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Uniform difference method for parameterized singularly perturbed delay differential equations

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Abstract This paper deals with the singularly perturbed initial value problem for quasilinear first-order delay differential equation depending on a parameter. A numerical method is constructed for this problem which involves an appropriate piecewise-uniform meshes on each time subinterval. The difference scheme is shown to converge to the continuous solution uniformly with respect to the perturbation parameter. Some numerical experiments illustrate in practice the result of convergence proved theoretically.

Keywords Delay differential equation · Parameterized problem · Singular perturbation · Piecewise-uniform mesh · Error estimates

1 Introduction

Consider the following singularly perturbed quasilinear delay differential problem depending on a parameter in the interval $\bar{I} = [0, T]$:

$$\varepsilon u'(t) + f(t, u(t), u(t-r), \lambda) = 0,\ t \in I = (0, T]\,,\ T > 0, \tag{1.1}$$

$$u(t) = \varphi(t), \quad t \in I_0, \quad u(T) = B, \tag{1.2}$$

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where $I = (0, T] = \bigcup_{p=1}^{m} I_p$, $I_p = \{t : r_{p-1} < t \leq r_p\}$, $1 \leq p \leq m$ and $r_s = sr$, for $0 \leq s \leq m$ and $I_0 = [-r, 0]$(for simplicity we suppose that T/r is integer; i.e., $T = mr$). $0 < \varepsilon \leq 1$ is the perturbation parameter and r is a constant delay, which is independent of ε. $\varphi(t)$ and $f(x, u, v, \lambda)$ are given sufficiently smooth functions satisfying certain regularity conditions in $\bar{I}$ and $\bar{I} \times \mathbb{R}^3$, respectively, and moreover,

$$0 < \alpha \leq \frac{\partial f}{\partial u} \leq a^*,$$

$$\left| \frac{\partial f}{\partial v} \right| \leq M_1^*,$$

$$m_1 \leq \left| \frac{\partial f}{\partial \lambda} \right| \leq M_1$$

By a solution of (1.1)–(1.2) we mean a pair $\{u(t), \lambda\} \in C^1(\bar{I}) \times \mathbb{R}$ for which problem (1.1)–(1.2) is satisfied. The function, $u(t)$, displays in general boundary layers on the right side of each point $t = r_s$ for small values of $\varepsilon (0 \leq s \leq m - 1)$ (see Section 2).

Multiple time scale is an important phenomenon in many physical and biological processes. Therefore the models describing these type of processes often appear to be singularly perturbed differential equations that share some common feature of the relaxation oscillation. Considered in this paper equation (1.1) has been serving as the model for many optical and biological problems (see [4–6, 8, 10, 11, 15–22] and references therein).

In [2, 3, 12–14, 22, 23, 25] have been considered some approximating aspects of first order delay differential equations. Problems with a parameter have also been considered for many years. For a discussion of existence and uniqueness results and for applications of parameterized equations see, [10, 15–17] and references therein. A number of papers devoted to the approximating techniques to initial and boundary value problems, see for example [1, 15, 17, 24]. But designed in the above-mentioned papers algorithms mainly focused on the stability of numerical methods to regular cases (i.e. when the boundary layers are absent). It is well known that standard discretization methods for solving singular perturbation problems are unstable and fail to give accurate results when the perturbation parameter ε is small. Therefore, it is important to develop suitable numerical methods to these problem, whose accuracy does not depend on the parameter value ε, i.e. methods that are convergent ε-uniformly [7, 9, 26]. One of the simplest ways to derive such methods consists of using a class of special piecewise uniform meshes (a Shishkin mesh) , (see, e.g., [9, 26] for motivation for this type of mesh), which are constructed a priori in function of sizes of parameter ε, the problem data and the number of corresponding mesh points.

In the present paper we discretize (1.1)–(1.2) using a numerical method, which is composed of an implicit finite difference scheme on piecewise-uniform

S-meshes on each time-subinterval. In Section 2, we state some important properties of the exact solution. In Section 3, we describe the finite difference discretization and introduce the piecewise uniform grid. In Section 4, we present the error analysis for approximate solution. Uniform convergence is proved in the discrete maximum norm. Finally, in Section 5, we formulate the iterative algorithm for solving the discrete problem and present numerical results which validate the theoretical analysis computationally. The technique to construct discrete problem and error analysis for approximate solution is similar to those in [1, 2].

Throughout the paper, C and c denote generic positive constants independent of ε and the mesh parameter. Some specific, fixed constants of this kind are indicated by their subscripting.

2 The continuous problem

Here we show some properties of the solution of (1.1)–(1.2), which are needed in later sections for the analysis of appropriate numerical solution. Let, for any continuous function g, $\|g\|_\infty$ denotes a continuous maximum norm on the corresponding interval, in particular we shall use $\|g\|_{\infty,p} = \max_{\bar{I}_p} |g(t)|$, $0 \leq p \leq m$.

Lemma 2.1 *Let*

$$\rho := m_1^{-1} M_1 \left(\left[1 + \alpha^{-1} M_1^* \right]^{m-1} - 1 \right) < 1.$$

Then the solution $\{u(t),\lambda\}$ of the problem (1.1)–(1.2) satisfies the following estimates

$$|\lambda| \leq C_0, \tag{2.1}$$

$$\|u(t)\|_{\infty,p} \leq C_p, \quad 1 \leq p \leq m, \tag{2.2}$$

where

$$C_0 = (1-\rho)^{-1} \left\{ \left(\frac{a^*}{m_1\,(1-\exp\,(-a^* T))} + m_1^{-1} M_1^* \left(1+\alpha^{-1} M_1^*\right)^{m-1} \right) \|\varphi\|_{\infty,0} + \frac{a^*}{m_1\,(1-\exp\,(-a^* T))} |B| + \left(m_1^{-1} + \frac{\left(1+\alpha^{-1} M_1^*\right)^{m-1} - 1}{M_1^*} \right) \|F\|_\infty \right\},$$

$$C_p = \|\varphi\|_\infty \left(1+\alpha^{-1} M_1^*\right)^p + \alpha^{-1} \sum_{s=1}^{p} \left(1+\alpha^{-1} M_1^*\right)^{p-s} \|F\|_{\infty,s} + \frac{\left(1+\alpha^{-1} M_1^*\right)^m - 1}{\alpha^{-1} M_1^*} M_1 C_0, \quad 1 \leq p \leq m,$$

$$F(t) = -f(t,0,0,0)$$

and

$$\left|u'(t)\right| \leq C \left\{1 + \frac{(t-r_{p-1})^{p-1}}{\varepsilon^p} \exp\left(-\frac{\alpha(t-r_{p-1})}{\varepsilon}\right)\right\}, \ t \in I_p, \ 1 \leq p \leq m, \quad (2.3)$$

provided $|\partial f/\partial t| \leq C$ *for* $t \in \bar{I}$ *and* $|\lambda| \leq C_0, \ |u|, |v| \leq C_m$.

Proof We rewrite (1.1) in the form

$$\varepsilon u'(t) + a(t)u(t) + b(t)u(t-r) = F(t) + \lambda c(t), \ t \in I \quad (2.4)$$

with

$$a(t) = \frac{\partial f}{\partial u}\left(t, \tilde{u}, \tilde{v}, \tilde{\lambda}\right),$$

$$b(t) = \frac{\partial f}{\partial v}\left(t, \tilde{u}, \tilde{v}, \tilde{\lambda}\right),$$

$$c(t) = -\frac{\partial f}{\partial \lambda}\left(t, \tilde{u}, \tilde{v}, \tilde{\lambda}\right),$$

$\tilde{u} = \gamma u, \ \tilde{v} = \gamma v, \tilde{\lambda} = \gamma\lambda (0 < \gamma < 1)$ −intermediate values.
From (2.4) we have

$$u(t) = \varphi(0) \exp\left(-\frac{1}{\varepsilon}\int_0^t a(\tau)\, d\tau\right) + \frac{1}{\varepsilon}\int_0^t \left[F(\tau) - b(\tau)\, u(\tau - r)\right] \exp\left(-\frac{1}{\varepsilon}\int_\tau^t a(\eta)\, d\eta\right) d\tau + \frac{1}{\varepsilon}\lambda\int_0^t c(\tau) \exp\left(-\frac{1}{\varepsilon}\int_\tau^t a(\eta)\, d\eta\right) d\tau,$$

from which, by setting the boundary condition $u(T) = B$, we get

$$\lambda = \frac{B - \varphi(0)\exp\left(-\frac{1}{\varepsilon}\int_0^T a(\tau)\, d\tau\right)}{\frac{1}{\varepsilon}\int_0^T c(\tau)\exp\left(-\frac{1}{\varepsilon}\int_\tau^T a(\eta)\, d\eta\right) d\tau} - \frac{\frac{1}{\varepsilon}\int_0^T \left[F(\tau) - b(\tau)\, u(\tau - r)\right]\exp\left(-\frac{1}{\varepsilon}\int_\tau^T a(\eta)\, d\eta\right) d\tau}{\frac{1}{\varepsilon}\int_0^T c(\tau)\exp\left(-\frac{1}{\varepsilon}\int_\tau^T a(\eta)\, d\eta\right) d\tau}. \quad (2.5)$$

Applying the mean value theorem for integrals, we deduce that

$$\frac{\left|\frac{1}{\varepsilon}\int_0^T \left[F(\tau) - b(\tau)\,u(\tau - r)\right] \exp\left(-\frac{1}{\varepsilon}\int_\tau^T a(\eta)\,d\eta\right) d\tau\right|}{\left|\frac{1}{\varepsilon}\int_0^T c(\tau)\exp\left(-\frac{1}{\varepsilon}\int_\tau^T a(\eta)\,d\eta\right) d\tau\right|}$$
$$\le m_1^{-1}\left(\|F\|_\infty + M_1^* \max_{0\le p\le m-1} \|u\|_{\infty,p}\right).$$

It then follows from (2.5) for $\varepsilon \le 1$ that

$$|\lambda| \le \frac{a^*}{m_1\left(1 - \exp(-a^* T)\right)}\left(|\varphi(0)| + |B|\right) + m_1^{-1}\left(\|F\|_\infty + M_1^* \max_{0\le p\le m-1}\|u\|_{\infty,p}\right). \tag{2.6}$$

Next from (2.4), the applying the maximum principle on I_p gives

$$\|u\|_{\infty,p} \le \left|u\left(r_{p-1}\right)\right| + \alpha^{-1}\left(\|b\|_{\infty,p}\|u\|_{\infty,p-1} + \|F\|_{\infty,p} + |\lambda|\,\|c\|_{\infty,p}\right)$$
$$\le \left(1 + \alpha^{-1}M_1^*\right)\|u\|_{\infty,p-1} + \alpha^{-1}(\|F\|_{\infty,p} + |\lambda|\,M_1),$$

which implies the first order difference inequality

$$w_p \le \mu w_{p-1} + \psi_p,$$

with

$$w_p = \|u\|_{\infty,p}, \quad \mu = 1 + \alpha^{-1}M_1^*, \quad \psi_p = \alpha^{-1}(\|F\|_{\infty,p} + |\lambda|\,M_1).$$

From the last inequality it follows that

$$w_p \le w_0\mu^p + \sum_{s=1}^{p}\mu^{p-s}\psi_s.$$

This inequality together with (2.6) leads to (2.1), (2.2).

After establishing the uniformly boundness in ε of $\|u(t)\|_\infty$and $|\lambda|$, the proof of (2.3) is almost identical to that of [2]. □

3 The difference scheme and mesh

Let $\bar{\omega}_{N_0}$ be any non-uniform mesh on $\bar{I}$:

$$\bar{\omega}_{N_0} = \left\{0 = t_0 < t_1 < \ldots < t_{N_0} = T, \tau_i = t_i - t_{i-1}\right\}$$

which contains by N mesh points at each subinterval $I_p (1 \leq p \leq m)$:

$$\omega_{N,p} = \{t_i : (p-1)N + 1 \leq i \leq pN\}, \quad 1 \leq p \leq m-1,$$
$$\omega_{N,m} = \{t_i : (m-1)N + 1 \leq i \leq N_0 = mN\},$$

and consequently

$$\omega_{N_0} = \bigcup_{p=1}^{m} \omega_{N,p}.$$

To simplify the notation we set $g_i = g(t_i)$ for any function $g(t)$, while g_i^N denotes an approximation of $g(t)$ at t_i. For any mesh function $\{w_i\}$ defined on ϖ_{N_0} we use

$$w_{\bar{t},i} = (w_i - w_{i-1})/\tau_i \ ,$$

$$\|w\|_{\infty,N,p} = \|w\|_{\infty,\omega_{N,p}} := \max_{(p-1)N \leq i \leq pN} |w_i| \, .$$

For the difference approximation to (1.1), we integrate (1.1) over (t_{i-1}, t_i) :

$$\varepsilon u_{\bar{t},i} + \tau^{-1} \int_{t_{i-1}}^{t_i} f(t, u(t), u(t-r), \lambda) dt = 0,$$

which yields the relation

$$\varepsilon u_{\bar{t},i} + f(t_i, u_i, u_{i-N}, \lambda) + R_i = 0, \qquad 1 \leq i \leq N_0, \tag{3.1}$$

with the local truncation error

$$R_i = -\tau_i^{-1} \int_{t_{i-1}}^{t_i} (t - t_{i-1}) \frac{d}{dt} f(t, u(t), u(t-r), \lambda) dt. \tag{3.2}$$

As a consequence of the (3.1), we propose the following difference scheme for approximating (1.1)–(1.2)

$$\varepsilon u_{\bar{t},i}^N + f\left(t_i, u_i^N, u_{i-N}^N, \lambda^N\right) = 0, \ \ 1 \leq i \leq N_0, \tag{3.3}$$

$$u_i^N = \varphi_i \ , \ \ -N \leq i \leq 0, \ u_N^N = B. \tag{3.4}$$

The difference scheme (3.3)–(3.4), in order to be ε- uniform convergent, we will use the Shishkin mesh. For the even number N, the piecewise uniform mesh $\omega_{N,p}$ divides each of the interval $[r_{p-1}, \sigma_p]$ and $[\sigma_p, r_p]$ into $N/2$ equidistant subinterval, where the transition point σ_p, which separates the fine and coarse portions of the mesh is obtained by

$$\sigma_p = r_{p-1} + \min\left\{r/2, \alpha^{-1}\theta_p \varepsilon \ln N\right\}, \ \text{for } 1 \leq p \leq m-1, \tag{3.5}$$

where $\theta_1 \geq 1$ and $\theta_p > 1$ $(2 \leq p \leq m)$ are some constants. Hence, if denote by $\tau_p^{(1)}$and $\tau_p^{(2)}$the stepsizes in $[r_{p-1}, \sigma_p]$ and $[\sigma_p, r_p]$ respectively, we have

$$\tau_p^{(1)} = 2\left(\sigma_p - r_{p-1}\right) N^{-1}, \quad \tau_p^{(2)} = 2\left(r_p - \sigma_p\right) N^{-1}, \quad 1 \leq p \leq m,$$

so

$$\bar{\omega}_{N,p} = \left\{ \begin{array}{l} t_i = r_{p-1} + (i - (p-1)N)\tau_p^{(1)}, i = (p-1)N, ..., (p-1/2)N; \\ t_i = \sigma_p + (i - (p-1/2)N)\tau_p^{(2)}, i = (p-1/2)N + 1, ..., pN \end{array} \right\},$$
$$1 \leq p \leq m.$$

In the rest of the paper we only consider this type of mesh.

4 Analysis of the scheme

To investigate the convergence of the method, note that the error functions $z_i^N = u_i^N - u_i$, $0 \leq i \leq N_0$, $\mu^N = \lambda^N - \lambda$ are the solution of the discrete probl

$$\varepsilon z_{\bar{t},i}^N + f\left(t_i, z_i^N + u_i, z_{i-N}^N + u_{i-N}, \mu^N + \lambda\right) - f(t_i, u_i, u_{i-N}, \lambda) = R_i, \ 1 \leq i \leq N_0, \quad (4.1)$$

$$z_i^N = 0, \quad -N \leq i \leq 0, \quad z_N^N = 0, \ fort_i \in I_p (p > 1),$$

where the truncation error R_i is given by (3.2).

Lemma 4.1 *Under the above smoothness conditions of Section 1, for the error function R, the following estimate holds*

$$\|R\|_{\infty, \omega_N, p} \leq CN^{-1} \ln N, \quad 1 \leq p \leq m.$$

Proof From explicit expression (3.2) for R_i, on an arbitrary mesh we have

$$|R_i| \leq h_i^{-1} \int_{t_{i-1}}^{t_i} (t - t_{i-1}) \left| \frac{\partial f}{\partial t} + \frac{\partial f}{\partial u} u'(t) + \frac{\partial f}{\partial v} u'(t-r) \right| dt, \quad 1 \leq i \leq N_0.$$

This inequality together with (2.1), (2.2), enables us to write

$$|R_i| \leq C \left\{ \tau_i + \int_{t_{i-1}}^{t_i} \left(\left|u'(t)\right| + \left|u'(t-r)\right| \right) dt \right\}, \quad 1 \leq i \leq N_0.$$

From here, in view of (2.3) it follows that

$$|R_i| \leq C \left\{ \tau_i + \frac{1}{\varepsilon} \int_{t_{i-1}}^{t_i} e^{-\frac{\alpha t}{\varepsilon}} dt \right\}, \ for\ 1 \leq i \leq N$$

and

$$|R_i| \leq C \left\{ \tau_i + \int_{t_{i-1}}^{t_i} \frac{(t-r_{p-1})^{p-1}}{\varepsilon^p} e^{-\frac{\alpha(t-r_{p-1})}{\varepsilon}} dt + \int_{t_{i-1}}^{t_i} \frac{(t-r_{p-1})^{p-2}}{\varepsilon^{p-1}} e^{-\frac{\alpha(t-r_{p-1})}{\varepsilon}} dt \right\},$$

The further part of the proof is similar to that of [2]. □

Lemma 4.2 *Under the above assumptions of Section 1 and Lemma 2.1, for the pair* $\{z_i^N, \mu^N\}$, *the following estimates hold*

$$\left\| z^N \right\|_{\infty,N,p} \leq C \sum_{k=1}^{p} \| R \|_{\infty,\omega_{N,k}}, \tag{4.2}$$

$$\left| \mu^N \right| \leq C \| R \|_{\infty,\omega_{N_0}}. \tag{4.3}$$

Proof The equation (4.1) can be written as

$$\varepsilon z_{\bar{t},i}^N + a_i z_i^N = b_i z_{i-N}^N + c_i \mu^N + R_i, \quad 1 \leq i \leq N_0 - 1, \tag{4.4}$$

where

$$a_i = \frac{\partial f}{\partial u} \left(t_i, u_i + \gamma z_i^N, u_{i-N} + \gamma z_{i-N}^N, \lambda + \gamma \mu^N \right),$$

$$b_i = -\frac{\partial f}{\partial v} \left(t_i, u_i + \gamma z_i^N, u_{i-N} + \gamma z_{i-N}^N, \lambda + \gamma \mu^N \right),$$

$$c_i = -\frac{\partial f}{\partial \lambda} \left(t_i, u_i + \gamma z_i^N, u_{i-N} + \gamma z_{i-N}^N, \lambda + \gamma \mu^N \right),$$

$$0 < \gamma < 1.$$

Hence

$$z_i^N = \frac{\varepsilon}{\varepsilon + a_i h_i} z_{i-1}^N + \frac{h_i b_i}{\varepsilon + h_i a_i} z_{i-N}^N + \frac{h_i c_i}{\varepsilon + h_i a_i} \mu^N + \frac{h_i R_i}{\varepsilon + a_i h_i}.$$

Solving the first-order difference equation with respect to z_i^N and setting the initial condition $z_0^N = 0$, we get

$$z_i^N = \sum_{k=1}^{i} \frac{h_k b_k z_{k-N}^N}{\varepsilon + h_k a_k} Q_{i-k} + \mu^N \sum_{k=1}^{i} \frac{h_k c_k}{\varepsilon + h_k a_k} Q_{i-k} + \sum_{k=1}^{i} \frac{h_k R_k}{\varepsilon + h_k a_k} Q_{i-k},$$

where

$$Q_{i-k} = \begin{cases} 1 & , \ k = i, \\ \prod_{j=k+1}^{i} \frac{\varepsilon}{\varepsilon + a_j h_j}, & 0 \leq k \leq i-1. \end{cases}$$

For $i = N$, taking into consideration that $z_N^N = 0$, we have

$$\mu^N = \frac{-\sum_{k=1}^{N} \frac{h_k b_k z_{k-N}^N}{\varepsilon + h_k a_k} Q_{N-k} - \sum_{k=1}^{N} \frac{h_k R_k}{\varepsilon + h_k a_k} Q_{N-k}}{\sum_{k=1}^{N} \frac{h_k c_k}{\varepsilon + h_k a_k} Q_{N-k}}.$$

Now, since $\varepsilon + a_i \tau_i > 0 (1 \le i \le N_0)$, by analogy with the proof of Lemma 2.1, we can obtain

$$\left|\mu^N\right| \le m_1^{-1} \left(\|R\|_{\infty,\omega_{N_0}} + M_1^* \max_{1 \le p \le m-1} \|z\|_{\infty,N,p} \right) \tag{4.5}$$

Next, the applying discrete maximum principle for the difference operator $\ell_N z_i := \varepsilon z_{\bar{t},i} + a_i z_i$, $(p-1)N < i \le pN$, to (4.4) yields

$$\begin{aligned} \|z\|_{\infty,p} &\le \left|z_{p-1}\right| + \alpha^{-1} \left(\|b\|_{\infty,p} \|z\|_{\infty,p-1} + \|R\|_{\infty,p} + \left|\mu^N\right| \|c\|_{\infty,p} \right) \\ &\le \left(1 + \alpha^{-1} M_1^*\right) \|z\|_{\infty,p-1} + \alpha^{-1} \left(\|R\|_{\infty,p} + \left|\mu^N\right| M_1 \right), \end{aligned}$$

therefore

$$\|z\|_{\infty,p} \le \alpha^{-1} \sum_{s=1}^{p} \left(1 + \alpha^{-1} M_1^*\right)^{p-s} \left(\|R\|_{\infty,p} + \left|\mu^N\right| M_1 \right), \quad 1 \le p \le m. \tag{4.6}$$

From (4.5) and (4.6) we easily obtain (4.2)–(4.3).

Combining the previous lemmas gives us the following convergence result. □

Theorem 4.3 *Let $\{u(t), \lambda\}$ be the solution of (1.1)–(2.2) and $\left\{u_i^N, \lambda^N\right\}$ the solution of (3.3)–(3.4). Then the following estimates hold*

$$\begin{gathered} \left|\lambda^N - \lambda\right| \le C N^{-1} \ln N, \\ \left\|u^N - u\right\|_{\infty,\bar{\omega}_{N,p}} \le C N^{-1} \ln N, \ 1 \le p \le m. \end{gathered}$$

5 Numerical results

In this section, we present some numerical experiments in order to illustrate the method described above.

We solve the nonlinear problem (3.3)–(3.4) using the following quasi-linearization technique:

$$\lambda^{(n)} = \lambda^{(n-1)} - \frac{\left(B - u_{N_0-1}^{(n-1)}\right)\rho_{N_0}^{-1} + f\left(T, B, u_{N_0-N}^{(n-1)}, \lambda^{(n-1)}\right)}{\frac{\partial f}{\partial \lambda}\left(T, B, u_{N_0-N}^{(n-1)}, \lambda^{(n-1)}\right)},$$

$$u_i^{(n)} = u_i^{(n-1)} - \frac{\left(u_i^{(n-1)} - u_{i-1}^{(n)}\right)\rho_i^{-1} + f\left(t_i, u_i^{(n-1)}, u_{i-N}^{(n)}, \lambda^{(n)}\right)}{\frac{\partial f}{\partial u}\left(t_i, u_i^{(n-1)}, u_{i-N}^{(n)}, \lambda^{(n)}\right) + \rho_i^{-1}},$$

$$i = 1, 2, ..., N_0;\ n = 1, 2, ...$$

$$u_i^{(n)} = \varphi_i,\ \ -N \le i \le 0,$$

where $\rho_i = h_i/\varepsilon$;$\lambda^{(0)}, u_i^{(0)}$ given.

Consider the test problem

$$\varepsilon u' + 2u - e^{-u} + t^2 + \lambda + \tanh(\lambda + x) + \frac{1}{2}u(t-1) = 0,\ 0 < t < 2$$

$$u(t) = 1,\ \ -1 \le t \le 0,\ u(2) = 0.$$

In the computations in this section we take $\alpha = 2$.The initial guess in the iteration process is taken as $u_i^{(0)} = 1 - x_i^2$, $\lambda^{(0)} = -0.4$ and the stopping criterion is

$$\max_i \left|u_i^{(n)} - u_i^{(n-1)}\right| \le 10^{-5},\ \ \left|\lambda^{(n)} - \lambda^{(n-1)}\right| \le 10^{-5}.$$

The exact solution of our test problem is not available. Therefore we use the double mesh principle to estimate the errors and compute the experimental rates of convergence in our computed solution, i.e. we compare the computed solution with the solution on a mesh that is twice as fine (see [1, 9]). The error estimates obtained in this way are denoted by

$$e_u^{\varepsilon,N,p} = \max_{\omega_{N,p}} \left|u^{\varepsilon,N} - \tilde{u}^{\varepsilon,2N}\right|,\ \ p = 1, 2,$$

$$e_\lambda^{\varepsilon,N} = \left|\lambda_\varepsilon^N - \tilde{\lambda}_\varepsilon^{2N}\right|,$$

where $\left\{\tilde{u}^{\varepsilon,2N}, \tilde{\lambda}_\varepsilon^{2N}\right\}$ is the approximate solution on a mesh which contains the mesh points of the original mesh $t_i \in \omega_{N_0}$ and also the mesh midpoints $t_{i+\frac{1}{2}} = (t_i + t_{i+1})/2, i = 0, 1, ..., N_0 - 1$.

The convergence rates are

$$r_u^{\varepsilon,N,p} = \ln\left(e_u^{\varepsilon,N,p}/e_u^{\varepsilon,2N,p}\right)/\ln 2$$

for u, and

$$r_\lambda^{\varepsilon,N} = \ln\left(e_\lambda^{\varepsilon,N}/e_\lambda^{\varepsilon,2N}\right)/\ln 2$$

Table 1 Errors $e_u^{\varepsilon,N}$, computed ε-uniform errors e_u^N and convergence rates r_u^N for $\theta_1 = 1$ on $\omega_{N,1}$

ε	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
2^{-2}	0.0112059	0.0056744	0.0028555	0.00143237	0.00071735
	0.98	0.99	0.99	0.99	
2^{-4}	0.0226994	0.0135156	0.0078274	0.00443734	0.00247624
	0.75	0.79	0.82	0.84	
2^{-6}	0.0226994	0.0135156	0.0078274	0.00443734	0.00247624
	0.75	0.79	0.82	0.84	
2^{-8}	0.0226994	0.0135156	0.0078274	0.00443734	0.00247624
	0.75	0.79	0.82	0.84	
2^{-10}	0.0226994	0.0135156	0.0078274	0.00443734	0.00247624
	0.75	0.79	0.82	0.84	
2^{-12}	0.0226994	0.0135156	0.0078274	0.00443734	0.00247624
	0.75	0.79	0.82	0.84	
2^{-14}	0.0226994	0.0135156	0.0078274	0.00443734	0.00247624
	0.75	0.79	0.82	0.84	
2^{-16}	0.0226994	0.0135156	0.0078274	0.00443734	0.00247624
	0.75	0.79	0.82	0.84	
e_u^N	0.0226994	0.0135156	0.0078274	0.00443734	0.00247624
r_u^N	0.75	0.79	0.82	0.84	

for λ. Approximations to the ε-uniform rates of convergence are estimated from

$$e_u^{N,p} = \max_{\varepsilon} e_u^{\varepsilon,N,p},$$

$$e_\lambda^N = \max_{\varepsilon} e_\lambda^{\varepsilon,N}.$$

Table 2 Errors $e_u^{\varepsilon,N}$, computed ε-uniform errors e_u^N and convergence rates r_u^N for $\theta_1 = 1, \theta_2 = 1.001$ on $\omega_{N,2}$

ε	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
2^{-2}	0.0095512	0.0041911	0.0027603	0.0017692	0.0005193
	0.78	0.82	0.92	0.95	
2^{-4}	0.0096645	0.0058090	0.0034149	0.0019899	0.0011661
	0.73	0.77	0.78	0.77	
2^{-6}	0.0101590	0.0058090	0.0034149	0.0019899	0.0011661
	0.80	0.77	0.78	0.77	
2^{-8}	0.0096644	0.0058090	0.0034149	0.0019899	0.0011661
	0.73	0.77	0.78	0.77	
2^{-10}	0.0096644	0.0058090	0.0034149	0.0019899	0.0011661
	0.73	0.77	0.78	0.77	
2^{-12}	0.0096644	0.0058090	0.0034149	0.0019899	0.0011661
	0.73	0.77	0.78	0.77	
2^{-14}	0.0096644	0.0058090	0.0034149	0.0019899	0.0011661
	0.73	0.77	0.78	0.77	
2^{-16}	0.0096644	0.0096644	0.0034149	0.0019899	0.0011661
	0.73	0.77	0.78	0.77	
e_u^N	0.0101590	0.0058090	0.0034149	0.0019899	0.0011661
r_u^N	0.73	0.77	0.78	0.77	

Table 3 Errors $e_\lambda^{\varepsilon,N}$, computed ε-uniform errors e_λ^N and convergence rates r_λ^N for $\theta_1 = 1, \theta_2 = 1.001$ on ω_{N_0}

ε	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
2^{-2}	0.0184354	0.0109542	0.0095513	0.0041912	0.0027604
	0.75	0.78	0.83	0.92	
2^{-4}	0.0222332	0.0138783	0.0086071	0.0054929	0.0031385
	0.68	0.69	0.75	0.81	
2^{-6}	0.0222332	0.0138806	0.0086145	0.0055013	0.0031443
	0.68	0.69	0.75	0.81	
2^{-8}	0.0222314	0.0138807	0.0086141	0.0055010	0.0031443
	0.68	0.69	0.75	0.81	
2^{-10}	0.0222315	0.0138806	0.0086141	0.0055010	0.0031443
	0.68	0.69	0.75	0.81	
2^{-12}	0.0222314	0.0138806	0.0086141	0.0055010	0.0031443
	0.68	0.69	0.75	0.81	
2^{-14}	0.0222314	0.0138807	0.0086141	0.0055010	0.0031443
	0.68	0.69	0.75	0.81	
2^{-16}	0.0222315	0.0138807	0.0086141	0.0055010	0.0031443
	0.68	0.69	0.75	0.81	
e_λ^N	0.0222332	0.0138807	0.0095513	0.0055013	0.0031443
r_λ^N	0.68	0.69	0.75	0.81	

The corresponding ε-uniform convergence rates are

$$r_u^{N,p} = \ln\left(e_u^{N,p}/e_u^{2N,p}\right)/\ln 2,$$

$$r_\lambda^N = \ln\left(e_\lambda^N/e_\lambda^{2N}\right)/\ln 2.$$

The resulting errors e_ε^N and the corresponding numbers e_ε^N for particular values of ε, N, are listed in the Tables 1, 2, and 3. It can be observed that they are essentially in agreement with the theoretical analysis described above.

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