (2.1) that

$$h(pu,qv) + h(pv,qu) = h(p,q)h(u,v), \tag{2.5}$$

that is, g and h both satisfy (1.11).

Thus we get from the general solution of (1.11) (see Chung et al [2]) that

$$\begin{aligned} g(p,q) &= M_5(p)M_6(q) + M_5(q)M_6(p) \quad p,q \in I, \\ h(p,q) &= M_3(p)M_4(q) + M_3(q)M_4(p) \quad p,q \in I, \end{aligned} \tag{2.6}$$

where $M_i : \mathbb{R}_+ \to \mathbb{C}, 3 \leq i \leq 6$, M_{2i-1} and M_{2i} are both real-valued or M_{2i} is the complex conjugate of $M_{2i-1}, i = 2, 3$. Using now the substitution for h we arrive at (2.4).

Now we treat the case $\lambda = 0$. Then we have to solve (1.11) and

$$f(pu,qv) + f(pv,qu) = g(u,v)f(p,q) + g(p,q)f(u,v). \tag{2.7}$$

The idea is to extend f and g simultaneously to functions $\bar{f}, \bar{g} : \mathbb{R}_+ \to \mathbb{R}$, where $\bar{f}, \bar{g}$ satisfy (1.11) and (1.2), too. Then it is possible to solve (1.11) and (2.7). It turns out that indeed it is only important to have the point (1,1) in the domain of f and g : putting $q = v = 1$ in (1.11) and (2.7) we get

$$\begin{aligned} g(p,u) &= g(p,1)g(u,1) - g(pu,1), \\ f(p,u) &= g(u,1)f(p,1) + g(p,1)f(u,1) - f(pu,1), \end{aligned}$$

respectively (so that it is sufficient to determine the functions $p \to g(p,1)$ and $p \to f(p,1)$).

Let us define

$$M : I \to \mathbb{R} \quad \text{by} \quad M(t) := \frac{1}{2}\, g(t,t) \quad , \quad t \in I \quad \text{and} \tag{2.8}$$

$$\bar{g} : \mathbb{R}_+ \to \mathbb{R} \quad , \quad \bar{g}(p,q) = \frac{g(tp,tq)}{M(t)}, \quad p,q \in \mathbb{R}_+ \tag{2.9}$$

(here (2.9) means that for given $p, q \in \mathbb{R}_+$ there is $t \in I$ such that $(tp, tq) \in I^2$). Then M is a multiplicative function which is different from zero everywhere. Moreover $\bar{g}$ is well-defined, is uniquely determined, is a continuation of g and satisfies (1.11) on $\mathbb{R}_+^2$ (see Chung et al [2]).

Before we define $\bar{f}$ we need to do some calculations first. Putting $u = v = t$ into (2.7) we obtain (with $G(t) := g(t,t) = 2M(t)$ and $F(t) := \frac{1}{2} f(t,t)$)

$$2f(tp, tq) = g(t,t)f(p,q) + g(p,q)f(t,t) = G(t)f(p,q) + 2F(t)g(p,q)$$

or

$$f(tp, tq) = M(t)f(p,q) + F(t)g(p,q), \qquad p, q \in I. \tag{2.10}$$

Substituting $p = q = t$ and $u = v = w$ into (2.7) we arrive at

$$F(tw) = F(t)M(w) + M(t)F(w), \qquad t, w \in I.$$

Then we get, defining $L(t) := \frac{F(t)}{M(t)}$ and dividing the last equation by $M(tw)$,

$$L(tw) = L(t) + L(w), \qquad t, w \in I. \tag{2.11}$$

Thus L is logarithmic. We now define the continuation $\bar{f} : \mathbb{R}_+ \to \mathbb{R}$ by

$$\bar{f}(p,q) = \frac{f(tp, tq)}{M(t)} - L(t)\bar{g}(p,q), \qquad p, q \in \mathbb{R}_+\,, \tag{2.12}$$

where for each $p, q \in \mathbb{R}_+$ we choose $t \in I$ such that $tp, tq \in I$.

In order to show that $\bar{f}$ is well-defined, we choose (for given $p, q \in \mathbb{R}_+$) $t, w \in$ $\in I$, $t \neq w$ such that $tp, tq, wp, wq \in I$. We have to prove that

$$\frac{f(tp, tq)}{M(t)} - L(t)\bar{g}(p,q) = \frac{f(wp, wq)}{M(w)} - L(w)\bar{g}(p,q)$$

or, equivalently

$$M(w)f(tp, tq) - F(t)M(w)\bar{g}(p,q) = M(t)f(wp, wq) - F(w)M(t)\bar{g}(p,q),$$
$$M(w)f(tp, tq) + F(w)g(tp, tq) = M(t)f(wp, wq) + F(t)g(wp, wq).$$

But the last equation is equivalent with the obvious identity (see (2.10))

$$f(w(tp), w(tq)) = f(t(wp), t(wq)).$$

The function $\bar{f}$ is indeed a continuation of f: Choose $t = p \in I$ to get

$$\bar{f}(p,q) = \frac{f(p^2, pq)}{M(p)} - L(p)\bar{g}(p,q) =$$
$$= \frac{1}{M(p)}(M(p)f(p,q) + F(p)g(p,q)) - \frac{F(p)}{M(p)}\bar{g}(p,q) = f(p,q)$$

from (2.12) and (2.10) for $q \in I$

We show that $\bar{f}$ and $\bar{g}$ satisfy (2.7) for all $p, q \in \mathbb{R}_+$. For $p, q, u, v \in \mathbb{R}_+$ choose $t \in I$ such that $tp, tq, tu, tv \in I$. Using (2.10) and (2.7) we get (using $M(t^2) = M(t)^2$ and $L(t^2) = 2L(t)$)

$$\bar{f}(pu, qv) + \bar{f}(pv, qu) =$$
$$= \frac{f(tptu, tqtv)}{M(t^2)} - L(t^2)\bar{g}(pu, qv) + \frac{f(tptv, tqtu)}{M(t^2)} - L(t^2)\bar{g}(pv, qu) =$$
$$= \bar{g}(u, v)(\frac{f(tp, tq)}{M(t)} - L(t)\bar{g}(p, q)) + \bar{g}(p, q)(\frac{f(tu, tv)}{M(t)} - L(t)\bar{g}(u, v)) =$$
$$= \bar{g}(u, v)\bar{f}(p, q) + \bar{g}(p, q)\bar{f}(u, v).$$

In order to prove, that f is uniquely determined, let us assume that $\tilde{f} : \mathbb{R}_+^2 \to \mathbb{R}$ is an extension of f satisfying also (2.7) for all $p, q, u, v \in \mathbb{R}_+$. Now choose for $p, q \in \mathbb{R}_+$ an element $t \in I$ such that $tp, tq \in I$ and put $u = v = t$ in (2.7). We get (since $\tilde{f} = f$ on I)

$$2\tilde{f}(tp, tq) = 2M(t)\tilde{f}(p, q) + 2\tilde{f}(t, t)g(p, q)$$

or, solving the last equation for $\tilde{f}(p, q)$ we see that

$$\tilde{f}(p, q) = \frac{\tilde{f}(tp, tq)}{M(t)} - g(p, q)\frac{F(t)}{M(t)} = \frac{f(tp, tq)}{M(t)} - L(t)g(p, q) = \bar{f}(p, q).$$

Simplifying the notation we don't distinguish f and $\bar{f}$, and g and $\bar{g}$ and suppose that f satisfies (2.7) for all $p, q, u, v \in \mathbb{R}_+$ and assume that g has the form

(2.13) $$g(p, q) = M_1(p)M_2(q) + M_1(q)M_2(p), \qquad p, q \in \mathbb{R}_+,$$

for some multiplicative functions $M_1, M_2 : \mathbb{R}_+ \to \mathbb{C}_+$, where M_1 and M_2 are both real-valued or M_2 is the complex conjugate of M_1.

Now we consider two cases: $M_1 \neq M_2$ and $M_1 = M_2 = M'$ in (2.13), respectively.

In the first case we get the solution (2.2) from Lemma 4 in Riedel and Sahoo [10] and in the second case we get the solution (2.3) from Lemma 2 in Riedel and Sahoo [10] (in these Lemmas the domain of the functions f, M, M_1, M_2 is $(0, 1]$ or $(0, 1]^2$ and the range is $\mathbb{C}$, but the proofs can be taken over directly for our domains and ranges).

Moreover the proofs of the two Lemmas show that the measurability of f and g imply the measurability of the functions M, L, L_i and M_i. ■

Note that f and g are both symmetric although it was not supposed.

Theorem 2.2. *All measurable, symmetrically weighted compositive sum form deviations* (M_n) *of additive-multiplicative type are given as follows: in the case* $\lambda = 0$ *by*

$$M_n(P,Q) = \sum_{i=1}^{n} [p_i^{\gamma} q_i^{\delta} (a \log p_i + b \log q_i) + p_i^{\delta} q_i^{\gamma} (a \log q_i + b \log p_i)] \tag{2.14}$$

or

$$M_n(P,Q) = \sum_{i=1}^{n} p_i^{\rho} q_i^{\rho} \left[c \log(p_i q_i) + d \left(\log \frac{p_i}{q_i} \right)^2 \right], \tag{2.15}$$

and in the case $\lambda \neq 0$ *by*

$$M_n(P,Q) = -\frac{1}{\lambda} \sum_{i=1}^{n} \left(p_i^{\gamma} q_i^{\delta} + p_i^{\delta} q_i^{\gamma} \right) \tag{2.16}$$

or

$$M_n(P,Q) = -\frac{1}{\lambda} \sum_{i=1}^{n} 2 p_i^{\rho} q_i^{\rho} \cos \left(\sigma \log \frac{p_i}{q_i} \right) \tag{2.17}$$

or

$$M_n(P,Q) = \frac{1}{\lambda} \sum_{i=1}^{n} \left[\left(p_i^{\alpha} q_i^{\beta} + p_i^{\beta} q_i^{\alpha} \right) - \left(p_i^{\gamma} q_i^{\delta} + p_i^{\delta} q_i^{\gamma} \right) \right] \tag{2.18}$$

or

$$M_n(P,Q) = \frac{1}{\lambda} \sum_{i=1}^{n} \left[2 p_i^{\rho} q_i^{\rho} \cos \left(\sigma \log \frac{p_i}{q_i} \right) - \left(p_i^{\gamma} q_i^{\delta} + p_i^{\delta} q_i^{\gamma} \right) \right], \tag{2.19}$$

where $a, b, c, d, \alpha, \beta, \gamma, \delta, \rho, \sigma$ *are arbitrary real constants with* $\alpha \neq \beta$ *and* $\gamma \neq \delta$*.*

Proof. We start from the fundamental equation (1.12) and substitute

$$h(p,q) = g(p,q) + \lambda f(p,q) \tag{2.20}$$

in the case $\lambda \neq 0$ into (1.12). Then (using (2.5)) equation (1.12) turns into $\sum_{j=1}^{n} F(u_j, v_j) = 0$ for all $U, V \in \Gamma_m$ and for all $m, n \geq 2$ where for fixed $P, Q \in \Gamma_n$

$$F(u,v) = \begin{cases} \sum\limits_{i=1}^{n} (f(p_i u, q_i v) + f(p_i v, q_i u) - g(u,v) f(p_i, q_i) - g(p_i, q_i) f(u,v)) \\ \text{if } \lambda = 0, \\ \sum\limits_{i=1}^{n} \left[h(p_i u, q_i v) + h(p_i v, q_i u) - h(u,v) h(p_i, q_i) \right] \text{ if } \lambda \neq 0. \end{cases}$$

The fact that $F : I^2 \to \mathbb{R}$ is measurable and satisfies $\sum_{j=1}^{n} F(u_j, v_j) = 0$ for all $U, V \in \Gamma_n$ and for all $n \geq 2$ implies

$$F(u, v) = a(u - v), \quad u, v \in I^2 \text{for some real constant } a.$$

Indeed, for $n = 2$ we get with $U = (u, 1 - u),\ V = (v, 1 - v) \in \Gamma_2$

$$F(u, v) + F(1 - u, 1 - v) = 0 \;\text{ for all }\; u, v \in I.$$

For $n = 3$ we get with $U = (u_1, u_2, 1-(u_1+u_2)),\ V = (v_1, v_2, 1-(v_1+v_2)) \in \Gamma_3$ that

$$F(u_1, v_1) + F(u_2, v_2) + F(1 - (u_1 + u_2), 1 - (v_1, v_2)) = 0.$$

But from last two equations result we obtain the 2-dimensional Cauchy-functional equation

$$F(u_1, v_1) + F(u_2, v_2) + F(u_1 + u_2, v_1 + v_2),$$

$$u_1, u_2, u_1 + u_2, v_1, v_2, v_1 + v_2 \in I.$$

Thus $F(u, v) = au + bv$ for some constants $a, b \in \mathbb{R}$. But then we obtain

$$\sum_{j=1}^{n} F(u_j, v_j) = \sum_{j=1}^{n} (au_j + bv_j) = a + b = 0.$$

Thus $a = -b$ and F has the form $F(u, v) = a(u - v)$. Since F is measurable and symmetric (since f and g are symmetric) we get $F(u, v) = a(u - v) = = F(v, u) = -a(u - v)$ for some constant a. Letting P, Q vary again we see that $a(P, Q) = -a(P, Q) = 0$ and so $F = 0$, too.

Now for fixed $u, v \in \Gamma_n$ we define

$$(2.21) \quad G(p, q) = \begin{cases} f(pu, qv) + f(pv, qu) - g(u, v)f(p, q) - g(p, q)f(u, v) \\ \text{if} \quad \lambda = 0, \\ \\ h(pu, qv) + h(pv, qu) - h(u, v)h(p, q) \quad \text{if} \quad \lambda \neq 0. \end{cases}$$

Again, G is measurable, symmetric and satisfies

$$(2.22) \qquad \sum_{i=1}^{n} G(p_i, q_i) = F(u, v) = 0,$$

and so that like above $G = 0$. This means that f satisfies

1. (2.7) (that is, g is given by (2.13)) and (1.11), or
2. $G(p, q) = 0$, where h satisfies (1.11) and g is given by (2.13) (see (2.20)).

Case 1. From (2.2) in Lemma 2.1 we obtain (using that L_1 and L_2 are measurable)

$$f(p,q) = p^\gamma q^\delta (a \log p + b \log q) + p^\delta q^\gamma (a \log q + b \log p) \ p, q \in I \tag{2.23}$$

for some constants $a, b, \gamma, \delta, \gamma \neq \delta$.

From (2.2) in Lemma 2.1 we get for arbitrary, but fixed p, q that

$$L_3(p) = c \log p, \ c \in \mathbb{R}, \qquad l(p,q) = d(q) \log p = l(q,p) = d(p) \log q \tag{2.24}$$

which implies $d(p) = d \log p$ for some $d \in \mathbb{R}$. Using this we arrive at

$$f(p,q) = p^\rho q^\rho \Big(c \log(p \cdot q) + d\Big(\log^2 p + \log^2 q - 2 \log p \log q \Big)\Big), \quad \rho \in \mathbb{R}. \tag{2.25}$$

Thus we get (2.14) and (2.15) by using the sum form of (M_n).

Case 2. From (2.20) we get $f(p,q) = \frac{1}{\lambda}(h(p,q) - g(p,q))$, so Lemma 2.1 implies the representation (2.4) for f. Like in Chung et al [2] we get

$$g(p,q) = p^\alpha q^\beta + q^\alpha p^\beta \quad \text{or} \quad g(p,q) = 2p^\rho q^\rho \cos(\sigma \log \frac{p_i}{q_i}), \tag{2.26}$$

$$h(p,q) = p^\gamma q^\delta + q^\gamma p^\delta \quad \text{or} \quad h(p,q) = 2p^\mu q^\mu \cos(\nu \log \frac{p_i}{q_i}) \tag{2.27}$$

for some constants $\alpha, \beta, (\alpha \neq \beta), \gamma, \delta, (\gamma \neq \delta), \rho, \sigma, \mu, \nu$. Then the cases $h = 0$ and $h \neq 0$ lead to the solutions in (2.16) - (2.19).

Reversely, all solutions, given by (2.14)–(2.19) satisfy (1.9). ■

Theorem 2.3. *A deviation (M_n) fulfills the conditions of Theorem 2.2 and satisfies $M_n(P,P) = 0$ iff*

$$M_n(P,Q) = a \sum_{i=1}^n \left(p_i^\gamma q_i^\delta - p_i^\delta q_i^\gamma \right) \log \frac{p_i}{q_i} \quad , \gamma \neq \delta, \quad \lambda = 0 \tag{2.28}$$

or

$$M_n(P,Q) = b \sum_{i=1}^n \left(\log \frac{p_i}{q_i} \right)^2, \qquad \lambda = 0 \tag{2.29}$$

or

$$M_n(P,Q) = \frac{1}{\lambda} \sum_{i=1}^n \left(p_i^\alpha q_i^\delta - q_i^\alpha p_i^\delta \right) \left(q_i^{\gamma-\alpha} - p_i^{\gamma-\alpha} \right) , \ \lambda \neq 0 \tag{2.30}$$

or

(2.31)
$$M_n(P,Q)=\frac{1}{\lambda}\sum_{i=1}^{n}\left(2p_i^{\frac{\gamma+\delta}{2}}q_i^{\frac{\gamma+\delta}{2}}\cos\left(\sigma\log\frac{p_i}{q_i}\right)-\left(p_i^{\gamma}q_i^{\delta}+p_i^{\delta}q_i^{\gamma}\right)\right),\ \lambda\neq 0,$$

where $a,b,\alpha,\gamma,\delta,\sigma$ *are arbitrary constants.*

Proof. We put $P=Q$ into (2.14)–(2.19) to obtain

$$M_n(P,P)=\sum_{i=1}^{n}2(a+b)p_i^{\gamma+\delta}\log p_i, \tag{2.32}$$

$$M_n(P,P)=\sum_{i=1}^{n}2c\cdot p_i^{2\rho}\log p_i, \tag{2.33}$$

$$M_n(P,P)=-\frac{2}{\lambda}\sum_{i=1}^{n}p_i^{\gamma+\delta}\neq 0, \tag{2.34}$$

$$M_n(P,P)=-\frac{2}{\lambda}\sum_{i=1}^{n}p_i^{2\rho}\neq 0, \tag{2.35}$$

$$M_n(P,P)=\frac{2}{\lambda}\sum_{i=1}^{n}\left(p_i^{\alpha+\beta}-p_i^{\gamma+\delta}\right), \tag{2.36}$$

$$\text{and}\qquad M_n(P,P)=\frac{2}{\lambda}\sum_{i=1}^{n}\left(p_i^{2\rho}-p_i^{\gamma+\delta}\right), \tag{2.37}$$

respectively. Now we consider $M_n(P,P)=0$ in all cases. We get $b=-a$ in (2.32) and $c=0$ in (2.33), which imply (2.28) and (2.29), respectively. Moreover, (2.34) and (2.35) lead to no solution, whereas (2.36) leads to $\alpha+\beta=$ $=\gamma+\delta$. Putting $\beta=\gamma+\delta-\alpha$ into (2.18) we have (2.30). Finally, $M_n(P,P)=0$ in (2.37) implies $2\rho=\gamma+\delta$ which gives (2.31). ■

The above distance measures contain many known measures as special case. Let us mention the following examples:

(a) $\delta=0$ in (2.28) gives

$$M_n(P,Q)=a2^{\gamma-1}L_n^{\gamma,\gamma}(P,Q).$$

(b) $\delta=0$ in (2.29) results in

$$M_n(P,Q)=\frac{2^{\alpha-1}-2^{\gamma-1}}{\lambda}L_n^{\alpha,\gamma}(P,Q).$$

(c) $\alpha = 0$ in (2.30) leads to

$$M_n(P,Q) = -\frac{1}{\lambda}\sum_{i=1}^{n}\left(\sqrt{p_i^{\gamma}q_i^{\delta}} - \sqrt{p_i^{\delta}q_i^{\gamma}}\right)^2.$$

(d) $(\gamma,\delta) \in (1,0),(0,1)$ in (c) yields

$$M_n(P,Q) = \frac{1}{\lambda}\sum_{i=1}^{n}\left(\sqrt{p_i} - \sqrt{q_i}\right)^2 = \frac{1}{\lambda}D_n^{\frac{1}{2}}(P,Q) \quad \text{(see (1.17))}.$$

(e) Note that

$$D_n^{\frac{1}{2}}(P,Q) = \frac{2}{\lambda}\Big[1 - B_n(P,Q)\Big],$$

where $B_n(P,Q) = \sum_{i=1}^{n}\sqrt{p_i q_i}$ is the Hellinger coefficient (see Hellinger [4]).

(f) If $\gamma = 2\alpha$ and $\delta = 1$ in (c) then we get

$$M_n(P,Q) = \frac{1}{\lambda}\sum_{i=1}^{n}\left(p_i^{\alpha} - q_i^{\alpha}\right)^2 = \frac{2^{2\alpha-1} - 2^{\alpha-1}}{\lambda}L_n^{\alpha,2\alpha}(P,Q) =$$
$$= \frac{1}{\lambda}D_n^{\frac{1}{2}}(P,Q).$$

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A NOTE ON DYADIC HARDY SPACES

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Dedicated to the 60th birthday of Professor Antal Járai

Abstract. The usual L^p-norms are trivially invariant with respect to multiplication by *Walsh* functions. The analogous question will be investigated in the dyadic Hardy space $\mathbf{H}$. We introduce an invariant subspace $\mathbf{H}_*$ of $\mathbf{H}$ in this sense and show some properties of $\mathbf{H}_*$. For example a function in $\mathbf{H}_*$ will be constructed the *Walsh–Fourier* series of which diverges in L^1-norm.

1. Introduction

Let w_n $(n \in \mathbf{N})$ be the *Walsh–Paley* system defined on the interval $[0,1)$. It is well-known that $w_n = \prod_{k=0}^{\infty} r_k^{n_k}$, where r_k is the k-th *Rademacher* function $(k \in \mathbf{N})$ and $n = \sum_{k=0}^{\infty} n_k 2^k$ $(n_k = 0$ or 1 for all k's) is the dyadic representation of n. If $n = \sum_{k=0}^{\infty} n_k 2^k$, $m = \sum_{k=0}^{\infty} m_k 2^k \in \mathbf{N}$ then $w_n w_m = w_{n \oplus m}$, where the operation $\oplus$ is defined by

$$n \oplus m := \sum_{k=0}^{\infty} |n_k - m_k| 2^k.$$

Thus it is clear that

$$2^n \oplus m = 2^n + m \quad (n \in \mathbf{N},\ m = 0, \dots, 2^n - 1),$$

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i.e. $r_n w_m = w_{2^n} w_m = w_{2^n+m}$. (For more details we refer to the book [1].) For $1 \le p \le \infty$ let $L^p := L^p[0,1)$ and let $\|.\|_p$ denote the usual *Lebesgue* space and norm. If $f \in L^1$, $n \in \mathbf{N}$ then let $S_n f$ be the n-th *Walsh–Fourier* partial sum of f, i.e. $S_n f = f * D_n$, where $D_n := \sum_{k=0}^{n-1} w_k$ and $*$ stands for dyadic convolution. We remark that $r_n D_{2^n} = D_{2^{n+1}} - D_{2^n} \quad (n \in \mathbf{N})$. The next famous property of D_{2^n}'s plays an important role in the Walsh analysis:

$$D_{2^n}(x) = \begin{cases} 2^n & (0 \le x < 2^{-n}) \\ 0 & (2^{-n} \le x < 1). \end{cases} \tag{1}$$

Therefore

$$S_{2^n} f(x) = 2^n \int\limits_{I_n(x)} f \qquad (x \in [0,1)).$$

Here $x \in I_n(x) := [j2^{-n}, (j+1)2^{-n})$ with a proper integer $j(x) = j = 0, \ldots, 2^n - 1$. Set $I_n := I_n(0)$.

We recall that

$$\sup_n \frac{\|D_n\|_1}{\log n} < \infty. \tag{2}$$

The dyadic maximal function f^* of $f \in L^1$ is defined as follows:

$$f^* := \sup_n |S_{2^n} f|.$$

Then for all $p > 1$ we have $\|f\|_p \le \|f^*\|_p \le C_p \|f\|_p$. (Here and later C_p, C will denote positive constants depending at most on p, although not always the same in different occurences.) The so-called dyadic *Hardy* space $\mathbf{H} := \mathbf{H}[0,1)$ is defined by means of the maximal function as follows:

$$\mathbf{H} := \{f \in L^1 : \|f\| := \|f^*\|_1 < \infty\}.$$

The atomic structure of $\mathbf{H}$ is very useful in many investigations. Namely, we call a function $a \in L^\infty$ (dyadic) atom if $\int_0^1 a = 0$ and there exists a dyadic interval $I_n(z) \quad (n \in \mathbf{N},\ z \in [0,1))$ such that $a(x) = 0 \quad (x \in [0,1) \setminus I_n(z))$ and $\|a\|_\infty \le 2^n$. Let supp $a := I_n(z)$. The characterization of $\mathbf{H}$ by means of atoms reads as follows:

$$f \in \mathbf{H} \iff f = \sum_{k=0}^{\infty} \alpha_k a_k,$$

where all a_k's are atoms and the coefficients α_k's have the next property: $\sum_{k=0}^{\infty} |\alpha_k| < \infty$. Furthermore,

$$\|f\| \sim \inf \sum_{k=0}^{\infty} |\alpha_k|,$$

where the infimum is taken over all atomic representations $\sum_{k=0}^{\infty} \alpha_k a_k$ of f. (For the martingale theoretic background we refer to [4].)

For example the functions $r_n D_{2^n}$ $(n \in \mathbf{N})$ are trivially atoms by (1). Thus

$$f := \sum_{n=0}^{\infty} \alpha_n r_{\nu_n} D_{2^{\nu_n}} \tag{3}$$

belongs to $\mathbf{H}$ if $\sum_{k=0}^{\infty} |\alpha_n| < \infty$ and the indices $\nu_0 < \nu_1 < \ldots$ are choosen arbitrarily. Moreover, $\|f\|_1 \le \|f\| \le \sum_{n=0}^{\infty} |\alpha_n|$.

It is not hard to see that the partial sums $S_{2^n} a$ $(n \in \mathbf{N})$ remain atoms if $a \in L^\infty$ is an atom. Indeed, if supp $a = I_N(z)$ $(N \in \mathbf{N},\ z \in [0,1))$ and $x \in [0,1) \setminus I_N(z)$ then for all $n \in \mathbf{N}$ the intervals $I_n(x)$ and $I_N(z)$ are disjoint or $I_n(x) \cap I_N(z) = I_N(z)$. Thus

$$|S_{2^n} a(x)| = \left| 2^n \int_{I_n(x)} a \right| = \left| 2^n \int_{I_n(x) \cap I_N(z)} a \right| \le \left| 2^n \int_{I_N(z)} a \right| = \left| 2^n \int_0^1 a \right| = 0,$$

thus $S_{2^n} a(x) = 0$. Furthermore, $\|S_{2^n} a\|_\infty \le \|a\|_\infty \le 2^N$, i.e. supp $S_{2^n} a = I_N(z)$ and $\int_0^1 S_{2^n} a = \int_0^1 a = 0$.

Therefore if $f = \sum_{k=0}^{\infty} \alpha_k a_k$ is an atomic representation of $f \in \mathbf{H}$ then $S_{2^n} f = \sum_{k=0}^{\infty} \alpha_k S_{2^n} a$ $(n \in \mathbf{N})$ is an atomic representation of $S_{2^n} f$. This means that $\|S_{2^n} f\| \le \sum_{k=0}^{\infty} |\alpha_k|$, i.e. $\|S_{2^n} f\| \le \|f\|$. (The last inequality follows also from the obvious estimation $(S_{2^n} f)^* \le f^*$.)

We remark that $\mathbf{H}$ can be defined also in another way. To this end let $f \in L^1$ and

$$Qf := \left(\sum_{n=-1}^{\infty} (\delta_n f)^2 \right)^{1/2}$$

be its quadratic variation, where $\delta_{-1} f := \int_0^1 f$, $\delta_n f := S_{2^{n+1}} f - S_{2^n} f = f * (r_n D_{2^n})$ $(n \in \mathbf{N})$. Then

$$\|f\| \sim \|Qf\|_1 \text{ , ill. } \|f\|_p \sim \|Qf\|_p \qquad (1 < p < \infty).$$

If $f \in L^1, n \in \mathbf{N}$ and $k = 0, \ldots, 2^n - 1$, then w_k is constant on $I_n(x)$ $(x \in [0,1))$, consequently $w_k(x) \int_{I_n(x)} f = \int_{I_n(x)} (f w_k)$. This means that $w_k S_{2^n} f = S_{2^n}(f w_k)$. Furthermore, if $2^n \le k \in \mathbf{N}$ is arbitrary then let us write $k = \sum_{j=0}^{N} k_j 2^j$ (with some $\mathbf{N} \ni N \ge n$). It is clear that

$$\delta_j(w_k S_{2^n} f) = \begin{cases} 0 & (j \ne N) \\ w_k S_{2^n} f & (j = N) \end{cases} \qquad (j \in \mathbf{N}).$$

From this it follows that $Q(w_k S_{2^n} f) = |S_{2^n} f|$, i.e. for all $k \in \mathbf{N}$ we have

$$\|w_k S_{2^n} f\| = \|S_{2^n}(f w_k)\| \ \ (k < 2^n) \quad \text{and}$$
$$\|w_k S_{2^n} f\| \leq C\|S_{2^n} f\|_1 \ \ (k \geq 2^n). \tag{4}$$

The *Walsh–Paley* system doesn't form a basis in L^1. Moreover, there exists $f \in \mathbf{H}$ such that

$$\sup_n \|S_n f\|_1 = \infty.$$

However (see [3]), if $f \in \mathbf{H}$ then

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{\|S_k f\|_1}{k} \to \|f\| \qquad (n \to \infty),$$

or equivalently

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{\|f - S_k f\|_1}{k} \to 0 \qquad (n \to \infty).$$

For the sake of the completeness and in order to demonstrate the usefulness of the atomic structure we sketch some examples. Namely we take the function given by (3). If $l_n = 0, 1, \ldots, 2^{\nu_n} - 1 \ \ (n \in \mathbf{N})$ then

$$\|S_{2^{\nu_n} + l_n} f - S_{2^{\nu_n}} f\|_1 = |\alpha_n| \|D_{l_n}\|_1. \tag{$*$}$$

It is well-known that $k_n \in \{0, 1, \ldots, 2^{\nu_n} - 1\}$ can be choosen so that

$$\|D_{k_n}\|_1 \geq C\nu_n \qquad (n \in \mathbf{N})$$

holds. Then we get

$$\|S_{2^{\nu_n} + k_n} f - S_{2^{\nu_n}} f\|_1 \geq C|\alpha_n|\nu_n \qquad (n \in \mathbf{N}).$$

If $\sup_n |\alpha_n|\nu_n = \infty$ then $\|S_{2^n} f\|_1 \leq \sum_{k=0}^{\infty} |\alpha_k| < \infty$ implies $\sup_n \|S_n f\|_1 = \infty$. It is obvious that $\alpha_n := 2^{-n}, \nu_n := 2^{n^2} \quad (n \in \mathbf{N})$ are suitable. (We remark that $\inf_n |\alpha_n|\nu_n > 0$ is trivially sufficient for the $\|.\|_1$ divergence of the *Walsh–Fourier* series of f.)

If $f \in \mathbf{H}$ is given by (3) then $\|S_n f - f\|_1 \to 0 \quad (n \to \infty)$ if and only if $\nu_n \alpha_n \to 0 \quad (n \to \infty)$. Indeed, if $l_n := k_n$'s are as above then $C\nu_n|\alpha_n| \leq$ $\leq |\alpha_n|\|D_{k_n}\|_1$ and $(*)$ proves necessity. It is known that $\|S_{2^n} g - g\|_1 \to 0$ $(n \to \infty)$ for all $g \in L^1$. Therefore (see (2)) $\|D_{l_n}\|_1 \leq C \log l_n \leq C\nu_n$ and $\nu_n \alpha_n \to 0 \ \ (n \to \infty)$ together with $(*)$ imply the $\|.\|_1$ convergence of the series (3).

Finally, we cite an example $f \in L^1 \setminus H$ such that $\|S_n f - f\|_1 \to 0 \quad (n \to \infty)$. To this end we take a special function $f := \sum_{n=0}^{\infty} \alpha_n r_n D_{2^n}$ in (3) such that the coefficients α_n form a null-sequence of bounded variation, i.e. $\sum_{n=0}^{\infty} |\alpha_n - -\alpha_{n+1}| < \infty$. It is well-known that this assumption on the coefficients implies the $\|.\|_1$-convergence of the series in question. Indeed, for all $n, m \in \mathbf{N}$, $n < m$ it follows by (1) that

$$\left\| \sum_{k=n}^{m} \alpha_k r_k D_{2^k} \right\|_1 = \left\| \sum_{k=n}^{m} \alpha_k (D_{2^{k+1}} - D_{2^k}) \right\|_1 =$$

$$= \left\| \sum_{k=n+1}^{m} (\alpha_{k-1} - \alpha_k) D_{2^k} + \alpha_m D_{2^m} - \alpha_n D_{2^n} \right\|_1 \leq$$

$$\leq \sum_{k=n+1}^{m} |\alpha_{k-1} - \alpha_k| \|D_{2^k}\|_1 + |\alpha_m| \|D_{2^m}\|_1 + |\alpha_n| \|D_{2^n}\|_1 =$$

$$= \sum_{k=n+1}^{m} |\alpha_{k-1} - \alpha_k| + |\alpha_m| + |\alpha_n| \to 0 \quad (n, m \to \infty).$$

Therefore $f \in L^1$. Furthermore, if $2^{-k-1} \leq x < 2^{-k} \quad (k \in \mathbf{N})$ then

$$Qf(x) = \sqrt{\sum_{n=0}^{\infty} \alpha_n^2 D_{2^n}^2(x)} = \sqrt{\sum_{n=0}^{k} \alpha_n^2 2^{2n}} \geq |\alpha_k| 2^k,$$

and

$$\|Qf\|_1 \geq \sum_{k=0}^{\infty} \int_{2^{-k-1}}^{2^{-k}} Qf \geq \sum_{k=0}^{\infty} \int_{2^{-k-1}}^{2^{-k}} |\alpha_k| 2^k = \frac{1}{2} \sum_{k=0}^{\infty} |\alpha_k|.$$

This means that $\|f\| = \infty$ if $\sum_{k=0}^{\infty} |\alpha_k| = \infty$. Now, we prove the $\|.\|_1$ convergence of the sequence $S_n f$. To this end let $1 \leq n \in \mathbf{N}$ and $m_n = 0, ..., 2^n - 1$. Then by (2) we have

$$\|S_{2^n + m_n} f - S_{2^n} f\|_1 = \|\alpha_n r_n D_{m_n}\|_1 = |\alpha_n| \|D_{m_n}\|_1 \leq C |\alpha_n| \log m_n \leq Cn|\alpha_n|.$$

Hence $n\alpha_n \to 0 \quad (n \to \infty)$ is implies to $\|S_{2^n + m_n} f - S_{2^n} f\|_1 \to 0 \quad (n \to \infty)$. Since $\|S_{2^n} f - f\|_1 \to 0 \quad (n \to \infty)$ we get $\|S_n f - f\|_1 \to 0 \quad (n \to \infty)$. A simple calculation shows that the sequence

$$\alpha_n := \frac{1}{(n+2) \log (n+2)} \qquad (n \in \mathbf{N})$$

satisfies all of the conditions above. By means of similar observations it can be proved that the assumption $\sum_{n=0}^{\infty} |\alpha_n| < \infty$ in (3) is necessary to $f \in \mathbf{H}$ in the general case as well.

2. Results

It is clear that for all $f \in L^p$ $(1 \le p \le \infty)$ and $n \in \mathbf{N}$ we have $fw_n \in L^p$ and $\|fw_n\|_p = \|f\|_p$. The situation in the case of $\mathbf{H}$ is more complicated. For example if we take the atoms $f_n := r_n D_{2^n} \in \mathbf{H}$ $(n \in \mathbf{N})$ then $\|f_n\| = 1$ and

$$\|r_n f_n\| = \|D_{2^n}\| = \|D^*_{2^n}\|_1 = \left\|\max_{k \le n} D_{2^k}\right\|_1,$$

where by (1)

$$\max_{k \le n} D_{2^k}(x) = \begin{cases} 2^k & (2^{-k-1} \le x < 2^{-k},\ k = 0, \dots, n-1) \\ 2^n & (0 \le x < 2^{-n}). \end{cases}$$

From this it follows immediately that $\|D_{2^n}\| = \dfrac{n+2}{2}$, i.e.

$$\|r_n f_n\| = \|w_{2^n} f_n\| = \frac{n+2}{2}\|f_n\|.$$

First we prove that an analogous relation holds in general.

Theorem 1. *Let $k \in \mathbf{N}$. Then there exists a constant C_k such that for all $f \in \mathbf{H}$ the product fw_k belongs to $\mathbf{H}$ and $\|fw_k\| \le C_k \|f\|$.*

Our example above shows that $C_{2^n} \ge \dfrac{n+2}{2}$ $(n \in \mathbf{N})$, i.e. $\sup_k C_k = \infty$. Since all *Walsh* functions are final products of *Rademacher* functions, we need to prove Theorem 1 only for $k = 2^n$ $(n \in \mathbf{N})$.

In this case let $f = \sum_{k=0}^{\infty} \alpha_k a_k$ be an atomic representation of $f \in \mathbf{H}$. Then

$$\|fw_{2^n}\| = \|fr_n\| = \|(fr_n)^*\|_1 \le \left\|\sum_{k=0}^{\infty} |\alpha_k| (a_k r_n)^*\right\|_1 \le$$

$$\le \sum_{k=0}^{\infty} |\alpha_k| \|(a_k r_n)^*\|_1 = \sum_{k=0}^{\infty} |\alpha_k| \|a_k r_n\|.$$

If we can show that

$$(**) \qquad A_n := \sup_a \|ar_n\| < \infty$$

(where the supremum is taken over all atoms a), then

$$\|(fr_n)^*\|_1 \le A_n \sum_{k=0}^{\infty} |\alpha_k|,$$

i.e. $\|fr_n\| \le A_n \|f\|$.

Proof of the inequality $(**)$. Let a be an atom, $k \in \mathbf{N}, x \in [0,1)$. In the case $k > n$ the n-th *Rademacher* function r_n is constant on the interval $I_k(x)$ and thus

$$S_{2^k}(ar_n)(x) = 2^k \int_{I_k(x)} ar_n = 2^k r_n(x) \int_{I_k(x)} a.$$

Therefore

$$(ar_n)^* = \sup_k |S_{2^k}(ar_n)| \le \max_{k \le n} |S_{2^k}(ar_n)| + \sup_{k>n} |S_{2^k} a| \le$$
$$\le \max_{k \le n} |S_{2^k}(ar_n)| + \sup_k |S_{2^k} a| = \max_{k \le n} |S_{2^k}(ar_n)| + a^* =: (ar_n)^{**} + a^*.$$

From this it follows that

$$\|ar_n\| = \|(ar_n)^*\|_1 \le \|(ar_n)^{**}\|_1 + \|a^*\|_1 =$$
$$= \|(ar_n)^{**}\|_1 + \|a\| \le \|(ar_n)^{**}\|_1 + 1.$$

This means that it is enough to show only

$$\sup_a \|(ar_n)^{**}\|_1 < \infty$$

(where the supremum is taken over all atoms a).

To this end let a be an atom. For the sake of simplicity we assume that $\operatorname{supp} a = I_N$ (with some $N \in \mathbf{N}$). Then

$$\|(ar_n)^{**}\|_1 = \int_{I_N} (ar_n)^{**} + \int_{2^{-N}}^{1} (ar_n)^{**} =: J_1(a) + J_2(a).$$

Hence by means of the *Cauchy* inequality and the properties of atoms it follows that

$$J_1(a) \le \left(\int_{I_N} ((ar_n)^{**})^2 \right)^{1/2} \cdot 2^{-N/2} \le 2^{-N/2} \|(ar_n)^{**}\|_2 \le C_2 2^{-N/2} \|ar_n\|_2 \le$$
$$\le C_2 2^{-N/2} \|a\|_\infty 2^{-N/2} \le C_2.$$

We will show that

$$\sup_a J_2(a) < \infty.$$

Indeed, if a is the atom as above and $n < N$, then $ar_n = a$, i.e.

$$J_2(a) \leq \|(ar_n)^{**}\|_1 = \| \max_{k\leq n} |S_{2^k} a|\|_1 \leq \|a^*\|_1 = \|a\| \leq 1.$$

Thus it can be assumed that $N \leq n$. Let $k = 0, ..., n$ and $2^{-N} \leq x < 1$. Then

$$S_{2^k}(ar_n)(x) = 2^k \int\limits_{I_k(x)} ar_n = 2^k \int\limits_{I_k(x)\cap I_N} ar_n,$$

where $I_k(x) \cap I_N \neq \emptyset$ exactly if $k \leq N-1$ and $x < 2^{-k}$ (in this case $I_k(x) = I_k$ and $I_k(x) \cap I_N = I_N$). This means that with the notation $k_0(x) := \max\{k = 0, ..., N-1 : x < 2^{-k}\}$ we get

$$(ar_n)^{**}(x) = \max_{k\leq k_0(x)} |S_{2^k}(ar_n)(x)| = \max_{k\leq k_0(x)} 2^k \left| \int\limits_{I_N} ar_n \right| \leq$$

$$\leq \max_{k\leq k_0(x)} 2^k \|a\|_1 \leq 2^{k_0(x)} \leq \frac{1}{x}.$$

Summarizing the above facts it follows that

$$J_2(a) = \int\limits_{2^{-N}}^{1} (ar_n)^{**} \leq \int\limits_{2^{-N}}^{1} \frac{dx}{x} \leq C \log_2 2^N = CN \leq Cn,$$

which proves Theorem 1. ■

Therefore it can be assumed that $\frac{n+2}{2} \leq C_{2^n} \leq C(n+1) \quad (n \in \mathbf{N})$. Furthermore, if $n = \sum_{j=0}^{\infty} n_j 2^j$ is the dyadic representation of $n \in \mathbf{N}$, then

$$\|fw_n\| \leq \|f\| \prod_{j=0}^{\infty} C_{2^j}^{n_j} \leq C^{|n|}[n]\|f\| \qquad (f \in \mathbf{H}),$$

where $|n| := \sum_{j=0}^{\infty} n_j$, and $[n] := \prod_{j=0}^{\infty} (j+1)^{n_j}$, and the above estimation cannot be improved. For example $|2^k| = 1$ and $[2^k] = k+1 \quad (k \in \mathbf{N})$.

Theorem 1 involves the next concept: if $f \in \mathbf{H}$ then let

$$\|f\|_* := \sup_n \|fw_n\|.$$

It follows immediately that $\|.\|_*$ is a norm, $\|.\| \leq \|.\|_*$ but (see the above remarks) $\|.\|_*, \|.\|$ are not equivalent. Moreover, it is not hard to construct $f \in \mathbf{H}$ such that $\|f\|_* = \infty$. Indeed, we take the function given in (3). Then for all $k \in \mathbf{N}$ we get

$$\|fr_{\nu_k}\| \geq |\alpha_k|\|D_{2^{\nu_k}}\| - \left\| \sum_{k\neq n=0}^{\infty} \alpha_n r_{\nu_k} r_{\nu_n} D_{2^{\nu_n}} \right\|.$$

It is clear that all products $r_{\nu_k} r_{\nu_n} D_{2^{\nu_n}} \quad (k \neq n \in \mathbf{N})$ are atoms, which implies

$$\left\| \sum_{k\neq n=0}^{\infty} \alpha_n r_{\nu_k} r_{\nu_n} D_{2^{\nu_n}} \right\| \leq \sum_{n=0}^{\infty} |\alpha_n| = q < \infty.$$

Then

$$\|f\|_* \geq \|fr_{\nu_k}\| \geq |\alpha_k|\|D_{2^{\nu_k}}\| - q = |\alpha_k|\frac{\nu_k + 2}{2} - q \to \infty \qquad (k \to \infty)$$

follows by means of a suitable choice of parameters.

F. Schipp (see [2]) introduced the following norms

$$\|f\|_{*p} := \|\sup_n Q(fw_n)\|_p \ , \ \ \|f\|^{*p} := \left\|\sup_{m,n} |S_{2^m}(fw_n)|\right\|_p$$
$$(f \in L^1, \ 1 \leq p < \infty),$$

and proved the non-trivial equivalence $\|f\|_{*p} \sim \|f\|_p \quad (1 < p < \infty)$. It is clear that these norms are shift invariant, i.e. for all $n \in \mathbf{N}$ the equalities $\|fw_n\|_{*p} = \|f\|_{*p}$, $\|fw_n\|^{*p} = \|f\|^{*p}$ hold. Furthermore, the inequality $\|.\|_* \leq \|.\|^{*1}$ follows immediately. Moreover, for all $k \in \mathbf{N}$ we get

$$\|fw_k\| \leq C\|Q(fw_k)\|_1 \leq C\|\sup_n Q(fw_n)\|_1 = C\|f\|_{*1},$$

i.e. $\|f\|_* \leq C\|f\|_{*1}$ holds, too. Schipp proved for $F := \sum_{n=0}^{\infty} 2^{-n/2} r_{2^n} D_{2^{2^n}}$ that $F \in \mathbf{H}$ but $\|F\|_{*1} = \infty$. (This example is a special case of (3).) Our example above along with $\|.\| \leq \|.\|_* \leq \|.\|^{*1}$ shows also the existence of $f \in \mathbf{H}$ such that $\|f\|^{*1} = \infty$. The question wheter the norm $\|.\|_{*1}$ and the norm $\|.\|^{*1}$ are equivalent or not remains open.

Let us introduce the space $\mathbf{H}_*$ as follows:

$$\mathbf{H}_* := \{f \in H : \|f\|_* < \infty\}.$$

Then $\mathbf{H}_*$ is a proper subspace of $\mathbf{H}$. For all $n, k \in \mathbf{N}$ it is clear that $1 = \|w_n\| = \|w_{k\oplus n}\| = \|w_k w_n\|$, i.e. $\|w_n\|_* = 1$. Thus $w_n \in \mathbf{H}_*$ and therefore every *Walsh* polynomial (finite linear combination of *Walsh* functions) belongs to $\mathbf{H}_*$. Furthermore, if $f \in \mathbf{H}_*$ then

$$\|fw_n\|_* = \sup_k \|fw_nw_k\| = \sup_k \|fw_{n\oplus k}\| = \sup_j \|fw_j\| = \|f\|_*.$$

In other words the norm $\|.\|_*$ is also invariant with respect to multiplication by *Walsh* functions.

Above we remarked that there exists $f \in \mathbf{H}$ such that its *Walsh–Fourier* series diverges in $\|.\|_1$ norm. We show that this result can be sharpened. Namely, the next theorem holds:

Theorem 2. *There exists $f \in \mathbf{H}_*$ with $\|.\|_1$-divergent Walsh–Fourier series.*

Proof. We take the function $f := \sum_{n=0}^{\infty} \alpha_n r_{\nu_n} D_{2^{\nu_n}}$ from (3). It was shown above (see $(*)$) that $q := \sum_{n=0}^{\infty} |\alpha_n| < \infty$ and $\inf_n |\alpha_n|\nu_n > 0$ imply the $\|.\|_1$ divergence of the *Walsh–Fourier* series of f.

To the proof of $f \in H_*$ let $k = \sum_{j=0}^{\infty} k_j 2^j$ be the dyadic representation of $k \in \mathbf{N}$. Then $w_k = \prod_{j=0}^{\infty} r_j^{k_j}$. Taking into account that

$$w_k r_s D_{2^s} = \prod_{j=s}^{\infty} r_j^{k_j} r_s D_{2^s} \qquad (s \in \mathbf{N})$$

is obviously an atom, provided $k_s = 0$ or $k_s = 1$, but there is $j \geq s+1$ such that $k_j = 1$. Let $\mathbf{N}_s$ be the set of such k's. Then $k \in \mathbf{N}^s := \mathbf{N} \setminus \mathbf{N}_s$ iff $k = 2^s + \sum_{j=0}^{s-1} k_j 2^s$, i.e. $\mathbf{N}^s = \mathbf{N} \cap [2^s, 2^{s+1})$. In this case $w_k r_s D_{2^s} = D_{2^s}$.

If $k \notin \bigcup_{n=0}^{\infty} \mathbf{N}^{\nu_n}$, then

$$fw_k = \sum_{n=0}^{\infty} \alpha_n w_k r_{\nu_n} D_{2^{\nu_n}}$$

is an atomic representation of fw_k and so $\|fw_k\| \leq \sum_{n=0}^{\infty} |\alpha_n| = q$.

If $k \in \bigcup_{n=0}^{\infty} \mathbf{N}^{\nu_n}$, then there is a unique $m \in \mathbf{N}$ such that $k \in \mathbf{N}^{\nu_m}$:

$$fw_k = \alpha_m D_{2^{\nu_m}} + \sum_{m \neq n=0}^{\infty} \alpha_n w_k r_{\nu_n} D_{2^{\nu_n}} =: \alpha_m D_{2^{\nu_m}} + f_0.$$

The above observations lead to $\|f_0\| \le \sum_{n=0}^\infty |\alpha_n| = q < \infty$ and

$$\|fw_k\| \le |\alpha_m| \|D_{2^{\nu_m}}\| + \|f_0\| \le C|\alpha_m|\nu_m + q.$$

We see that the assumption $\sup_n |\alpha_n|\nu_n < \infty$ is sufficient to

$$\sup_k \|fw_k\| \le C \sup_n |\alpha_n|\nu_n + q < \infty.$$

In this case $f \in H_*$. For example if $\alpha_n := 2^{-n}, \nu_n := 2^n \quad (n \in \mathbf{N})$, then the function $f = \sum_{n=0}^\infty 2^{-n} r_{2^{2^n}} D_{2^{2^n}}$ proves Theorem 2. ■

If $f \in \mathbf{H}$ then $Qf \in L^1$, i.e. $Qf = \left(\sum_{k=-1}^\infty (\delta_k f)^2\right)^{1/2} < \infty$ a.e. Thus $\left(\sum_{k=n}^\infty (\delta_k f)^2\right)^{1/2} \to 0 \quad (n \to \infty)$ a.e. and we get by *Lebesgue*'s theorem that

$$\|f - S_{2^n} f\| \le C\|Q(f - S_{2^n})\|_1 = C \left\| \left(\sum_{k=n}^\infty (\delta_k f)^2 \right)^{1/2} \right\|_1 \to 0 \qquad (n \to \infty).$$

However, this last convergence property doesn't hold true if the norm $\|.\|$ will be replaced by $\|.\|_*$. Indeed, taking the function $f \in \mathbf{H}_*$ from the proof of Theorem 2 we get analogously that

$$\|f - S_{2^{\nu_n}} f\|_* = \left\| \sum_{k=n}^\infty \alpha_k r_{\nu_k} D_{2^{\nu_k}} \right\|_* \ge C \inf_{k \ge n} |\alpha_k|\nu_k - q \qquad (n \in \mathbf{N}).$$

Let $\alpha_k := 2^{-k}, \nu_k := 2^{k+s} \ (k \in \mathbf{N})$, where $s \in \mathbf{N}$ is defined by $2^s C > 2$. Then $q = \sum_{k=0}^\infty |\alpha_k| = 2$ and $\|f - S_{2^{\nu_n}} f\|_* \ge 2^s C - 2 \quad (n \in \mathbf{N})$, i.e. $\|f - S_{2^n} f\|_*$ doesn't tend to zero if $n \to \infty$.

We recall that $\|S_{2^n} f\|_1 \le \|f\|_1 \ (f \in L^1)$, $\|S_{2^n} f\| \le \|f\| \ (f \in \mathbf{H}, n \in \mathbf{N})$. Applying (4) it is not hard to prove that an analogous inequality holds if we replace the norm $\|.\|$ by $\|.\|_*$. Indeed,

$$\|S_{2^n} f\|_* = \sup_k \|w_k S_{2^n} f\| = \max\left\{ \sup_{k<2^n} \|w_k S_{2^n} f\|, \sup_{k \ge 2^n} \|w_k S_{2^n} f\| \right\} \le$$

$$\le \max\left\{ \sup_{k<2^n} \|fw_k\|, C\|S_{2^n} f\|_1 \right\} \le \max\left\{ \sup_k \|fw_k\|, C\|f\|_1 \right\} \le C\|f\|_*.$$

Hence if $f \in L^1$ then

$$\|f\|_* = \sup_n \|(fw_n)^*\|_1 = \sup_n \| \sup_m |S_{2^m}(fw_n)| \|_1.$$

Let $p>1$ and $f\in L^p$. Then for arbitrary $n\in\mathbf{N}$ we can write

$$\|fw_n\|=\|(fw_n)^*\|_1\le\|(fw_n)^*\|_p\le C_p\|fw_n\|_p=C_p\|f\|_p,$$

i.e. $\|f\|_*\le C_p\|f\|_p$. Thus $L^p\subset H_*$. In other words $\bigcup_{p>1}L^p\subset\mathbf{H}_*$. We will show that the next statement holds:

Theorem 3. $H_*\setminus\left(\bigcup_{p>1}L^p\right)\ne\emptyset$.

Proof. Let $1<p<\infty$ and take the function $f=\sum_{n=0}^\infty 2^{-n}r_{2^{2^n}}D_{2^{2^n}}=:$ $=:\sum_{n=0}^\infty \alpha_n r_{\nu_n}D_{2^{\nu_n}}$ as in the proof of Theorem 2. Then $f\in H_*$. On the other hand

$$\|f\|_p^p\ge C_p\|Qf\|_p^p\ge C_p\left\|\sqrt{\sum_{n=0}^\infty\alpha_n^2D_{2^{\nu_n}}^2}\right\|_p^p\ge C_p\sum_{k=0}^\infty\int_{2^{-\nu_k-1}}^{2^{-\nu_k}}\left(\sum_{n=0}^k\alpha_n^2D_{2^{\nu_n}}^2\right)^{p/2}=$$

$$=C_p\sum_{k=0}^\infty2^{-\nu_k}\left(\sum_{n=0}^k\alpha_n^22^{2\nu_n}\right)^{p/2}\ge C_p\sum_{k=0}^\infty\alpha_k^p2^{(p-1)\nu_k}=\infty.\ \blacksquare$$

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FOURIER TRANSFORM FOR MEAN PERIODIC FUNCTIONS

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Dedicated to the 60th birthday of Professor Antal Járai

Abstract. Mean periodic functions are natural generalizations of periodic functions. There are different transforms - like Fourier transforms - defined for these types of functions. In this note we introduce some transforms and compare them with the usual Fourier transform.

1. Introduction

In this paper $\mathcal{C}(\mathbb{R})$ denotes the locally convex topological vector space of all continuous complex valued functions on the reals, equipped with the linear operations and the topology of uniform convergence on compact sets. Any closed translation invariant subspace of $\mathcal{C}(\mathbb{R})$ is called a *variety*. The smallest variety containing a given f in $\mathcal{C}(\mathbb{R})$ is called the *variety generated by* f and it is denoted by $\tau(f)$. If this is different from $\mathcal{C}(\mathbb{R})$, then f is called *mean periodic*. In other words, a function f in $\mathcal{C}(\mathbb{R})$ is mean periodic if and only if there exists a nonzero continuous linear functional μ on $\mathcal{C}(\mathbb{R})$ such that

$$f * \mu = 0 \tag{1}$$

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holds. In this case sometimes we say that f is *mean periodic with respect to* μ. As any continuous linear functional on $\mathcal{C}(\mathbb{R})$ can be identified with a compactly supported complex Borel measure on $\mathbb{R}$, equation (1) has the form

$$\int f(x-y)\,d\mu(y) = 0 \tag{2}$$

for each x in $\mathbb{R}$. The dual of $\mathcal{C}(\mathbb{R})$ will be denoted by $\mathcal{M}_C(\mathbb{R})$. As the convolution of two nonzero compactly supported complex Borel measures is a nonzero compactly supported Borel measure as well, all mean periodic functions form a linear subspace in $\mathcal{C}(\mathbb{R})$. We equip this space with the following topology. For each nonzero μ from the dual of $\mathcal{C}(\mathbb{R})$ let $V(\mu)$ denote the solution space of (1). Clearly, $V(\mu)$ is a variety and the set of all mean periodic functions is equal to the union of all these varieties. We equip this union with the inductive limit of the topologies of the varieties $V(\mu)$ for all nonzero μ from the dual of $\mathcal{C}(\mathbb{R})$. The locally convex topological vector space obtained in this way will be denoted by $\mathcal{MP}(\mathbb{R})$, the space of mean periodic functions.

An important class of mean periodic functions is formed by the exponential polynomials. We call a function of the form

$$\varphi(x) = p(x)\,e^{\lambda x} \tag{3}$$

an *exponential monomial*, where p is a complex polynomial and λ is a complex number. If $p \equiv 1$, then the corresponding exponential monomial $x \mapsto e^{\lambda x}$ is called an *exponential.* Exponential monomials of the form

$$\varphi_k(x) = x^k\,e^{\lambda x} \tag{4}$$

with some natural number k and complex number λ, are called *special exponential monomials.*

Linear combinations of exponential monomials are called *exponential polynomials.* To see that the special exponential monomial in (3) is mean periodic one considers the measure

$$\mu_k = (e^{\lambda}\,\delta_1 - \delta_0)^{k+1}\,, \tag{5}$$

where δ_y is the Dirac–measure concentrated at the number y for each real y, and the $k+1$-th power is meant in convolution-sense. It is easy to see that

$$\varphi_k * \mu_k = 0$$

holds. Sometimes we write 1 for δ_0.

Exponential polynomials are typical mean periodic functions in the sense that any mean periodic function f in $V(\mu)$ is the uniform limit on compact sets of a sequence of linear combinations of exponential monomials, which belong to $V(\mu)$, too. More precisely, the following theorem holds (see [9]).

Theorem 1 (L. Schwartz, 1947). *In any variety of $\mathcal{C}(\mathbb{R})$ the linear hull of all exponential monomials is dense.*

A similar theorem in $\mathcal{C}(R^n)$ fails to hold for $n \geq 2$ as it has been shown in [4] by D. I. Gurevich. Moreover, he gave examples for nonzero varieties in $\mathcal{C}(R^2)$ which do not contain nonzero exponential monomials at all. However, as it has been shown by L. Ehrenpreis in [1], Theorem 1 can be extended to varieties of the form $V(\mu)$ in $\mathcal{C}(R^n)$ for any positive integer n.

Another important result in [9] is the following (Théorème 7, on p. 881.):

Theorem 2. *In any proper variety of $\mathcal{C}(\mathbb{R})$ no special exponential monomial is contained in the closed linear hull of all other special exponential monomials in the variety.*

In other words, if a variety $V \neq \{0\}$ in $\mathcal{C}(\mathbb{R})$ is given, then for each special exponential monomial φ_0 in V there exists a measure μ in $\mathcal{M}_C(\mathbb{R})$ such that $\mu(\varphi_0) = 1$ and $\mu(\varphi) = 0$ for each special exponential monomial $\varphi \neq \varphi_0$ in V.

2. A mean operator for mean periodic functions

Based on Theorems 1 and 2 by L. Schwartz we introduced a mean operator on the space $\mathcal{MP}(\mathbb{R})$ in the following way (see also [10], pp. 64–65.).

For each x, y in $\mathbb{R}$ and f in $\mathcal{C}(\mathbb{R})$ let

$$\tau_y f(x) = f(x+y)\,,$$

and call $\tau_y f$ the *translate of f by y*. The continuous linear operator τ_y on $\mathcal{C}(\mathbb{R})$ is called *translation operator*. The operator τ_0 will be denoted by 1. Clearly, the continuous function f is a polynomial of degree at most k if and only if

$$(\tau_y - 1)^{k+1} f(x) = 0 \tag{6}$$

holds for each x, y in $\mathbb{R}$ and for $k = 0, 1, \dots$. The set $\mathcal{P}(\mathbb{R})$ of all polynomials is a subspace of $\mathcal{MP}(\mathbb{R})$, which we equip with the topology inherited from $\mathcal{MP}(\mathbb{R})$.

Theorem 3. *The subspace $\mathcal{P}(\mathbb{R})$ is closed in $\mathcal{MP}(\mathbb{R})$.*

Proof. First we show that the set of the degrees of all polynomials in any proper variety is bounded from above. By the Taylor–formula it follows that the derivative of a polynomial is a linear combination of its translates, hence if a polynomial belongs to a variety then all of its derivatives belong to the same variety, too. Therefore, if the set of the degrees of all polynomials in a proper variety is not bounded from above, then all polynomials belong to this variety. But, in this case, by the Stone–Weierstrass–theorem, all continuous functions must belong to the variety, hence it cannot be proper.

Suppose now that $(p_i)_{i\in I}$ is a net of polynomials which converges in $\mathcal{MP}(\mathbb{R})$ to the continuous function f. By the definition of the inductive limit topology there exists a nonzero μ in $\mathcal{M}_c(\mathbb{R})$ such that p_i belongs to $V(\mu)$ for each i in I. By our previous consideration, for the degrees we have $\deg p_i \leq k$ for some positive integer k. By (6) this means that

$$(\tau_y - 1)^{k+1} p_i(x) = 0$$

holds for each x, y in $\mathbb{R}$. This implies that the same holds for f, hence f is a polynomial of degree at most k, too. The theorem is proved. ■

Theorem 4. *There exists a unique continuous linear operator*

$$M : \mathcal{MP}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$$

satisfying the properties

1) $M(\tau_y f) = \tau_y M(f)$,
2) $M(p) = p$

for each f in $\mathcal{MP}(\mathbb{R})$, p in $\mathcal{P}(\mathbb{R})$ and y in $\mathbb{R}$.

Proof. First we prove uniqueness. By Theorem 1, it is enough to show that the properties of M determine M on the set of all special exponential monomials. Let $m \neq 1$ be any nonzero continuous complex exponential. Then we have

$$M(m) = M\big[m(-y)\tau_y m\big] = m(-y)M(\tau_y m) = m(-y)\tau_y M(m),$$

which implies that either $M(m) = 0$ or m is a polynomial. Hence $M(m) = 0$. Suppose that we have proved for $j = 0, 1, \dots, k-1$ that

$$M\big[x^j m(x)\big] = 0$$

for any continuous complex exponential $m \neq 1$. Then we have

$$M\big[(x+y)^k m(x+y)\big] = M\Big[\sum_{j=0}^{k} \binom{k}{j} x^j y^{k-j} m(x)m(y)\Big] =$$

$$= \sum_{j=0}^{k} \binom{k}{j} y^{k-j} m(y) M\big[x^j m(x)\big] = m(y) M\big[x^k m(x)\big],$$

which implies, as above, that $M\big[x^k m(x)\big] = 0$. This proves the uniqueness.

In order to prove existence, first we notice, that, by Theorem 2, for any nonzero μ in $\mathcal{M}_c(\mathbb{R})$, the exponential 1 is not contained in the closed linear subspace of $\mathcal{C}(\mathbb{R})$ spanned by all special exponential monomials in $V(\mu)$ different from 1. This implies the existence of a measure μ_0 in $\mathcal{M}_c(\mathbb{R})$ such that $\mu_0(1) = 1$, further $\mu_0(\varphi) = 0$ for any special exponential monomial $\varphi \neq 1$ in $V(\mu)$.

From this fact it follows, that $x^k m(x) * \mu_0 = 0$ for each positive integer k and exponential $m \neq 1$ in $V(\mu)$, further $x^k * \mu_0 = x^k$. This shows, that $\varphi * \mu_0$ is a polynomial in $V(\mu)$ for any exponential polynomial φ in $V(\mu)$. On the other hand, as in the proof of Theorem 3, it follows that if a polynomial of degree n belongs to $V(\mu)$, then all the functions $1, x, x^2, \ldots, x^n$ also belong to $V(\mu)$. Hence, all polynomials in $V(\mu)$ have a degree smaller than some fixed positive integer. Now, if f is arbitrary in $V(\mu)$, then by Theorem 1, there exist exponential polynomials φ_i in $V(\mu)$ such that $f = \lim \varphi_i$. Then we have $f * \mu_0 = \lim(\varphi_i * \mu_0)$, hence also $f * \mu_0$ is a polynomial.

Suppose now, that f belongs also to some $V(\nu)$ with some nonzero ν. Then $f * \mu_0$ also belongs to $V(\nu)$, and it is a polynomial. Hence we have $f * \mu_0 = f * \mu_0 * \nu_0$. Similarly, $f * \nu_0 = f * \nu_0 * \mu_0$. Hence $f * \mu_0$ does not depend on the special choice of μ_0. On the other hand, each f in $\mathcal{MP}(\mathbb{R})$ is contained in some $V(\mu)$ with $\mu \neq 0$, and we can define

$$M(f) = f * \mu_0$$

with any μ_0 in $\mathcal{M}_c(\mathbb{R})$ satisfying the previous properties. The continuity and linearity of M follows from the definition, 1) follows from the properties of convolution, and 2) has been proved. ■

3. The Fourier transform

For each f in $\mathcal{C}(\mathbb{R})$ we define $\check{f}$ by the formula

$$\check{f}(x) = f(-x) \tag{7}$$

for any x in $\mathbb{R}$. It is obvious, that $f\check{m}$ is mean periodic for any f in $\mathcal{MP}(\mathbb{R})$ and for any continuous complex exponential m. Hence we may define $\hat{f}$ as follows:

$$\hat{f}(m) = M(f\check{m}) \tag{8}$$

for any nonzero continuous exponential m.

Theorem 5. *The map $f \mapsto \hat{f}$ defined above is linear and has the following properties:*

1) $\hat{p}(m) = 0$ *for* $m \neq 1$ *and* $\hat{p}(1) = p$,
2) $(p\,f)\hat{}(m) = p\hat{f}(m)$,
3) $(\tau_y f)\hat{}(m) = m(y)\tau_y\big(\hat{f}(m)\big)$,
4) $(\check{f})\hat{}(m) = \big[\hat{f}(\check{m})\big]\check{}$

for any f in $\mathcal{MP}(\mathbb{R})$, for any p in $\mathcal{P}(\mathbb{R})$ and for each y in $\mathbb{R}$, whenever $p\,f$ is mean periodic.

Proof. In the proof of Theorem 4 we have seen that $M(p\,m) = 0$ for each polynomial p and exponential $m \neq 1$. This means that if the exponential polynomial φ has the form

$$\varphi(x) = p_0(x) + \sum_{i=1}^{k} p_i(x)\,m_i(x) \tag{9}$$

for each real x, where k is a nonnegative integer, $p_0, p_1, \ldots, p_k$ are polynomials and $m_1, m_2, \ldots, m_k$ are different exponentials, then we have

$$M(\varphi) = p_0\,. \tag{10}$$

Clearly, this implies 1) – 4) for any exponential polynomial $f = \varphi$. Then, by Theorem 1, our statements follow for any mean periodic f. ■

Theorem 6 ("Uniqueness Theorem"). *For any f in $\mathcal{MP}(\mathbb{R})$, if $\hat{f} = 0$, then $f = 0$.*

Proof. From the previous theorem it follows by linearity and continuity, that $\hat{\varphi} = 0$ for all φ in $\tau(f)$. In particular, $\hat{\varphi} = 0$ for any exponential polynomial φ in $\tau(f)$, hence, by (9), we have that the only exponential polynomial in $\tau(f)$ is 0. Now our statement is a consequence of Theorem 1. ■

As the exponentials of the additive group of $\mathbb{R}$ can be identified with complex numbers, there is a one to one mapping between $\mathbb{C}$ and the set of all exponentials. Hence, instead of $\hat{f}(m)$ we can write $\hat{f}(\lambda)$, where λ is the unique complex number corresponding to the exponential m. By Theorem 5 the Fourier transform of the mean periodic function f is a polynomial-valued mapping $\hat{f}$, which is defined on $\mathbb{C}$, the set of complex numbers, having the properties listed in 5. On the other hand, the Fourier transformation $f \mapsto \hat{f}$ is an injective, linear mapping of $\mathcal{MP}(\mathbb{R})$ into the set of all polynomial-valued mappings of $\mathbb{C}$ into $\mathcal{P}(\mathbb{R})$, having the properties listed in 5. If f is a bounded mean periodic function, then $\tau(f)$ consists of bounded functions, in particular, each exponential is a character and each polynomial in $\tau(f)$ is constant. Hence, in this case $M(f)$ is a constant, and $\hat{f}(m)$ is constant, for each character m of $\mathbb{R}$. In particular, using the results in [8] we have the following theorem.

Theorem 7. *For any almost periodic f in $\mathcal{MP}(\mathbb{R})$, the function $\hat{f}$ coincides with the Fourier transform of f as an almost periodic function in the sense of Bohr.*

For exponential polynomials we have the following immediate "Inversion Theorem".

Theorem 8. *Let f be an exponential polynomial. Then*

$$f(x) = \sum_{\lambda \in \mathbb{C}} \hat{f}(\lambda)(x)\, e^{\lambda x} \tag{11}$$

holds for each x in $\mathbb{R}$.

For any mean periodic f we call the *spectrum of f* the set $sp(f)$ of all complex numbers λ for which the exponential $x \mapsto e^{\lambda x}$ belongs to the variety $\tau(f)$ generated by f. The following theorem is easy to prove.

Theorem 9. *A mean periodic function is a polynomial if and only if its spectrum is $\{0\}$. It is an exponential monomial if and only if its spectrum is a singleton and it is an exponential polynomial if and only if its spectrum is finite.*

4. The Carleman transform

As we have seen in the previous section it is possible to introduce a Fourier–like transform for mean periodic functions on $\mathbb{R}$ which enjoys several useful properties similar to the classical Fourier transform. However, this transform yields a polynomial-valued function, hence the role of classical Fourier coefficients are played by polynomials. The existence of this transform depends on the mean operator, which is a kind of mean value, but it takes polynomials as values, instead of numbers. The most important property of this mean — besides linearity and continuity — is that it commutes with translations: instead of translation invariance we have translation covariance, which — obviously — reduces to translation invariance in case of constant functions. The Fourier transform, based on this mean operator, can be realized in case of exponential polynomials as follows: if the exponential polynomial φ has the canonical representation (9) for each real x, where k is a nonnegative integer, $p_0, p_1, \ldots, p_k$ are polynomials and $m_1, m_2, \ldots, m_k$ are different exponentials, then the mean operator M takes the value p_0 on φ, and, more generally, the Fourier transform of φ at λ is the polynomial p_λ, which is the coefficient of the exponential $x \mapsto e^{\lambda x}$ in the canonical representation of φ. As spectral analysis and spectral synthesis hold in $\mathbb{R}$ by [9], heuristically, the support of $\hat{f}$ consists of those λ's which take part in the spectral analysis of f in the sense, that the corresponding exponentials $x \mapsto e^{\lambda x}$ belong to the spectrum of f, and the value $\hat{f}(\lambda) = M\big[f(x) \cdot e^{-\lambda x}\big]$, which is a polynomial, shows, to what content this exponential takes part in the reconstruction process of f from its spectrum: in the spectral synthesis of f.

As the existence of the Fourier transform introduced above is a result of a transfinite procedure, depending on Hahn–Banach-theorem, it is not clear how to determine the value of $\hat{f}$ at some complex number λ, how to compute it, if a general mean periodic function f is given, which is not necessarily an exponential polynomial. In other words, it is not clear how to compute the coefficients of the polynomial $\hat{f}(\lambda)$ for a general mean periodic function f. On the other hand, an "Inversion Theorem"-like result would be highly welcome, for which, as usual, different estimates on the "Fourier–like coefficients" were necessary.

In his fundamental work [6] (see also [5]) J. P. Kahane used another transform based on the Carleman transform (see [3]). Here we present the details.

Let f be a mean periodic function in $\mathcal{MP}(\mathbb{R})$ and we put

$$f^-(x) = \begin{cases} 0, & x \geq 0 \\ f(x), & x < 0\,. \end{cases} \tag{12}$$

As f is mean periodic, there exists a nonzero compactly supported Borel measure in $\mathcal{M}_c(\mathbb{R})$ such that

$$f * \mu = 0 \tag{13}$$

holds. Denote μ any of such measures and we put

$$g = f^- * \mu\,. \tag{14}$$

It is easy to see, that the support of g is compact (see [6], Lemma on p. 20). The *Carleman transform* of f is defined as

$$\mathcal{C}(f)(w) = \frac{\hat{g}(w)}{\hat{\mu}(w)} \tag{15}$$

for each w in $\mathbb{C}$ which is not a zero of $\hat{\mu}$. By the Paley–Wiener-theorem (see e.g. [11]) $\hat{g}$ and $\hat{\mu}$ are entire functions of exponential type, hence $\mathcal{C}(f)$ is meromorphic. Originally Carleman in [3] introduced this transform for functions which are not very rapidly increasing at infinity, but Kahane observed that it works also for mean periodic functions.

We present a simple example for the computation of this transform. Let

$$f(x) = x$$

for each x in $\mathbb{R}$. Then f is mean periodic and $\tau(f)$ is annihilated by the measure

$$\mu = (\delta_1 - 1)^2 .$$

The Fourier transform of μ is as follows:

$$\hat{\mu}(w) = \int e^{-iwx}\, d\mu(x) = \int e^{-iwx}\, d(\delta_1 - 1)^2(x) = \int e^{-iwx}\, d(\delta_2 - 2\delta_1 + 1)(x) =$$

$$= e^{-2iw} - 2e^{-iw} + 1 = (e^{-iw} - 1)^2$$

for each w in $\mathbb{C}$. The next step is to form the function f^- (see (12)). Hence, we have, by (14)

$$g(x) = \big(f^- * \mu\big)(x) = \int f^-(x-y)\, d\mu(y) = \int f^-(x-y)\, d(\delta_2 - 2\delta_1 + 1)(y) =$$

$$= f^-(x-2) + 2f^-(x-1) + f^-(x) =$$

$$= \begin{cases} 0, & x \geq 2 \\ x-2, & 2 > x \geq 1 \\ -x, & 1 > x \geq 0 \\ 0, & 0 > x\,. \end{cases}$$

The Fourier transform of g is

$$\hat{g}(w) = \int g(x) e^{-iwx}\, dx = \int\limits_0^2 g(x) e^{-iwx}\, dx =$$

$$= \int\limits_1^2 (x-2) e^{-iwx}\, dx - \int\limits_0^1 x e^{-iwx}\, dx =$$

$$= -\frac{1}{iw} e^{-iw} - \frac{1}{(iw)^2}\Big(e^{-2iw} - e^{-iw}\Big) + \frac{1}{iw} e^{-iw} + \frac{1}{(iw)^2}\Big(e^{-iw} - 1\Big) =$$

$$= -\frac{1}{(iw)^2}(e^{-iw} - 1)^2 .$$

From this we have

$$\mathcal{C}(f)(w) = \frac{-\frac{1}{(iw)^2}(e^{-iw}-1)^2}{(e^{-iw}-1)^2} = -\frac{1}{(iw)^2}$$

for each w in $\mathbb{C}$ which is not a zero of $\hat{\mu}$.

At this moment one cannot see any relation between $\mathcal{C}(f)$ and $\hat{f}$. Consider another easy example. Let

$$f(x) = x^3\, e^{\lambda x}\,,$$

where x is real and λ is a complex number. In this case we can take

$$\mu = (e^{\lambda} - 1)^4\,,$$

and

$$\hat{\mu}(w) = \left(e^{-(iw-\lambda)} - 1\right)^4,$$

further

$$\hat{g}(w) = -\frac{3!}{(iw-\lambda)^4}\left(e^{-(iw-\lambda)} - 1\right)^4,$$

and finally

$$\mathcal{C}(f)(w) = \frac{\hat{g}(w)}{\hat{\mu}(w)} = -\frac{3!}{(iw-\lambda)^4}\,.$$

We shall see that there is an intimate relation between the Carleman transform and the Fourier transform of exponential monomials. First we need the following theorem.

Theorem 10. *For each x in $\mathbb{R}$ let*

$$f(x) = p(x)e^{\lambda x}\,, \tag{16}$$

where p is a polynomial and λ is a complex number. Then we have

$$\mathcal{C}(f)(w) = -\sum_{k=0}^{\infty} \frac{p^{(k)}(0)}{(iw-\lambda)^{k+1}}\,, \tag{17}$$

where the sum is actually finite.

Proof. Let

$$f_k(x) = x^k e^{\lambda x}$$

for each nonnegative integer k and complex number λ. Then f_k is mean periodic and $\tau(f)$ is annihilated by the finitely supported measure

$$\mu_k = (e^{\lambda}\delta_1 - 1)^{k+1}\,.$$

Indeed, we have for each x in $\mathbb{R}$

$$f_k * \mu_k(x) = \int f_k(x-y)\,d\mu(y) = \sum_{j=0}^{k+1} \binom{k+1}{j} (-1)^{k+1-j} e^{\lambda j} (x-j)^k e^{\lambda x - \lambda j} =$$

$$= e^{\lambda x} \sum_{j=0}^{k+1} \binom{k+1}{j} (-1)^{k+1-j} (x-j)^k = e^{\lambda x} \left(\tau_{-1} - 1\right)^{k+1} \varphi_k(x) = 0$$

by (6), where

$$\varphi_k(x) = x^k$$

for x in $\mathbb{R}$.

Let w be a complex number. For the sake of simplicity set

$$T = iw - \lambda\,.$$

The Fourier transform of μ_k at w in $\mathbb{C}$ is

$$\hat{\mu}_k(w) = \int e^{-iwx}\,d\mu_k(x) = \sum_{j=0}^{k+1} \binom{k+1}{j} (-1)^{k+1-j} e^{\lambda j} e^{-iwj} = \left(e^{-T} - 1\right)^{k+1}.$$

As

$$f_k^-(x) = \begin{cases} 0, & x \geq 0 \\ f_k(x), & x < 0 \end{cases}$$

it follows for $l = 0, 1, \ldots, k$

$$g_k(x) = f_k^- * \mu_k(x) = \int f_k^-(x-y)\,d\mu_k(y) =$$

$$= \begin{cases} 0, & k+1 \leq x; \\ e^{\lambda x} \sum_{j=l+1}^{k+1} \binom{k+1}{j} (-1)^{k+1-j} (x-j)^k, & l \leq x < l+1; \\ 0, & x < 0. \end{cases}$$

By definition, the Fourier transform of g_k at w in $\mathbb{C}$ is

$$\hat{g}_k(w) = \int e^{-iwx} g_k(x)\,dx = \sum_{l=0}^{k} \int_l^{l+1} e^{-iwx} g_k(x)\,dx =$$

$$= \sum_{l=0}^{k} \sum_{j=l+1}^{k+1} \binom{k+1}{j} (-1)^{k+1-j} \int_l^{l+1} (x-j)^k e^{-Tx}\,dx\,.$$

Using the fact, like above, that

$$\sum_{j=0}^{k+1} \binom{k+1}{j} (-1)^{k+1-j} (x-j)^k = 0\,,$$

we have

$$\hat{g}_k(w) = \sum_{l=0}^{k} \sum_{j=l+1}^{k+1} \binom{k+1}{j} (-1)^{k+1-j} \int\limits_{l}^{l+1} (x-j)^k e^{-Tx}\, dx =$$

$$= \sum_{l=0}^{k} \sum_{j=0}^{k+1} \binom{k+1}{j} (-1)^{k+1-j} \int\limits_{l}^{l+1} (x-j)^k e^{-Tx}\, dx -$$

$$- \sum_{l=0}^{k} \sum_{j=0}^{l} \binom{k+1}{j} (-1)^{k+1-j} \int\limits_{l}^{l+1} (x-j)^k e^{-Tx}\, dx =$$

$$= \sum_{l=0}^{k} \int\limits_{l}^{l+1} \left[\sum_{j=0}^{k+1} \binom{k+1}{j} (-1)^{k+1-j} (x-j)^k \right] e^{-Tx}\, dx -$$

$$- \sum_{l=0}^{k} \sum_{j=0}^{l} \binom{k+1}{j} (-1)^{k+1-j} \int\limits_{l}^{l+1} (x-j)^k e^{-Tx}\, dx =$$

$$- \sum_{l=0}^{k} \sum_{j=0}^{l} \binom{k+1}{j} (-1)^{k+1-j} \int\limits_{l}^{l+1} (x-j)^k e^{-Tx}\, dx =$$

$$= (-1)^k \sum_{j=0}^{k} \binom{k+1}{j} (-1)^j \sum_{l=j}^{k} \int\limits_{l}^{l+1} (x-j)^k e^{-Tx}\, dx =$$

$$= (-1)^k \sum_{j=0}^{k} \binom{k+1}{j} (-1)^j \int\limits_{j}^{k+1} (x-j)^k e^{-Tx}\, dx =$$

$$= - \sum_{j=0}^{k+1} \binom{k+1}{j} (-1)^{k+1-j} \int\limits_{j}^{k+1} (x-j)^k e^{-Tx}\, dx\,.$$

Integration by parts yields

$$\int_j^{k+1} (x-j)^k e^{-Tx}\,dx = \left[\frac{(x-j)^k e^{-Tx}}{-T}\right]_j^{k+1} + \frac{k}{T}\int_j^{k+1} (x-j)^{k-1} e^{-Tx}\,dx =$$

$$= \frac{(k+1-j)^k e^{-(k+1)T}}{-T} + \frac{k}{T}\int_j^{k+1} (x-j)^{k-1} e^{-Tx}\,dx\,,$$

for $k \geq 1$. Continuing this process we arrive at

$$\hat{g}(w) = \sum_{j=0}^{k+1} \binom{k+1}{j} (-1)^{k+1-j} \sum_{i=0}^{k} \frac{k!}{(k-i)!} \frac{(k+1-j)^{k-i} e^{-(k+1)T}}{T^{i+1}} -$$

$$-\sum_{j=0}^{k+1} \binom{k+1}{j} (-1)^{k+1-j} \frac{k!}{T^{k+1}} e^{-jT} =$$

$$= \sum_{i=0}^{k} \frac{k!}{(k-i)!} \frac{1}{T^{i+1}} e^{-(k+1)T} \sum_{j=0}^{k+1} \binom{k+1}{j} (-1)^{k+1-j} (k+1-j)^{k-i} -$$

$$-\frac{k!}{T^{k+1}} \sum_{j=0}^{k+1} \binom{k+1}{j} (-1)^{k+1-j} \left(e^{-T}\right)^j = -\frac{k!}{T^{k+1}} \left(e^{-T} - 1\right)^{k+1}.$$

Here we used again, that by (6)

$$\sum_{j=0}^{k+1} \binom{k+1}{j} (-1)^{k+1-j} (k+1-j)^{k-i} = 0\,.$$

Returning to the original notation we have that

$$\mathcal{C}(f_k)(w) = -\frac{k!}{(iw-\lambda)^{k+1}}\,, \tag{18}$$

and this implies our statement. The theorem is proved. ■

5. Relation between the Carleman transform and the Fourier transform

Using the initial of the name of Kahane here we introduce the K-mean of a mean periodic function f. In [6] it is proved that for a complex number λ the exponential monomial $x \mapsto p(x)e^{\lambda x}$ belongs to $\tau(f)$ if and only if λ is a pole of order at least n of $\mathcal{C}(f)$, where n is the degree of the polynomial p. As $\mathcal{C}(f)$ is meromorphic, each pole of it is of finite order. Consider the case $\lambda = 0$. If 0 is not a pole of $\mathcal{C}(f)$, then no nonzero polynomial belongs to $\tau(f)$. In particular, the function 1 does not belong to $\tau(f)$. In this case let $\mathcal{K}(f) = 0$, the zero polynomial. Suppose now that 0 is a pole of $\mathcal{C}(f)$. Let $n \geq 1$ denote the order of this pole, and define the polynomial $\mathcal{K}(f)$ of degree $n-1$ as follows: for each real x let

$$\mathcal{K}(f)(x) = -\sum_{k=0}^{n-1} \frac{i^{k+1}\, c_{k+1}}{k!}\, x^k \,, \tag{19}$$

where c_k denotes the coefficient of w^{-k} in the polar part of the Laurent series expansion of $\mathcal{C}(f)$ around $w = 0$ $(k = 0, 1, \ldots, n-1)$.

By Theorem 10. we have the following basic result.

Theorem 11. *For each polynomial p we have*

$$\mathcal{K}(p) = p\,. \tag{20}$$

Proof. Formula (18) gives the result with $\lambda = 0$ for the polynomial $x \mapsto x^k$ for each natural number k. The general case follows by linearity. ■

Using again equation (18) and linearity we have the extension of the previous theorem.

Theorem 12. *Let φ be an exponential polynomial of the form* (9)*. Then we have*

$$\mathcal{K}(\varphi) = p_0\,. \tag{21}$$

Another basic property of the K-transform is expressed by the following theorem.

Theorem 13. *The K-transformation is a continuous linear mapping from $\mathcal{MP}(\mathbb{R})$ into $\mathcal{P}(\mathbb{R})$, which commutes with all translations.*

Proof. By the definition of $\mathcal{C}(f)$ the K-transformation is clearly linear.

For the proof of continuity we remark that the mapping $f \mapsto f^-$ and hence also $f \mapsto g$ and $f \mapsto \mathcal{C}(f)$ are continuous on $\mathcal{MP}(\mathbb{R})$. Finally, the coefficients c_k of the Laurent expansion of $\mathcal{C}(f)$ can be expressed — by Cauchy's integral formulas — by path integrals which can be interchanged with taking uniform limits over compact sets. Hence the K-transformation is continuous from $\mathcal{MP}(\mathbb{R})$ into $\mathcal{P}(\mathbb{R})$.

Let φ be an exponential polynomial of the form (9) and y be real number. Then, by Theorems 11. and 12., we have

$$\tau_y K(\varphi)(x) = K(\varphi)(x+y) = p_0(x+y) = K(\tau_y\varphi)(x)$$

for each real x. Hence the K-transformation commutes with all translations on the exponential polynomials. By the spectral synthesis result Theorem 1, exponential polynomials form a dense subset in $\tau(f)$ for each mean periodic f, hence, by continuity, the theorem is proved. ■

Our main theorem follows.

Theorem 14. *For each mean periodic function f we have*

$$\mathcal{K}(f) = M(f)\,. \tag{22}$$

Proof. In [10] we have shown (see Theorem 4.2.5 on p. 64) that linearity and continuity together with the property of commuting with translations and leaving polynomials fixed characterize the operator M among the mappings from $\mathcal{MP}(\mathbb{R})$ into $\mathcal{P}(\mathbb{R})$. As we have seen in the previous theorems the operator $\mathcal{K}$ shares these properties with M, hence they are identical. ■

6. Fourier series and convergence

In (11) we have seen that if f is an exponential polynomial, then we have the representation

$$f(x) = \sum_{\lambda\in\mathbb{C}} \hat{f}(\lambda)(x)\, e^{\lambda x}\,. \tag{23}$$

This is a finite sum because $\hat{f}(\lambda) = 0$ if λ does not belong to the spectrum of f, and the spectrum is finite. The question arises: if f is an arbitrary mean periodic function, does a similar - not necessarily finite - sum converge to f in some sense? The answer is clearly negative even in the case of periodic functions but still we can get a kind of convergence in a special class of measures.

The measure (or compactly supported distribution) μ is called *slowly decreasing* if there are constants $A, B, \varepsilon > 0$ such that

$$\max\{|\hat{\mu}(y)| :\, y \in \mathbb{R},\ |x-y| \le A\ln(2+|x|)\} \ge \varepsilon(1+|x|)^{-B}\,.$$

For instance, if $\hat{\mu}$ is a nonzero exponential polynomial, then μ is slowly decreasing.

We shall formulate a convergence theorem for another class of mean periodic functions, namely for C^∞-mean periodic functions. Let $\mathcal{E}(\mathbb{R})$ denote the space $C^\infty(\mathbb{R})$ with the usual topology of uniform convergence of all derivatives over compact subsets. This is a locally convex topological vector space and its dual is the space of all compactly supported distributions. If μ is a compactly supported distribution and f is in $\mathcal{E}(\mathbb{R})$ satisfying

$$f * \mu = 0\,,$$

then f is called *mean periodic with respect to* μ, or simply *mean periodic*. Now we can formulate a convergence theorem for Fourier series.

Theorem 15 (L. Ehrenpreis, 1960)*. Let μ be a slowly decreasing compactly supported distribution and let f be a mean periodic function with respect to μ in $\mathcal{E}(\mathbb{R})$. Then there are finite disjoint subsets V_k $(k = 1, 2, \dots)$ of $sp(f)$ such that $\bigcup_k V_k = sp(f)$ and the series*

$$\sum_{k=1}^{\infty} \sum_{\lambda \in V_k} \hat{f}(\lambda)(x)\, e^{\lambda x}$$

converges to f in $\mathcal{E}(\mathbb{R})$.

We note that continuous mean periodic functions can be approximated very well by mean periodic functions in $\mathcal{E}(\mathbb{R})$. Indeed, let

$$\chi_\varepsilon(x) = \frac{1}{\varepsilon}\chi\Big(\frac{x}{\varepsilon}\Big)\,,$$

where χ is a compactly supported C^∞ function. Then $f_\varepsilon = \chi_\varepsilon * f$ tends to f in $\mathcal{E}(\mathbb{R})$. Further f_ε satisfies the same equation as f:

$$f_\varepsilon * \mu = (\chi_\varepsilon * f) * \mu = \chi_\varepsilon * (f * \mu) = 0\,.$$

Hence the theory of continuous mean periodic functions can be reduced to the theory of C^∞-mean periodic functions.

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ON FEJÉR TYPE SUMMABILITY OF WEIGHTED LAGRANGE INTERPOLATION ON THE LAGUERRE ROOTS

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Dedicated to Professor Antal Járai on his 60th and Professor Péter Vértesi on his 70th birthdays

Abstract. The sequence of certain arithmetic means of the Lagrange interpolation on the roots of Laguerre polynomials is shown to be uniformly convergent in suitable weighted function spaces.

1. Introduction

Let $w_\alpha(x) := x^\alpha e^{-x}$ $(x \in \mathbb{R}^+ := (0, +\infty),\ \alpha > -1)$ be a Laguerre weight and denote by $U_n(w_\alpha)$ $(n \in \mathbb{N} := \{1, 2, \ldots\})$ the root system of $p_n(w_\alpha)$ $(n \in \in \mathbb{N}_0 := \{0, 1, \ldots\})$ (orthonormal polynomials with respect to the weight w_α). We shall consider a Fejér type summation of Lagrange interpolation on $U_n(w_\alpha)$ $(n \in \mathbb{N})$. The corresponding polynomials will be denoted by $\sigma_n\big(f, U_n(w_\alpha), \cdot\big)$ (see (2.8)).

The goal of this paper is to give conditions for the parameters $\alpha > -1, \gamma \geq 0$ ensuring

$$\lim_{n\to+\infty} \big\| \big(f - \sigma_n\big(f, U_n(w_\alpha), \cdot\big)\big)\sqrt{w_\gamma}\big\|_\infty = 0$$

for all $f \in C_{\sqrt{w_\gamma}}$ (see Section 2.1), where $\|\cdot\|_\infty$ denotes the maximum norm.

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2. Notations and preliminaries

We are going to summarize definitions and statements on function spaces, weighted approximation, weighted Lagrange interpolation, which we shall need in the following sections.

2.1. Some weighted uniform spaces. Setting

$$w_\gamma(x) := x^\gamma e^{-x} \qquad (x \in \mathbb{R}_0^+ := [0, +\infty),\ \gamma \geq 0),$$

we define the weighted functional space $C_{\sqrt{w_\gamma}}$ as follows:

i) for $\gamma > 0$, $f \in C_{\sqrt{w_\gamma}}$ iff f is a continuous function in any segment $[a, b] \subset \mathbb{R}^+$ and

$$\lim_{x\to 0+0} f(x)\sqrt{w_\gamma(x)} = 0 = \lim_{x\to +\infty} f(x)\sqrt{w_\gamma(x)};$$

ii) for $\gamma = 0$, $f \in C_{\sqrt{w_0}}$ iff f is continuous in $[0, +\infty)$ and

$$\lim_{x\to +\infty} f(x)\sqrt{w_0(x)} = 0.$$

In other words, when $\gamma > 0$, the function f in $C_{\sqrt{w_\gamma}}$ could take very large values, with polynomial growth, as x approaches zero from the right, and could have an exponential growth as $x \to +\infty$.

If we introduce the norm

$$\|f\|_{\sqrt{w_\gamma}} := \big\|f\sqrt{w_\gamma}\big\|_\infty := \max_{x\in\mathbb{R}_0^+} \big|f(x)\big|\sqrt{w_\gamma(x)},$$

in $C_{\sqrt{w_\gamma}}$, $\gamma \geq 0$, then we get the Banach space $\big(C_{\sqrt{w_\gamma}}, \|\cdot\|_{\sqrt{w_\gamma}}\big)$.

2.2. Weighted polynomial approximation. We recall two fundamental results with respect to the polynomial approximation in the function space $\big(C_{\sqrt{w_\gamma}}, \|\cdot\|_{\sqrt{w_\gamma}}\big)$.

The first fact is that *the set of polynomials are dense in the function space* $\big(C_{\sqrt{w_\gamma}}, \|\cdot\|_{\sqrt{w_\gamma}}\big)$. More precisely, if we denote by $\mathcal{P}_n$ the linear space of all polynomials of degree at most n and by

$$E_n(f, \sqrt{w_\gamma}) := \inf_{P\in\mathcal{P}_n} \|(f - P)\sqrt{w_\gamma}\|_\infty = \inf_{P\in\mathcal{P}_n} \|f - P\|_{w_\gamma}$$

the best polynomial approximation of the function $f \in C_{\sqrt{w_\gamma}}$, then we have

$$\lim_{n\to+\infty} E_n(f, \sqrt{w_\gamma}) = 0$$

(see for example [9, p. 11] and [1, p. 186]).

The second fact is associated with *the Mhaskar–Rahmanov–Saff number*: For every $\gamma \geq 0$ and $n \in \mathbb{N}$ there are positive real numbers $a_n := a_n(\gamma)$ and $b_n := b_n(\gamma)$ such that for any polynomial $P \in \mathcal{P}_n$ we get

$$\|P\|_{\sqrt{w_\gamma}} = \|P\sqrt{w_\gamma}\|_\infty = \max_{x\in\mathbb{R}_0^+} \left|P(x)\right|\sqrt{w_\gamma(x)} = \max_{a_n\leq x\leq b_n} \left|P(x)\right|\sqrt{w_\gamma(x)} \tag{2.1}$$

and

$$\|P\sqrt{w_\gamma}\|_\infty > \left|P(x)\right|\sqrt{w_\gamma(x)} \qquad \text{for all } 0 \leq x < a_n \text{ and } b_n < x.$$

Moreover, for every $\gamma \geq 0$ and $n \in \mathbb{N}$ we have

$$\begin{aligned} a_n :&= a_n(\gamma) = (2n+\gamma)\left(1 - \sqrt{1 - \frac{\gamma^2}{(\gamma+2n)^2}}\right) > \frac{\gamma^2}{4n+2\gamma}, \\ b_n :&= b_n(\gamma) = (2n+\gamma)\left(1 + \sqrt{1 - \frac{\gamma^2}{(\gamma+2n)^2}}\right) = 4n + 2\gamma + \frac{C}{n} \end{aligned} \tag{2.2}$$

with a constant $C > 0$ independent of n (see for example [6, (2.1)]).

2.3. Weighted Lagrange interpolation. Let

$$p_n(w_\alpha, x) \qquad (x \in \mathbb{R}_0^+,\ n \in \mathbb{N}_0,\ \alpha > -1)$$

be the sequence of *orthonormal* Laguerre polynomials with positive leading coefficients. Let us denote by

$$U_n(w_\alpha) := \{y_{k,n} := y_{k,n}(w_\alpha) \mid k = 1, 2, \dots, n\} \quad (n \in \mathbb{N}) \tag{2.3}$$

the n different roots of $p_n(w_\alpha, \cdot)$. We index them as

$$0 < y_{1,n}(w_\alpha) < y_{2,n}(w_\alpha) < \dots < y_{n-1,n}(w_\alpha) < y_{n,n}(w_\alpha) < \infty.$$

For a given function $f : \mathbb{R}_0^+ \to \mathbb{R}$ we denote by $L_n(f, U_n(w_\alpha), \cdot)$ *the Lagrange interpolatory polynomial* of degree $\leq n-1$ at the zeros of $p_n(w_\alpha)$, i.e.

$$L_n\left(f, U_n(w_\alpha), y_{k,n}\right) = f(y_{k,n}) \qquad (k = 1, 2, \dots, n).$$

We have

$$L_n\,(f, U_n(w_\alpha), x) = \sum_{k=1}^{n} f(y_{k,n})\ell_{k,n}\,(U_n(w_\alpha), x)$$
$$(x \in \mathbb{R}_0^+, \quad n \in \mathbb{N}),$$

where

$$\ell_{k,n}\,(U_n(w_\alpha), x) = \frac{p_n(w_\alpha, x)}{p_n'(w_\alpha, y_{k,n})(x - y_{k,n})}$$
$$(x \in \mathbb{R}_0^+;\; k = 1, 2, \ldots, n;\; n \in \mathbb{N})$$

are *the fundamental polynomials* associated with the nodes $U_n(w_\alpha)$.

Consider the (uniform) convergence of the sequence $L_n\big(f, U_n(w_\alpha), \cdot\big)$ $(n \in \mathbb{N})$ in the Banach space $\big(C_{\sqrt{w_\gamma}}, \|\cdot\|_{\sqrt{w_\gamma}}\big)$. In other words, for a function $f \in C_{\sqrt{w_\gamma}}$ we have to investigate the real sequence

$$\varrho_n(f) := \Big\|\Big(f - L_n\big(f, U_n(w_\alpha), \cdot\big)\Big)\sqrt{w_\gamma}\Big\|_\infty \quad (n \in \mathbb{N}).$$

In other words, we approximate the function $f\sqrt{w_\gamma}$ by *the weighted Lagrange interpolatory polynomials*

$$L_n\big(f, U_n(w_\alpha), x\big)\sqrt{w_\gamma(x)} \quad (x \in \mathbb{R}_0^+,\; n \in \mathbb{N}). \tag{2.4}$$

The main question is: is it true that $\varrho_n(f) \to 0$ $(n \to +\infty)$ for all $f \in C_{\sqrt{w_\gamma}}$ or not?

The classical Lebesgue estimate for the weighted Lagrange interpolation is the following: take the best uniform approximation $P_{n-1}(f)$ to $f \in C_{w_\gamma}$ (the existence of such a $P_{n-1}(f)$ is obvious), and consider

$$\Big|f(x) - L_n\big(f, U_n(w_\alpha), x\big)\Big|\sqrt{w_\gamma(x)} \leq$$
$$\leq \Big|f(x) - P_{n-1}(f, x)\Big|\sqrt{w_\gamma(x)} + \Big|L_n\Big(f - P_{n-1}(f), U_n(w_\alpha), x\Big)\Big|\sqrt{w_\gamma(x)} \leq$$
$$\leq E_{n-1}\big(f, \sqrt{w_\gamma}\big)\left(1 + \sum_{k=1}^{n}\Big|\ell_{k,n}\big(U_n(w_\alpha, x)\big)\Big|\frac{\sqrt{w_\gamma(x)}}{\sqrt{w_\gamma(y_{k,n})}}\right).$$

This estimate shows that the pointwise/uniform convergence of the sequence (2.4) depends on the orders of *the weighted Lebesgue functions*:

$$\lambda_n\big(U_n(w_\alpha), \sqrt{w_\gamma}, x\big) := \sum_{k=1}^{n}\Big|\ell_{k,n}\big(U_n(w_\alpha, x)\big)\Big|\frac{\sqrt{w_\gamma(x)}}{\sqrt{w_\gamma(y_{k,n})}}$$
$$(x \in \mathbb{R}_0^+,\; n \in \mathbb{N}),$$

and on the orders of *the weighted Lebesgue constants*:

$$\Lambda_n\big(U_n(w_\alpha),\sqrt{w_\gamma}\big) := \sup_{x\in\mathbb{R}_0^+} \lambda_n\big(U_n(w_\alpha),\sqrt{w_\gamma},x\big) \quad (n\in\mathbb{N}).$$

It is clear that for all $\gamma\geq 0$, $\alpha>-1$ and $n\in\mathbb{N}$

$$\begin{aligned}\mathcal{L}_n(\cdot,U_n(w_\alpha),\sqrt{w_\gamma}) \ :\ &\big(C_{\sqrt{w_\gamma}},\|\cdot\|_{\sqrt{w_\gamma}}\big)\to\mathcal{P}_{n-1}\subset\big(C_{\sqrt{w_\gamma}},\|\cdot\|_{\sqrt{w_\gamma}}\big)\\ &\mathcal{L}_n(f,U_n(w_\alpha),\sqrt{w_\gamma}) := L_n\big(f,U_n(w_\alpha),\cdot\big)\end{aligned} \tag{2.5}$$

is a bounded linear operator and its norm is

$$\begin{aligned}\|\mathcal{L}_n(\cdot,U_n(w_\alpha),\sqrt{w_\gamma})\| :&= \sup_{0\not\equiv f\in C_{\sqrt{w_\gamma}}} \frac{\|\mathcal{L}_n(f,U_n(w_\alpha),\sqrt{w_\gamma})\|_{\sqrt{w_\gamma}}}{\|f\|_{\sqrt{w_\gamma}}} =\\ &= \sup_{0\not\equiv f\in C_{w_\gamma}} \frac{\|L_n\big(f,U_n(w_\alpha),\cdot\big)\sqrt{w_\gamma}\|_\infty}{\|f\sqrt{w_\gamma}\|_\infty}.\end{aligned}$$

Since

$$\begin{aligned}L_n\big(f,U_n(w_\alpha),x\big) &= \sum_{k=1}^{n} f(y_{k,n})\ell_{k,n}\big(U_n(w_\alpha),x\big) =\\ &= \sum_{k=1}^{n} f(y_{k,n})\sqrt{w_\gamma(y_{k,n})}\cdot\ell_{k,n}\big(U_n(w_\alpha),x\big)\cdot\frac{1}{\sqrt{w_\gamma(y_{k,n})}},\end{aligned}$$

thus by a usual argument we have that the norm of the operator (2.5) equals to the n-th Lebesgue constant, i.e.

$$\big\|\mathcal{L}_n(\cdot,U_n(w_\alpha),\sqrt{w_\gamma})\big\| = \Lambda_n\big(U_n(w_\alpha),\sqrt{w_\gamma}\big) \quad (n\in\mathbb{N}).$$

The pointwise/uniform convergence of $L_n(f,U_n(w_\alpha),\cdot)$ $(n\in\mathbb{N})$ in different function spaces were investigated by several authors (see [3], [8], [6]). For example in 2001, G. Mastroianni and D. Occorsio showed that for arbitrary $\gamma\geq 0$ and $\alpha>-1$ the order of the norm of the operator $\mathcal{L}_n(\cdot,U_n(w_\alpha),\sqrt{w_\gamma})$ is $n^{1/6}$ (see [6, Theorem 3.3]), i.e.

$$\|\mathcal{L}_n(\cdot,U_n(w_\alpha),\sqrt{w_\gamma})\| \sim n^{1/6} \quad (n\in\mathbb{N}).$$

Here and in the sequel, if A and B are two expressions depending on certain indices and variables, then we write

$$A\sim B, \qquad \text{if and only if} \qquad 0<C_1\leq\left|\frac{A}{B}\right|\leq C_2$$

uniformly for the indices and variables considered.

From results of P. Vértesi it follows that for *any* interpolatory matrix $X_n \subset \subset \mathbb{R}_0^+$ $(n \in \mathbb{N})$ the order of the corresponding weighted Lebesgue constants is at least $\log n$, i.e. if $\gamma \geq 0$ and $X_n \subset \mathbb{R}^+$ $(n \in \mathbb{N})$ is an arbitrary interpolatory matrix then there exists a constant $C > 0$ independent of n such that

$$\Lambda_n(X_n, \sqrt{w_\gamma}) = \|\mathcal{L}_n(\cdot, X_n, \sqrt{w_\gamma})\| \geq C \, \log n \quad (n \in \mathbb{N}).$$

See [16, Theorem 7.2], [14] and [15]. Thus using the Banach–Steinhaus theorem we obtain the following Faber type result:

Theorem A. *If $\gamma \geq 0$ and $X_n \subset \mathbb{R}^+$ $(n \in \mathbb{N})$ is an arbitrary interpolatory matrix then there exists a function $f \in C_{\sqrt{w_\gamma}}$ for which the relation*

$$\left\|(f - L_n(f, X_n, \cdot))\sqrt{w_\gamma}\right\|_\infty \to 0 \quad as \quad n \to +\infty$$

does not hold.

In [6] G. Mastroianni and D. Occorsio also proved that there exist point systems for which the optimal Lebesgue constants can be attained. We recall only the following result:

Theorem B (see [6, Theorem 3.4]). *If $\mathcal{V}_{n+1} := U_n(w_\alpha) \cup \{4n\}$, then*

$$\|\mathcal{L}_{n+1}\big(\cdot, \mathcal{V}_{n+1}, \sqrt{w_\gamma}\big)\| = \Lambda_{n+1}\big(\mathcal{V}_{n+1}, \sqrt{w_\gamma}\big) \sim \log n \quad (n \in \mathbb{N})$$

if and only if the parameters $\alpha > -1$ and $\gamma \geq 0$ satisfy the additional conditions:

$$\frac{\alpha}{2} + \frac{1}{4} \leq \gamma \leq \frac{\alpha}{2} + \frac{5}{4}.$$

2.4. Fejér type sums. Using the Christoffel–Darboux formula [12, Theorem 3.2.2] we write the Lagrange interpolatory polynomials as

$$\text{(2.6)} \qquad L_n\big(f, U_n(w_\alpha), x\big) = \sum_{l=0}^{n-1} c_{l,n}(f) p_l(w_\alpha, x) \quad (x \in \mathbb{R}_0^+, \ n \in \mathbb{N}),$$

where

$$\text{(2.7)} \qquad c_{l,n}(f) := \sum_{k=1}^{n} f(y_{k,n}) p_l\left(w_\alpha, y_{k,n}\right) \lambda_{k,n} \quad (l = 0, 1, \ldots, n-1, \ n \in \mathbb{N}).$$

Here and in the sequel $\lambda_{k,n} := \lambda_{k,n}(w_\alpha)$ $(k = 1, 2, \ldots, n, \ n \in \mathbb{N})$ denote the Christoffel numbers with respect to the weight w_α (cf. [12, (15.3.5)]).

Using (2.6) and (2.7) we have

$$L_n(f,x) := L_n\big(U_n(w_\alpha), f, x\big) = \sum_{l=0}^{n-1} c_{l,n}(f) p_l(w_\alpha, x) =$$
$$= \sum_{k=1}^{n} f\left(y_{k,n}\right) K_{n-1}\left(x, y_{k,n}\right) \lambda_{k,n},$$

where

$$K_{n-1}(x,y) := \sum_{l=0}^{n-1} p_l\left(w_\alpha, x\right) p_l\left(w_\alpha, y\right)$$
$$\big(x, y \in \mathbb{R}_0^+,\ n \in \mathbb{N}\big).$$

Let

$$L_{n,m}(f,x) := \sum_{l=0}^{m} c_{l,n}(f) p_l(w_\alpha, x) = \sum_{k=1}^{n} f\left(y_{k,n}\right) K_m\left(x, y_{k,n}\right) \lambda_{k,n}$$
$$\big(x \in \mathbb{R}_0^+,\ m = 0, 1, \ldots, n-1,\ n \in \mathbb{N}\big).$$

The *Fejér means of the Lagrange interpolation* of the function $f : \mathbb{R}_0^+ \to \mathbb{R}$ are defined as the arithmetic means of the sums $L_{n,0}$, $L_{n,1}$, …, $L_{n,n-1}$, i.e.

$$\sigma_n(f,x) := \sigma_n\big(f, U_n(w_\alpha), x\big) :=$$

(2.8)
$$:= \frac{L_{n,0}(f,x) + L_{n,1}(f,x) + \cdots + L_{n,n-1}(f,x)}{n}$$
$$(x \in \mathbb{R}_0^+,\ n \in \mathbb{N}).$$

From the above formulas we have

$$\sigma_n(f,x) = \sum_{l=0}^{n-1} \left(1 - \frac{l}{n}\right) c_{l,n}(f) p_l(w_\alpha, x) =$$

(2.9)
$$= \sum_{k=1}^{n} f(y_{k,n}) \Big\{ \frac{1}{n} \sum_{m=0}^{n-1} K_m(x, y_{k,n}) \Big\} \lambda_{k,n} =$$
$$= \sum_{k=1}^{n} f(y_{k,n}) K_n^{(1)}(x, y_{k,n}) \lambda_{k,n},$$

where

(2.10)
$$K_n^{(1)}(x,y) := \frac{1}{n} \sum_{m=0}^{n-1} K_m(x,y) = \sum_{l=0}^{n-1} \left(1 - \frac{l}{n}\right) p_l(w_\alpha, x) p_l(w_\alpha, y)$$
$$\big(x, y \in \mathbb{R}_0^+,\ n \in \mathbb{N}\big).$$

Remark 1. It is important to observe that we defined the Fejér means of Lagrange interpolation by considering the means (2.8) and *not* by the means

$$\frac{L_0(f,x)+L_1(f,x)+\cdots+L_{n-1}(f,x)}{n} \quad (x\in\mathbb{R}_0^+,\ n\in\mathbb{N}). \tag{2.11}$$

Several earlier results suggest that the two methods (2.8) and (2.11) have different behavior with respect to uniform convergence.

For example in the trigonometric case J. Marcinkiewicz [4] proved that the method corresponding to (2.8) is uniformly convergent in $C_{2\pi}$ (the Banach space of 2π periodic continuous functions defined on $\mathbb{R}$ endowed with the maximum norm), moreover there exists a function $f\in C_{2\pi}$ such that the sequence corresponding to (2.11) diverges at a point. In other words we have an analogue of the classical theorem of L. Fejér about the uniform convergence of the $(C,1)$ means of the partial sums of the trigonometric Fourier series *only for suitable* arithmetic means of the Lagrange interpolation.

The situation is similar if we consider the Lagrange interpolation on the roots of the Chebyshev polynomials of the first kind. In [13] A.K. Varma and T.M. Mills showed that the (2.8) type means of the Lagrange interpolation uniformly convergent for every $f\in C[-1,1]$. Moreover in [2] P. Erdős and G. Halász proved that there exists a continuous function for which the (2.11) type means are almost everywhere divergent on the interval $[-1,1]$.

3. Uniform convergence of suitable arithmetic means

The main goal of this paper is to show that the (2.8) type arithmetic means of the Lagrange interpolation on the roots of Laguerre polynomials is uniformly convergent in suitable weighted function spaces.

Theorem. *Let* $\alpha>-1$ *and* $0\le\gamma=:\alpha+2r$*, i.e.* $\sqrt{w_\gamma(x)}=\sqrt{w_\alpha(x)}x^r$ $(x\in\mathbb{R}^+)$*. If*

$$-\min\left(\frac{\alpha}{2},\frac{1}{4}\right)<r\le\frac{7}{6}, \tag{3.1}$$

then

$$\lim_{n\to+\infty}\left\|\Big(f-\sigma_n(f,U_n(w_\alpha),\cdot)\Big)\sqrt{w_\gamma}\right\|_\infty=0 \tag{3.2}$$

holds for all $f\in C_{\sqrt{w_\gamma}}$*.*

Remark 2. We intend to investigate the convergence of the method (2.11) in a subsequent paper.

Remark 3. The formulas (2.6) and (2.7) show that the Lagrange interpolation polynomials on the roots of Laguerre polynomials can be considered as a discrete version of partial sums of the Fourier series with respect to the system of Laguerre polynomials. In [9] E.L. Poiani proved (among other things) that the sequence of the $(C,1)$ means of the Laguerre series of an arbitrary function $f \in C_{w_\gamma}$ $(\gamma = 2r+\alpha,\ \alpha > -1)$ converges to f in the space $\left(C_{w_\gamma, \|\cdot\|_{w_\gamma}}\right)$, if

$$-\min\left(\frac{\alpha}{2},\frac{1}{2}\right) < r < 1 + \min\left(\frac{\alpha}{2},\frac{1}{4}\right) \quad \text{and} \quad -\frac{1}{2} \le r \le \frac{7}{6}.$$

4. Proof of the Theorem

4.1. Laguerre polynomials. We mention some relations with respect to the Laguerre polynomials which will be used later. Let $\{p_n(w_\alpha)\}$, $\alpha > -1$, be the sequence of *orthonormal* Laguerre polynomials with positive leading coefficients. The zeros $y_{k,n} := y_{k,n}(w_\alpha)$ of $p_n(w_\alpha)$, $n \ge 1$ satisfy

$$\frac{C_1}{n} < y_{1,n} < y_{2,n} < \ldots < y_{n,n} = 4n + 2\alpha + 2 - C_2\sqrt[3]{4n}, \tag{4.1}$$

$$y_{k,n} \sim \frac{k^2}{n} \qquad (k = 1,2,\ldots,n,\ n \in \mathbb{N}) \tag{4.2}$$

(see [12, Section 6.32] and [5, Section 2.3.5]).

Here and what follows $C, C_1, \ldots$ will always denote positive constants (not necessarily the same at each occurrence) being independent of parameters k and n.

It is known that

$$\triangle y_{k,n} := y_{k+1,n} - y_{k,n} \sim \sqrt{\frac{y_{k,n}}{4n - y_{k,n}}} \tag{4.3}$$
$$(k = 1,2,\ldots,n-1,\ n \in \mathbb{N}),$$

and for $y_{k,n} \le x \le y_{k+1,n}$ $(k = 1,2,\ldots,n-1)$ we have

$$\sqrt{\frac{y_{k,n}}{4n - y_{k,n}}} \sim \sqrt{\frac{x}{4n - x}} \sim \sqrt{\frac{y_{k+1,n}}{4n - y_{k+1,n}}}$$

uniformly in k and n (see [6, (2.4) and (2.5)]). From (4.2) and (4.3) it follows that

$$|y_{j,n} - y_{k,n}| \geq C\, \frac{|j^2 - k^2|}{n} \qquad (j, k = 1, 2, \dots, n). \tag{4.4}$$

For the Christoffel numbers we have

$$\lambda_{k,n} := \lambda_{k,n}(w_\alpha) \sim w_\alpha(y_{k,n}) \sqrt{\frac{y_{k,n}}{4n - y_{k,n}}} \sim w_\alpha(y_{k,n}) \triangle y_{k,n} \tag{4.5}$$

uniformly in $k = 1, 2, \dots, n$ and $n \in \mathbb{N}$ (see [6, (2.7)]).

In an article of B. Muckenhoupt and D.W. Webb [7] there is a pointwise upper estimate for the kernel of (C, δ) $(\delta > 0)$ Cesàro means of Laguerre–Fourier series (see also [17]). We shall use this result only with respect to $(C, 1)$ means, that is for the kernel function $K_n^{(1)}(x, y)$ (see (2.10)): Let $\alpha > -1$. Then we have

$$\left| K_n^{(1)}(x, y) \right| \leq \frac{C}{\sqrt{w_\alpha(x)}\, \sqrt{w_\alpha(y)}}\, G_n(x, y) \qquad (0 < x, y < \nu(n) + \sqrt[3]{\nu(n)},\ n \in \mathbb{N}), \tag{4.6}$$

where $\nu := \nu(n) := 4n + 2\alpha + 2$,

$$G_n(x,y) := \frac{1}{\nu}\, \mathcal{M}_n(x) \mathcal{M}_n(y) \frac{(x + y)\left[\nu^{1/3} + |x - \nu| + |y - \nu|\right]^2}{(x + y) + (x - y)^2 \left[\nu^{1/3} + |x - \nu| + |y - \nu|\right]} \tag{4.7}$$

and

$$\mathcal{M}_n(x) := \frac{x^{\alpha/2}\left(x + \frac{1}{\nu}\right)^{-\alpha/2 - 1/4}}{\sqrt[4]{\nu^{1/3} + |x - \nu|}} \tag{4.8}$$

(see [7, p. 1124]).

Denote by $y_{j,n}$ one of the closest root(s) to x (shortly $x \approx y_{j,n}, j = j(n)$). Using the above relations we obtain that

$$\mathcal{M}_n(x) \sim \mathcal{M}_n(y_{j,n}) \sim \begin{cases} \dfrac{1}{\sqrt{j}}, & \text{if } \dfrac{c}{n} \leq x \leq \dfrac{\nu}{2} \\ \dfrac{1}{\sqrt[4]{n |y_{j,n} - \nu|}}, & \text{if } \dfrac{\nu}{2} \leq x \leq \nu - \sqrt[3]{\nu} \\ \dfrac{1}{\sqrt[3]{n}}, & \text{if } \nu - \sqrt[3]{\nu} \leq x \leq \nu + \sqrt[3]{\nu} \end{cases} \tag{4.9}$$

for $x \in [c/n, \nu + \sqrt[3]{\nu}]$.

4.2. Uniform boundedness. Let us consider for every $n \in \mathbb{N}$ the bounded linear operator

$$\mathcal{F}_n : \big(C_{\sqrt{w_\gamma}}, \|\cdot\|_{\sqrt{w_\gamma}}\big) \to \mathcal{P}_n \subset \big(C_{\sqrt{w_\gamma}}, \|\cdot\|_{\sqrt{w_\gamma}}\big)$$

$$\mathcal{F}_n f := \sigma_n(f, U_n(w_\alpha), \cdot).$$

For the norm of the operator $\mathcal{F}_n$ we obtain that (see (2.9))

$$\|\mathcal{F}_n\| := \sup_{0 \not\equiv f \in C_{\sqrt{w_\gamma}}} \frac{\|\mathcal{F}_n f\|_{\sqrt{w_\gamma}}}{\|f\|_{\sqrt{w_\gamma}}} = \sup_{0 \not\equiv f \in C_{\sqrt{w_\gamma}}} \frac{\big\|\sigma_n\big(f, U_n(w_\alpha), \cdot\big)\sqrt{w_\gamma}\big\|_\infty}{\big\|f\sqrt{w_\gamma}\big\|_\infty} =$$

$$= \max_{x \in \mathbb{R}_0^+} \sum_{k=1}^{n} \big|K_n^{(1)}(x, y_{k,n})\big| \frac{\sqrt{w_\gamma(x)}}{\sqrt{w_\gamma(y_{k,n})}} \lambda_{k,n}.$$

The core of the proof of the Theorem is contained in the following lemma, which states the uniform boundedness of the operator sequence $(\mathcal{F}_n)$.

Lemma 4.1. *Let $\alpha > -1$ and r satisfy the inequality* (3.1). *Then there exists a constant $C > 0$ independent of $n \in \mathbb{N}$ such that*

$$\|\mathcal{F}_n\| = \max_{x \in \mathbb{R}_0^+} \sum_{k=1}^{n} \big|K_n^{(1)}(x, y_{k,n})\big| \frac{\sqrt{w_\alpha(x)}}{\sqrt{w_\alpha(y_{k,n})}} \Big(\frac{x}{y_{k,n}}\Big)^r \lambda_{k,n} \le C. \tag{4.10}$$

Proof. We shall use the following important equality (see [11, Lemma 1]): If $\gamma \ge 0$, $m \le n \in \mathbb{N}$ and $q_k \in \mathcal{P}_n$ $(k = 1, 2, \dots, m)$ are arbitrary polynomials then

$$\max_{x \in \mathbb{R}_0^+} \Big[\sqrt{w_\gamma}(x) \sum_{k=1}^{m} |q_k(x)|\Big] = \max_{a_n \le x \le b_n} \Big[\sqrt{w_\gamma}(x) \sum_{k=1}^{m} |q_k(x)|\Big].$$

Therefore by (4.5)–(4.7) it is enough to show that

$$\max_{a_n \le x \le b_n} \sum_{k=1}^{n} G_n(x, y_{k,n}) \left(\frac{x}{y_{k,n}}\right)^r \sqrt{\frac{y_{k,n}}{4n - y_{k,n}}} \le C, \tag{4.11}$$

where

$$\frac{c}{n} \le a_n = a_n(\gamma) \le x \le b_n = b_n(\gamma) < \nu + \sqrt[3]{\nu}.$$

In order to prove (4.11), we distinguish several cases.

CASE 1: Let $x \in [a_n, \frac{\nu}{2}]$ and

$$\sum_{k=1}^{n} G_n(x, y_{k,n}) \left(\frac{x}{y_{k,n}}\right)^r \sqrt{\frac{y_{k,n}}{4n - y_{k,n}}} = \tag{4.12}$$

$$\sum_{y_{k,n} \le \frac{\nu}{2}} \ldots + \sum_{y_{k,n} > \frac{\nu}{2}} \ldots =: A_n^{(1)}(x) + A_n^{(2)}(x).$$

Since $\nu^{1/3} + |x - \nu| + |y_{k,n} - \nu| \sim n$ $(k = 1, 2, \ldots, n,\ n \in \mathbb{N})$ thus by (4.2), (4.4), (4.7) and (4.9) we have

$$A_n^{(1)}(x) \le C_1 \sum_{y_{k,n} \le \frac{\nu}{2}} n \frac{j^2 + k^2}{j^2 + k^2 + |j^2 - k^2|^2} \frac{1}{\sqrt{kj}} \left(\frac{j}{k}\right)^{2r} \frac{k}{n} \le$$

$$\le C_2 \left\{ \sum_{k \le \frac{j}{2}} \frac{j^{2r-5/2}}{k^{2r-1/2}} + \sum_{\frac{j}{2} \le k \le 2j} \frac{1}{1 + (k - j)^2} + \sum_{k \ge 2j} \frac{j^{2r-1/2}}{k^{2r+3/2}} \right\}.$$

The second sum is bounded. For the first sum we obtain that

$$\sum_{k \le j/2} \frac{j^{2r-5/2}}{k^{2r-1/2}} \sim \begin{cases} \dfrac{\log j}{j}, & \text{if } r = \dfrac{3}{4} \\ j^{2r-5/2}, & \text{if } r > \dfrac{3}{4} \\ \dfrac{1}{j}, & \text{if } r < \dfrac{3}{4} \end{cases}$$

and these expressions are bounded (independently of j and n), if $r \le 5/4$. Moreover by

$$\sum_{k=j}^{n} \frac{1}{k^s} \sim \begin{cases} \log \frac{n}{j}, & \text{if } s = 1 \\ \left|n^{-s+1} - j^{-s+1}\right|, & \text{if } s \ne 1 \end{cases}$$

we have

$$\sum_{k \ge 2j} \frac{j^{2r-1/2}}{k^{2r+3/2}} \sim \begin{cases} \dfrac{\log \frac{n}{j}}{j}, & \text{if } r = -\dfrac{1}{4} \\ \dfrac{1}{j}\left|\left(\dfrac{j}{n}\right)^{2r+1/2} - 1\right|, & \text{if } r \ne -\dfrac{1}{4} \end{cases}$$

whence the third sum is bounded (independently of j and n), if $r > -\frac{1}{4}$. Therefore

$$A_n^{(1)}(x) \le C \quad \left(x \in [a_n, \tfrac{\nu}{2}],\ n \in \mathbb{N}\right), \quad \text{if } -\tfrac{1}{4} < r \le \tfrac{5}{4}. \tag{4.13}$$

Let us consider $A_n^{(2)}(x)$. Since $y_{k,n} \geq \frac{\nu}{2}$ thus by (4.2), (4.4), (4.7) and (4.9) we have

$$A_n^{(2)}(x) \leq$$

$$\leq C_1 \sum_{y_{k,n} \geq \frac{\nu}{2}} \frac{n}{1+|y_{j,n}-y_{k,n}|^2} \frac{1}{\sqrt{j}} \frac{1}{\sqrt[4]{n|y_{k,n}-\nu|}} \left(\frac{j}{k}\right)^{2r} \sqrt{\frac{y_{k,n}}{4n-y_{k,n}}} \leq$$

$$\leq C_2 \left\{ \sum_{\frac{\nu}{2} \leq y_{k,n} \leq \frac{x+y_{n,n}}{2}} \cdots + \sum_{\frac{x+y_{n,n}}{2} < y_{k,n}} \cdots \right\} =: \tag{4.14}$$

$$=: A_n^{(21)}(x) + A_n^{(22)}(x).$$

If $\frac{\nu}{2} \leq y_{k,n} \leq \frac{x+y_{n,n}}{2}$ then $|y_{k,n}-\nu| \sim n$ thus by (4.2) and (4.4) we obtain that

$$A_n^{(21)}(x) \leq C_1 \left(\frac{j}{n}\right)^{2r-1/2} \sum_{\frac{\nu}{2} \leq y_{k,n} \leq \frac{x+y_{n,n}}{2}} \frac{1}{1+|k-j|^2}.$$

If $x \approx y_{j,n} \leq \frac{\nu}{4}$ and $y_{k,n} \geq \frac{\nu}{2}$ then $|k-j| \geq cn$ therefore in this case

$$A_n^{(21)}(x) \leq C \frac{1}{j} \left(\frac{j}{n}\right)^{2r+1/2},$$

which is bounded (independently of j and n), if $r \geq -\frac{1}{4}$. Moreover, if $x \approx y_{j,n} \geq \geq \frac{\nu}{4}$ then $j \sim n$ hence $A_n^{(21)}$ is bounded for all r.

If $y_{k,n} \geq (x+y_{n,n})/2$ then $|y_{j,n}-y_{k,n}| \sim n$ thus by (4.3) and (4.14) we obtain that

$$A_n^{(22)}(x) \leq C_1 \frac{j^{2r-1/2}}{n^{2r+5/4}} \sum_{\frac{x+y_{n,n}}{2} \leq y_{k,n}} \frac{\triangle y_{k,n}}{\sqrt[4]{|y_{k,n}-\nu|}} \leq$$

$$\leq C_2 \frac{j^{2r-1/2}}{n^{2r+5/4}} \int_{\nu/2}^{y_{n,n}} \frac{1}{\sqrt[4]{\nu-t}} dt \leq$$

$$\leq C_3 \frac{j^{2r-1/2}}{n^{2r+5/4}} n^{3/4} = C_3 \left(\frac{j}{n}\right)^{2r+1/2} \frac{1}{j},$$

and this is bounded, if $r \geq -\frac{1}{4}$. Consequently

$$A_n^{(2)}(x) \leq C \quad \left(x \in \left[a_n, \tfrac{\nu}{2}\right],\ n \in \mathbb{N}\right), \quad \text{if } -\tfrac{1}{4} \leq r. \tag{4.15}$$

By (4.13)–(4.15) we get: there exists a constant $C > 0$ independent of x and n such that

$$A_n^{(1)}(x) + A_n^{(2)}(x) \le C \quad \left(x \in \left[a_n, \tfrac{\nu}{2}\right],\ n \in \mathbb{N}\right) \quad \text{if } -\tfrac{1}{4} < r \le \tfrac{5}{4}. \tag{4.16}$$

Case 2: Let $x \in [\frac{1}{2}\nu, \frac{3}{4}\nu]$ and

$$\sum_{k=1}^{n} G_n(x, y_{k,n}) \left(\frac{x}{y_{k,n}}\right)^r \sqrt{\frac{y_{k,n}}{4n - y_{k,n}}} =$$

$$= \sum_{y_{k,n} \le \frac{\nu}{4}} \dots + \sum_{\frac{\nu}{4} < y_{k,n} \le \frac{7}{8}\nu} \dots + \sum_{\frac{7}{8}\nu < y_{k,n}} \dots =: \tag{4.17}$$

$$:= B_n^{(1)}(x) + B_n^{(2)}(x) + B_n^{(3)}(x).$$

If $x \in [\frac{\nu}{2}, \frac{3}{4}\nu]$ and $y_{k,n} \le \frac{\nu}{4}$ then $|x - y_{k,n}| \sim n$ therefore by (4.7) and (4.9) we get

$$B_n^{(1)}(x) \le C_1 \sum_{y_{k,n} \le \frac{\nu}{4}} \frac{1}{n} \frac{nn^2}{n + n^2 n} \frac{1}{\sqrt{n}} \frac{1}{\sqrt{k}} \left(\frac{n}{k}\right)^{2r} \frac{k}{n} \le$$

$$\le C_2 \sum_{k=1}^{n} \frac{n^{2r-5/2}}{k^{2r-1/2}} \sim \begin{cases} \dfrac{\log n}{n}, & \text{if } r = \dfrac{3}{4} \\ n^{2r-5/2}, & \text{if } r > \dfrac{3}{4} \\ \dfrac{1}{n}, & \text{if } r < \dfrac{3}{4} \end{cases}$$

and this is bounded, if $r \le \frac{5}{4}$.

If $x \in [\frac{\nu}{2}, \frac{3}{4}\nu]$ and $\frac{\nu}{4} \le y_{k,n} \le \frac{7}{8}\nu$ then

$$|x - y_{k,n}| \ge c_1 \frac{|j^2 - k^2|}{n} \ge c_2 |j - k|$$

(see (4.4)) thus by (4.7) and (4.9) we have

$$B_n^{(2)}(x) \le C_1 \sum_{\frac{\nu}{4} \le y_{k,n} \le \frac{7}{8}\nu} \frac{1}{n} \frac{nn^2}{n + |j-k|^2 n} \frac{1}{\sqrt{n}} \frac{1}{\sqrt{n}} \le C_2 \sum_{k=1}^{n} \frac{1}{1 + |j-k|^2},$$

i.e. this term is bounded for all r.

If $x \in [\frac{1}{2}\nu, \frac{3}{4}\nu]$ and $y_{k,n} \geq \frac{7}{8}\nu$ then $|x - y_{k,n}| \geq cn$ thus

$$B_n^{(3)}(x) \leq C_1 \sum_{\frac{7}{8}\nu \leq y_{k,n}} \frac{1}{n} \frac{nn^2}{n + n^2 n} \frac{1}{\sqrt{n}} \frac{1}{\sqrt[4]{n|y_{k,n} - \nu|}} \sqrt{\frac{y_{k,n}}{4n - y_{k,n}}} \leq$$

$$\leq \frac{C_2}{n^{7/4}} \int_{\frac{7}{8}\nu}^{y_{n,n}} \frac{1}{\sqrt[4]{\nu - t}} dt \leq C_3 \frac{n^{3/4}}{n^{7/4}}$$

which means that this term is also bounded for all r.

Consequently there exists a constant $C > 0$ independent of x and n such that

$$B_n^{(1)}(x) + B_n^{(2)}(x) + B_n^{(3)}(x) \leq C \quad \left(x \in \left[\tfrac{1}{2}\nu, \tfrac{3}{4}\nu\right],\ n \in \mathbb{N}\right), \quad \text{if } r \leq \tfrac{5}{4}. \tag{4.18}$$

Case 3: Let $x \in [\frac{3}{4}\nu, y_{n,n}]$ and

$$\sum_{k=1}^{n} G_n(x, y_{k,n}) \left(\frac{x}{y_{k,n}}\right)^r \sqrt{\frac{y_{k,n}}{4n - y_{k,n}}} =$$

$$= \sum_{y_{k,n} \leq \frac{5\nu}{8}} \ldots + \sum_{\frac{5\nu}{8} < y_{k,n} < y_{j-1,n}} \ldots + \sum_{k=j-1}^{j+1} \ldots + \sum_{y_{j+1,n} < y_{k,n} < \frac{x + y_{n,n}}{2}} \ldots + \tag{4.19}$$

$$+ \sum_{\frac{x + y_{n,n}}{2} \leq y_{k,n}} \ldots =:$$

$$=: D_n^{(1)}(x) + D_n^{(2)}(x) + D_n^{(3)}(x) + D_n^{(4)}(x) + D_n^{(5)}(x).$$

If $y_{k,n} \leq \frac{5}{8}\nu$ then $|x - y_{k,n}| \sim n$ and $|y_{j,n} - \nu| \geq c\sqrt[3]{n}$ therefore by (4.7) and (4.9) we get

$$D_n^{(1)}(x) \leq C_1 \sum_{y_{k,n} \leq \frac{5\nu}{8}} \frac{1}{n} \frac{nn^2}{n + n^2 n} \frac{1}{\sqrt[4]{n|y_{j,n} - \nu|}} \frac{1}{\sqrt{k}} \left(\frac{n}{k}\right)^{2r} \frac{k}{n} \leq$$

$$\leq C_2 \sum_{k=1}^{n} \frac{n^{2r - 7/3}}{k^{2r - 1/2}} \sim \begin{cases} \dfrac{\log n}{n^{5/6}}, & \text{if } r = \dfrac{3}{4} \\ n^{2r - 7/3}, & \text{if } r > \dfrac{3}{4} \\ n^{-5/6}, & \text{if } r < \dfrac{3}{4} \end{cases}$$

and this is bounded, if $r \leq \frac{7}{6}$.

If $x \in [\frac{3}{4}\nu, y_{n,n}]$ and $\frac{5}{8}\nu \le y_{k,n} < y_{j-1,n}$ then

$$\nu^{1/3} + |x-\nu| + |y_{k,n}-\nu| \sim |y_{k,n}-\nu| \ge c|y_{j,n}-\nu|,$$

$$|y_{k,n}-\nu| \le cn,$$

$$x - y_{j-2,n} \ge \triangle y_{j-1,n} \sim \triangle y_{j,n} \sim \sqrt{\frac{n}{\nu - y_{j,n}}}$$

thus

$$D_n^{(2)}(x) \le C_1 \sum_{\frac{5\nu}{8} \le y_{k,n} < y_{j-1,n}} \frac{1}{n} \frac{n|y_{k,n}-\nu|^2}{n + (x-y_{k,n})^2 |y_{k,n}-\nu|} \times$$

$$\times \frac{1}{\sqrt[4]{n|y_{j,n}-\nu|}} \frac{1}{\sqrt[4]{n|y_{k,n}-\nu|}} \sqrt{\frac{y_{k,n}}{4n - y_{k,n}}} \le$$

$$\le C_2 \sqrt{\frac{n}{|y_{j,n}-\nu|}} \sum_{\frac{5\nu}{8} \le y_{k,n} < y_{j-1,n}} \frac{\triangle y_{k,n}}{(x-y_{k,n})^2} \le$$

$$\le C_3 \sqrt{\frac{n}{|y_{j,n}-\nu|}} \int_{\frac{5\nu}{8}}^{y_{j-2,n}} \frac{1}{(x-t)^2} dt \le C_3 \sqrt{\frac{n}{|y_{j,n}-\nu|}} \frac{1}{x - y_{j-2,n}} \le C_4,$$

which holds for all r.

Let us consider $D_n^{(3)}(x)$. Using that $\nu^{1/3} + |x-\nu| + |y_{j,n}-\nu| \sim |y_{j,n}-\nu|$ we get

$$\frac{1}{n} \frac{n|y_{j,n}-\nu|^2}{n + (x-y_{k,n})^2 |y_{j,n}-\nu|} \frac{1}{\sqrt{n|y_{j,n}-\nu|}} \sqrt{\frac{y_{j,n}}{4n - y_{j,n}}} \le$$
$$\le C_1 \frac{|y_{j,n}-\nu|^2}{n^{3/2}} \frac{\sqrt{n}}{|y_{j,n}-\nu|} \le C_2$$

hence $D_n^{(3)}(x)$ is bounded for all r.

If $x \in [\frac{3}{4}\nu, y_{n,n}]$ and $y_{j+1,n} < y_{k,n} < \frac{x+y_{n,n}}{2}$ then

$$|y_{j,n} - y_{k,n}| \ge c_1 \frac{|j^2 - k^2|}{n} \ge c_2 |j-k|, \quad |x-\nu| \sim |y_{k,n}-\nu| \sim |y_{j,n}-\nu|$$

thus

$$D_n^{(4)}(x) \le$$

$$\le C_1 \sum_{y_{j+1,n}<y_{k,n}\le\frac{x+y_{n,n}}{2}} \frac{1}{n}\frac{n|x-\nu|^2}{n+(y_{j,n}-y_{k,n})^2|x-\nu|}\frac{1}{\sqrt{n|x-\nu|}}\sqrt{\frac{n}{|x-\nu|}} \le$$

$$\le C_2 \sum_{k=j+1}^{n} \frac{1}{(j-k)^2} \le C_3,$$

which holds for all r.

Finally let $x \in [\frac{3}{4}\nu, y_{n,n}]$ and $\frac{x+y_{n,n}}{2} \le y_{k,n}$. Then

$$\nu^{1/3}+|x-\nu|+|y_{k,n}-\nu| \sim |x-\nu|, |x-y_{k,n}| \ge \frac{|x-\nu|}{2}, |y_{j,n}-\nu| \ge c\sqrt[3]{n}.$$

Thus

$$D_n^{(5)}(x) \le C_1 \sum_{\frac{x+y_{n,n}}{2}\le y_{k,n}} \frac{1}{n}\frac{n|x-\nu|^2}{n+(x-y_{k,n})^2|x-\nu|}\times$$

$$\times \frac{1}{\sqrt[4]{n|y_{j,n}-\nu|}}\frac{1}{\sqrt[4]{n|y_{k,n}-\nu|}}\sqrt{\frac{y_{k,n}}{4n-y_{k,n}}} \le$$

$$\le C_2 \frac{1}{\sqrt{n}|y_{j,n}-\nu|^{5/4}} \sum_{\frac{x+y_{n,n}}{2}\le y_{k,n}} \frac{\triangle y_{k,n}}{\sqrt[4]{\nu-y_{k,n}}} \le$$

$$\le \frac{C_3}{n^{11/12}} \int_{\frac{x+y_{n,n}}{2}}^{y_{n,n}} \frac{1}{\sqrt[4]{\nu-t}}dt \le C_4 \frac{n^{3/4}}{n^{11/12}} \le C_5$$

for all r.

Consequently there exists a constant $C>0$ independent of x and n such that

$$\sum_{k=1}^{5} D_n^{(k)}(x) \le C \quad \left(x\in\left[\tfrac{3}{4}\nu, y_{n,n}\right],\ n\in\mathbb{N}\right), \quad \text{if } r\le \tfrac{7}{6}. \tag{4.20}$$

Case 4: Let $y_{n,n} \le x \le b_n(\gamma) \le \nu+\sqrt[3]{\nu}$ and

$$\begin{aligned}(4.21)\qquad &\sum_{k=1}^{n} G_n(x,y_{k,n})\left(\frac{x}{y_{k,n}}\right)^r\sqrt{\frac{y_{k,n}}{4n-y_{k,n}}} = \\ &= \sum_{y_{k,n}\le\frac{\nu}{2}} \ldots + \sum_{\frac{\nu}{2}<y_{k,n}} \ldots =: E_n^{(1)}(x)+E_n^{(2)}(x).\end{aligned}$$

If $y_{k,n} \le \frac{\nu}{2}$ then (4.2), (4.7) and (4.9) yields

$$E_n^{(1)}(x) \le C_1 \sum_{y_{k,n}\le\frac{\nu}{2}} \frac{1}{n}\frac{n\cdot n^2}{n+n^2\cdot n}\frac{1}{\sqrt[3]{n}}\frac{1}{\sqrt{k}}\left(\frac{n}{k}\right)^{2r}\frac{k}{n} \le$$

$$\le C_2 \sum_{k=1}^{n} \frac{n^{2r-7/3}}{k^{2r-1/2}} \sim \begin{cases} \dfrac{\log n}{n^{5/6}}, & \text{if } r = \dfrac{3}{4} \\ n^{2r-7/3}, & \text{if } r > \dfrac{3}{4} \\ n^{-5/6}, & \text{if } r < \dfrac{3}{4} \end{cases}$$

which is bounded if $r \le \frac{7}{6}$.

Now let $\frac{\nu}{2} \le y_{k,n} < y_{n,n}$ and $x \in [y_{n,n}, \nu + \sqrt[3]{\nu}]$. Then

$$|x - y_{k,n}| \ge c|y_{k,n} - \nu|.$$

Indeed, this is obvious if $x \ge \nu$. Moreover if $x \in [y_{n,n}, \nu]$ then by (4.2) and (4.3) we have

$$|y_{k,n} - \nu| = |x - y_{k,n}| + |x - \nu| \le |x - y_{k,n}| + c_1\sqrt[3]{n} \le$$

$$\le |x - y_{k,n}| + c_2|x - y_{n-1,n}| \le c_3|x - y_{k,n}|.$$

Therefore

$$E_n^{(2)}(x) \le C_1 \sum_{\frac{\nu}{2}\le y_{k,n}<y_{n,n}} \frac{1}{n}\frac{n|y_{k,n}-\nu|^2}{n+|x-y_{k,n}|^2|y_{k,n}-\nu|}\frac{1}{\sqrt[3]{n}}\times$$

$$\times \frac{1}{\sqrt[4]{n|y_{k,n}-\nu|}}\sqrt{\frac{y_{k,n}}{4n-y_{k,n}}} + C_2\frac{1}{n}\frac{nn^{2/3}}{n}\frac{1}{\sqrt[3]{n}}\frac{1}{\sqrt[4]{nn^{1/3}}}\sqrt[3]{n} \le$$

$$\le C_3 n^{-7/12} \sum_{\frac{\nu}{2}\le y_{k,n}} \frac{\triangle y_{k,n}}{|y_{k,n}-\nu|^{5/4}} + C_4 \le$$

$$\le C_5 n^{-7/12} \int_{\nu/2}^{y_{n,n}} \frac{1}{(\nu-t)^{5/4}}dt + C_6 \le C_7.$$

From the above relations it follows that there exists a constant $C > 0$ independent of x and n such that

$$E_n^{(1)}(x) + E_n^{(2)}(x) \le C \quad \big(x \in [y_{n,n}, b_n],\ n \in \mathbb{N}\big), \quad \text{if } r \le \tfrac{7}{6}. \tag{4.22}$$

Combining (4.12)–(4.22) we get (4.11) so Lemma 4.1 is proved. ■

4.3. Finishing the proof. For the proof of the Theorem we use the Banach–Steinhaus theorem.

Lemma 4.1 states that the sequence of the norm of operators $\mathcal{F}_n$ $(n \in \mathbb{N})$ is uniformly bounded.

Now we show that (3.2) holds for every polynomial. It is enough to prove that for all fixed $j = 0, 1, 2, \ldots$

$$\lim_{n\to+\infty} \Big\| \Big(p_j(w_\alpha) - \sigma_n(p_j(w_\alpha), U_n(w_\alpha), \cdot)\Big)\sqrt{w_\gamma}\Big\|_\infty = 0. \tag{4.23}$$

Using the quadrature formula for $\{p_j := p_j(w_\alpha)\}$ (see [12, Section 3.1]) we have

$$c_{l,n}(p_j) = \sum_{k=1}^{n} p_j(y_{k,n})p_l(y_{k,n})\lambda_{k,n} = \delta_{l,j}$$
$$\big(l, j = 0, 1, 2, \ldots, n-1,\ n \in \mathbb{N}\big).$$

Thus

$$p_j - \sigma_n\big(p_j, U_n(w_\alpha)\big) = \left(1 - \frac{j}{n}\right) p_j,$$

which proves (4.23).

Since the polynomials are dense in the Banach space $\big(C_{\sqrt{w_\gamma}}, \|\cdot\|_{\sqrt{w_\gamma}}\big)$ (see Section 2.2) thus the conditions of the Banach–Steinhaus theorem hold, so the Theorem is proved. ■

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RESTRICTED SUMMABILITY OF MULTI-DIMENSIONAL VILENKIN–FOURIER SERIES

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Dedicated to Professor Antal Járai on his 60th birthday

Abstract. It is proved that the maximal operator of the (C, α) $(\alpha = (\alpha_1, \ldots, \alpha_d))$ and Riesz means of a multi-dimensional Vilenkin–Fourier series is bounded from H_p to L_p $(1/(\alpha_k + 1) < p < \infty)$ and is of weak type $(1, 1)$, provided that the supremum in the maximal operator is taken over a cone-like set. As a consequence we obtain the a.e. convergence of the summability means of a function $f \in L_1$ to f.

1. Introduction

It can be found in Zygmund [16] (Vol. I, p.94) that the trigonometric Cesàro or (C, α) means $\sigma_n^\alpha f$ $(\alpha > 0)$ of a one-dimensional function $f \in L_1(\mathbb{T})$ converge a.e. to f as $n \to \infty$. Moreover, it is known (see Zygmund [16, Vol. I, pp.

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154-156]) that the maximal operator of the (C,α) means $\sigma_*^\alpha := \sup_{n\in\mathbb{N}} |\sigma_n^\alpha|$ is of weak type $(1,1)$, i.e.

$$\sup_{\rho>0} \rho\,\lambda(\sigma_*^\alpha f > \rho) \le C\|f\|_1 \qquad (f \in L_1(\mathbb{T})).$$

For two-dimensional trigonometric Fourier series Marcinkiewicz and Zygmund [6] proved that the Fejér means $\sigma_n^1 f$ of a function $f \in L_1(\mathbb{T}^2)$ converge a.e. to f as $n \to \infty$ in the restricted sense. This means that n must be in a positive cone, i.e., $2^{-\tau} \le n_i/n_j \le 2^\tau$ for every $i,j = 1,2$ and for some $\tau \ge 0$. The author [13] extended this result to the (C,α) and Riesz means of the trigonometric Fourier series for higher dimensions, too. We proved also that the restricted maximal operator

$$\sigma_*^\alpha := \sup_{\substack{2^{-\tau} \le n_i/n_j \le 2^\tau \\ i,j=1,\ldots,d}} |\sigma_n^\alpha|$$

is bounded from H_p to L_p for $\max\{1/(\alpha_j+1)\} < p < \infty$ where $\alpha = (\alpha_1,\ldots,\alpha_d)$. By interpolation we obtained the weak $(1,1)$ inequality for σ_*^α which guarantees the preceding convergence results. Recently Gát [4] introduced more general sets than cones, the so called cone-like sets, and proved the preceding convergence theorem for two-dimensional Fejér means. The author [15] extended this result to higher dimensions, to Cesàro and Riesz means and proved also the above maximal inequality.

For one-dimensional Walsh–Fourier series the convergence result is due to Fine [2] and the weak $(1,1)$ inequality for $\alpha = 1$ to Schipp [7]. Fujii [3] proved that σ_*^1 is bounded from H_1 to L_1 (see also Schipp, Simon [8]). For Vilenkin–Fourier series the results are due to Simon [10]. The author [12, 14] proved the convergence theorem and the maximal inequality mentioned above for multi-dimensional Cesàro and Riesz means of Vilenkin–Fourier series, provided that the n is in a cone.

More recently Gát and Nagy [5] extended the convergence for cone-like sets and for two-dimensional Fejér means of Walsh-Fourier series. In this paper we generalize the preceding results and prove the convergence and maximal inequality for cone-like sets and for Cesàro and Riesz means of more-dimensional Vilenkin–Fourier series.

2. Martingale Hardy spaces and cone-like sets

For a set $\mathbb{X} \ne \emptyset$ let $\mathbb{X}^d$ be its Cartesian product $\mathbb{X} \times \ldots \times \mathbb{X}$ taken with itself d-times. To define the d-dimensional Vilenkin systems we need a sequence

$p := (p_n, n \in \mathbb{N})$ of natural numbers whose terms are at least 2. We suppose always that this sequence is bounded. Introduce the notations $P_0 = 1$ and

$$P_{n+1} := \prod_{k=0}^{n} p_k, \qquad (n \in \mathbb{N}).$$

By a *Vilenkin interval* we mean one of the form $[k/P_n, (k+1)/P_n)$ for some $k, n \in \mathbb{N}$, $0 \le k < P_n$. Given $n \in \mathbb{N}$ and $x \in [0,1)$ let $I_n(x)$ denote the Vilenkin interval of length $1/P_n$ which contains x. Clearly, the Vilenkin rectangle of area $1/P_{n_1} \times \ldots \times 1/P_{n_d}$ containing $x \in [0,1)^d$ is given by $I_n(x) := := I_{n_1}(x_1) \times \ldots \times I_{n_d}(x_d)$. For $n := (n_1, \ldots, n_d) \in \mathbb{N}^d$ the σ-algebra generated by the Vilenkin rectangles $\{I_n(x), x \in [0,1)^d\}$ will be denoted by $\mathcal{F}_n$. The conditional expectation operators relative to $\mathcal{F}_n$ are denoted by E_n. We briefly write L_p instead of the $L_p([0,1)^d, \lambda)$ space. The Lebesgue measure is denoted by λ in any dimension. We denote the Lebesgue measure of a set H also by $|H|$.

Suppose that for all $j = 2, \ldots, d$, $\gamma_j : \mathbb{R}_+ \to \mathbb{R}_+$ are strictly increasing and continuous functions such that $\lim_\infty \gamma_j = \infty$. Moreover, suppose that there exist $c_{j,1}, c_{j,2}, \xi > 1$ such that

$$c_{j,1}\gamma_j(x) \le \gamma_j(\xi x) \le c_{j,2}\gamma_j(x) \qquad (x > 0). \tag{1}$$

Let $c_{j,1} = \xi^{\tau_{j,1}}$ and $c_{j,2} = \xi^{\tau_{j,2}}$ $(j = 2, \ldots, d)$. For convenience we extend the notations for $j = 1$ by $\gamma_1 := \mathcal{I}$, $c_{1,1} = c_{1,2} = \xi$ and $\tau_{1,1} = \tau_{1,2} = 1$. Let $\gamma = (\gamma_1, \ldots, \gamma_d)$ and $\delta = (\delta_1, \ldots, \delta_d)$ with $\delta_1 = 1$ and fixed $\delta_j \ge 1$ $(j = 2, \ldots, d)$. We will investigate the maximal operator of the summability means and the convergence over a *cone-like set* (with respect to the first dimension)

$$L := \{n \in \mathbb{N}^d : \delta_j^{-1}\gamma_j(n_1) \le n_j \le \delta_j\gamma_j(n_1), j = 2, \ldots, d\}. \tag{2}$$

Cone-like sets were introduced and investigated first by Gát [4]. The condition on γ_j seems to be natural, because he [4] proved in the two-dimensional case that to each cone-like set with respect to the first dimension there exists a larger cone-like set with respect to the second dimension and reversely, if and only if (1) holds.

To consider summability means over a cone-like set we need to define new martingale Hardy spaces depending on γ. Given $n_1 \in \mathbb{N}$ we define $n_2, \ldots, n_d$ by $\gamma_j^0(P_{n_1}) := P_{n_j}$, where $P_{n_j} \le \gamma_j(P_{n_1}) < P_{n_j+1}$ $(j = 2, \ldots, d)$. Let $\overline{n}_1 := := (n_1, n_2, \ldots, n_d)$. Since the functions γ_j are increasing, the sequence $(\overline{n}_1, n_1 \in \in \mathbb{N})$ is increasing, too. We investigate the class of (*one-parameter*) *martingales* $f = (f_{\overline{n}_1}, n_1 \in \mathbb{N})$ with respect to $(\mathcal{F}_{\overline{n}_1}, n_1 \in \mathbb{N})$.

For $0 < p \le \infty$ the *martingale Hardy space* $H_p^\gamma([0,1)^d) = H_p^\gamma$ consists of all martingales for which

$$\|f\|_{H_p^\gamma} := \| \sup_{n_1 \in \mathbb{N}} |f_{\overline{n}_1}| \|_p < \infty.$$

It is known (see e.g. Weisz [13]) that $H_p^\gamma \sim L_p$ for $1 < p \le \infty$ where $\sim$ denotes the equivalence of the norms and spaces.

3. Cesàro and Riesz summability of Vilenkin–Fourier series

Every point $x \in [0,1)$ can be written in the following way:

$$x = \sum_{k=0}^{\infty} \frac{x_k}{P_{k+1}}, \qquad 0 \le x_k < p_k,\ x_k \in \mathbb{N}.$$

If there are two different forms, choose the one for which $\lim_{k\to\infty} x_k = 0$. The functions

$$r_n(x) := \exp \frac{2\pi \imath x_n}{p_n} \qquad (n \in \mathbb{N})$$

are called *generalized Rademacher functions*, where $\imath = \sqrt{-1}$. The functions corresponding to the sequence $(2, 2, \ldots)$ are called Rademacher functions.

The product system generated by the generalized Rademacher functions is the *one-dimensional Vilenkin system*:

$$w_n(x) := \prod_{k=0}^{\infty} r_k(x)^{n_k}$$

where $n = \sum_{k=0}^{\infty} n_k P_k$, $0 \le n_k < p_k$. The product system corresponding to the Rademacher functions is called *Walsh system* (see Vilenkin [11] or Schipp, Wade, Simon and Pál [9]).

The Kronecker product $(w_n; n \in \mathbb{N}^d)$ of d Vilenkin systems is said to be the *d-dimensional Vilenkin system.* Thus

$$w_n(x) := w_{n_1}(x_1) \cdots w_{n_d}(x_d)$$

where $n = (n_1, \ldots, n_d) \in \mathbb{N}^d$, $x = (x_1, \ldots, x_d) \in [0,1)^d$. If we consider in each coordinate a different sequence $(p_n^{(j)}, n \in \mathbb{N})$ and a different Vilenkin system

$(w_n^{(j)}; n \in \mathbb{N}^d)$ $(j = 1, \ldots, d)$, then the same results hold. For simplicity we suppose that each Vilenkin system is the same.

If $f \in L_1$ then the number $\hat{f}(n) := \int_{[0,1)^d} f w_n \, d\lambda$ $(n \in \mathbb{N}^d)$ is said to be the nth *Vilenkin–Fourier coefficients* of f. We can extend this definition to martingales in the usual way (see Weisz [13]).

Let $\alpha = (\alpha_1, \ldots, \alpha_d)$ with $0 < \alpha_k \le 1$ $(k = 1, \ldots, d)$ and let

$$A_j^\beta := \binom{j+\beta}{j} = \frac{(\beta+1)(\beta+2)\ldots(\beta+j)}{j!} \qquad (j \in \mathbb{N}; \beta \ne -1, -2, \ldots).$$

It is known that $A_j^\beta \sim O(j^\beta)$ $(j \in \mathbb{N})$ (see Zygmund [16]). The (C, α) *or Cesàro means* and the *Riesz means* of a martingale f are defined by

$$\sigma_n^\alpha f := \frac{1}{\prod_{i=1}^d A_{n_i-1}^{\alpha_i}} \sum_{j=1}^d \sum_{m_j=0}^{n_j-1} \Big(\prod_{i=1}^d A_{n_i-m_i-1}^{\alpha_i} \Big) \hat{f}(m) w_m$$

and

$$\sigma_n^{\alpha,\beta} f := \frac{1}{\prod_{i=1}^d {n_i}^{\alpha_i \beta_i}} \sum_{j=1}^d \sum_{m_j=0}^{n_j-1} \Big(\prod_{i=1}^d (n_i^{\beta_i} - m_i^{\beta_i})^{\alpha_i} \Big) \hat{f}(m) w_m,$$

where $\beta = (\beta_1, \ldots, \beta_d)$ and $0 < \alpha_k \le 1 \le \beta_k$ $(k = 1, \ldots, d)$. The functions

$$K_n^\alpha := \frac{1}{A_{n-1}^\alpha} \sum_{k=0}^{n-1} A_{n-k-1}^\alpha w_k, \quad \text{and} \quad K_n^{\alpha,\beta} := \frac{1}{n^{\alpha\beta}} \sum_{k=0}^{n-1} (n^\beta - k^\beta)^\alpha w_k$$

are the one-dimensional *Cesàro* and *Riesz kernels*. If $\alpha = 1$ or $\alpha = \beta = 1$ then we obtain the *Fejér means*

$$\sigma_n^1 f := \sum_{j=1}^d \sum_{m_j=0}^{n_j-1} \Big(\prod_{i=1}^d (1 - \frac{m_i}{n_i}) \Big) \hat{f}(m) w_m = \frac{1}{\prod_{i=1}^d n_i} \sum_{j=1}^d \sum_{m_j=0}^{n_j-1} s_m f.$$

Since the results of this paper are independent of β, both the (C, α) and Riesz kernels will be denoted by K_n^α and the corresponding summability means by σ_n^α. It is simple to show that

$$\sigma_n^\alpha f(x) = \int_{[0,1)^d} f(t) (K_{n_1}^{\alpha_1}(x_1 \dot{-} t_1) \cdots K_{n_d}^{\alpha_d}(x_d \dot{-} t_d)) \, dt \qquad (n \in \mathbb{N}^d)$$

if $f \in L_1$. Note that the group operations $\dot{+}$ and $\dot{-}$ were defined in Vilenkin [11] or in Schipp, Wade, Simon, Pál [9].

For a given γ, δ satisfying the above conditions the *restricted maximal operator* is defined by

$$\sigma_\gamma^\alpha f := \sup_{n \in L} |\sigma_n^\alpha f|,$$

where the cone-like set L is defined in (2). If $\gamma_j = \mathcal{I}$ for all $j = 2, \ldots, d$ then we get a cone.

4. Estimations of the (C, α) and Riesz kernels

Recall (see Fine [1] and Vilenkin [11]) that the *Vilenkin-Dirichlet kernels* $D_k := \sum_{j=0}^{k-1} w_j$ satisfy

$$D_{P_k}(x) = \begin{cases} P_k, & \text{if } x \in [0, P_k^{-1}) \\ 0, & \text{if } x \in [P_k^{-1}, 1) \end{cases} \qquad (k \in \mathbb{N}). \tag{3}$$

If we write n in the form $n = r_1 P_{n_1} + r_2 P_{n_2} + \ldots + r_v P_{n_v}$ with $n_1 > n_2 > \ldots > n_v \geq 0$ and $0 < r_i < p_i$ $(i = 1, \ldots, v)$, then let $n^{(0)} := n$ and $n^{(i)} := n^{(i-1)} - r_i P_{n_i}$. We have estimated the (C, α) and Riesz kernels in [14].

Theorem 1 ([14]) *For $0 < \alpha \leq 1 \leq \beta$ we have*

$$|K_n^\alpha(x)| \leq C n^{-\alpha} \sum_{k=1}^{v} \sum_{j=0}^{n_k} \sum_{i=j}^{n_k} \sum_{h=0}^{p_j - 1} P_i^{\alpha-1} P_j D_{P_i}(x \dot{+} h P_{j+1}^{-1}), \qquad (n \in \mathbb{N}). \tag{4}$$

The uniform boundedness of the integrals of the kernel functions follows easily from this (see [14]): for $0 < \alpha \leq 1 \leq \beta$ we have

$$\int_0^1 |K_n^\alpha| \, d\lambda \leq C, \qquad (n \in \mathbb{N}). \tag{5}$$

Lemma 1. *If $1 \leq s \leq K$, $0 < \alpha \leq 1 \leq \beta$ and $1/(\alpha + 1) < p \leq 1$ then*

$$\int_{P_{K-s}^{-1}}^{1} \sup_{n \geq P_{K-s}} \Big(\int_0^{P_K^{-1}} |K_n^\alpha(x \dot{+} t)| \, dt \Big)^p \, dx \leq C_p P_K^{-1},$$

where C_p is depending on s, p and α.

Proof. If $j \geq K-s$ and $x \notin [0, P_{K-s}^{-1})$ then $x \dot{+} h P_{j+1}^{-1} \notin [0, P_{K-s}^{-1})$. Thus

$$\int_0^{P_K^{-1}} D_{P_i}(x \dot{+} h P_{j+1}^{-1} \dot{+} t)\, dt = 0$$

for $x \notin [0, P_{K-s}^{-1})$, $i \geq j \geq K-s$ and $h = 0, \dots, p_j - 1$. Applying (4) we conclude

$$\begin{aligned}
&\int_0^{P_K^{-1}} |K_n^\alpha(x \dot{+} t)|\, dt \leq \\
\leq\ & C n^{-\alpha} \sum_{\substack{k=1 \\ n_k < K-s}}^{v} \sum_{j=0}^{n_k} \sum_{i=j}^{n_k} \sum_{h=0}^{p_j-1} P_i^{\alpha-1} P_j \int_0^{P_K^{-1}} D_{P_i}(x \dot{+} h P_{j+1}^{-1} \dot{+} t)\, dt + \\
& + C n^{-\alpha} \sum_{\substack{k=1 \\ n_k \geq K-s}}^{v} \sum_{j=0}^{K-s-1} \sum_{i=j}^{K-1} \sum_{h=0}^{p_j-1} P_i^{\alpha-1} P_j \int_0^{P_K^{-1}} D_{P_i}(x \dot{+} h P_{j+1}^{-1} \dot{+} t)\, dt + \\
& + C n^{-\alpha} \sum_{\substack{k=1 \\ n_k \geq K-s}}^{v} \sum_{j=0}^{K-s-1} \sum_{i=K}^{n_k} \sum_{h=0}^{p_j-1} P_i^{\alpha-1} P_j \int_0^{P_K^{-1}} D_{P_i}(x \dot{+} h P_{j+1}^{-1} \dot{+} t)\, dt = \\
=\ & (A_n) + (B_n) + (C_n).
\end{aligned}$$

It is easy to see, that equality (3) implies

$$\int_0^{P_K^{-1}} D_{P_i}(x \dot{+} h P_{j+1}^{-1} \dot{+} t)\, dt = P_i P_K^{-1} 1_{[h P_{j+1}^{-1}, h P_{j+1}^{-1} \dot{+} P_i^{-1})}(x)$$

for $j \leq i \leq K-1$. Thus

$$(A_n) \leq C P_{K-s}^{-\alpha} \sum_{l=1}^{K-s-1} \sum_{j=0}^{l} \sum_{i=j}^{K-1} \sum_{h=0}^{p_j-1} P_i^{\alpha-1} P_j P_i P_K^{-1} 1_{[h P_{j+1}^{-1}, h P_{j+1}^{-1} \dot{+} P_i^{-1})}(x).$$

Consequently, if $p > 1/(\alpha+1)$ and $\alpha p \neq 1$ then

$$\begin{aligned}\int\limits_{P_{K-s}^{-1}}^{1} \sup_{n\geq P_{K-s}} (A_n)^p \, d\lambda &\leq C_p P_K^{-\alpha p - p} \sum_{l=1}^{K-s-1} \sum_{j=0}^{l} \sum_{i=j}^{K-1} P_i^{\alpha p-1} P_j^p \leq \\ &\leq C_p P_K^{-\alpha p - p} \sum_{l=1}^{K-s-1} \sum_{j=0}^{l} P_j^{\alpha p + p - 1} \leq \\ &\leq C_p P_K^{-\alpha p - p} \sum_{l=1}^{K-s-1} P_l^{\alpha p + p - 1} \leq \\ &\leq C_p P_K^{-1}.\end{aligned}$$

Recall that the sequence (p_j) is bounded. If $\alpha p = 1$, in other words, if $\alpha = p = 1$ then

$$\begin{aligned}\int\limits_{P_{K-s}^{-1}}^{1} \sup_{n\geq P_{K-s}} (A_n)^p \, d\lambda &\leq C_p P_K^{-\alpha p - p} \sum_{l=1}^{K-s-1} \sum_{j=0}^{l} (K-j) P_j^p \leq \\ &\leq C_1 P_K^{-1} \sum_{j=1}^{K-s-1} (K-j)^2 P_j P_K^{-1} \leq \\ &\leq C_1 P_K^{-1} \sum_{j=1}^{K-s-1} (K-j)^2 2^{j-K} \leq \\ &\leq C_1 P_K^{-1}.\end{aligned}$$

Since $P_{n_1}^{-\alpha} P_{K-s-1}^{\alpha} (n_1 - K + s + 1) \leq 2^{-\alpha(n_1-K+s+1)}(n_1 - K + s + 1)$, which is bounded, we obtain

$$\begin{aligned}&(B_n) \leq \\ &\leq C P_{n_1}^{-\alpha}(n_1 - K + s + 1) \sum_{j=0}^{K-s-1} \sum_{i=j}^{K-1} \sum_{h=0}^{p_j-1} P_i^{\alpha-1} P_j \int\limits_0^{P_K^{-1}} D_{P_i}(x \dot{+} h P_{j+1}^{-1} \dot{+} t) \, dt \leq \\ &\leq C P_{K-s-1}^{-\alpha} \sum_{j=0}^{K-s-1} \sum_{i=j}^{K-1} \sum_{h=0}^{p_j-1} P_i^{\alpha-1} P_j P_i P_K^{-1} 1_{[hP_{j+1}^{-1}, hP_{j+1}^{-1} \dot{+} P_i^{-1})}(x).\end{aligned}$$

Hence

$$\int\limits_{P_{K-s}^{-1}}^{1} \sup_{n\geq P_{K-s}} (B_n)^p \, d\lambda \leq C_p P_K^{-\alpha p - p} \sum_{j=0}^{K-s-1} \sum_{i=j}^{K-1} P_i^{\alpha p - 1} P_j^p \leq C_p P_K^{-1}$$

as before. The case $\alpha = p = 1$ can be handled similarly.

If $i \geq K$ then (3) implies

$$\int_0^{P_K^{-1}} D_{P_i}(x \dot{+} hP_{j+1}^{-1} \dot{+} t)\, dt = 1_{[hP_{j+1}^{-1}, hP_{j+1}^{-1} \dot{+} P_K^{-1})}(x).$$

Similarly as above we can see that

$$(C_n) \leq$$

$$\leq Cn^{-\alpha/3} \sum_{\substack{k=1 \\ n_k \geq K-s}}^{v} \sum_{j=0}^{K-s-1} \sum_{i=K}^{n_k} \sum_{h=0}^{p_j-1} P_i^{\alpha/3-1} P_j \int_0^{P_K^{-1}} D_{P_i}(x \dot{+} hP_{j+1}^{-1} \dot{+} t)\, dt \leq$$

$$\leq CP_{n_1}^{-\alpha/3}(n_1 - K + s + 1) \sum_{j=0}^{K-s-1} \sum_{i=K}^{\infty} \sum_{h=0}^{p_j-1} P_i^{\alpha/3-1} P_j 1_{[hP_{j+1}^{-1}, hP_{j+1}^{-1} \dot{+} P_K^{-1})}(x) \leq$$

$$\leq CP_{K-s-1}^{-\alpha/3} \sum_{j=0}^{K-s-1} \sum_{i=K}^{\infty} \sum_{h=0}^{p_j-1} P_i^{\alpha/3-1} P_j 1_{[hP_{j+1}^{-1}, hP_{j+1}^{-1} \dot{+} P_K^{-1})}(x).$$

Consequently,

$$\int_{P_{K-s}^{-1}}^{1} \sup_{n \geq P_{K-s}} (C_n)^p \, d\lambda \leq C_p P_K^{-\alpha p/3} \sum_{j=0}^{K-s-1} \sum_{i=K}^{\infty} P_i^{(\alpha/3-1)p} P_j^p P_K^{-1} \leq C_p P_K^{-1},$$

which shows the lemma. ■

5. The boundedness of the maximal operators on Hardy spaces

A bounded measurable function a is a *p-atom* if there exists a Vilenkin rectangle $I \in \mathcal{F}_{\overline{n}_1}$ such that

(i) $\operatorname{supp} a \subset I$,

(ii) $\|a\|_\infty \leq |I|^{-1/p}$,

(iii) $\int_I a \, d\lambda = 0$.

Theorem 2. *Suppose that*

$$\max\{1/(\alpha_j+1), j=1,\dots,d\} =: p_0 < p < \infty$$

and $0<\alpha_j \le 1 \le \beta_j$ $(j=1,\dots,d)$. *Then*

$$\|\sigma_\gamma^\alpha f\|_p \le C_p \|f\|_{H_p} \qquad (f \in H_p). \tag{6}$$

In particular, if $f \in L_1$ *then*

$$\sup_{\rho>0} \rho\, \lambda(\sigma_\gamma^\alpha f > \rho) \le C\|f\|_1. \tag{7}$$

Proof. We have to show that the operator σ_γ^α is bounded from L_∞ to L_∞ and

$$\int_{[0,1)^d} |\sigma_\gamma^\alpha a|^p \, d\lambda \le C_p \tag{8}$$

for every p-atom a (see Weisz [13]).

The boundedness follows from (5). Let a be an arbitrary p-atom with support $I = I_1 \times \ldots \times I_d$ and $|I_1| = P_K^{-1}$, $|I_j| = \gamma_j^0(P_K)^{-1}$ $(j=2,\dots,d;$ $K \in \mathbb{N})$. Recall that $\gamma_1^0 = \mathcal{I}$ and $\gamma_j^0(P_K) := P_{K_j}$, if $P_{K_j} \le \gamma_j(P_K) < P_{K_j+1}$ $(j=2,\dots,d; K, K_j \in \mathbb{N})$. We can assume that $I_j = [0, P_{K_j}^{-1})$ $(j=1,\dots,d)$. It is easy to see that $\hat{a}(n) = 0$ if $n_j < \gamma_j^0(P_K)$ for all $j=1,\dots,d$. In this case $\sigma_n^\alpha a = 0$.

Suppose that $n_1 < P_{K-r}$ for some $r \in \mathbb{N}$. Let $\delta_j = \xi^{\mu_j}$ and $a_j \tau_{j,1} \le \mu_j <$ $< (a_j+1)\tau_{j,1}$ for some $a_j \in \mathbb{N}$. By the definition of the cone-like set and by (1) we have

$$n_j \le \xi^{\mu_j} \gamma_j(n_1) \le \xi^{(a_j+1)\tau_{j,1}} \gamma_j(P_{K-r}) \le \gamma_j(\xi^{a_j+1} P_{K-r}).$$

Choose $a, b_j \in \mathbb{N}$ such that $\xi \le 2^a$ and $m = \sup_{j\in\mathbb{N}} p_j \le \xi^{\tau_{j,1} b_j}$. Then

$$\begin{aligned} n_j &\le \xi^{-\tau_{j,1} b_j} \gamma_j(\xi^{a_j+1+b_j} P_{K-r}) \le \frac{1}{m} \gamma_j(2^{a(a_j+1+b_j)} P_{K-r}) \le \\ &\le \frac{1}{m}\gamma_j(2^r P_{K-r}) \le \frac{1}{m}\gamma_j(P_K) \le \gamma_j^0(P_K) \end{aligned}$$

for all $j=2,\dots,d$, where let $r := \max_{j=2,\dots,d}\{a(a_j+1+b_j)\}$. In this case $\sigma_n^\alpha a = 0$.

Thus we can suppose that $n_1 \ge P_{K-r}$. By the right hand side of (1),

$$\begin{aligned} n_j &\ge \xi^{-(a_j+1)\tau_{j,1}} \gamma_j(P_{K-r}) \ge \xi^{-(a_j+1)\tau_{j,1}} \xi^{-\tau_{j,2} br} \gamma_j(P_{K-r}\xi^{br}) \ge \\ &\ge \xi^{-(a_j+1)\tau_{j,1} - \tau_{j,2} br} \gamma_j(P_{K-r} m^r) \ge 2^{-a((a_j+1)\tau_{j,1} + \tau_{j,2} br)} \gamma_j(P_K) \ge \\ &\ge 2^{-s} P_{K_j} \ge P_{K_j - s}, \end{aligned}$$

where $b, s \in \mathbb{N}$ are chosen such that $m \leq \xi^b$ and

$$\max_{j=2,\ldots,d} \{a((a_j+1)\tau_{j,1}+\tau_{j,2}br)\} \leq s.$$

We can suppose that $s \geq r$. Therefore

$$\sigma_\gamma^\alpha a \leq \sup_{n_j \geq P_{K_j-s}, j=1,\ldots,d} |\sigma_n^\alpha a|.$$

By the L_∞ boundedness of σ_γ^α we conclude

$$\int_{\prod_{j=1}^d [0,P_{K_j-s}^{-1})} |\sigma_\gamma^\alpha a|^p \, d\lambda \leq C_p \|a\|_\infty^p \prod_{j=1}^d P_{K_j-s}^{-1} \leq C_p \prod_{j=1}^d P_{K_j} \prod_{j=1}^d P_{K_j-s}^{-1} \leq C_p.$$

To compute the integral over $[0,1)^d \setminus \prod_{j=1}^d [0, P_{K_j-s}^{-1})$ it is enough to integrate over

$$H_k := [0,1) \setminus [0, P_{K_1-s}^{-1}) \times \ldots \times [0,1) \setminus [0, P_{K_k-s}^{-1}) \times [0, P_{K_{k+1}-s}^{-1}) \times \ldots \times [0, P_{K_d-s}^{-1})$$

for $k = 1, \ldots, d$. Using (5) and the definition of the atom we can see that

$$\begin{aligned} |\sigma_n^\alpha a(x)| &\leq \int_{\prod_{j=1}^d [0,P_{K_j}^{-1})} |a(t)| (|K_{n_1}^{\alpha_1}(x_1 \dot{+} t_1)| \times \cdots \times |K_{n_d}^{\alpha_d}(x_d \dot{+} t_d)|) \, dt \leq \\ &\leq C \Big(\prod_{j=1}^d P_{K_j}^{1/p} \Big) \prod_{j=1}^k \int_{[0,P_{K_j}^{-1})} |K_{n_j}^{\alpha_j}(x_j \dot{+} t_j)| \, dt_j. \end{aligned}$$

Lemma 1 implies that

$$\int_{H_k} |\sigma_\gamma^\alpha a(x)|^p \, dx \leq C_p \prod_{j=1}^d P_{K_j} \prod_{j=1}^k P_{K_j}^{-1} \prod_{j=k+1}^d P_{K_j-s}^{-1} = C_p$$

which verifies (8) as well as (6) for each $p_0 < p \leq 1$. The weak type $(1,1)$ inequality in (7) follows by interpolation. ■

This theorem was proved by the author in [12, 14] for cones, i.e. if each $\gamma_j = \mathcal{I}$, and in [15] for trigonometric Fourier series.

Observe that the set of the Vilenkin polynomials is dense in L_1. The weak type $(1,1)$ inequality in Theorem 2 and the usual density argument of Marcinkiewicz and Zygmund [6] imply

Corollary 1. *If* $0 < \alpha_j \le 1 \le \beta_j$ $(j = 1, \dots, d)$ *and* $f \in L_1$ *then*

$$\lim_{n\to\infty, n\in L} \sigma_n^\alpha f = f \qquad a.e.$$

The a.e. convergence of $\sigma_n^\alpha f$ was proved by Gát and Nagy [5] for two-dimensional Fejér means.

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