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ON THE ENUMERATION OF HYPERMAPS WHICH ARE SELF-EQUIVALENT WITH RESPECT TO REVERSING THE COLORS OF VERTICES

Abstract. A *map* (S, G) is a closed Riemann surface S with an embedded graph G such that $S \setminus G$ is the disjoint union of connected components, called *faces*, each of which is homeomorphic to an open disk. Tutte began a systematic study of maps in the 1960s, and contemporary authors are actively developing it. We recall the concept of a circular map introduced by the author and Mednykh and demonstrate a relationship between bipartite maps and circular maps through the concept of the duality of maps. We thus obtain an enumeration formula for the number of bipartite maps with a given number of edges. A *hypermap* is a map whose vertices are colored black and white in such a way that every edge connects vertices of different colors. Hypermaps are also known as dessins d'enfants (or Grothendieck's dessins).

A *hypermap is self-equivalent with respect to reversing the colors of vertices* if it is equivalent to the hypermap obtained by reversing the colors of its vertices.

The main result of this paper is an enumeration formula for the number of unrooted hypermaps, regardless of genus, which have n edges and are self-equivalent with respect to reversing the colors of vertices.

§1. Preliminaries

Definition 1. *A* map (S, G) *is a closed Riemann surface S with an embedded graph G such that $S \setminus G$ is the disjoint union of connected components, called* faces*, each of which is homeomorphic to an open disk.*

Key words and phrases: unrooted maps, dessins d'enfants, Riemann surface, two-colored maps, bipartite maps, hypermaps, hypermaps which are self-equivalent with respect to reversing the colors of vertices.

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Definition 2. *Two maps* (S, G) *and* (S_1, G_1) *are called* equivalent *whenever there exists an orientation-preserving homeomorphism* $h : S \to S_1$ *with* $h(G) = G_1$.

Definition 3. *A map is called a* bipartite map *if its vertices can be colored with two colors (black and white) in such a way that every edge connects vertices of different colors.*

Denote by $\mathrm{Bip}(n)$ the number of bipartite maps with n edges, up to equivalence.

Definition 4. *A* hypermap *is a map whose vertices are colored black and white in such a way that every edge connects vertices of different colors.*

Note that a hypermap was defined by R. Cori to be a pair of permutations σ and α on a finite set B, such that the group generated by σ and α is transitive on B (see [1]). T. R. S. Walsh demonstrated a one-to-one correspondence [10] between hypermaps and the set of (oriented) 2-coloured bipartite maps. For convenience, we use it as a definition.

Definition 5. *Two hypermaps* (S, G) *and* (S_1, G_1) *are called* equivalent *whenever there exists an orientation-preserving homeomorphism* $h : S \to S_1$ *with* $h(G) = G_1$ *and* h *taking black and white vertices of* (S, G) *to black and white vertices of* (S_1, G_1)*, respectively.*

Denote by $\mathrm{Hyp}(n)$ the number of hypermaps with n edges, up to equivalence of hypermaps.

Definition 6. *One says that a* hypermap is self-equivalent with respect to reversing the colors of vertices*, if it is equivalent to the hypermap obtained by reversing the colors of its vertices.*

Denote the number of such hypermaps with n edges by $\mathrm{Shyp}(n)$. Examples of such hypermaps with $n = 4$ edges can be seen at the end of this paper.

Remark 1. Reversing the colors has a natural interpretation in terms of Belyi functions. Recall that a Belyi function is a non-constant meromorphic function $f : S \to \overline{\mathbb{C}}$ unramified outside $\{0, 1, \infty\}$, where S is a compact Riemann surface. There is a connection between Belyi functions f and hypermaps on S. The black (resp. white) vertices of the hypermap are the preimages of 0 (resp. of 1). (For more details see, for instance, 1.8 and 2.1 in [5]). When we reverse colors, preimages of 0 become preimages of 1, and preimages of 1 become preimages of 0.

For a hypermap to be self-equivalent with respect to reversing the colors of vertices it means that f and $f-1$ have the same number of roots with the same multiplicities.

The main problem of this paper is to calculate the number of hypermaps, with a given number of edges, which are self-equivalent with respect to reversing the colors of vertices.

Note that the vertices of a bipartite map can be properly colored black and white in two different ways unless the map is self-equivalent with respect to reversing the colors of vertices.

Thus we have

$$\mathrm{Shyp}(n) = 2\,\mathrm{Bip}(n) - \mathrm{Hyp}(n). \tag{1}$$

§2. Enumeration of hypermaps with a given number of edges, regardless of genus

Let us recall [8] that each hypermap with n edges corresponds to a conjugacy class of subgroups of index n in the group $\Delta = \langle x, y, z : xyz = 1\rangle$, a free group of rank 2 acting on the hyperbolic plane H^2 by orientation-preserving isometries.

The number of conjugacy classes of subgroups of index n in a free group of rank r was calculated by Liskovets [13]. In our case $r = 2$ and we have the following proposition.

Proposition 2.1. *The number* $\mathrm{Hyp}(n)$ *of hypermaps with a given number* n *of edges can be calculated from the formula*

$$\mathrm{Hyp}(n) = \frac{1}{n} \sum_{\substack{l|n \\ lm=n}} s^{+}(m,0)\, \varphi_{m+1}(l),$$

where $\varphi_{m+1}(l) = \sum_{d|l} \mu\left(\frac{l}{d}\right) d^{m+1}$ *is the Jordan totient function,* $\mu(n)$ *is the Möbius function, and* $s^{+}(m,0)$ *is calculated using Hall's recurrence formula ([3]) for the number of subgroups of index* n *in a free group of* rank $r = 2$ *:*

$$s^{+}(n,0) = (n+1)! - \sum_{k=1}^{n} k!\, s^{+}(n-k,0),$$

$s^{+}(0,0) = 1$.

The sequence $\mathrm{Hyp}(n)$ coincides with the sequence A057005 in [11].

§3. Enumeration of bipartite maps with a given number of edges, regardless of genus

Following ([5, pp. 51–52]) let us recall the definition of the *dual map*. We put a new vertex inside each face of the original map (the "center" of the face). Then, for each edge of the original map, we draw a new edge which intersects it in its midpoint, and which connects the centers of the two faces adjacent to this original edge. If these two faces coincide, the new edge thus obtained is a loop. (See Fig. 1). It is easy to see that the dual of the dual of a map is the original map itself. The faces of the original map are in bijective correspondence with the vertices of the dual map, and also the degrees of the new vertices are equal to the degrees of the old faces. The same may be said about the new faces and the old vertices. Clearly, the dual map has the same number of edges as the original map.

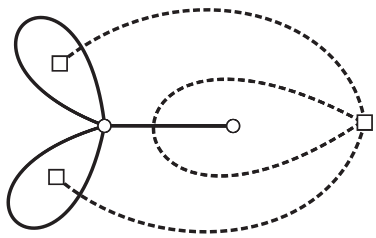

Fig. 1. A pair of mutually dual spherical maps.

Following [12] we define a circular map as follows.

Definition 7. *Define an* elementary circular map $(S_\circ, G_\circ)$ *to be a map on the sphere* $S_\circ$ *with one edge, one vertex, and two faces* (*inner and outer*).

Definition 8. *Define a* circular map *to be a map covering an elementary circular map. In other words,* (S, G) *is a circular map if there exists a branched covering* $f : (S, G) \to (S_\circ, G_\circ)$ *ramified only over the centers of the faces and the vertex of* $G_\circ$*, and such that* $f(G) = G_\circ$.

Let us remark that the dual of a bipartite map is a map whose faces can be colored with two colors in such a way that every edge divides faces of two different colors. According to Lemma 1.1 of [12] a map whose faces can be colored properly with two colors is a circular map and conversely. Thus the number of bipartite maps with a given number of edges coincides with the number of circular maps with that number of edges. This last number is calculated in [12]. As a result we obtain the following theorem.

Theorem 3.1. *The number* $\mathrm{Bip}(n)$ *of bipartite maps with a given number* n *of edges can be calculated from the formula*

$$\mathrm{Bip}(n)=\frac{1}{2n}\sum_{\substack{l\mid n\\ lm=n}}\Bigg(s^{+}(m,0)\varphi_{m+1}(l)+\mathrm{Int}\left(\frac{m}{2}\right)\left(s\left(\frac{m}{2},0\right)-s^{+}\left(\frac{m}{2},0\right)\right)\varphi^{\mathrm{odd}}_{\frac{m}{2}+1}(l)$$

$$+\sum_{H=1}^{m}\mathrm{Int}\left(\frac{m-H}{2}\right)\frac{T(m,H)}{(m-1)!}\varphi_{\frac{m-H}{2}+1}(l)\Bigg),$$

where $\varphi_m(l)$ *is the Jordan totient function,* $\varphi^{\mathrm{odd}}_{m+1}(l)$ *is the odd Jordan totient function, which is equal to*

$$\varphi^{\mathrm{odd}}_{m+1}(l)=\sum_{\substack{d\mid l\\ \frac{l}{d}\ \mathrm{odd}}}\mu\left(\frac{l}{d}\right)d^{m+1},$$

and $s(m,0)$*, and* $s^{+}(m,0)$ *are calculated using the recurrences*

$$s(n,0)=(2n+1)!!-\sum_{k=1}^{n}(2k-1)!!\,s(n-k,0),$$

$s(0,0)=1$*, and*

$$s^{+}(n,0)=(n+1)!-\sum_{k=1}^{n}k!\,s^{+}(n-k,0),$$

$s^{+}(0,0)=1$*, respectively.*

The function $T(m,H)$ *is given by the recurrence formula:*

$$T(m,H)=B(m,H)-\sum_{h=0}^{H}\sum_{i=1}^{m-1}\binom{m-1}{m-i}T(i,h)\,B(m-i,H-h)\,,$$

where

$$B(i,j)=i!\,\frac{i!}{1^{j}\,j!\,2^{\frac{i-j}{2}}\,\left(\frac{i-j}{2}\right)!}\,\mathrm{Int}\left(\frac{i-j}{2}\right),$$

$B(0,0)=1$*,* $T(0,0)=0$*, and*

$$\mathrm{Int}(x)=\begin{cases}1 & \textit{if } x\in\mathbb{Z} \textit{ and } x\geqslant 0,\\ 0 & \textit{otherwise.}\end{cases}$$

Remark 2. Note that it follows from Proposition 1.1 of [12] that a planar map is a circular map if and only if it is an Euler map, i.e., every vertex has even valency. This property fails for Riemann surfaces of higher genus. Hence the number $\mathrm{Bip}_0(n)$ of planar bipartite maps with n edges coincides with the number of planar Euler maps with n edges. This last number was calculated in [7].

The sequences $\mathrm{Bip}(n)$ and $\mathrm{Bip}_0(n)$ coincide with the sequences A234278 and A069727 in [11], respectively.

§4. Enumeration of hypermaps which are self-equivalent with respect to reversing the colors of vertices

Taking into account (1), Proposition 2.1 and Theorem 3.1, we obtain the following theorem.

Theorem 4.1. *The number* $\mathrm{Shyp}(n)$ *of hypermaps which are self-equivalent with respect to reversing the colors of vertices and which have a given number* n *of edges can be calculated from the formula*

$$\mathrm{Shyp}(n) = \frac{1}{n}\sum_{\substack{l|n\\ lm=n}}\left(\mathrm{Int}\left(\frac{m}{2}\right)\left(s\left(\frac{m}{2},0\right)-s^{+}\left(\frac{m}{2},0\right)\right)\varphi^{\mathrm{odd}}_{\frac{m}{2}+1}(l)\right.$$
$$\left.+\sum_{H=1}^{m}\mathrm{Int}\left(\frac{m-H}{2}\right)\frac{T(m,H)}{(m-1)!}\,\varphi_{\frac{m-H}{2}+1}(l)\right),$$

where $\varphi_m(l)$, $\varphi^{\mathrm{odd}}_{m+1}(l)$, $s(m,0)$, $s^{+}(m,0)$, $T(m,H)$ *and* $\mathrm{Int}(x)$ *are as in Theorem* 3.1.

Table 1 contains the numbers of hypermaps which are self-equivalent with respect to reversing the colors of vertices, regardless of genus, up to 16 edges.

Remark 3. Note that the number $\mathrm{Shyp}_0(n)$ of planar hypermaps which have n edges and are self-equivalent with respect to reversing the colors of vertices was obtained in [6]. The sequence $\mathrm{Shyp}_0(n)$ coincides with the sequence A090375 in [11].

§5. Some examples

In this section we illustrate the hypermaps with $n = 4$ edges which are self-equivalent with respect to reversing the colors of vertices. From

n	Shyp(n)	$\mathrm{Shyp}_0(n)$
1	1	1
2	1	1
3	3	2
4	6	4
5	15	8
6	42	17
7	131	40
8	442	93
9	1551	224
10	5723	538
11	22171	1344
12	89156	3352
13	370199	8448
14	1589240	21573
15	7020127	54912
16	31906974	143037

Table 1

Table 1 it follows that there are four such hypermaps on the sphere (see Figs. 2–5) and two on the torus (see Figs. 6, 7). Let us use the notation for maps from the catalogue [4]. (Note that the maps in [4] are not colored.)

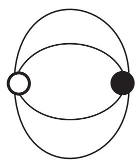

Fig. 2. Map 0.37.

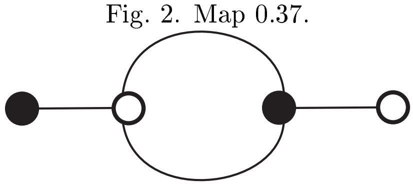

Fig. 3. Map 0.69.

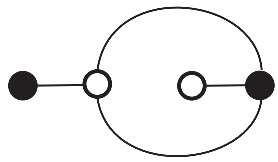

Fig. 4. Map 0.70.

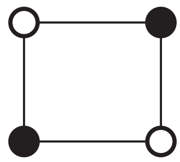

Fig. 5. Map 0.74.

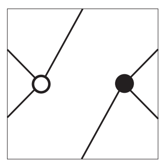

Fig. 6. Map 1.40 (The torus is a rectangle with opposite sides identified.)

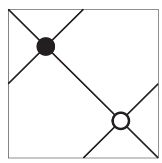

Fig. 7. Map 1.41 (The torus is a rectangle with opposite sides identified.)

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