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ENUMERATIVE COMBINATORICS OF $XX0$ HEISENBERG CHAIN

Abstract. In the present paper the enumeration of a certain class of directed lattice paths is based on the analysis of dynamical correlation functions of the exactly solvable $XX0$ model. This model is the zero anisotropy limit of one of the basic models of the theory of integrable systems, the XXZ Heisenberg magnet. We demonstrate that the considered correlation functions under different boundary conditions are the exponential generating functions of the different types of paths, Dyck and Motzkin in particular.

§1. Introduction

In recent years lattice paths have received a lot of attention in different fields, such as, computer science, biology,chemistry, physics, and many more [1, 2]. Their enumeration is an active branch of combinatorics [3–5]. Different methods were developed for the studies of the *directed lattice paths* [6–10]. Dyck and Motzkin paths are the most often considered not only in enumerative combinatorics [11, 12] but also in the theory of quantum computations [13, 14]. Finding integral representations of counting numbers is also a topic of interest [15–17].

Our approach to the study of lattice paths is based on the analyzes of correlation functions of integrable models [18–25]. The main goal of this paper is to describe the alternative method of the derivation of mainly known results in the theory of *simple directed lattice paths*. We will study the dynamical correlation function over the state with all spins up of the $XX0$ model which is the zero anisotropy limit of the XXZ Haisenberg magnet. Depending on the boundary conditions this correlation function is considered to be the exponential generating function of the directed walks of different type.

The paper is organized as follows. We begin with notations, basic definitions and background results. The spin $XX0$ model is introduced and

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the dynamical correlation function over the state with all spins up is considered. The interpretation of correlation function in terms of the directed lattice paths is given. In Sec. 3 the unconstrained paths and in Sec. 4 the constrained ones are considered. We conclude with final remarks.

§2. Heisenberg $XX0$ model and weighted random walks.

The inhomogeneous asymmetric $XX0$ Heisenberg model of interacting $\frac{1}{2}$ – spins in an external field h_n on a one dimensional lattice (chain) is defined by the Hamiltonian

$$\widehat{H} = \sum_{n,m} \Delta_{nm} \sigma_n^- \sigma_m^+ = \sum_n \left\{ \sigma_{n-1}^- \sigma_n^+ + \sigma_{n+1}^- \sigma_n^+ + h \varsigma_n \right\}. \tag{1}$$

The 2×2 spin Pauli matrices $\sigma_n^\pm, \sigma_n^z$ satisfy commutation relations

$$\begin{aligned} [\sigma_n^+, \sigma_m^-] &= \sigma_n^z \delta_{nm}, \\ [\sigma_n^z, \sigma_m^\pm] &= \pm 2 \sigma_n^\pm \delta_{nm}, \end{aligned} \tag{2}$$

and $\varsigma_m = \frac{1}{2}(1 - \sigma_m^z)$ is a projector. The interaction of spins is defined by matrix Δ:

$$\Delta_{nm} = \delta_{n+1,m} + \delta_{n-1,m} + h \delta_{n,m}. \tag{3}$$

The state with all spins up

$$| \Uparrow \rangle = \otimes_n |\uparrow\rangle_n \equiv \otimes_n \begin{pmatrix} 1 \\ 0 \end{pmatrix}_n, \tag{4}$$

where the product is taken over all lattice sites is called the ferromagnetic state. The ferromagnetic state is annihilated by the Hamiltonian (1):

$$\widehat{H} \, |\Uparrow\rangle = 0 \tag{5}$$

and satisfies the following properties. This state is normalized: $\langle \Uparrow \mid \Uparrow \rangle = 1$. It is annihilated by the operator σ_m^+ with the arbitrary m: $\sigma_m^+ | \Uparrow\rangle = 0$; and is an eigenvector of the σ_m^z matrix: $\sigma_m^z | \Uparrow\rangle = | \Uparrow\rangle$. The matrix σ_m^- flips the m-th spin in the product (4): $\sigma_m^- | \uparrow\rangle_m = | \downarrow\rangle_m$.

The continuous temporal evolution of the states, obtained by selective flipping of the spin $\sigma_m^- \, |\Uparrow\rangle$, is defined by the one-particle correlation function

$$G_h \left(j, m | t \right) \equiv \langle \Uparrow | \, \sigma_j^+ e^{t \widehat{H}} \sigma_m^- \, |\Uparrow\rangle. \tag{6}$$

Differentiating $G_h\,(j,m|t)$ with respect to t and applying the commutation relation

$$[\widehat{H},\sigma_m^-]=\sum_n \Delta_{nm}\sigma_n^-\sigma_m^z=\sigma_{m-1}^-\sigma_m^z+\sigma_{m+1}^-\sigma_m^z+h\sigma_m^-, \tag{7}$$

where Δ_{nm} is the matrix element (3), we obtain the equality

$$\begin{aligned}\frac{d}{dt}G_h\,(j,m|t)&=\langle\Uparrow|\,\sigma_j^+e^{t\widehat{H}}\widehat{H}\sigma_m^-\,|\Uparrow\rangle\\&=\sum_n\Delta_{nm}\langle\Uparrow|\,\sigma_j^+e^{t\widehat{H}}\sigma_n^-\,|\Uparrow\rangle=\sum_n\Delta_{nm}G_h\,(j,n|t)\,.\end{aligned}$$

and hence the correlation function satisfies the difference equation:

$$\frac{d}{dt}G_h\,(j,m|t)=G_h\,(j-1,m|t)+G_h\,(j+1,m|t)+hG_h\,(j,m|t) \tag{8}$$

for the fixed index m, and the same equation for the index m with the fixed j. The initial condition is defined by the equality $G_h\,(j,m|0)=\delta_{jm}$.

The expansion of correlation function in powers of t gives:

$$G_h\,(j,m|t)=\sum_K\frac{t^K}{K!}\langle\Uparrow|\,\sigma_j^+(\widehat{H})^K\sigma_m^-\,|\Uparrow\rangle. \tag{9}$$

Applying then the commutation relation (7), one obtains

$$\begin{aligned}\widehat{H}^K\sigma_m^-\,|\Uparrow\rangle&=\widehat{H}^{K-1}[\widehat{H},\sigma_m^-]\,|\Uparrow\rangle=\widehat{H}^{K-1}\sum_{n_1}\Delta_{n_1m}\sigma_{n_1}^-\,|\Uparrow\rangle\\&=\sum_{n_1,\dots,n_K}\Delta_{n_Kn_{K-1}}\dots\Delta_{n_2n_1}\Delta_{n_1m}\sigma_{n_K}^-\,|\Uparrow\rangle.\end{aligned} \tag{10}$$

The multiplication of the equality (10) from the left on the state $\langle\Uparrow|\,\sigma_j^+$ leads to the equality:

$$\langle\Uparrow|\,\sigma_j^+(\widehat{H})^K\sigma_m^-\,|\Uparrow\rangle\equiv\mathfrak{G}_h(j,m|K)=\sum_{n_1,\dots,n_{K-1}}\Delta_{jn_{K-1}}\dots\Delta_{n_2n_1}\Delta_{n_1m}. \tag{11}$$

From the definition (11) and the expansion (9) it follows that

$$G_h\,(j,m|t)=\sum_K\frac{t^K}{K!}\mathfrak{G}_h(j,m|K). \tag{12}$$

From Eqs. (8) and (9) it follows that the discreet correlation function $\mathfrak{G}_h(j,m|K)$ (11) satisfies equation:

$$\mathfrak{G}_h(j,m|K+1)=\mathfrak{G}_h(j-1,m|K)+\mathfrak{G}_h(j+1,m|K)+h\mathfrak{G}_h(j,m|K), \tag{13}$$

with the initial condition $\mathfrak{G}_h(j,m|0)=\delta_{jm}$. From definition (11) it follows that the following equation is also valid:

$$\begin{aligned}\mathfrak{G}_h(j,m|K+2) &= \mathfrak{G}_h(j-2,m|K)+\mathfrak{G}_h(j+2,m|K)+2\mathfrak{G}_h(j,m|K)\\ &+2h\left[\mathfrak{G}_h(j-1,m|K)+\mathfrak{G}_h(j+1,m|K)\right]\\ &+h^2\mathfrak{G}_h(j,m|K).\end{aligned} \tag{14}$$

To connect the discussed model with the theory of random walks let us notice that moves of the single walker on a chain may be expressed by the matrix Δ (3). This choice of the matrix Δ means that the walker can step up either down from an arbitrary cite m or stay at it with a weight h. A random weighted lattice path made by a walker is given by a sequence of edges Δ_{ij} which joins a sequence of lattice cites:

$$\Delta_{jn_{K-1}}\dots\Delta_{n_2n_1}\Delta_{n_1m}. \tag{15}$$

The generating function of all admissible weighted lattice paths running from m to j of K steps is expressed as

$$\sum_{n_1,\dots,n_{K-1}}\Delta_{jn_{K-1}}\dots\Delta_{n_2n_1}\Delta_{n_1m}, \tag{16}$$

and thus is given by the function $\mathfrak{G}_h(j,m|K)$ (11). The evaluation of $\mathfrak{G}_h(j,m|K)$ consists of all walks of K steps from m to j. In the case when $h=1$ this function gives the number of all lattice paths of the length K from m into j.

The introduced one dimensional walks may be considered as random *directed* paths in the plane integer lattice $\mathbb{Z}^2$ defined by a series of equal length steps. The paths made by a walker consist of up-steps $(1,1)$, down-steps $(1,-1)$ and level-steps $(1,0)$ with a weight h. The path of a length K runs from a site $(0,m)$ to a site (K,j). The directed paths are defined by the fact, that for each step (x,y) one has $x\geqslant 0$. We consider only *simple paths*, where every element in the step set S is of the form $(1,a)$. We use the abbreviation $S=(a_1,a_2,\dots,a_K)$ in this case. As illustrated in Fig. 1, a walk can be visualized by its geometric realization.

The correlation function (6) may be considered as the exponential generating function (12) of directed walks. In the theory of random walks the following generating function of all paths from m to j is mainly used

$$F_h(j,m|z)=\sum_K z^K\mathfrak{G}_h(j,m|K)=\sum_K z^K\langle\Uparrow|\,\sigma_j^+(\widehat{H})^K\sigma_m^-\,|\Uparrow\rangle, \tag{17}$$

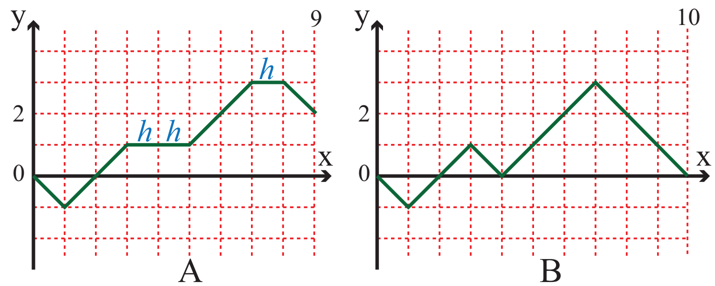

Figure 1. A: The *weighted directed path* of 9 steps from $(0,0)$ to $(9,2)$ with a step set $S=(-1,1,1,0,0,1,1,0,-1)$ that corresponds to the one dimensional walk $\Delta_{0,-1}\Delta_{-1,0}\Delta_{0,1}\Delta_{1,0}\Delta_{0,1}\Delta_{1,2}\Delta_{2,3}\Delta_{3,3}\Delta_{3,2}$. B: A *bridge* with a step set $S=(-1,1,1,-1,1,1,1,-1,-1,-1)$.

which is the Laplace transformation of correlation function (9):

$$\begin{aligned}\int\limits_0^\infty e^{-\frac{t}{z}} G_h(j,m|t)dt &= \int\limits_0^\infty e^{-\frac{t}{z}} \langle\Uparrow|\, \sigma_j^+ e^{t\widehat{H}} \sigma_m^-\, |\Uparrow\rangle dt \\ &= z\langle\Uparrow|\, \sigma_j^+ \left(\frac{1}{1-z\widehat{H}}\right) \sigma_m^-\, |\Uparrow\rangle = zF_h(j,m|z).\end{aligned} \tag{18}$$

Applying the Theorem 4.7.2 in [3] we obtain the determinant representation of generating function:

$$F_h(j,m|z) = (-1)^{j+m} \frac{\det[I-z\Delta : m,j]}{\det[I-z\Delta]},$$

where $(A : m,j)$ denotes the matrix obtained by removing the mth row and jth column.

§3. Unconstrained directed paths

A directed path that starts the origin and ends anywhere is called a *path* (see Fig. 1A). If the path ends on the x-axis then it is called a *bridge* (see Fig. 1B).

To study the unconstrained walks we consider the $XX0$ model on an infinite chain. For the one-dimensional walks from 0 to $2j$ the solution of the equation (8) with $h = 0$ is the modified Bessel function

$$G\,(2j,0|t) = I_{2j}(2t) = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} e^{2t\cos\theta} e^{2ij\theta} d\theta. \tag{19}$$

Using the decomposition of modified Bessel function

$$I_n(2z) = \sum_{k=0}^{\infty} \frac{z^{n+2k}}{k!(k+n)!}, \tag{20}$$

we find out that

$$G(2j,0|t) = \sum_{K\geqslant j}^{\infty} \frac{t^{2K}}{(K-j)!(K+j)!}. \tag{21}$$

From the expansion (12) follows that

$$D_K(j) \equiv \mathfrak{G}(2j,0|2K) = \frac{2K!}{(K-j)!\,(K+j)!} = \binom{2K}{K-j}. \tag{22}$$

It is a well known binomial formula for a number of all lattice paths from $(0,0)$ to $(2K,2j)$ of length $2K$ on $\mathbb{Z}^2$. The number of bridges is given by $D_K(0)$.

The generating function for this kind of paths is

$$\begin{aligned} D(j|z) \equiv F(2j,0|z) &= z^{-1}\int\limits_0^{\infty} e^{-\frac{t}{z}} G(2j,0|t) dt \\ &= z^{-1}\int\limits_0^{\infty} e^{-\frac{t}{z}} I_{2j}(2t) dt = \frac{\left(1-\sqrt{1-4z^2}\right)^{2j}}{2^{2j} z^{2j}\sqrt{1-4z^2}}. \end{aligned} \tag{23}$$

If besides the up and down steps we admit the level steps with the weight h we have to consider the equation (8) with $h \neq 0$. In this case the exponential generating function of the one-dimensional walks from 0 to j is equal to

$$\begin{aligned} G_h\,(j,0|t) = e^{th} I_j(2t) &= \sum_{K\geqslant j}^{\infty} \frac{t^K}{K!} \sum_{k=0}^{\left[\frac{K-j}{2}\right]} \frac{h^{K-2k-j} K!}{(K-2k-j)!k!(k+j))!} \\ = \sum_{K\geqslant j}^{\infty} \frac{t^K}{K!} \sum_{k=0}^{\left[\frac{K-j}{2}\right]} h^{K-2k-j} \binom{K}{2k+j}\binom{2k+j}{k} &= \sum_{K\geqslant j}^{\infty} \frac{t^K}{K!} \mathfrak{G}_h(j,0|K). \end{aligned} \tag{24}$$

The generating function of *weighted paths* in K steps from $(0,0)$ to (K,j) with the level steps carrying the weight h is given by the equality:

$$R_K(j|h) \equiv \mathfrak{G}_h(j,0|K) = \sum_{k=0}^{\left[\frac{K-j}{2}\right]} h^{K-2k-j} \binom{K}{2k+j} \binom{2k+j}{k}. \tag{25}$$

From this expression it follows that the number of lattice paths of K steps with L level steps is:

$$R_K^L(j) = \binom{K}{K-L} \binom{K-L}{\frac{K-L-j}{2}}. \tag{26}$$

The total number of the introduced lattice paths is given by:

$$R_K(j|1) = \sum_{k=0}^{\left[\frac{K-j}{2}\right]} \binom{K}{2k+j} \binom{2k+j}{k}. \tag{27}$$

The number of correspondent bridges is given by $R_K(0|1)$ and is equal to the central trinomial coefficient

$$\binom{K}{0}_2 = \sum_{k=0}^{\left[\frac{K}{2}\right]} \binom{K}{2k} \binom{2k}{k}. \tag{28}$$

These numbers are listed in [26].

The generating function of weighted paths is

$$\begin{aligned} R(j|z;h) \equiv F_h(j,0|z) &= z^{-1} \int_0^\infty e^{-(\frac{1}{z}-h)t} I_j(2t) dt \\ &= \frac{\left(1 - zh - \sqrt{1-2hz-z^2(4-h^2)}\right)^j}{2^j z^j \sqrt{1-2hz-z^2(4-h^2)}}. \end{aligned} \tag{29}$$

The representation of the modified Bessel function

$$I_j(2t) = \frac{t^j}{\sqrt{\pi}\Gamma(j+\frac{1}{2})} \int_{-1}^{1} e^{2ts} (1-s^2)^{j-\frac{1}{2}} ds \tag{30}$$

allows us to obtain the decomposition of the exponential generating function (24) in the form

$$
\begin{aligned}
G_h(j,0|t) &= e^{ht} I_j(2t) \\
&= \frac{2^{2j} j!}{\pi (2j)!} \sum_{K \geqslant j} \frac{t^K}{(K-j)!} \int_{-1}^{1} (2s+h)^{K-j} (1-s^2)^{j-\frac{1}{2}} ds,
\end{aligned} \tag{31}
$$

from which, the integral representation for the generating function (25) follows

$$
R_K(j|h) = \frac{2^{2j}}{\pi} \frac{\binom{K}{j}}{\binom{2j}{j}} \int_{-1}^{1} (2s+h)^{K-j} (1-s^2)^{j-\frac{1}{2}} ds. \tag{32}
$$

§4. Constrained paths

The constrained path moves from left to right never dipping below the height it began on. Among these paths most known are Dyck and Motzkin paths.

A *Dyck path* is a path constructed from the step set $(1,1), (1,-1)$, which starts at the origin, never passes below the x-axis and ends on the x-axis. A *Dyck path of altitude* m is a Dyck path that terminates at altitude m.

A *Motzkin path* is a Dyck path with the level steps $(1,0)$ allowed. Correspondingly, a *Motzkin path of altitude* m is a Motzkin path that terminates at altitude m. The *weighted Motzkin paths* are paths the level steps of which are equipped with a wight h.

Dyck and Motzkin paths belong to a class of lattice paths known as *excursions*, while Dyck and Motzkin paths of altitude j are *meanders*.

Since excursions and meanders never drop below x-axis we have to consider the one-dimensional walks on the semiaxis $(0 \leqslant j < \infty)$. Their exponential generating function $G(j,0|t)$ is given by the equation (8) with the boundary condition $G(2j,0|t) = 0$ for $j = -1$.

4.1. Dyck paths and Catalan numbers. Let us consider Dyck paths of altitude $2j$ (Fig. 2B). The solution of the equation (8) on the semiaxis with $h = 0$ is equal to:

$$
\begin{aligned}
G(2j, 0|t) &= \frac{1}{\pi} \int_{-\pi}^{\pi} e^{2t\cos\theta} \sin\left[(2j+1)\theta\right] \sin\theta d\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{2t\cos\theta} \left\{\cos(2j\theta) - \cos\left[2(j+1)\theta\right]\right\} d\theta \\
&= I_{2j}(2t) - I_{2j+2}(2t) = \frac{2j+1}{t} I_{2j+1}(2t).
\end{aligned} \tag{33}
$$

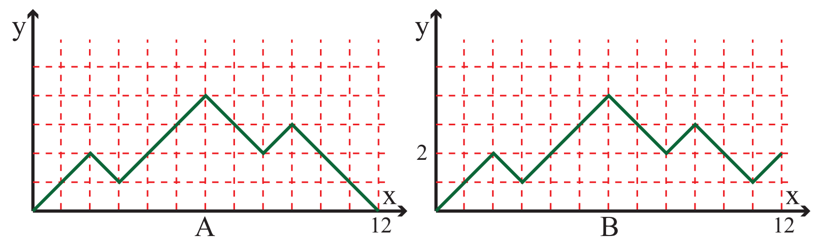

Figure 2. A: Dyck excursion in 12 steps. B: Dyck meander of altitude 2 in 12 steps.

Using the decomposition of modified Bessel function (20) we find out that

$$
\begin{aligned}
G(2j, 0|t) &= \sum_{K \geqslant j}^{\infty} (2j+1) \frac{t^{2K}}{(K+j+1)!(K-j)!} \\
&= \sum_{K \geqslant j}^{\infty} \frac{t^{2K}}{2K!} \mathfrak{G}(2j, 0|2K),
\end{aligned} \tag{34}
$$

where

$$
C_K(j) \equiv \mathfrak{G}(2j, 0|2K) = \frac{2j+1}{K+j+1} \binom{2K}{K-j}; \quad K \geqslant j \tag{35}
$$

is the number of Dyck paths (Fig. 2A) of altitude $2j$ in $2K$-steps from $(0, 0)$ to $(2j, 2K)$. The number of Dyck paths in $2K$-steps from $(0, 0)$ to $(0, 2K)$ is equal to Catalan number C_K:

$$
C_K \equiv C_K(0) = \frac{1}{K+1} \binom{2K}{K}. \tag{36}
$$

The recurrence relation on numbers $C_K(j)$ follows from the equation (14):

$$C_{K+2}(j) = C_K(j-2) + C_K(j+2) + 2C_K(j); \quad C_K(-1) = 0. \tag{37}$$

It can be expressed in terms of generating function of Catalan numbers (36):

$$C(z) \equiv C(0|z) = \frac{1-\sqrt{1-4z^2}}{2z^2}. \tag{38}$$

We obtain that

$$C(j|z) = \frac{z^{2j}}{(2j+1)} \left(C(z)\right)^{2j+1}. \tag{39}$$

The generating function $D(j|z)$ of unconstrained walks (22) may be expressed through the generating function of Catalan numbers:

$$D(j|z) = \frac{\left(zC(z)\right)^{2j}}{1-2z^2C(z)}. \tag{40}$$

4.2. Motzkin paths and Motzkin numbers. The weighted Motzkin paths of altitude j (Fig. 3B) are described by the solution $G(j,0|t)$ of the equation (8) on the semiaxis with $h \neq 0$:

$$\begin{aligned} G_h(j,0|t) &= e^{th}(j+1)\frac{I_{j+1}(2t)}{t} \\ &= \sum_{K\geqslant j}^{\infty} \frac{t^K}{K!} \sum_{k=0}^{\left[\frac{K-j}{2}\right]} \frac{(j+1)K!}{(K-2k-j)!k!(k+j+1))!} h^{K-2k-j} \\ &= \sum_{K=0}^{\infty} \frac{t^K}{K!} \mathfrak{G}_h(j,0|K). \end{aligned} \tag{41}$$

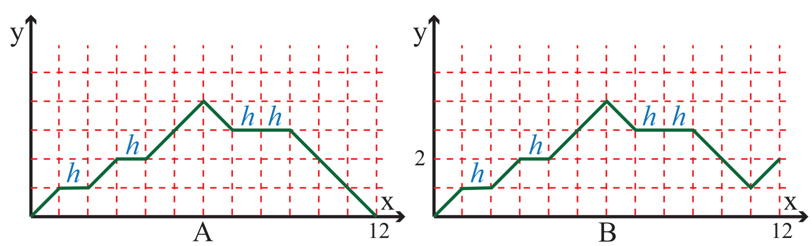

Figure 3. A: weighted Motzkin excursion in 12 steps. B: weighted Motzkin meander of altitude 2 in 12 steps.

The generating function of weighted Motzkin paths of altitude j and length K is given by the equality:

$$\begin{aligned} M_K(j|h) &\equiv \mathfrak{G}_h(j,0|K) \\ &= \sum_{k=0}^{\left[\frac{K-j}{2}\right]} \frac{j+1}{k+j+1} \binom{K}{2k+j} \binom{2k+j}{k} h^{K-2k-j}. \end{aligned} \tag{42}$$

From this expression it follows that the number of Motzkin paths of altitude j of length K with L level steps is:

$$M_K^L(j) = \frac{2(j+1)}{K-L+j+2} \binom{K}{K-L} \binom{K-L}{\frac{K-L-j}{2}}. \tag{43}$$

The number of the Motzkin paths of altitude j is:

$$M_K(j|1) = \sum_{k=0}^{\left[\frac{K-j}{2}\right]} \frac{j+1}{k+j+1} \binom{K}{2k+j} \binom{2k+j}{k}. \tag{44}$$

From this formula follows that the number of Motzkin paths of the length K is defined by the equality

$$M_K \equiv M_K(0|1) = \sum_{k=0}^{\left[\frac{K}{2}\right]} \binom{K}{2k} C_k, \tag{45}$$

where C_K is Catalan number (36), and M_K is a Motzkin number.

The recurrence relation on the $M_K(j|1)$ numbers follows from the equation (14):

$$\begin{aligned} M_{K+2}(j|1) &= M_K(j-2|1) + M_K(j+2|1) \\ &\quad + 2M_K(j-1|1) + 2M_K(j+1|1) + 3M_K(j|1); \\ M_K(-2|1) &= M_K(-1|1) = 0. \end{aligned} \tag{46}$$

The generating function of generalized Motzkin paths is:

$$M(j|z,h) = z^{-1} \int_0^\infty e^{-\left(\frac{1}{z}-h\right)t} \frac{I_{2j+1}(2t)}{t} dt = \frac{z^{2j}}{2j+1} \left(M(z,h)\right)^{2j+1}, \tag{47}$$

where $M(z,h)$ is the generating function of Motzkin paths:

$$M(z,h) \equiv M(0|z,h) = \frac{1}{2z^2} \left(1 - zh - \sqrt{1-2hz-z^2(4-h^2)}\right). \tag{48}$$

The generating function $R(j|z;h)$ of unconstrained walks (29) is expressed in terms of the generating function of weighted Motzkin paths:

$$R(j|z;h) = \frac{(zM(z,h))^j}{1 - zh - 2z^2M(z,h)}. \tag{49}$$

The representation (30) allows us to obtain integral representation of generating function $M_K(j|h)$. We have

$$M_K(j|h) = \frac{2^{2(j+1)}}{\pi} \frac{\binom{K}{j}}{\binom{2(j+1)}{j+1}} \int_{-1}^{1} (2s+h)^{K-j}(1-s^2)^{j+\frac{1}{2}} ds. \tag{50}$$

From this equality follows the ontegral representation for the Motzkin numbers (see [17] and refs. there):

$$M_K = \frac{2}{\pi} \int_{-1}^{1} (2s+1)^K \sqrt{1-s^2} ds. \tag{51}$$

If we change $K \to 2K$ and $j \to 2j$ and put $h = 0$ in (50) we obtain the representation of the numbers $C_K(j)$ (35):

$$C_K(j) = \frac{2^{2(K+j+1)}}{\pi} \frac{\binom{2K}{2j}}{\binom{2(2j+1)}{2j+1}} \int_{-1}^{1} s^{2(K-j)}(1-s^2)^{2j+\frac{1}{2}} ds. \tag{52}$$

For $j = 0$ we obtain integral representation of Catalan numbers [15, 16]:

$$C_K = \frac{2^{2K+1}}{\pi} \int_{-1}^{1} s^{2K} \sqrt{1-s^2} ds. \tag{53}$$

§5. Conclusion

Generating functions play an important role in mathematics and physics [27]. We have demonstrated that the main results in the theory of the simple directed paths may be obtained from the exponential generating function of the directed paths which in turn is the dynamical correlation function of the $XX0$ model. The obtained results may be easily generalized on the case when the path starts not from the origin.

It is worth noting that the correlation function (6) of the $XX0$ model with the nearest- and next-to-nearest neighbours interactions:

$$\widehat{H}_{NNN} = \sum_n \left\{\sigma^-_{n-1}\sigma^+_n + \sigma^-_{n+1}\sigma^+_n + \sigma^-_{n-2}\sigma^+_n + \sigma^-_{n+2}\sigma^+_n\right\}$$

describes a class of lattice paths constructed from the step set $(1,1)$, $(1,2)$, $(1,-2)$, $(1,-1)$. These paths possess rich combinatorial properties and are known as the basketball walks [7, 8]. Basketball walks with the step set $S = (2,-1,1,1,2,-1,-2,-1,1,1,-2,-1)$ are presented in (Fig. 4). We

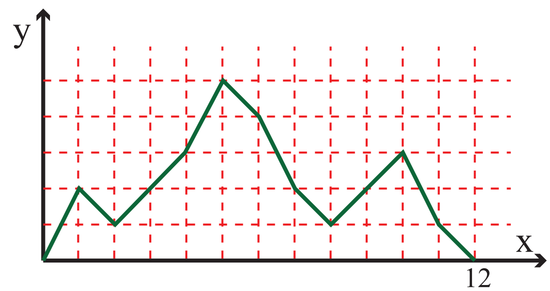

Figure 4. Basketball walks in 12 steps.

shall study this type of walks in the further publication.

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